Deterministic and stochastic surface growth with generalized nonlinearity

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The scaling behavior of the interface width for the generalized Kardar-Parisi-Zhang (KPZ) equation $\partial h/\partial t = \nu\nabla^2 h + \lambda|\nabla h|^\mu + \eta(x,t)$ is studied in two dimensions as a function of μ with and without the additive noise term η. In the case of additive noise, the scaling of the surface width is found for all μ to be the same as for the ordinary KPZ equation ($\mu=2$) in contrast to a previous conjecture. This appears to be due to the combined action of the nonlinearity $|\nabla h|^\mu$ and the noise under renormalization, which together induce a $|\nabla h|^2$ term. For the deterministic case corresponding to the smoothing of an initially rough interface with roughness exponent α, good agreement with the scaling relation $z=\min[2,\alpha(1-\mu)+\mu]$ is obtained for $\mu \geq 1$. However, for $\mu<1$, an instability is observed, which leads to a fluctuating grooved surface. For an asymptotically large system, the roughness exponent is 1 and the growth exponent is approximately equal to $\frac{1}{2}$. The evolution of two slightly different surfaces is studied in order to determine a Lyapunov exponent characterizing this instability.

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I. INTRODUCTION

The central feature of many growth processes is the formation of a surface which evolves in time [1]. The spatiotemporal characteristics of a growing surface may play an important role in determining bulk properties as well. For this reason there is considerable interest in understanding the dynamics and the morphology of growing surfaces and interfaces [1].

In addition to extensive computer simulations of simple models, the morphology of surface growth has been studied experimentally in a wide variety of systems, including recrystallization of amorphous semiconductor films [2], thin-film growth by molecular-beam epitaxy [3,4], vapor deposition [5,6], and sputtering [7], two-fluid displacement in porous media [8,9], settling of granular material [10], biological growth [11,12], tearing [13], and burning [14] of paper, and electrochemical deposition [15]. Much of the progress in describing surface growth has been based on the observation that surface fluctuations exhibit scaling behavior in both time and space. In particular, assuming an initially flat interface, the scaling of the interface width $w(L,t)$ on length scale L at time t is expected to be of the form [16], $w(L,t)=L^\alpha f(t/L^z)$, where $f(x)\sim x^\beta$ for $x \ll 1$, $f(x)\to$const for $x \gg 1$, and $z=\alpha/\beta$. This scaling behavior has been found in simulations of a wide variety of surface growth models [16–19] as well as experiments [2–14].

From the theoretical point of view the main approach for describing the growth of surfaces and interfaces is based on coarse-grained Langevin-type equations [20,21]. In this approach, the interface fluctuations are assumed to depend on an interplay between smoothing effects and random noise which tends to roughen the surface. The simplest such equation, appropriate for a fluctuating interface in equilibrium (also studied as a model of random surface deposition with diffusion under gravity), is the Edwards-Wilkinson equation [20],

$$\frac{\partial h}{\partial t} = \nu\nabla^2 h + \eta(\mathbf{x},t) , \tag{1}$$

where $h(\mathbf{x},t)$ is the interface height (in d dimensions) above a $(d-1)$-dimensional plane, the Laplacian represents a surface-tension term which tends to smooth the surface, and the roughening is caused by the noise $\eta(\mathbf{x},t)$, which is typically assumed to be δ-function correlated in space and time, i.e., $\langle \eta(\mathbf{x},t)\eta(\mathbf{x}',t')\rangle = D\delta^{d-1}(\mathbf{x}-\mathbf{x}')\delta(t-t')$. This equation is linear, and the surface width scaling exponents may be shown to have the form $\alpha=(3-d)/2$, $\beta=(3-d)/4$, and $z=2$, which implies in $d=2$, $\alpha=\frac{1}{2}$, and $\beta=\frac{1}{4}$.

For the case of a growing surface, Kardar, Parisi, and Zhang (KPZ) [21] have argued that due to sideways growth, the correct equation at large length scales should also include a term proportional to the square of the local gradient. This leads to a nonlinear interface equation of the form

$$\frac{\partial h}{\partial t} = \nu\nabla^2 h + \lambda|\nabla h|^2 + \eta(\mathbf{x},t) . \tag{2}$$

The nonlinear term may also be explained as the first relevant term in an analytic expansion of the dependence of growth rate on the local tilt. In two dimensions $(d=2)$ the scaling exponents for the KPZ equation have been shown from a renormalization-group analysis [21,22] and symmetry arguments [23] to be $\alpha=\frac{1}{2}$, $\beta=\frac{1}{3}$, $z=\frac{3}{2}$. In three and higher dimensions the exponents are not exactly known, however, the general scaling relation $\alpha+z=2$ holds [19,21,24].

Scaling results from computer simulations of a number of simple stochastic growth models [16,19,25,26] are in excellent agreement with those obtained for the KPZ equation [21,22,27]. This supports the assumption of sideways growth made by Kardar, Parisi, and Zhang. However, in general surface growth processes the dependence of the local growth velocity on tilt is expected to be different from the quadratic form assumed by KPZ. Therefore, it is of interest to investigate the scaling behavior of surface growth equations in which a more general slope dependence is included.

Recently, Krug and Spohn [28] considered a generalized form of Eq. (2) without additive noise of the form

$$\frac{\partial h}{\partial t} = \nu \nabla^2 h + \lambda |\nabla h|^\mu , \quad (3)$$

where $\mu \geq 1$. They argued that such an equation might be appropriate to describe the smoothing of an initially rough interface under deterministic growth. For the case in which the initial interface has roughness exponent α they used scaling arguments to derive the scaling relation

$$z = \min[2, \mu(1-\alpha)+\alpha] . \quad (4)$$

This result can also be obtained by equating the scaling [24,29] of the $|\nabla h|^\mu$ term with the $\partial h/\partial t$ term under the scale change $h \to b^\alpha h$, $x \to bx$, and $t \to b^z t$, and is consistent with the KPZ scaling relation $\alpha + z = 2$ for $\mu = 2$. In addition, it implies the existence of a critical value $\mu_c = (2-\alpha)/(1-\alpha)$, beyond which $z = 2$ and the nonlinearity becomes irrelevant. For $\alpha = \frac{1}{2}$, relation (4) predicts $\mu_c = 3$.

In order to test this relation, Krug and Spohn [28] studied several deterministic discrete surface growth models in two dimensions. They argued that for these models the appropriate continuum description is Eq. (3) with $\mu = 1$. Starting from an initially rough interface with $\alpha = \frac{1}{2}$, they obtained $z = 1$ from their simulations, in agreement with (4). However, no attempt was made to test relation (4) for general μ. In addition, the stochastic version of Eq. (3) with additive noise has not been studied for $\mu \neq 2$. Recently Wolf [30] has argued that the scaling relation (4) should hold in the stochastic case as well. Accordingly, we decided to directly study Eq. (3) with and without the addition of noise, as a function of μ.

In this paper we present the results of a systematic study of a surface growth equation [see Eq. (5) below] with a generalized nonlinear dependence on tilt. We consider both the case of stochastic growth (with additive noise) as well as the case of deterministic growth starting from an initially rough interface. In Sec. II we define the model in more detail and describe our numerical solution technique. The results of numerical integration of the stochastic equation are described in Sec. III while our results for the deterministic equation are presented in Sec. IV. In the deterministic case, an interesting instability is found for $\mu < 1$. This instability is further discussed in the context of a Lyapunov exponent in Sec. V. In Sec. VI, we summarize our results and present our conclusions.

II. SOLUTION OF GENERALIZED NONLINEAR EQUATION

We have numerically solved the generalized nonlinear equation

$$\frac{\partial h}{\partial t} = \nu \nabla^2 h + \lambda |\nabla h|^\mu + \eta(x,t) , \quad (5)$$

in $d = 2$ dimensions, for $\frac{1}{2} \leq \mu \leq 4$. In our simulations Eq. (5) was integrated using a finite-difference scheme similar to that used in previous numerical studies of the KPZ equation [27,31]. Using a lattice with grid spacing $\Delta x = 1$ and system size L and assuming periodic boundary conditions, we rewrite (5) in the discrete representation

$$\begin{aligned} h(i,t+1) = {} & h(i,t) + \nu\Delta t[h(i+1,t) - 2h(i,t) + h(i-1,t)] \\ & + \lambda\Delta t \left[\frac{|h(i+1,t) - h(i-1,t)|}{2} \right]^\mu \\ & + \sqrt{2d\Delta t}\, \xi(i,t) . \end{aligned} \quad (6)$$

For the stochastic case, $\xi(i,t)$ was taken to be an independent random variable with either a uniform or a Gaussian distribution, and with unit strength, while for the deterministic case $\xi(i,t) = 0$. The time step Δt was decreased until the results were essentially independent of Δt.

In the stochastic case corresponding to the roughening of an interface, Eq. (5) was integrated with additive noise starting from a flat interface until saturation, and the scaling exponents α and β were determined from the scaling of the interface width. To study the deterministic case, we began with an additive noise and integrated (5) until the surface reached the steady state. We then "turned off" the noise ($\xi = 0$) and continued the integration in order to study the smoothing behavior of the interface and determine the dynamic exponent z for this case. Averages were taken over many (40–60) runs and for several system sizes L. In order to study the behavior for $\mu < 1$ and beyond μ_c, roughening and smoothing of the interface was studied for values of μ equal to $\frac{1}{2}$, 1, 2, 3, and 4. Pictures of the interface profile during smoothing were also obtained.

For both the stochastic and the deterministic case, the exponents α and z were determined from scaling plots of the form $w(L,t)/L^\alpha$ versus t/L^z for different values of L. For the stochastic case, the exponents α and β were also determined by studying the scaling of the correlation function $G(x) = \langle [h(x) - h(0)]^2 \rangle$ (which should scale as $x^{2\alpha}$) as well as the early-time behavior of the width ($w \sim t^\beta$) for very large system sizes $L = 65\,536$–$131\,072$. These results were always in agreement with our scaling function plots.

III. STOCHASTIC GROWTH

Figures 1 and 2 show our results for the scaling function $f(t/L^z)$ for the surface width, obtained from numerical integration of (5) with additive noise, for $\mu = 1$ and $\mu = 3$, respectively. In both cases we find, somewhat surprisingly, good agreement with the KPZ values $\alpha = \frac{1}{2}$, $\beta = \frac{1}{3}$, $z = \frac{3}{2}$, which correspond to $\mu = 2$. Simulations were also conducted for both values of μ for large system sizes, with slightly lower values of λ than shown in Figs. 1 and

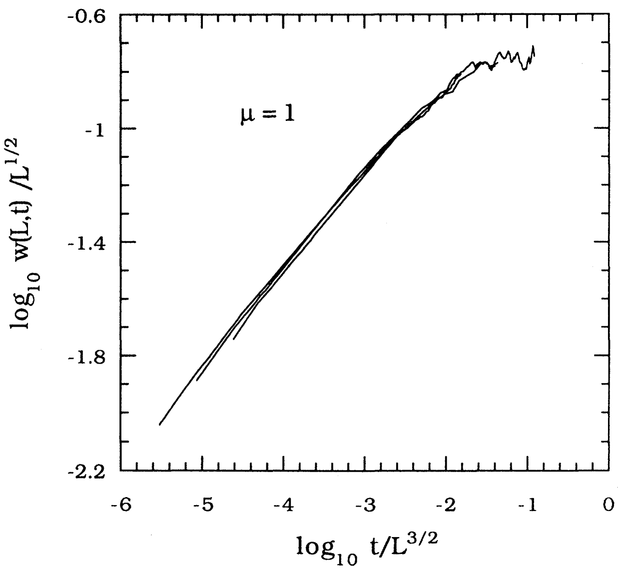

FIG. 1. Scaling plot for surface width $w(L,t)$ in stochastic case with $\mu=1$, L $-$256, 512, and 1024, and $z=\frac{3}{2}$. Integration parameters are $\nu=1.0$, $D=0.5$, $\Delta t=0.005$, and $\lambda=5.0$. Slope of fit in early-time region is $\beta=0.31\pm0.01$.

2, and a slow crossover from Edwards-Wilkinson behavior ($\beta=\frac{1}{4}$) to KPZ behavior ($\beta=\frac{1}{3}$) at much later times was observed.

We now consider the case $\mu=4$ with additive noise. This case is particularly interesting, since it corresponds to the next term after the $|\nabla h|^2$ term in an analytic expansion of the dependence of the growth velocity on tilt (assuming an even function). Consequently, it might be expected to yield a new universality class corresponding

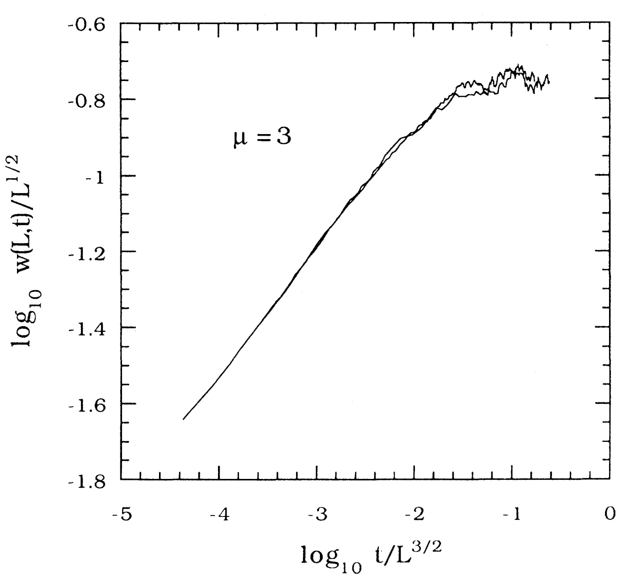

FIG 2. Scaling plot for $\mu=3$ with additive noise for $L=256$ and $L=512$, and $z=\frac{3}{2}$. Integration parameters are $\nu=1.0$, $D=0.5$, $\Delta t=0.005$, and $\lambda=3.0$. Slope of fit in early-time region is $\beta=0.34\pm0.01$.

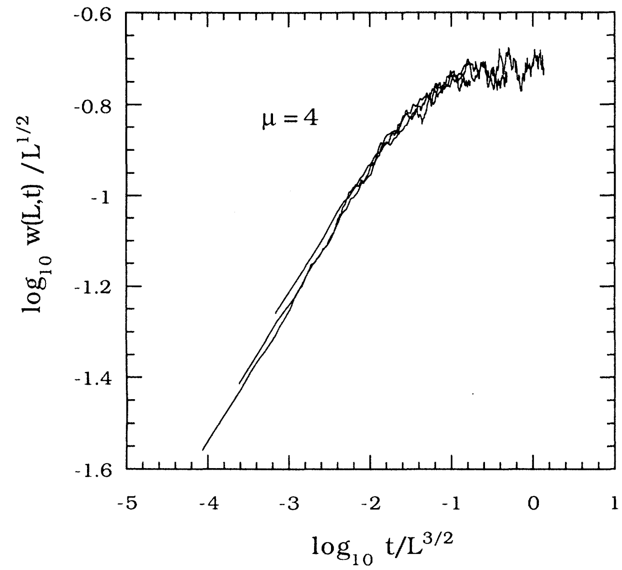

FIG. 3. Scaling plot for $\mu=4$ with additive noise for $L=128$, 256, and 512, and $z=\frac{3}{2}$. Integration parameters are $\nu=1.0$, $D=0.5$, $\Delta t=0.001$, and $\lambda=1.0$. Linear fit at early time has slope $\beta=0.32\pm0.02$.

to the case where the $|\nabla h|^2$ term is zero. Simple power-counting arguments imply that for $\mu>3$, the $|\nabla h|^\mu$ term becomes irrelevant and therefore one expects Edwards-Wilkinson ($\alpha=\frac{1}{2}$, $\beta=\frac{1}{4}$, $z=2$) behavior.

The case $\mu=4$ is relatively difficult to integrate, due to the high order of the nonlinearity. However, for an intermediate value of λ ($\lambda=1.0$) we found, in contrast to the power-counting arguments, that the scaling behavior was the same as for the KPZ equation as shown in Fig. 3.

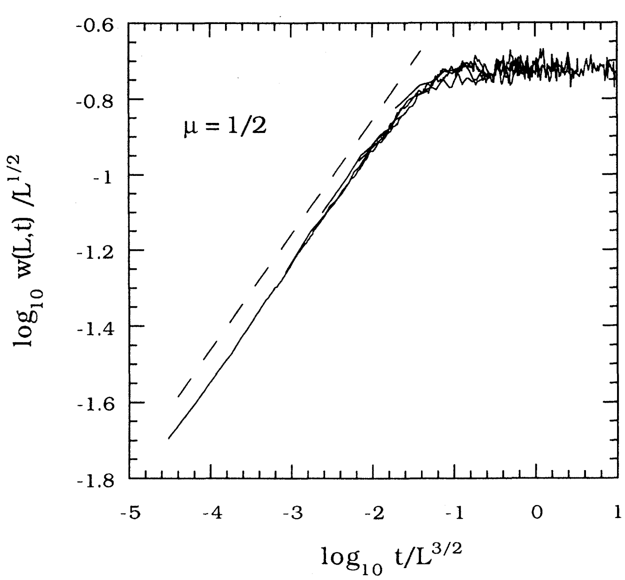

FIG. 4. Scaling plot for $\mu=\frac{1}{2}$ with additive noise for $L=64-1024$ in multiples of 2, and $z=\frac{3}{2}$. Integration parameters are $\nu=1.0$, $D=0.5$, $\Delta t=0.005$, and $\lambda=2.65$. Dashed line has slope of $\frac{1}{3}$.

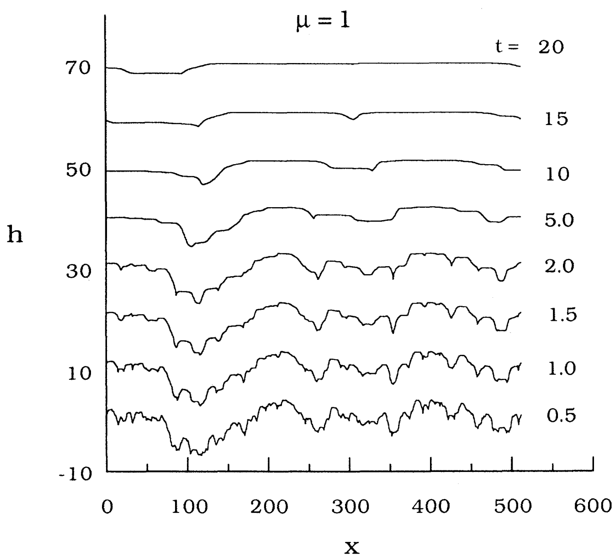

FIG. 5. Pictures of interface during smoothing at eight different times (starting from a fully roughened interface at $t=0$) for $\mu=1$. System size is $L=512$. Interfaces have been shifted for clarity.

For smaller values of λ we obtained somewhat smaller values for β (intermediate between $\frac{1}{4}$ and $\frac{1}{3}$), which are most likely due to a slow crossover to KPZ behavior. For somewhat larger values of λ, the integration was found to be numerically unstable. Thus, we conclude that for $\mu=4$, the scaling behavior of the interface width is the same as for the KPZ case.

Finally, we consider the stochastic case with $\mu=\frac{1}{2}$. For this case we again find, as shown in Fig. 4, KPZ scaling exponents for the interface width. (As we discuss below, this case becomes unstable when the additive noise is removed.) Thus for all values of μ with additive noise, we find KPZ scaling behavior for the interface width.

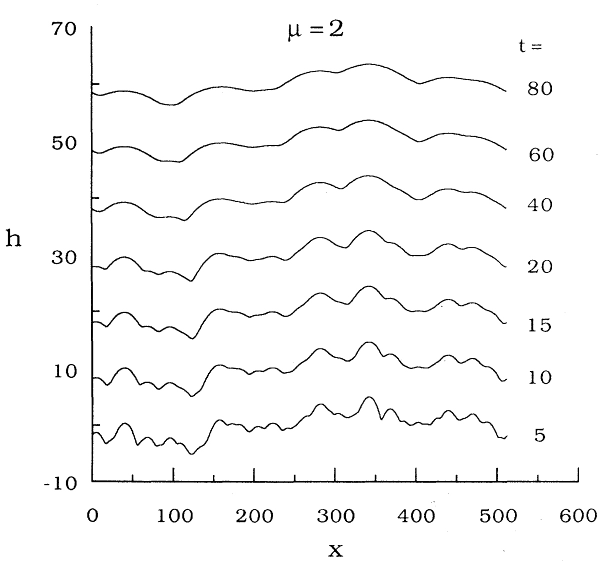

FIG. 6. Pictures of interface during smoothing for $\mu=2$.

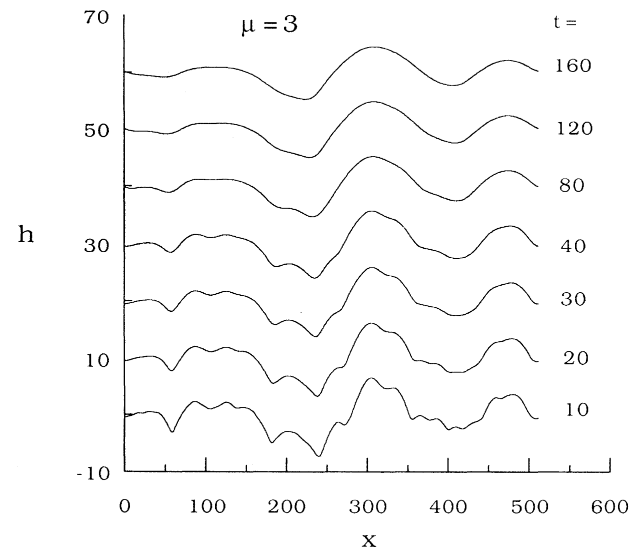

FIG. 7. Pictures of interface during smoothing for $\mu=3$.

IV. DETERMINISTIC GROWTH

We now turn to the generalized KPZ equation without additive noise. Figure 5 shows a picture of the smoothing of the interface for $\mu=1$, which looks quite similar to the interfaces in the deterministic models studied by Krug and Spohn [28]. The case $\mu=2$ corresponds to Burgers equation, as has already been pointed out by a number of authors [21,28], and so one obtains the familiar cusps (see Fig. 6) in this case. Figure 7 shows similar pictures for $\mu=3$.

A scaling plot of our results for the surface width for $\mu=1$, of the form $w(L,t)$ versus t/L^z with $z=1$ and $\alpha=\frac{1}{2}$, is shown in Fig. 8. As one can see, the scaling is very good, in agreement with (4) and the previous results

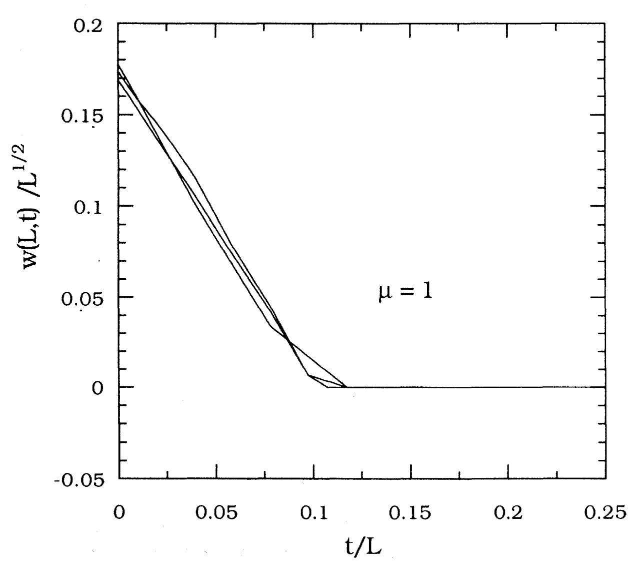

FIG. 8. Scaling plot for deterministic equation with $\mu=1$ for $L=256$, 512, and 1024 using $z=1$.

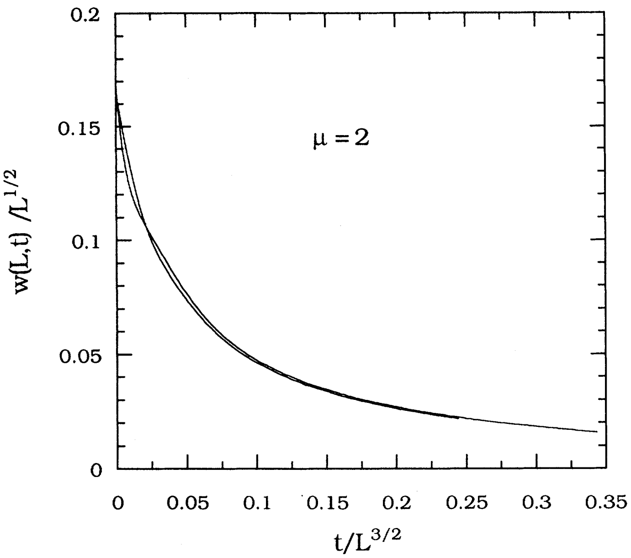

FIG. 9. Scaling plot for deterministic equation with $\mu=2$, $L=256$ and 512, and $z=\frac{3}{2}$.

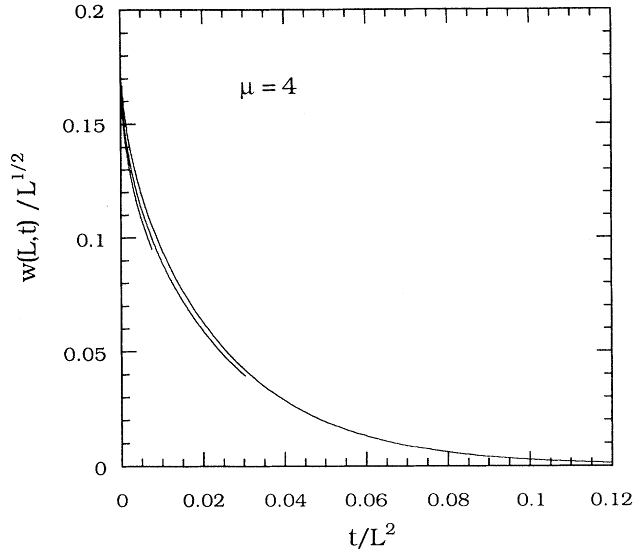

FIG. 11. Scaling plot for deterministic equation with $\mu=4$, $L=128$, 256, and 512, and $z=2$.

of Ref. [28]. Figure 9 shows a similar scaling plot for $\mu=2$ with $z=\frac{3}{2}$, the same as for the stochastic case. Figures 10 and 11 show scaling plots of our results for $\mu=3$ and 4, respectively, with $z=2$ for both. Thus, the nonlinearity appears to be irrelevant for the deterministic case for $\mu\geq 3$ and relation (4) appears to hold for $\mu\geq 1$. We note that for the $\mu=4$ case there is a small discrepancy in the scaled values of the width at early time. However, this is most likely due to fluctuations and should disappear with averages over a larger number of runs.

One interesting question is the shape of the "decay curve" and its dependence on μ and/or z. For the smoothing of a rough interface following the linear equation, one may show that the scaling form $w(L,t)=L^{\alpha}f(t/L^{z})$ holds, with the shape of the decay curve given by $f(u)\sim 1-u^{1/z}$ for $u\ll 1$, with $\alpha=\frac{1}{2}$ and $z=2$ in $d=2$. We conjecture that this scaling form holds for the general nonlinear case as well, with the appropriate value of z. Scaling plots of the form $w(L,t)/L^{\alpha}$ versus $u=t^{1/z}/L$ (not shown) show linear behavior for small u, in approximate agreement with this form.

We now consider the deterministic case for $\mu=\frac{1}{2}$. Figure 12 shows a scaling plot with $z=1$ for this case. Somewhat surprisingly although the decay of the surface width is quite rapid, it does not decay completely to zero, but in fact saturates at a finite value at late times. In ad-

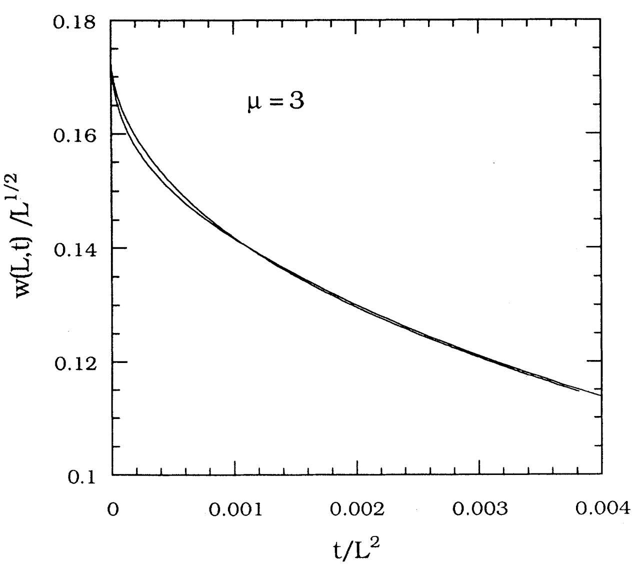

FIG. 10. Scaling plot for deterministic equation with $\mu=3$, $L=256$ and 512, and $z=2$.

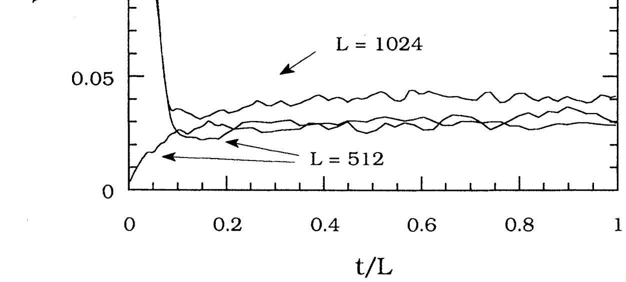

FIG. 12. Scaling plot for deterministic equation with $\mu=\frac{1}{2}$, $L=512$ and 1024, and $z=1$, starting from an initially saturated interface (upper curves). Lower curve is for an initially random interface as described in text.

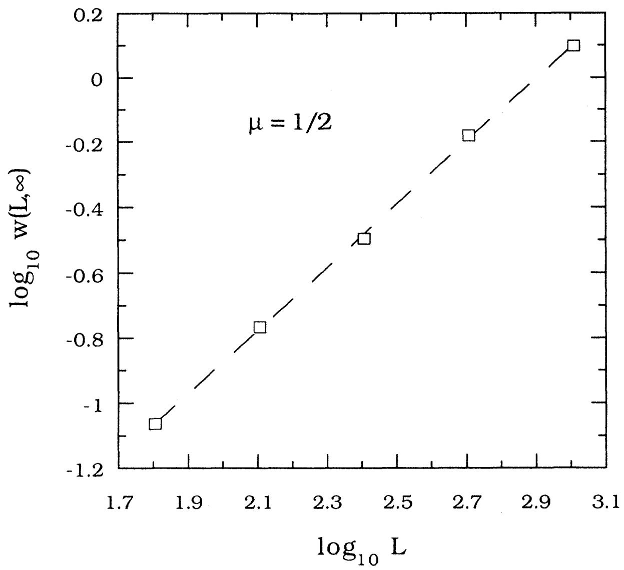

FIG. 13. Log-log plot of saturation width $w(L,\infty)$ vs L for an initially rough interface with $\mu=\frac{1}{2}$ and no additive noise. Slope of fit is 0.96.

dition, while the early "decay" part of the curve scales well with $z=1$ (and $\alpha=\frac{1}{2}$), the saturation value for the width at late times scales with a higher power of L. We note that Eq. (4) would predict $z=\frac{3}{4}$ and thus does not hold in this case. Also shown is the "growth" curve for a system of size $L=512$ (again with no additive noise), starting from an initially random surface corresponding to heights randomly distributed from -0.005 to 0.005. The final state is the same as for the case of "smoothing" starting with a saturated surface. Thus, the late-time

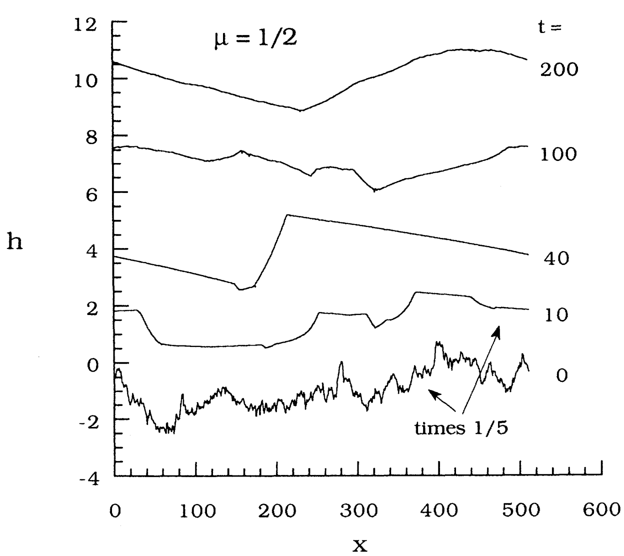

FIG. 14. Pictures of evolving interface for deterministic growth with $\mu=\frac{1}{2}$ starting with a saturated interface at $t=0$ ($L=512$). First two pictures have been scaled by a factor of $\frac{1}{5}$ for clarity.

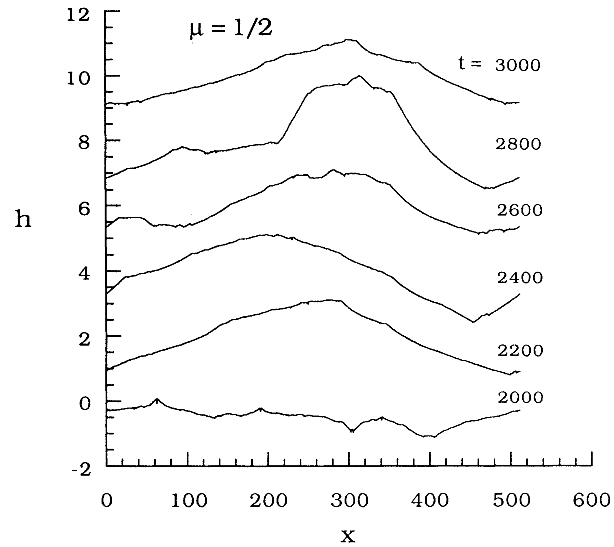

FIG. 15. Pictures showing further evolution of the interface shown in Fig. 14.

evolution of the fluctuating interface appears to be independent of initial conditions.

The existence of a finite interface width at long times in the absence of additive noise appears to be due to an instability which occurs for $\mu<1$, so that the flat solution with zero slope becomes nonlinearly unstable. Due to this instability any slightly random initial interface will form a rough surface despite the absence of additive noise. Figure 13 shows a log-log plot of saturation width as a function of system size L for $\mu=\frac{1}{2}$, for the deterministic case. From Fig. 13 we see that the roughness ex-

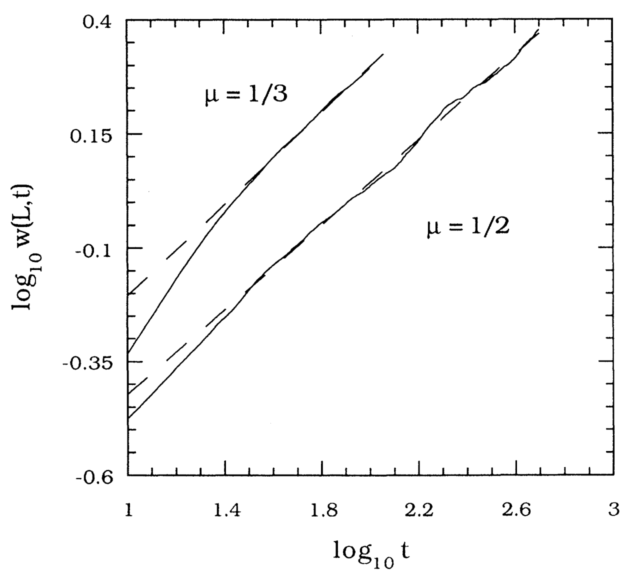

FIG. 16. Log-log plots of $w(L,t)$ vs t for $\mu=\frac{1}{3}$ (upper curve) and $\mu=\frac{1}{2}$ (lower curve) in absence of additive noise starting with an initially random configuration of amplitude 0.005 and with $L=262\,144$. Dashed-line fits have slopes 0.47 and 0.50, respectively.

ponent is $\alpha \simeq 1$. A similar result has been obtained from the correlation function $G(x)=\langle [h(x)-h(0)]^2 \rangle$, which was found to scale as $G(x) \sim x^{2\alpha} \sim x^2$. Thus a "grooved" state appears to be formed.

Figures 14 and 15 show the development and subsequent evolution of this grooved state, starting with the interface at saturation in the presence of noise at $t=0$ for a system of size $L=512$. Figure 16 shows a log-log plot of $w(L,t)$ versus t for $L=262\,144$, starting from a random interface with heights randomly distributed from -0.005 to 0.005. The fit at late time indicates a growth exponent $\beta \simeq \frac{1}{2}$, which implies $z=2$ for this case. Also shown are results for $\mu=\frac{1}{3}$ for which approximately the same value of β is found.

One striking feature of our results for the deterministic case with $\mu=\frac{1}{2}$, is that the roughness exponent α is larger ($\alpha=1$) than for the stochastic case for which $\alpha=\frac{1}{2}$. This implies that for large enough system size L (significantly larger than we have studied here), the surface width will be larger for the deterministic case than for the stochastic case for $\mu=\frac{1}{2}$. Thus, this may be a rather interesting example of noise-induced smoothing. Alternatively, there may be some mechanism which induces a crossover (for fixed nonlinearity strength λ) from $\alpha=1$ to $\alpha=\frac{1}{2}$ at long length scales. Further work will be needed to determine if this is the case. Another interesting aspect of our results is that the existence of a finite saturation width for this case appears to be due to some sort of deterministic chaos or sensitivity to initial conditions which does not allow the interface to settle down to the flat state. This is considered in the following section.

V. LYAPUNOV EXPONENTS FOR SURFACE GROWTH

In order to study the sensitivity to initial conditions for $\mu=\frac{1}{2}$, we consider two different interfaces h_1 and h_2 of size L with slightly different initial values, $h_1(i,0)$ and $h_2(i,0)$, and follow their evolution in time. As a measure of the difference between the two interfaces we consider the quantity,

$$\Delta(t)=\frac{1}{L}\sum_{i=1}^{L}|h_1(i,t)-h_2(i,t)| \; . \tag{7}$$

In analogy with nonlinear dynamics, we expect that for early times, the deviation $\Delta(t)$ will show an exponential behavior of the form $e^{\gamma t}$, where $\gamma>0$ is a Lyapunov-like exponent characterizing the surface dynamics. We take $h_1(i,0)$ to have uniform independent random values (ranging typically from -0.1 to 0.1) for each value of i, and set $h_2(i,0)=h_1(i,0)+\delta r(i)$ where $r(i)$ is a uniform random number from -0.5 to $+0.5$, and δ is a small number (typically less than 0.01). As above, we integrate Eq. (5) ($\nu=1.0$, $\lambda=2.65$, $\Delta t=0.005$) without noise for both initial interfaces h_1 and h_2 and calculate $\Delta(t)$. Figure 17 shows a semilogarithmic scaling plot of $\Delta(t)/\delta$ versus t for different values of δ for systems of size $L=512$ without noise (top curves). We see that for early times the scaled interface difference $\Delta(t)/\delta$ has the same exponential behavior $e^{\gamma t}$ for all δ, with $\gamma \simeq 3$. As δ decreases, the range of time for this exponential behavior increases. Also shown in Fig. 17 (lower curve) is the scaled deviation $[\Delta(t)/\delta]$ for the stochastic case with an additive noise of strength $D=0.5$. For this case the deviation between the two interfaces decreases, indicating a negative Lyapunov exponent. Similar behavior has also been observed for $\mu=\frac{1}{3}$ and we conjecture that a similar instability occurs for all $\mu<1$. This is supported by the results shown in Fig. 18. For $\mu=2$, $\Delta(t)$ decreases rapidly with time, while for the "marginal" case $\mu=1$, Δ first decreases, then increases before slowly decaying with

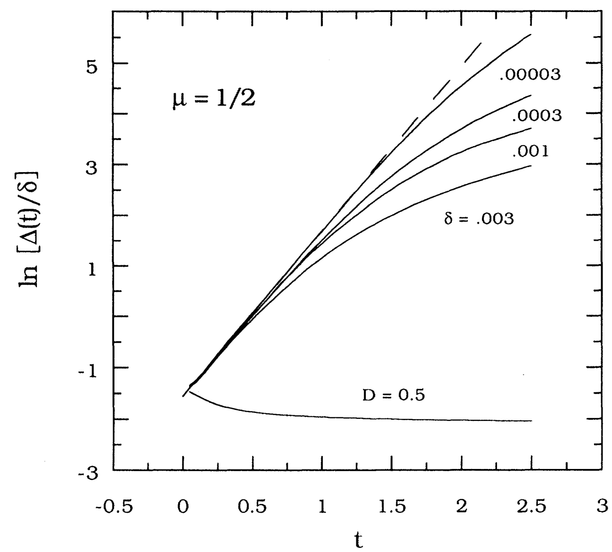

FIG. 17. Semilog plot of $\Delta(t)/\delta$ vs t for both deterministic (upper curves) and stochastic (lower curve) growth with $\mu=\frac{1}{2}$. Dashed-line fit has slope $\gamma \simeq 3.2$.

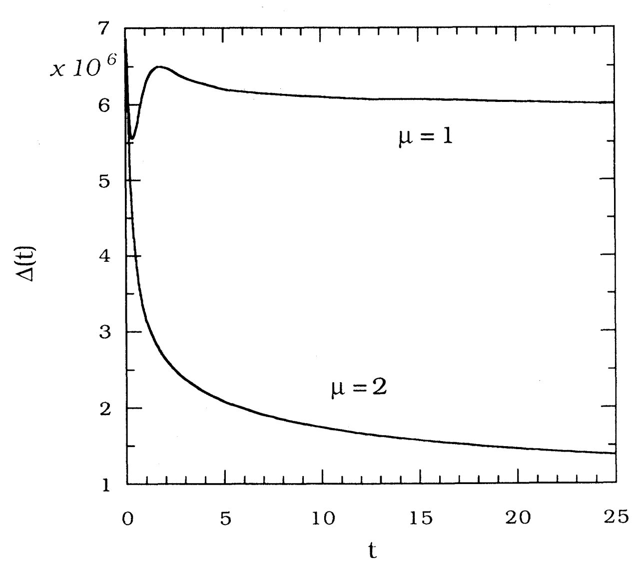

FIG. 18. $\Delta(t)$ vs t for deterministic growth with $\mu=1$ and $\mu=2$ ($L=131\,072$) starting from an initially random interface with $\delta=0.000\,03$.

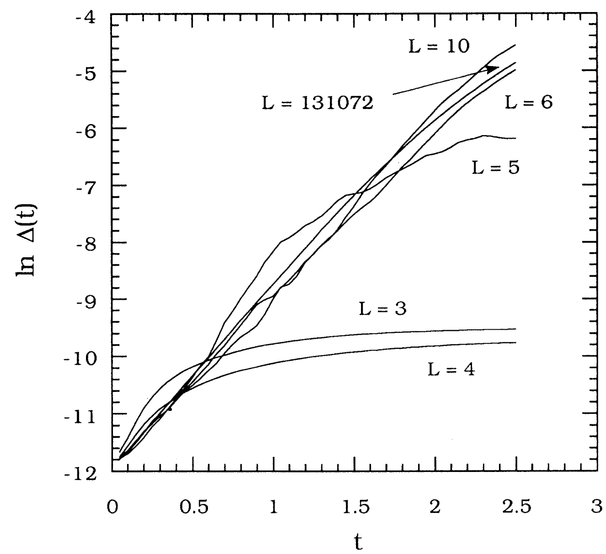

FIG. 19. $\Delta(t)$ vs t for deterministic growth with $\mu=\frac{1}{2}$ ($\delta=0.000\,03$) and $L=3$, 4, 5, 6, 10, and 131 072.

time. Similar results have been obtained for $\lambda<0$, so that our results do not appear to depend on the sign of λ.

While the origin of the unusual behavior for $\mu<1$ is not completely understood, one promising approach is to consider our numerical discretization of Eq. 5 for $\mu<1$, as a set of L coupled nonlinear deterministic equations or maps for a given Δt and Δx. The question then is, for what value of L (how many sets of coupled equations) does the exponential behavior occur? Figure 19 shows a semilogarithmic plot of $\Delta(t)$ versus g for $\mu=\frac{1}{2}$ (using $\delta=0.000\,03$) for different system sizes from $L=3$ to $L=131\,072$. We see that while for $L=3$ and 4 there is no clear evidence of exponential behavior, by $L=6$ the results already approximate the asymptotic behavior for a large system.

VI. DISCUSSION

We have studied the generalized KPZ Eq. (5) in two dimensions both with and without noise for values of μ ranging from $\mu=\frac{1}{2}$ to $\mu=4$. For the case in which additive noise is included we find, in contrast to the conjecture of Wolf [30], KPZ scaling behavior for all values of μ. Somewhat surprisingly, this occurs even for $\mu<2$ (i.e., $\mu=\frac{1}{2}$ and $\mu=1$), even though from scaling arguments such a term might be expected to be more "relevant" than the $|\nabla h|^2$ term. It also occurs for the $\mu=4$ case, which from scaling arguments should be an irrelevant correction to the linear equation.

Our results explain why all discrete stochastic surface growth models studied so far have been observed to have either no dependence on $|\nabla h|$ in the long-wavelength limit ($\lambda=0$), or to have a linear dependence of the growth velocity on $|\nabla h|^2$ at long length scales. This is most likely due to the combination of the additive noise η and the $|\nabla h|^{\mu}$ nonlinearity, which upon renormalization together induce a nonlinear $|\nabla h|^2$ term at larger length scales. Previously no concrete theoretical calculations or numerical results had been presented to support this finding.

For the deterministic case corresponding to the smoothing of an initially rough interfaces with roughness exponent α, we have obtained agreement with scaling relation (4) for $\mu\geq 1$. The validity of this scaling relation appears to be due to the fact that there is no renormalization of the nonlinearity in this case. In addition, we have found approximate agreement with a conjecture of our own regarding the shape of the smoothing curve.

Finally, for the deterministic case with $\mu<1$, we have found an interesting instability which makes the surface rough in the absence of additive noise. Starting from an initially slightly rough interface, the surface evolves in the absence of noise into a fluctuating grooved state. For the case $\mu=\frac{1}{2}$, we find $\alpha=1$ and $\beta\simeq\frac{1}{2}$. In addition, we have measured an effective positive Lyapunov-like exponent which characterizes the surface dynamics in this case. Based on similar results which have been obtained for $\mu=\frac{1}{3}$, we conjecture that this instability occurs for any $\mu<1$. It would be of interest to further investigate the dynamics of this class of nonlinear deterministic growth models, as well as the effects of small amounts of additive noise.

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