Ordering kinetics in systems with long-range interactions

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The growth kinetics following a quench from high temperatures to zero temperature is studied using the time-dependent Ginzburg-Landau model. We investigate d-dimensional systems with n-component order parameter and assume that the interactions decay with distance r as $\hat{V}(r) \sim r^{-d-\sigma}$ with $0<\sigma<2$. The spherical limit ($n=\infty$) is solved for both conserved and nonconserved order-parameter dynamics and the scaling properties of the structure factor are calculated. We find scaling features (including multiscaling in the conserved case) that are similar to those of systems with short-range interactions. The essential difference is that the short-range value of the dynamic critical exponent z_s is replaced by $z=z_s-2+\sigma$ and the form of the scaling function is modified. We also study the general n case for nonconserved order-parameter dynamics and calculate the structure factor in an approximate scheme with the results that (i) the spherical-limit value of z remains unchanged as n is decreased down to $n=1$ and (ii) the spatial correlations decay at large distances as $r^{-d-\sigma}$.

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I. INTRODUCTION

When a system is rapidly quenched from a high-temperature disordered state to a temperature well below its order-disorder point, phase separation occurs and domains of ordered regions grow in time [1]. For a system with $O(n)$ symmetric order parameter, the excess energy is accumulated at the defects such as domain walls ($n=1$), vortex strings ($n=2$), and monopoles ($n=3$), etc. [2], and, as a result, the late stages of phase ordering is governed by defect motion. Such processes of phase ordering have been the subject of many theoretical and experimental investigations [1].

At the late stages of phase ordering, the concept of the dynamical scaling becomes important in describing the growth of order. In the case of nonconserved order-parameter dynamics, the system has a single characteristic length, the average domain size $l(t)$, which grows as a power law in time, $l(t) \sim t^{1/z}$. The growth exponent in systems with short-range interactions seems to be independent of the dimensionality of the order parameter [3–7] and $z=2$ for spatial dimensions $d \geq 2$. On the other hand, growth in a system with a conserved order parameter is more complicated. It is generally accepted that one has simple scaling with $z=3$ for scalar order parameters [5–8]. Coniglio and Zanetti [9], however, solved the spherical limit ($n=\infty$) of the time-dependent Landau-Ginzburg (TDGL) model, and found two length scales diverging with an exponent ($z=4$) that is different from the $n=1$ value. As a consequence of the two distinct length scales, they also found a "multiscaling" form for the structure factor. Using general renormalization-group arguments [5] as well as approximate methods [7], it has been shown that the $z=4$ result is probably valid for all $n \geq 2$, and multiscaling appears only in the $n=\infty$ limit.

When analyzing an experiment, one has to be aware of the possible presence of long-range forces. Although most theoretical work on growth of order has been carried out to characterize systems with short-range interactions, long-range forces such as dipole-dipole interactions, elastic forces, or Rudermann-Kittel-Kasuya-Yoshida (RKKY) -type interactions may play an important role. In particular, just as in the case of critical phenomena, both the critical exponents and the scaling functions may change if the forces are sufficiently long ranged [10]. In fact, the time evolution of a block-copolymer system [11,12] that can be described by a scalar conserved model with long-range interactions displays a very slow growth with an exponent that is close to zero. Thus it is of interest to study how long-range interactions affect the process of phase ordering.

In this paper, we use the TDGL model to study the dynamics of phase ordering in the presence of attractive long-range interactions that decay with distance r as $\hat{V}(r) \sim r^{-d-\sigma}$. After introducing the model in Sec. II, the spherical limit ($n=\infty$) is solved for both conserved and nonconserved dynamics (Sec. III). We find that the growth exponent z can be expressed through the corresponding short-range value z_s and the potential parameter σ as $z=z_s-2+\sigma$ ($0<\sigma<2$). An explicit form of the structure factor is also obtained, and we find multiscaling in the case of conserved dynamics. In Sec. IV, we consider the case of nonconserved dynamics for general n and derive the correlation function in an approximation developed by Bray and Puri [3] and Toyoki [4]. This approximate theory leads to the conclusions that (i) z is equal to the spherical-limit value for all n, (ii) an explicit form of the spatial correlation function can be obtained, and (iii) the spatial correlations have a power-law tail at large distances.

II. THE MODEL

We consider the time-dependent Ginzburg-Landau model [13] in which the time evolution of an n-

component, spatially varying order parameter $\mathbf{S}(\mathbf{r},\mathbf{t})\equiv(S^1,S^2,\ldots,S^n)$ is given by the following Langevin equation:

$$\frac{\partial \mathbf{S}(\mathbf{r},t)}{\partial t}=-L(\nabla^2)\frac{\partial F}{\delta \mathbf{S}}+\boldsymbol{\eta}(\mathbf{r},t) \ . \tag{1}$$

Here $L(\nabla^2)$ is the kinetic coefficient that is a constant Γ_0 for nonconserved dynamics while $L(\nabla^2)=-\lambda\nabla^2$ in the case of conserved dynamics (in the following we set $\Gamma_0=1$ and $\lambda=1$ by suitable choice of time scale). The noise $\boldsymbol{\eta}\equiv(\eta^1,\eta^2,\ldots,\eta^n)$ is a Gaussian-Markovian random force with zero average and with a correlation function of the form

$$\langle \eta^i(\mathbf{r},t)\eta^j(\mathbf{r}',t')\rangle=2L(\nabla^2)T\delta_{ij}\delta(\mathbf{r}-\mathbf{r}')\delta(t-t') \ . \tag{2}$$

The system governed by Eq. (1) relaxes to an equilibrium state described by the free-energy functional $F(\{\mathbf{S}\})$. Since we are interested in the effect of long-range forces, the usual short-range part of the free-energy functional is supplemented with a long-range part,

$$F\{\mathbf{S}\}=\int d^dr\left[-\frac{r_0}{2}|\mathbf{S}|^2+\frac{u}{4}|\mathbf{S}|^4+\tfrac{1}{2}|\nabla\mathbf{S}|^2\right]-\frac{\beta}{2}\int d^dr\int d^dr'\mathbf{S}(\mathbf{r})\hat{V}(\mathbf{r}-\mathbf{r}')\mathbf{S}(\mathbf{r}') \ , \tag{3}$$

where the integrals are over d-dimensional space, u is a constant, $r_0\propto T_c-T>0$, with T_c being the mean-field critical temperature of the short-range system while T is the final temperature of the quench. The term with $\hat{V}(\mathbf{r})$ represents the long-range interaction. The large-distance behavior of $\hat{V}(\mathbf{r})$ is assumed to be $\hat{V}(\mathbf{r})\sim r^{-d-\sigma}$.

Introducing the Fourier transform $S^i_{\mathbf{k}}$, of $S^i(\mathbf{r},t)$, Eq. (1) is transformed into

$$\frac{\partial S^i_{\mathbf{k}}(t)}{\partial t}=L_{\mathbf{k}}\left[\gamma(\mathbf{k})S^i_{\mathbf{k}}(t)-u\sum_{j=1}^{n}\int_{\mathbf{k}'}\int_{\mathbf{k}''}S^i_{\mathbf{k}'}S^j_{\mathbf{k}''}S^j_{\mathbf{k}-\mathbf{k}'-\mathbf{k}''}\right]+\eta^i_{\mathbf{k}}(t) \ , \tag{4}$$

where the integrals $\int_{\mathbf{k}}=\int d^dk/(2\pi)^d$ are over d-dimensional spheres of radius Λ and $\eta^i_{\mathbf{k}}$ is the noise in the Fourier space. The kinetic coefficient is $L_{\mathbf{k}}=1$ for nonconserved dynamics, while it is $L_{\mathbf{k}}=k^2$ for the conserved order parameter. In (4), we introduced

$$\gamma(\mathbf{k})=r_0-k^2+\beta V(\mathbf{k}) \ , \tag{5}$$

where $\Lambda(\mathbf{k})$ is the Fourier component of $\hat{V}(\mathbf{r})$, and its asymptotic form for small k is given by $V(|\mathbf{k}|=k)\sim 1-ck^\sigma$. In the following, we shall absorb the constant $\beta V(0)=\beta$ into r_0, thus renormalizing the mean-field critical temperature T_c, and we shall write $V(k)\sim -k^\sigma$.

Before turning to the solution of Eq. (4), we discuss physical examples where the short-range interactions may be dominated by long-range forces: (i) The dipole-dipole interaction is an important example of $\hat{V}(\mathbf{r})$ with $\sigma=0$ but it is directional dependent, while we treat only rotationally symmetric potentials. (ii) RKKY-type interactions play an important role in spin glasses [14]. The replica Hamiltonian obtained from RKKY interaction contains a cubic nonlinearity and a long-range interaction characterized by $\sigma=3$. (iii) Roland and Desai [15] analyzed a model of nonconserved dynamics for a uniaxial ferromagnetic thin film that can be described by repulsive interaction with $\sigma=0$ and $n=1$. (iv) Onuki [16] derived an effective repulsive long-range interaction that arises from elastic fields in a solid that undergoes phase separation. In his model, a scalar order parameter is the conserved quantity and $\sigma=0$. Nishimori and Onuki [17] simulated phase separation in the presence of elastic fields and found very slow growth of patterns. Their results may also be interpreted as freezing of patterns. (v) Ohta and Kawasaki [11] derived the effective free energy to discuss the phase separation of block copolymers. They found that the free energy has an effective repulsive long-range force with $\sigma=-2$. The dynamics of this system has been discussed by Bahiana and Oono [12], who found again a very slow growth of order.

As we can see from the above list, many well-known examples of long-range interactions are repulsive and suppress domain growth. Attractive cases should also be investigated, however, since it would be interesting to find ways of accelerating domain growth. One has learned from the dynamics of near-equilibrium critical phenomena that attractive long-range interactions may lead to an acceleration of critical relaxation [18] in the sense that the dynamic critical exponent z becomes smaller than the value of z in the corresponding short-range system. This motivated us to investigate the phase ordering in the presence of attractive long-range interactions with σ in the range of $0<\sigma<2$.

In closing this section, we should comment about the range of σ. First, when $\sigma>2$, the attractive long-range interaction is irrelevant because $V(k)\sim -k^\sigma$ can be regarded as a correction to the short-range interaction k^2 term in a long-wavelength expansion. Consequently, one expects that for $\sigma>2$ the essential features of domain growth are described by the short-range limit. Second, we set a lower bound of σ as 0, since the energies are divergent in an infinite system when $\sigma\le 0$. In the examples with $\sigma<0$ that were mentioned above, there should exist a cutoff length that prevents the divergencies in the thermodynamic limit.

III. SPHERICAL LIMIT OF THE TDGL MODEL WITH LONG-RANGE INTERACTIONS

Growth of order in the spherical limit ($n\to\infty$) of the short-range interaction TDGL model has been much investigated [7,9,19–21]. An exact solution of the nonconserved case verified the conventional scaling picture and led to the conjecture that the growth exponent $z=2$ is independent of n. The solution of the conserved dynamics, on the other hand, produced a growth exponent $z=4$ that is different from accepted value ($z=3$) in the case of the scalar order parameter and, furthermore, the structure factor was shown to display an interesting "multiscaling" form. In this section, we shall investigate the $n=\infty$ limit of the TDGL model with long-range interaction and obtain results that are rather similar to the short-range case.

In the spherical limit, one has $u \sim 1/n$, and the model becomes solvable because the nonlinear evolution equation (4) can be linearized, since the fluctuations in $u\sum_j S_{\mathbf{k}}^j(t)S_{\mathbf{k}'}^j(t)$ may be neglected. This quantity may be replaced by [21,22]

$$u \sum_{j=1}^{n} \langle S_{\mathbf{k}}^j(t)S_{\mathbf{k}'}^j(t)\rangle = unC(k,t)\delta(\mathbf{k}+\mathbf{k}') , \tag{6}$$

where the brackets $\langle\ \rangle$ denote averaging over both the initial conditions and the noise $\eta_k(t)$ and the dynamic structure factor $C(k,t)=\langle S_{\mathbf{k}}^j(t)S_{-\mathbf{k}}^j(t)\rangle$ is taken to be independent of both j and the direction of $\mathbf{k}$. This means that we restrict our studies to initial conditions that are rotationally invariant both in the spin and coordinate space. More general initial conditions can be treated [19–21] by separating the transversal and longitudinal components of $C(k,t)$ and following the time evolution of the magnetization as well. Our conclusions remain valid for more general initial conditions, provided the initial correlations have no long-range part [20].

Using (6), one solves Eq. (4) and derives the following self-consistency equation [22,23] for $C(k,t)$:

$$C(k,t)=\Delta \exp\left[\int_0^t R(k,s)ds\right] + 2L_k T \int_0^t dt' \exp\left[\int_0^{t'} R(k,s)ds\right] , \tag{7}$$

where $\Delta = C(k,0)$ is the initial structure factor of a disordered state (the quench is from $T=\infty$), and $R(k,t)$ is given by

$$R(k,t)=2L_k\left[r_0-k^\sigma-un\int_{k'} C(k',t)\right] . \tag{8}$$

Note that we have omitted the k^2 term, since it is negligible compared to k^σ in the long-wavelength limit, and we set $\beta c = 1 > 0$ in (5), since we consider only attractive long-range forces.

This discussion is considerably simplified if we assume that the quench is to $T=0$, since then the second term on the right-hand side of Eq. (7) disappears. This assumption is not really needed, since the same results can be derived for finite-temperature quenches following the steps described in Ref. [20]. Since it is quite well established that the temperature is an irrelevant variable in domain growth problems [5,20], we present only the calculations for the $T=0$ limit.

At this point, the discussion of nonconserved and conserved dynamics becomes different, and we shall first consider the nonconserved dynamics. In that case, we have a constant the kinetic coefficient, $K_k=\Gamma_0=1$, and the scaling properties of $C(k,t)$ can be obtained by deriving a differential equation for

$$\Phi(t)=un\int_k C(k,t) . \tag{9}$$

Setting $T=0$ in (7), then integrating the equation by k and taking a time derivative of the integral, we find the following equation for $\Phi(t)$:

$$\dot{\Phi}=\left[2r_0-\frac{d}{\sigma t}\right]\Phi-2\Phi^2 . \tag{10}$$

The above equation should contain an extra term of the order $O(\Phi\exp(-2\Lambda^\sigma t))$. Since we are interested in the large-time limit, this term can be neglected. Equation (10) is linear in Φ^{-1}, so it can be solved exactly. The long-time behavior, however, can be obtained easily from the above form by an expansion in t^{-1}. The result is

$$\Phi(t)=r_0-\frac{d}{2\sigma}t^{-1}+O(t^{-2}) . \tag{11}$$

Substituting this expression into Eq. (7), we find the standard scaling form for the structure factor

$$C(k,t)\approx l^d(t)f(kl(t)) , \tag{12}$$

where

$$l(t)\sim(2t)^{1/z} , \tag{13}$$

with $z=\sigma$ and the scaling function $f(x)$ given by

$$f(x)=\exp(-x^z) . \tag{14}$$

These results should be contrasted with those known [19–21] for the short-range interaction case: $l(t)\sim(2t)^{1/z_s}$ and $f_s(x)=\exp(-x^{z_s})$, where $z_s=2$. We can see that the scaling properties of the structure factor in the long-range interaction case can be obtained from the results for short-range interactions by simply replacing z_s by $z=z_s-2+\sigma$.

We now turn to the discussion of conserved dynamics. The k dependence of the kinetic coefficient, $L_k=k^2$, makes the calculations rather involved, so we are able to show only that a "self-consistent" solution with the expected scaling features does exist. The argument follows the line developed for the case of short-range interactions [9]. We start with the observation that the solution of Eq. (7) should have the properties characteristic of growth of order with the conserved order parameter. Namely, for fixed t, $C(k,t)$ should have a maximum as a function of k at $k_m(t)$ and, since this peak should evolve with time into a Bragg peak, $C(k_m,t)$ should scale with k_m as $C(k_m,t)\sim k_m^{-d}$. In order to see how these properties may arise from Eq. (7), let us write it in the form

$$C(k,t)=\Delta\exp[-2k^{2+\sigma}t+2k^2Q(t)] , \tag{15}$$

where the function $Q(t)$ is defined as

$$Q(t)=\int_0^t\left[r_0-un\int_k C(k,s)\right]ds . \tag{16}$$

It is clear from Eq. (15) that $C(k,t)$ can have a maximum at some k_m only if the two terms in the exponents have different signs, i.e., if $Q(t)>0$. One can see from (14) that $Q(t)$ can be positive if the initial correlations (Δ) are sufficiently small. Indeed, then the second term in the time integral is negligible, and since $r_0>0$ we have $Q(t)>0$ for short times and one expects that Q remains positive as $un\int C(k,t)$ approaches its stationary value $-r_0$ in the $t\to\infty$ limit. The position of the maximum of $C(k,t)$ can be expressed in terms of $Q(t)$ as

$$k_m(t)=\left[\frac{2}{2+\sigma}\frac{Q(t)}{t}\right]^{1/\sigma}, \tag{17}$$

and then the condition $C(k_m,t)\sim k_m^{-d}$ gives the following self-consistency equation:

$$k_m^{-d}\approx\exp(\sigma k_m^{2+\sigma}t) . \tag{18}$$

For $t\to\infty$, the asymptotic solution of the above equation is given by

$$k_m\approx\left[\frac{d}{\sigma(2+\sigma)}\frac{\ln t}{t}\right]^{1/(2+\sigma)} . \tag{19}$$

Using Eqs. (15), (17), and (19), we can show that the structure factor obeys generalized scaling (termed "multiscaling" by Coniglio and Zanetti [9]) just as in the case of short-range interactions:

$$C(k,t)=[l^d(t)]^{\varphi(k/k_m(t))}, \tag{20}$$

where $l(t)=t^{1/(2+\sigma)}$ and the "exponent" function $\varphi(x)$ is slightly more complicated than in the case of short-range interactions $[\varphi_s(x)=1-(1-x^2)^2]$, namely

$$\varphi(x)=1+\frac{2}{\sigma}(1-x^{2+\sigma})-\frac{2+\sigma}{\sigma}(1-x^2) . \tag{21}$$

The emergence of multiscaling is the result of the presence of two length scales diverging slightly differently $[l(t)\sim t^{1/z}$ and $k_m^{-1}(t)\sim(t/\ln t)^{1/z}$, where $z=2+\sigma]$. The above calculation shows that this feature of the conserved dynamics in the spherical limit remains unchanged when long-range interactions are introduced. It should be noted, however, that Bray and Humayun [7] have shown that there may be problems with the order of the limits $n\to\infty$ and $t\to\infty$, and the multiscaling may be an artifact of the exchange of limits. Although they arrived at this result for the case of short-range forces and they used an approximate scheme, the argument is rather compelling, and we feel that multiscaling may not be a feature of the solution of finite-n systems. A feature of the spherical limit that we believe pertains to systems with finite n is the value of the scaling exponent that is changed from the short-range value $z=z_s=4$ to $z=z_s-2+\sigma$.

IV. AN APPROXIMATE SOLUTION OF THE TDGL MODEL WITH LONG-RANGE INTERACTIONS

In this section, we apply an approximate method developed recently by Bray and Puri [3] and by Toyoki [4] for solving the n-component TDGL equation with nonconserved order-parameter dynamics. Since this method gives reasonable results for models with short-range interaction, we assume that it works in the long-range case, where the fluctuations are expected to be smaller.

As in the previous section, we neglect the effects of noise, and Eqs. (4) and (5) are supplemented by the initial condition for the order-parameter field. Since we are interested in a quench from the high-temperature phase, the components of $\mathbf{S}(\mathbf{r},0)$ are assumed to be independent Gaussian variables with $\langle S^i(\mathbf{r},0)\rangle_{\text{in}}=0$, where $\langle\ \rangle_{\text{in}}$ is the average by the initial configuration. The method of solution we use was originally developed by Kawasaki, Yablik, and Gunton (KYG) [24] for treating scalar order parameter. They used a singular perturbation method for describing relaxation for an unstable state. The KYG method [24] as applied to an n-component model [3] assumes that the asymptotic solution of (4) without noise is given by

$$\mathbf{S}(\mathbf{r},t)\simeq\frac{\mathbf{v}(\mathbf{r},t)}{[1+M_e^{-2}\mathbf{v}(\mathbf{r},t)^2]^{1/2}}\simeq M_e\frac{\mathbf{v}}{|\mathbf{v}|}, \tag{22}$$

where $M_e=\sqrt{r_0/u}$ is the equilibrium value of $|\mathbf{S}|$ and an auxiliary field $\mathbf{v}(\mathbf{r},t)$ is introduced as the Fourier transform of the noninteracting solution of (4):

$$\mathbf{v}_\mathbf{k}(t)=\exp[t\gamma(\mathbf{k})]\mathbf{S}_\mathbf{k}(\mathbf{0}) . \tag{23}$$

The expression $M_e\mathbf{v}/|\mathbf{v}|$ in (22) is valid for $t\to\infty$. The derivation of (22) and (23) is completely parallel to that in a system with short-range interaction. The difference exists only in $\gamma(\mathbf{k})$ in (5). If (22) and (23) are correct, we can guess that the characteristic length $l(t)$ obeys $l(t)\sim t^{1/\sigma}$ because $\gamma(\mathbf{k})=r_0-k^\sigma$. If $n\le d$, then Eqs. (22) and (23) capture the essential features of the assembly of topological defects [25] seeded at $t=0$: Except for the "defects" ($\mathbf{S}=\mathbf{v}=\mathbf{0}$), $|\mathbf{S}|$ approaches its equilibrium value M_e as $t\to\infty$. The method in its original form, however, does not assume the existence of defects, and it gives correct results (as far as scaling exponents and scaling functions are concerned) even in the $n\to\infty$ limit, where no topological defects exist. It is not obvious that the method should perform well for arbitrary n, since there are examples of unexpected growth laws in models with short-range interactions. It is known [26] that kink-antikink dynamics results in a logarithmically increasing $l(t)$ for $n=d=1$. Furthermore, Newman, Bray, and Moore [27] simulated one-dimensional hard spin models and obtained a growth exponent z that is $z=2$ for $n\ge 3$ but $z=4$ for $n=2$. For $n=d=2$, Bray and Humayun [28] reported an anomalous growth law $z=4$ from their hard spin simulation. Although most of these examples are in some sense special (zero-critical-temperature or Kosterlitz-Thouless-type ordering transitions), they warn us about the possible breakdown of the validity of (22) and (23) for special values of n and d. We expect that similar problems may arise when long-range forces are present.

The field $\mathbf{v}$ is a Gaussian variable, so its correlation function can be written as

$$\langle v^i(\mathbf{r},t)v^j(\mathbf{r}',t)\rangle_{\text{in}}=\delta_{ij}h_\sigma(|\mathbf{r}-\mathbf{r}'|,t) , \tag{24}$$

where $h_\sigma(r,t)=\Delta\int_\mathbf{k}e^{2\gamma(k)t+i\mathbf{k}\cdot\mathbf{r}}$ is calculated from (23). Now the correlation function of the order parameter can be obtained as

$$C(r,t)=\langle \mathbf{S}(\mathbf{r}_0)\cdot\mathbf{S}(\mathbf{r}_0+\mathbf{r})\rangle_{\text{in}} \simeq M_e^2\left\langle \frac{\mathbf{v}(\mathbf{r}_0)\cdot\mathbf{v}(\mathbf{r}_0+\mathbf{r})}{|\mathbf{v}(\mathbf{r}_0)||\mathbf{v}(\mathbf{r}_0+\mathbf{r})|}\right\rangle_{\text{in}} . \tag{25}$$

Using $|\mathbf{v}|^{-1}=A_n\int d^na/(2\pi)^{n/2}e^{i\mathbf{v}\cdot\mathbf{a}}a^{1-n}$ with $A_n=2^{(n-2)/2}\pi^{-1/2}\Gamma((n-1)/2)$, where $\Gamma(x)$ is the Γ function, we can rewrite $C(r,t)$ as

$$C(r,t)=A_n^2\int\frac{d^na}{(2\pi)^{n/2}}\int\frac{d^nb}{(2\pi)^{n/2}}(ab)^{-n+1}Q(\mathbf{a},\mathbf{b}) . \tag{26}$$

Here

$$\begin{aligned}Q(\mathbf{a},\mathbf{b})&=\langle \mathbf{v}(\mathbf{r}_0)\cdot\mathbf{v}(\mathbf{r}_0+\mathbf{r})e^{i\mathbf{a}\cdot\mathbf{v}(\mathbf{r}_0)+i\mathbf{b}\cdot\mathbf{v}(\mathbf{r}+\mathbf{r}_0)}\rangle_{\text{in}}\\&=\{h_\sigma(r)-h_\sigma(r)h_\sigma(0)(a^2+b^2)\\&\quad-[h_\sigma(r)^2+h_\sigma(0)^2]ab\cos\theta\}\\&\quad\times\exp\left[-\frac{h_\sigma(0)}{2}(a^2+b^2)-abh_\sigma(r)\right] ,\end{aligned} \tag{27}$$

with θ being the angle between $\mathbf{a}$ and $\mathbf{b}$ and, in obtaining the second equality, we used the Gaussian property of the $\mathbf{v}$ field, $\langle e^{i\mathbf{a}\cdot\mathbf{v}(\mathbf{r}_0)+i\mathbf{b}\cdot\mathbf{v}(\mathbf{r}+\mathbf{r}_0)}\rangle_{\text{in}}=e^{-(a^2+b^2)h_\sigma(0)/2-abh_\sigma(r)}$. It is clear that the effects of long-range interaction appear only through $\gamma(\mathbf{k})$ in $h_\sigma(r)$. Therefore, we can integrate Eq. (26) as in the case of the short-range model [4,29], and find that

$$C(r,t)=M_e^2\frac{[\Gamma((n+1)/2)]^2}{\Gamma(n/2)\Gamma(n/2+1)}s_\sigma^{1/2}\times F\left[\frac{1}{2},\frac{1}{2};\frac{n+2}{2};s_\sigma\right] , \tag{28}$$

where $s_\sigma=[h_\sigma(r)/h_\sigma(0)]^2$ and $F(a,b,c;s_\sigma)=\sum_{m=0}^{\infty}[(a)_m(b)_m/(c)_m](s_\sigma^m/m!)$, where $(a)_m=\Gamma(a+m)/\Gamma(a)$ is the Gaussian hypergeometric function [29].

The most important properties of the long-range interaction that appear in s_σ can be expressed as

$$s_\sigma^{1/2}=\frac{h_\sigma(r)}{h_\sigma(0)}=C_d\frac{r^{1-d/2}\int_0^\infty dk\, k^{d/2}e^{2\gamma(k)t}J_{d/2-1}(kr)}{\int_0^\infty dk\, k^{d-1}e^{2\gamma(k)t}} \tag{29}$$

where $C_d=2^{(d-2)/2}\Gamma(d/2)$ and $J_{d/2-1}(x)$ is the Bessel function. Up to this point, the long-range nature of the interaction did not impose any restriction on the generality of the results.

We now restrict ourselves to long-range attractive interactions with $0<\sigma<2$. If the short-range effects are neglected as in the previous section, and the explicit form of $\gamma(k)=r_0-\mathbf{k}^\sigma$ is used, we obtain

$$\begin{aligned}s_\sigma^{1/2}&=\frac{2^{(d-2)/2}\Gamma(d/2)}{\Gamma(d/\sigma)}\times\int_0^\infty dx\, x^{(d+2)/2\sigma-1}e^{-x}\times J_{d/2-1}\left[\frac{x^{1/\sigma}r}{l(t)}\right]\left[\frac{r}{l(t)}\right]^{1-d/2}\\&=\frac{\Gamma(d/2)}{\Gamma(d/\sigma)}\sum_{m=0}^{\infty}\frac{(-1)^m\Gamma((2m+d)/\sigma)}{m!\Gamma(d/2+m)}\times\left[\frac{r}{2l(t)}\right]^{2m} .\end{aligned} \tag{30}$$

In obtaining the series we used the series representation of the Bessel function and carried out a term-by-term integration. Equation (30) contains a single characteristic length $l(t)$ that increases with time as

$$l(t)=(2t)^{1/\sigma} . \tag{31}$$

If we substitute $\sigma=2$ in (30) we recover the usual result $s_{\sigma=2}^{1/2}=e^{-r^2/8t}$, i.e., $l(t)\sim t^{1/2}$. The correlation function has a scaling form $C(r,t)=\hat{C}(r/l(t))$. This result is valid, however only for $1<\sigma<2$, since the radius of convergence of the series in Eq. (30) is zero for $0<\sigma<1$. In the regime $0<\sigma\leq 1$ a series expansion is more difficult, but, at least for $d=1$, we can derive a series expansion even for $0<\sigma<1$ using the method of Montroll and West [30]:

$$s_\sigma^{1/2}=\frac{\sigma^2}{\Gamma(1/\sigma)}\sum_{n=0}^{\infty}\frac{(-1)^n\Gamma(\sigma(n+1))}{n!z^{\sigma(n+1)+1}}\times\sin[\pi\sigma(n+1)/2] , \tag{32}$$

where $z=r/l(t)$. Note that the convergent series in (32) is still scaled by $l(t)\sim t^{1/\sigma}$. Therefore, we believe that Eq. (31) is valid for all $0<\sigma<2$.

Several limiting cases of (31) are of importance. For $n=1$, which is the Ising limit, the identify $F(\frac{1}{2},\frac{1}{2};\frac{3}{2};z^2)=\arcsin(z)/z$ gives the well-known result [4,29]

$$C(r,t)=\frac{2M_e^2}{\pi}\arcsin(s_\sigma^{1/2}) . \tag{33}$$

For $n\to\infty$, using the asymptotic formula $n^{b-a}\Gamma(a+b)/\Gamma(n+b)=1+O(n^{-1})$ and $F(a,b;c;s_\sigma)=\sum_{n=0}^{m}[(a)_n(b)_n/(c)_n](s_\sigma^n/n!)+O(|c|^{-m-1})$ for $|c|\gg 1$ with $(a)_n=\Gamma(a+n)/\Gamma(a)$, we obtain

$$C(r,t)=M_e^2s_\sigma^{1/2} , \tag{34}$$

which can be recognized as the Fourier transform of the result (14), once it is noted that $s_\sigma^{1/2}$ in (34) is proportional to $\int_k d^dk\, e^{i\mathbf{k}\cdot\mathbf{r}}C(k,t)$ and that $C(k,t)\propto\exp(-2k^\sigma t)$.

Let us now discuss the spatial dependence of the correlation function. The large-scale $\{X^2/2=[r/2l(t)]^2\gg 1\}$ behavior is well described by the lowest-order term in the series expression of the Gaussian hypergeometric function [29], and we find

$$C(r,t)=\hat{C}(X)\simeq\frac{[\Gamma((n+1)/2)]^2}{\Gamma(n/2)\Gamma(n/2+1)}s_\sigma^{1/2}(X) . \tag{35}$$

Since the numerator and the denominator in (29) have asymptotic forms [30,31] $h(r,t) \sim e^{2r_0 t} t r^{-d-\sigma}$ and $h(0,t) \sim e^{2r_0 t} t^{-d/\sigma}$, the large argument limit of $s_\sigma(X \to \infty)$ is given by

$$\hat{C}(X) \sim s_\sigma^{1/2}(X) \sim X^{-d-\sigma} . \quad (36)$$

Thus, in contrast to the short-range case, long-range interactions generate a power-law tail in the scaling function.

On the other hand, the short-distance behavior ($X \ll 1$) in our model is similar to that of the short-range model [3,4]. The asymptotic form for $1 < \sigma < 2$ can be described as

$$\hat{C}(X) \simeq \begin{cases} 1 - \dfrac{2}{\pi}\alpha(d,\sigma)^{1/2} X & \text{at } n=1 \\ 1 - \alpha(d,\sigma)\{\ln 2 - \frac{1}{2}\ln X - \frac{1}{4} - \frac{1}{2}\ln[\alpha(d,\sigma)]\} X^2 & \text{at } n=2 \\ 1 - \dfrac{\alpha(d,\sigma)}{2}\left[1 + \dfrac{1}{n-2}\right] X^2 & \text{at } n \geq 3 , \end{cases} \quad (37)$$

where $\alpha(d,\sigma) = 2\Gamma((2+d)/\sigma)/d\Gamma(d/\sigma)$. Equation (37) is reduced to the short-range result if we substitute $\alpha(d,\sigma=2)=1$. The linear decay in the $n=1$ case represents Porod's law [32]. It is easy to show that the leading singularity of $\hat{C}(X)$ appears in

$$\hat{C}_{\text{sing}}(X) = \frac{n}{2\pi}\alpha(d,\sigma)^{n/2} \frac{\Gamma((n+1)/2)^2 \Gamma(-n/2)}{\Gamma((n+2)/2)} X^n \quad (38)$$

for odd n and

$$\hat{C}_{\text{sing}}(X) = (-1)^{m-1} \frac{\alpha(d,\sigma)^m \Gamma(m+\frac{1}{2})^2}{\pi m!(m-1)!} X^m \ln X \quad (39)$$

for even $n=2m$. Therefore, the large argument limit of the scaling function in the structure factor $C(k,t) \sim l(t)^d f(kl(t))$ obeys the generalized Porod's law [3,4]

$$f(x) \sim x^{-d-n} \quad (40)$$

for $1 < \sigma < 2$ and finite n just as in the case of short-range interactions.

In summary, we find that phase ordering of the general n-vector model with long-range interactions obeys usual scaling and the characteristic length grows with time as $l(t) \sim t^{1/\sigma}$. The scaling function of the spatial correlations, however, has an interesting power-law tail at large arguments.

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[1] J. D. Gunton, M. S. Miguel, and P. S. Sahni, in *Phase Transitions and Critical Phenomena,* edited by C. Domb and J. L. Lebowitz (Academic, London, 1983), Vol. 8; K. Binder, Rep. Prog. Phys. **50**, 783 (1987); S. Komura, Phase Transitions **12**, 3 (1988).
[2] K. Kawasaki, Ann. Phys. (N.Y.) **151**, 319 (1984); A. Vilenkin, Phys. Rep. **121**, 263 (1985).
[3] A. J. Bray and S. Puri, Phys. Rev. Lett. **67**, 2670 (1991).
[4] H. Toyoki, Phys. Rev. B **45**, 1965 (1992).
[5] A. J. Bray, Phys. Rev. B **41**, 6724 (1990).
[6] G. F. Mazenko, Phys. Rev. B **42**, 4487 (1990); **43**, 5747 (1991).
[7] A. J. Bray and K. Humayun, Phys. Rev. Lett. **68**, 1559 (1992).
[8] I. M. Lifshitz and V. V. Slyozov, J. Phys. Chem. Solids **19**, 35 (1961).
[9] A. Coniglio and M. Zanetti, Europhys. Lett. **10**, 575 (1989).
[10] M. E. Fisher, S. K. Ma, and B. G. Nickel, Phys. Rev. Lett. **29**, 917 (1972); J. Sak, Phys. Rev. B **8**, 281 (1973); M. Suzuki, Phys. Lett. **43A**, 245 (1973).
[11] T. Ohta and K. Kawasaki, Macromolecules **19**, 2621 (1986).
[12] M. Bahiana and Y. Oono, Phys. Rev. A **41**, 6763 (1990).
[13] P. C. Hohenberg and B. I. Halperin, Rev. Mod. Phys. **49**, 435 (1977).
[14] M. C. Chang and J. Sak, Phys. Rev. Lett. **29**, 2652 (1972); A. J. Bray, M. A. Moore and A. P. Young, *ibid.* **56**, 2641 (1986).
[15] C. Roland and R. C. Desai, Phys. Rev. B **42**, 6658 (1990).
[16] A. Onuki, J. Phys. Soc. Jpn. **58**, 3069 (1989).
[17] H. Nishimori and A. Onuki, Phys. Rev. B **42**, 980 (1990); J. Phys. Soc. Jpn. **60**, 1208 (1991); A. Onuki and H. Nishimori, Phys. Rev. B **43**, 13 649 (1991).
[18] M. Suzuki and G. Igarashi, Prog. Theor. Phys. **49**, 1070 (1973).
[19] G. F. Mazenko and M. Zanetti, Phys. Rev. B **32**, 4565 (1985).
[20] T. J. Newman and A. J. Bray, J. Phys. A **23**, 4491 (1990).

[21] Z. Rácz and T. Tél, Phys. Lett. **60A**, 3 (1977).
[22] Note that the equation for $C(q,t)$ can be obtained by field-theoretic methods by iterating Eq. (2). In the $n=\infty$ limit one has to keep track of the closed loops, since $u \sim n^{-1}$ and each loop generates a power of n.
[23] F. de Pasquale and P. Tartaglia, Phys. Rev B **33**, 2081 (1986).
[24] K. Kawasaki, M. C. Yablik, and J. D. Gunton, Phys. Rev. A **17**, 455 (1978).
[25] T. Ohta, D. Jasnow, and K. Kawasaki, Phys. Rev. Lett. **49**, 1223 (1982); Toyoki [4] used Eq. (22) as the starting point of his analysis of defect dynamics.
[26] T. Nagai and K. Kawasaki, Physica A **120**, 587 (1983); K. Kawasaki and T. Nagai, *ibid.* **121**, 175 (1983).
[27] T. J. Newman, A. J. Bray, and M. A. Moore, Phys. Rev. B **42**, 4514 (1990).
[28] A. J. Bray and K. Humayun, J. Phys. A **23**, 5897 (1990).
[29] See, e.g., *Handbook of Mathematical Functions,* edited by M. Abramowitz and I. A. Stegun (Dover, New York, 1970).
[30] E. W. Montroll and B. J. West, in *Fluctuation Phenomena,* edited by E. W. Montroll and J. L. Lebowitz (North-Holland, Amsterdam, 1979).
[31] V. Seshadri and B. J. West, Proc. Natl. Acad. Sci. U.S.A. **70**, 4501 (1982).
[32] G. Porod, in *Small Angle X-ray Scattering,* edited by O. Glatter and L. Krattky (Academic, New York, 1983).