Spatially periodic orbits in coupled-map lattices

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We obtain the conditions that ensure the stability of spatially and temporally periodic orbits of coupled-map lattices. The stability matrices can be put in a circulant and block circulant form. This allows us to reduce the problem to smaller matrices corresponding to the building blocks of spatial periodicity. We find that additional conditions are imposed as we expand the size of the lattice. For the traveling-wave solution the analysis is considerably simplified. We have analyzed both the one-dimensional and higher-dimensional lattices.

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I. INTRODUCTION

Observation of routes to chaos in hydrodynamic experiments has been one of the achievements of nonlinear physics. However, not much is known about the spatially extended systems with higher degrees of freedom. Some attempts have been made recently to enhance our understanding in spatially extended systems. Considerable attention that these systems have received during recent times is due to their wide range of applications such as turbulence, pattern formation in natural systems, solitons, etc. [1–4]. They also exhibit a very rich phenomenology including a wide variety of both spatial as well as temporal periodic structures, intermittency, chaos, domain walls, kink dynamics, etc. Spatially extended systems have been modeled using various types of models like cellular automata, coupled oscillator arrays, and coupled-map lattices.

Among the above-mentioned models the model of the coupled-map lattice has been quite popular recently and various studies have been carried out on it. The reason for its popularity are simplicity in analysis and simulation. This model is tractable, easy to handle numerically as well as analytically, and is sometimes able to capture the essential qualitative features of physical systems. Detailed numerical studies show that this model gives rise to a variety of rich spatial and temporal structures [1]. It has been successful in modeling some of the phenomena in spatially extended systems. For example, it has been used to model the real-life phenomenon like spatiotemporal intermittency and spiral waves [5,6]. There has also been a recent proposal of a coupled-map lattice model for crystal growth [7]. This model has been used in contexts other than pattern formation and has been successful in modeling the dynamics in a computationally more efficient manner. As an example it has been developed as an efficient scheme of simulating the kinetics of important equations in phase-ordering processes such as Cahn–Hilliard-Cook (CHC) and time-dependent Ginzburg-Landau (TDGL) equations [8,9].

The problem that we will be dealing with here is of the type where the spatial correlation is maintained throughout the lattice [10]. The easiest example is the wave kind of patterns on lattices which are basically spatially periodic patterns. Such patterns are seen in various physical systems and it is not necessary that a spatial periodicity can exist only in temporally periodic systems (e.g., [11]). The model explored for studying these is that of a coupled-map lattice. We note that Waller and Kapral [12] and Oppo and Kapral [13] have considered a similar problem for some very specific maps and couplings and for simple homogeneous and small period solutions. Here we analyze the problem in a very general way and obtain the conditions for the stability of the spatially extended solutions. Periodic-orbit analysis for coupled-map lattices is also recently given by Politi and Torcini [14].

In Sec. II we develop the formalism for analyzing the stability properties of spatially and temporally periodic structures in one-dimensional coupled-map lattices. We are able to reduce the problem to that of the analysis of the building blocks of spatial periodicity. In Sec. III we consider some illustrative examples. Some further simplifications are possible for the traveling-wave solution. This is discussed in Sec. IV. In Sec. V we extend the analysis to the coupled-map lattices in higher dimensions. We conclude with a discussion in the last section.

II. SPATIALLY AND TEMPORALLY PERIODIC ORBITS

In this section we address the problem of stability of spatial and temporal periodic structures. We specifically consider coupled-map lattices with nearest-neighbor couplings. Consider following the general model,

$$x_{t+1}(i)=h_0 f_0(x_t(i))+h_1 f_1(x_t(i+1)) +h_{-1} f_{-1}(x_t(i-1)) , \tag{1}$$

where $x_t(i)$ is the variable associated with the ith lattice point at time t taking values in a suitably bounded phase space. The maps f_0, f_1, f_{-1} are some maps, such as logistic map, which describe the evolution of an otherwise isolated system. The parameters h_0, h_1, and h_{-1} represent the coupling strengths and are chosen so that $x_{t+1}(i)$ lies in the same phase space (e.g., [0,1] for the logistic map $f(x)=\mu x(1-x)$, $0\leq\mu\leq 4$). Henceforth we assume that h_0, h_1, h_{-1} are positive. However, almost all our results are valid even otherwise.

Let $\mathcal{C}_N$ denote a closed chain of N lattice points in which the right-hand neighbor of the Nth point is the first lattice point. We note that for $N=1$ the chain $\mathcal{C}_1$ consists of a single point which is to be understood as a neighbor of itself. Let $R_t=(x_t(1),\ldots,x_t(N))$ denote the state of the system for the chain $\mathcal{C}_N$ at time t. Let $S_\tau(N,1)$ denote a solution of Eq. (1) with temporal periodicity τ for the chain $\mathcal{C}_N$, i.e.,

$$S_\tau(N,1)=\{R_1,R_2,\ldots,R_\tau,R_1,R_2,\ldots\}\ .$$

Now consider a closed chain of twice the length, i.e., $\mathcal{C}_{2N}\equiv\mathcal{C}_{2\times N}$. Obviously, the spatially periodic sequence

$$S_\tau(N,2)=\{\langle R_1R_1\rangle_2,\langle R_2R_2\rangle_2,\ldots,\langle R_\tau R_\tau\rangle_2,\langle R_1R_1\rangle_2,\ldots\}$$

of wavelength N built from the states $\{R_t\}$ as the building blocks, is a solution of Eq. (1) for the closed chain $\mathcal{C}_{2N}$ with temporal periodicity τ. Here the ordered pair $\langle R_tR_t\rangle_2$ represents the state $[x_t(1),\ldots,x_t(N),x_t(N+1),\ldots,x_t(2N)]$, with $x_t(N+i)=x_t(i)$, $i=1,2,\ldots N$, which is made up of two replicas of the state R_t.

Thus from the above discussion it is clear that, in general, the sequence

$$S_\tau(N,k)=\{\langle R_1,\ldots,R_1\rangle_k,\ldots,\langle R_\tau,\ldots,R_\tau\rangle_k,\langle R_1,\ldots,R_1\rangle_k,\ldots\}$$

represents a solution of Eq. (1) for the closed chain $\mathcal{C}_{k\times N}$ with temporal periodicity τ and wavelength N. Here the ordered pair $\langle R_t,\ldots,R_t\rangle_k$ represents a state made up of k replicas of the state R_t. We call $S_\tau(N,k)$ the k replica solution of $S_\tau(N,1)$. We address the problem of what can be stated about the stability properties of such spatially and temporally periodic solutions $S_\tau(N,k)$, from the analysis of the stability matrices for $S_\tau(N,1)$ of the building blocks [10]. In other words the question is, What is the effect of enlargement of phase space and the couplings on the stability of the replica solutions?

A. Homogeneous case

We begin with the simplest case of $N=1$ so that R_t consists of a single lattice point $x_t(1)=x_t$ and consequently we suppress the lattice index. The replica solution $S_\tau(1,k)$ for the chain $\mathcal{C}_k=\mathcal{C}_{k\times 1}$ is a homogeneous solution with $\{x_t\}$ as building blocks [15]. Now the solution $S_\tau(1,1)=\{x_1,x_2,\ldots,x_\tau,x_1,x_2,\ldots\}$ for the building block is a stable solution provided

$$|f'(x_1)f'(x_2)\cdots f'(x_\tau)|<1\ , \tag{2}$$

where

$$f(x)=h_0f_0(x)+h_1f_1(x)+h_{-1}f_{-1}(x)$$

and

$$f'(x)=\frac{df(x)}{dx}\ .$$

For the homogeneous solution $S_\tau(1,k)$, the stability condition is that modulus of all eigenvalues of the $k\times k$ stability matrix $J=J_\tau\cdots J_2J_1$ have magnitude less than one. Here J_t is a $k\times k$ Jacobian matrix given by

$$J_t=\begin{bmatrix} h_0f'_0(x_t) & h_1f'_1(x_t) & 0 & \cdots & 0 & 0 & h_{-1}f'_{-1}(x_t)\\ h_{-1}f'_{-1}(x_t) & h_0f'_0(x_t) & h_1f'_1(x_t) & \cdots & 0 & 0 & 0\\ 0 & h_{-1}f'_{-1}(x_t) & h_0f'_0(x_t) & \cdots & 0 & 0 & 0\\ \vdots & & & & & & \vdots\\ 0 & 0 & 0 & \cdots & h_{-1}f'_{-1}(x_t) & h_0f'_0(x_t) & h_1f'_1(x_t)\\ h_1f'_1(x_t) & 0 & 0 & \cdots & 0 & h_{-1}f'_{-1}(x_t) & h_0f'_0(x_t) \end{bmatrix}\ . \tag{3}$$

The matrix J_t is a circulant matrix [16] and may be written as

$$J_t=\mathrm{circ}(h_0f'_1(x_t),h_1f'_1(x_t),0,\ldots,0,h_{-1}f'_{-1}(x_t))\ . \tag{4}$$

The eigenvalues of J_t are given by [16]

$$\lambda_{t,r}=[h_0f'_0(x_t)+\omega_rh_1f'_1(x_t)+\omega_r^{k-1}h_{-1}f'_{-1}(x_t)]\ ,\quad r=1,2,\ldots,k \tag{5}$$

where ω_r is a kth root of unity given by

$$\omega_r=e^{i[2\pi(r-1)/k]}\ . \tag{6}$$

Thus the eigenvalues of the stability matrix J are

$$\lambda_r=\prod_{t=1}^{\tau}\lambda_{t,r}$$
$$=\prod_{t=1}^{\tau}[h_0f'_0(x_t)+\omega_r h_1 f'_1(x_t)+\omega_r^{k-1}h_{-1}f'_{-1}(x_t)] \ . \tag{7}$$

Now $|\lambda_r|<1$, for all r, ensures the stability of the homogeneous solution $S_\tau(1,k)$.

Consider the special case when all the maps are the same, i.e.,

$$f_0(x)=f_1(x)=f_{-1}(x)=f(x) \ . \tag{8}$$

For the stability of a single-point solution, i.e., for $S_\tau(1,1)$ for the chain $\mathcal{C}_1$ we should have

$$\left|(h_0+h_1+h_{-1})^\tau\prod_{t=1}^{\tau}f'(x_t(1))\right|<1 \ . \tag{9}$$

The homogeneous solution $S_\tau(1,k)$ for the chain $\mathcal{C}_k$ is stable if

$$|(h_0+\omega_r h_1+\omega_r^{k-1}h_{-1})^\tau\prod_{t=1}^{\tau}f'(x_t(1))|\le 1$$
$$\text{where } r=1,2,\ldots,k \ . \tag{10}$$

Using triangle inequality and the fact that couplings are positive, it can be proved that condition (10) can be satisfied provided condition (9) is satisfied. Thus the stability of the homogeneous solution $S_\tau(1,k)$ is guaranteed by the stability of the single-point solution $S_\tau(1,1)$ for the same parameters of the map exhibiting no effect of enlargement of phase space and the couplings.

B. Case of higher spatial periods

Now we turn to the case of higher values of N for the one-dimensional model given by Eq. (1). Consider the solution $S_\tau(N,1)$ for the closed chain $\mathcal{C}_N$. Stability of the solution is determined by the eigenvalue with largest magnitude of $N\times N$ matrix,

$$j=j_\tau j_{\tau-1}\cdots j_1 \ , \tag{11}$$

where j_t is the Jacobian matrix given by

$$j_t=A_t+B_t+C_t \ . \tag{12}$$

Here A_t is a tridiagonal matrix given by

$$A_t=\begin{bmatrix} h_0f'_0(x_t(1)) & h_1f'_1(x_t(2)) & 0 & \cdots \\ h_{-1}f'_{-1}(x_t(1)) & h_0f'_0(x_t(2)) & h_1f'_1(x_t(3)) & \cdots \\ & & & \\ \vdots & & & \vdots \end{bmatrix} \tag{13}$$

and the matrices B_t and C_t have only a single nonzero element and are given by

$$(B_t)_{ij}=h_1f'_1(x_t(1))\delta_{iN}\delta_{j1} \ , \tag{14}$$

$$(C_t)_{ij}=h_{-1}f'_{-1}(x_t(N))\delta_{i1}\delta_{jN} \ . \tag{15}$$

Let us now consider the solution $S_\tau(N,k)$ of the closed chain $\mathcal{C}_{k\times N}$ which is obtained by k replicas of the solution $S_\tau(N,1)$ for $\mathcal{C}_N$. The stability of $S_\tau(N,k)$ is determined by the eigenvalues of $kN\times kN$ stability matrix $J=J_\tau J_{\tau-1}\cdots J_1$ where J_t is a $kN\times kN$ Jacobian matrix given by

$$J_t=\begin{bmatrix} A_t & B_t & 0 & \cdots & 0 & C_t \\ C_t & A_t & B_t & \cdots & 0 & 0 \\ 0 & C_t & A_t & \cdots & 0 & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & \cdots & A_t & B_t \\ B_t & 0 & 0 & \cdots & C_t & A_t \end{bmatrix} \tag{16}$$

for $k>2$. For $k=2$, J_t is

$$J_t=\begin{bmatrix} A_t & B_t+C_t \\ B_t+C_t & A_t \end{bmatrix} , \tag{17}$$

and for $k=1$, $J_t=A_t+B_t+C_t=j_t$. We note that Jacobian matrix J_t [Eqs. (16) and (17)] is a block circulant matrix where each block is a $N\times N$ matrix [16] and may be written as

$$J_t=b\ \mathrm{circ}(A_t,B_t,0,\ldots,0,C_t) \ . \tag{18}$$

This observation is crucial for our analysis of stability properties. A block circulant matrix can be put into a block-diagonal form by a unitary transformation. The block-diagonal form is [16]

$$D_t=\begin{bmatrix} M_t^1 & 0 & \cdots & 0 \\ 0 & M_t^2 & \cdots & 0 \\ \vdots & & \cdots & \vdots \\ 0 & 0 & & M_t^k \end{bmatrix} , \tag{19}$$

where the matrices, M_t^r, $r=1,\ldots,k$, are $N\times N$ matrices given by

$$M_t^r=A_t+\omega_r B_t+\omega_r^{k-1}C_t \ . \tag{20}$$

Note that this form is a generalization of Eq. (5). The matrix M_t^1 is the same as the matrix j_t of Eq. (11) since $\omega_1=1$. The unitary matrix which affects the above block diagonalization is a direct product of Fourier matrices of sizes $k\times k$ and $N\times N$ [16]. The elements of Fourier matrices are only roots of unity and thus are independent of the matrix being diagonalized. Consequently, the same unitary matrix block diagonalizes the product of J_t's. Thus the block-diagonal form of the product matrix $J=J_\tau\cdots J_2J_1$ is given by

$$D=\begin{bmatrix} \prod_{t=1}^{\tau}M_t^1 & 0 & \cdots & \cdots & 0 \\ 0 & \prod_{t=1}^{\tau}M_t^2 & \cdots & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & \cdots & \prod_{t=1}^{\tau}M_t^k \end{bmatrix} . \tag{21}$$

The first block $\prod_{t=1}^{\tau} M_t^1$ is the same as the matrix $j=j_\tau j_{\tau-1}\cdots j_1$ of Eq. (11). The stability properties of the solution $S_\tau(N,k)$ are determined by the eigenvalues of the matrix given by Eq. (21) of which j is only one constituent block. In addition to the eigenvalues of j, we now must look at the eigenvalues of the remaining $k-1$ blocks of Eq. (21). Thus the effects on the stability due to the enlargement of the phase space and couplings manifest themselves through the eigenvalues of the additional blocks. A general block, $M(\theta)$ (size $N\times N$) has the following structure:

$$M(\theta)=\prod_{t=1}^{\tau}(A_t+e^{i\theta}B_t+e^{-i\theta}C_t)\ , \tag{22}$$

where we have set $\omega_r=e^{i\theta}$ and $\omega_\tau^{k-1}=e^{-i\theta}$.

We note that the elements of $M(\theta)$ are just combinations of entries of j [Eqs. (11)–(15)]. Thus the problem of stability analysis of larger orbits is reduced to that of the entries of j. This corresponds to the reduction of the analysis of $kN\times kN$ matrices to that of $N\times N$ matrices.

Returning to Eq. (22), it is clear that if we check the eigenvalues of $M(\theta)$ for θ between 0 and 2π, it ensures the stability for all values of k. Actually, it is sufficient to check for $0\le\theta\le\pi$. Of course, for a given value of k it is sufficient to check for a maximum of $[(k/2)+1]$ values of θ [17].

One can get further simplifications for $\tau=1$, i.e., for a fixed-point solution. In this case what is important is the characteristic polynomial $\mathcal{P}$ of $M(\theta)$, given by the determinant of the matrix $M(\theta)-\lambda I$,

$$\mathcal{P}=\mathrm{Det}[(A_1+e^{i\theta}B_1+e^{-i\theta}C_1)-\lambda I]\ , \tag{23}$$

where $i=0,1,\ldots,k-1$, i.e., the determinant of

$$\begin{bmatrix} h_0f_0'(x_1(1))-\lambda & h_1f_1'(x_1(2)) & 0 & \cdots & h_{-1}f_{-1}'(x_1(N))e^{i\theta} \\ h_{-1}f_{-1}'(x_1(1)) & h_0f_0'(x_1(2))-\lambda & h_1f_1'(x(3)) & \cdots & 0 \\ 0 & h_{-1}f_{-1}'x_1(2) & h_0f_0'(x_1(3))-\lambda & & 0 \\ \vdots & & & & \vdots \\ h_1f_1'(x_1(1))e^{-i\theta} & 0 & 0 & & h_0f_0'(x_1(N))-\lambda \end{bmatrix} . \tag{24}$$

Let the characteristic polynomial for the above equation be

$$a_N\lambda^N+a_{N-1}\lambda^{N-1}+\cdots+a_1\lambda+a_0=0\ . \tag{25}$$

From Eq. (24), it can be seen that the only θ-dependent term in the characteristic polynomial is

$$a_0=G_+e^{i\theta}+G_-e^{-i\theta}+R\ ,$$

where

$$G_+=-\left[h_{-1}^N\left[\prod_{i=1}^{N}f_{-1}(x(i))\right]\right]$$

and

$$G_-=-\left[h_1^N\left[\prod_{i=1}^{N}f_1(x(i))\right]\right]\ ,$$

and R is some real constant. One can see that if derivatives of the coupling functions are symmetric

$$h_1f_1'=h_{-1}f_{-1}'\ ,$$

then this term a_0 is real. One more case in which the term a_0 is real is when any of the eigenvalues λ is real for some $\theta\in(0,\pi)$ (0 and π are excluded since polynomial is any way real in these cases). In both the cases a theorem in analysis by Pólya and Szegö [18] is useful. The theorem implies that if all the roots lie in the complex unit circle for extremum values of a_0, i.e., for $\theta=0$ and π then they lie in the complex unit circle for the values in between. Thus in these two cases, the stability for $k=1$ and $k=2$ implies the stability for any value of k, i.e., only two values of θ, 0 and π, need to be studied.

III. EXAMPLES

Here we illustrate the above formalism using a few examples.

A. Homogeneous case

As a specific example, for the special case when all the maps are the same [Eq. (8)], we take the logistic map,

$$f(x)=\mu x(1-x)\ , \tag{26}$$

where $0\le\mu\le4$ and $x\in[0,1]$. This map has several stable periodic orbits depending on the value of μ [19]. In particular, it shows a period-doubling structure leading to a period-doubling attractor [19]. The analysis of the preceding section shows that for the coupled logistic map the entire period-doubling structure and the structure of other periodic windows will be lifted to the chain $\mathcal{C}_k$ for the same values of μ together with the same stability properties for all k.

The second example is that considered by Waller and Kapral [12]. They consider the maps

$$\begin{aligned} h_0f_0(x)&=\mu x(1-x)-2\gamma x\ , \\ h_1f_1(x)&=h_{-1}f_{-1}(x)=\gamma x\ . \end{aligned} \tag{27}$$

Using Eq. (7) for the fixed point and the condition $\lambda=\pm1$, i.e., the condition for marginal stability, we obtain the boundaries of the stability region of the fixed point and the periodic solution in the μ-γ plane. Our results coincide with those of Ref. [12]. For example, for the fixed-point homogeneous solution $x^*=0$, the stability criterion using Eq. (7) is given by

$$\mu(1-2x^*)-2\gamma+\gamma(e^{i\theta}+e^{-i\theta})=\pm 1 ,$$

which means

$$\mu=\pm 1+4\gamma \sin^2(\theta/2) \tag{28}$$

where $\theta=2\pi j/k, j=1,2,\ldots,k$. This coincides with Eq. (2.6) of Ref. [12].

B. Stability of higher spatial periods

Now we will illustrate the procedure for higher N with coupled logistic maps with $\mu=4$, i.e., $f(x)=\mu x(1-x)$ and

$$x_{t+1}(i)=(1-\epsilon)f(x_t(i)) + \frac{\epsilon}{2}[f(x_t(i+1))+f(x_t(i-1))] , \tag{29}$$

where $0\le\epsilon\le 1$. We discuss the stability of the following solutions.

(a) First, consider a fixed-point solution $S_1(2,1)=(x_+,x_-)$ of Eq. (29) for the chain $\mathcal{C}_2$ with $x_+\neq x_-$. The solution is

$$x_\pm=\frac{[8\epsilon-3\pm(32\epsilon^2-36\epsilon+9)^{1/2}]}{8(2\epsilon-1)} . \tag{30}$$

The stability of the k replica solution $S_1(2,k)$ can be studied using Eq. (22). The criterion for the stability of a k-replica solution is that any of the eigenvalues of the Jacobian should not have modulus grater than unity. Using Eq. (22), one can say that the solution $S_1(2,k)$ is stable if none of the k matrices $M(\theta)$ have an eigenvalue with modulus greater than unity. These matrices $M(\theta)$ are given by

$$M(\theta)=\begin{bmatrix} (1-\epsilon)f'(x_+) & \frac{\epsilon}{2}(1+e^{i\theta})f'(x_-) \\ \frac{\epsilon}{2}(1+e^{-i\theta})f'(x_+) & (1-\epsilon)f'(x_-) \end{bmatrix} , \tag{31}$$

where $\theta=0,2\pi/k,\ldots,(k-1)2\pi/k$. For the stability of original solution $S_1(2,1)$ we have to consider the eigenvalues of $M(0)$. The eigenvalue equation for $M(0)$ is given by

$$(2\epsilon-1)\lambda^2+2(1-\epsilon)\lambda+(32\epsilon^2-36\epsilon+8)=0 . \tag{32}$$

This equation has solution $\lambda=-1$ when $\epsilon=(4\pm\sqrt{6})/8=0.8061\ldots$ or $\epsilon=0.1939\ldots$ and similarly $\lambda=1$ when $\epsilon=0.75$. The eigenvalue is complex if

$$[2(1-\epsilon)]^2-4(2\epsilon-1)(32\epsilon^2-36\epsilon+8)<0 ,$$

and has modulus unity when

$$2\epsilon-1=+32\epsilon^2-36\epsilon+8 ,$$

i.e.,

$$\epsilon=(19\pm\sqrt{73})/32=0.8608\ldots \text{ and } 0.3268\ldots .$$

Thus the solution given by Eq. (30) is stable for the values of ϵ in the range $(4+\sqrt{6})/8=0.806\ldots$ to $(19+\sqrt{73})/32=0.860\ldots$. At $\epsilon=0.806\ldots$, eigenvalue crosses -1, and at $\epsilon=0.860\ldots$, the eigenvalue crosses the unit circle at complex conjugate values. As expected, one gets period doubling in the first case for $\epsilon<0.806\ldots$, and in the second case for $\epsilon>0.860\ldots$ a Hopf bifurcation is observed.

Since this is a fixed-point solution with symmetric coupling, we use the criterion noted in the preceding section. To check the stability of the solution $S_1(2,k)$ for the chain $\mathcal{C}_{k\times 2}$ obtained by k replicas of the solution $S_1(2,1)$, it is sufficient to consider only two values of θ, namely $\theta=0$ and π in Eq. (19). The condition for stability for $\theta=0$ is the same as that for the solution $S_1(2,1)$ and one needs to check only for $\theta=\pi$ additionally. The eigenvalues for the matrix $M(\pi)$ are $(1-\epsilon)f'(x_+)$ and $(1-\epsilon)f'(x_-)$. Calculations on the lines of $M(0)$ show that no further condition is imposed for the stability in the range in which $S_1(2,1)$ solution is stable. Thus the solution $S_1(2,k)$ remains stable for the same range of ϵ values for all k.

(b) We consider a period two solution of Eq. (29) for the closed chain $\mathcal{C}_2$, namely $S_2(2,1)=\{R_1,R_2,R_1,\ldots\}$ where $R_1=(x_1(1),x_1(2))$ and $R_2=(x_2(1)=x_1(2),x_2(2)=x_1(1))$ with $x_1(1)\neq x_1(2)$. With some algebra it can be seen that the solution is analytically given by

$$x_1(1)=[8\epsilon-5+(32\epsilon^2-28\epsilon+5)^{1/2}]/[8(2\epsilon-1)] , \tag{33}$$

$$x_1(2)=[8\epsilon-5-(32\epsilon^2-28\epsilon+5)^{1/2}]/[8(2\epsilon-1)] .$$

These solutions exist when

$$32\epsilon^2-28\epsilon+5>0 , \tag{34}$$

i.e., if $\epsilon>\frac{5}{8}=0.625$ or $\epsilon<\frac{1}{4}=0.25$.

As in the previous case, using Eq. (22), we can say that the solution $S_2(2,k)$, is stable if none of the k matrices $M(\theta)$ have an eigenvalue with modulus greater than unity. In the present case matrices $M(\theta)$ are given by

$$M(\theta)=\prod_{t=1}^{2}\begin{bmatrix} (1-\epsilon)f'(x_t(1)) & \frac{\epsilon}{2}(1+e^{i\theta})f'(x_t(2)) \\ \frac{\epsilon}{2}(1+e^{-i\theta})f'(x_t(1)) & (1-\epsilon)f'(x_t(2)) \end{bmatrix} , \tag{35}$$

where $\theta=0,2\pi/k,\ldots,(k-1)2\pi/k$. For the stability of original solution $S_2(2,1)$ we have to consider the eigenvalues of $M(0)$. The eigenvalue equation for $M(0)$ is given by

$$\lambda^2(4\epsilon^2-4\epsilon+1)+\lambda(-128\epsilon^3+172\epsilon^2-72\epsilon+8) + 1024\epsilon^4-1792\epsilon^3+1040\epsilon^2-224\epsilon+16=0 . \tag{36}$$

Let us rewrite this equation as

$$\lambda^2(2\epsilon-1)^2-\lambda[(2\epsilon)^2-2(2\epsilon-1)(-32\epsilon^2+28\epsilon-4)] +(-32\epsilon^2+28\epsilon-4)^2=0 \ . \quad (37)$$

Let us introduce a variable Z given by $Z=-\sqrt{\lambda}$. With some algebra it can be seen that Z satisfies the equation

$$(2\epsilon-1)Z^2-2\epsilon Z-32\epsilon^2+28\epsilon-4=0 \ . \quad (38)$$

This equation in Z is just like Eq. (32) in λ which has been considered in detail in the previous example, except that one has replaced ϵ by $1-\epsilon$. Thus this solution is stable for the values of ϵ in the range $(4-\sqrt{6})/8=0.1938\ldots$ to $(13-\sqrt{73})/32=0.1392\ldots$.

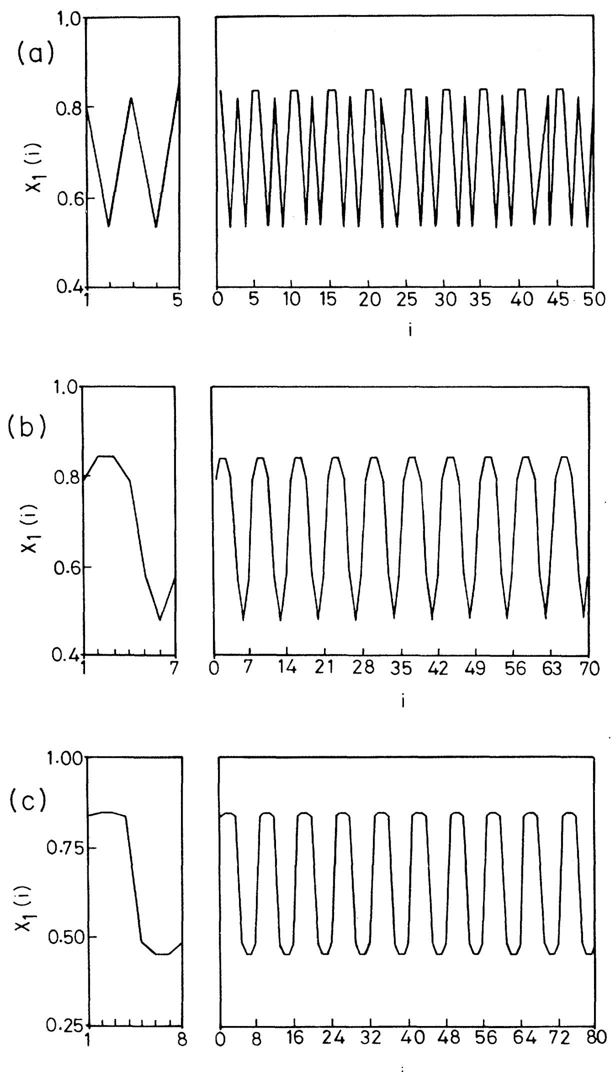

FIG. 1. (a) Kink-type solution for $N=5$ [$S_2(5,1)$] on the left-hand side and its replica solutions for $k=10$ [$S_2(5,10)$] on the right-hand side. (b) and (c) show similar kink-type solutions for $N=7$ and $N=8$, respectively.

At $(4-\sqrt{6})/8$ the eigenvalue in the previous case was -1 which is $+1$ now, since the actual eigenvalues [Eq. (36)] are squares of the solutions of Eq. (38).

The stability of the k replica solution $S_2(2,k)$ has been verified numerically using Eq. (22). For even k, the lower bound shifts to 0.140 37. . . . For odd k, the lower bound approaches this value according to the sequence 0.140 09. . . for $k=3$; 0.140 26. . . for $k=5$; 0.140 31. . . for $k=7$, etc.

(c) Our next example is the kink-type solutions [4] to Eq. (29). We have considered several kink-type solutions. Here we discuss the following solutions for $\mu=3.41$.

(i) $N=5$. Consider the basic unit $S_2(5,1)$ shown in Fig. 1(a). Figure 1(a) also shows the replica solution with $k=10$. The basic unit $S_2(5,1)$ is stable in the ϵ range from 0 to 0.0967. . . . We use Eq. (22) to determine the stability of the replica solutions. The higher-order solutions are stable in the same limits within computational accuracy. This has been confirmed by actual numerical simulations for replica solutions with many k values.

(ii) $N=7$. Consider the basic unit $S_2(7,1)$ shown in Fig. 1(b). Figure 1(b) also shows the replica solution with $k=10$. The basic unit $S_2(7,1)$ is stable in the ϵ range from 0 to 0.337 72. . . . We analyze the stability of replica solutions using Eq. (22). We find that for even k, the stability range reduces to $\epsilon=0$ to $\epsilon=0.337\,62\ldots$. For odd k the lower limit remains the same, i.e., $\epsilon=0$ and the upper limit approaches 0.337 62. . . by the sequence 0.337 66. . . for $k=3$, 0.337 64. . . for $k=5$, etc. Again, this result has been confirmed by actual numerical simulations.

(iii) $N=6$ and 8. We consider the kink solutions with an equal number of consecutive points in the upper and lower branches, i.e., 3 and 4 points for $N=6$ and 8, respectively. The basic unit $S_2(8,1)$ for $N=8$ and its replica solution with $k=10$ are shown in Fig. 1(c). In this case using Eq. (22) and also by actual numerical simulations we find that the stability of the replica solutions remains unchanged by enlargement of the phase space.

IV. TRAVELING-WAVE SOLUTION

Traveling-wave solutions have always been of interest to the pattern formation community. For example, they can be seen in convection patterns in annular cells [20]. The stability criterion undergoes one more simplification in this case. Let us consider a solution of wavelength N repeated k times traveling to the right with velocity 1, i.e., moving on sites to the right at each time step. Obviously, it is temporally periodic with period N. Note that time period and space period are the same in this case. Let the solution for $k=1$ be $R_1=(x_1(1),x_1(2),\ldots,x_1(N))$ at time 1. Obviously, $R_2=(x_2(1)=x_1(2),x_2(2)=x_1(3),\ldots,x_2(N)=x_1(1))$ at time 2 and $R_t=(x_t(1)=x_1(t),x_t(2)=x_1(t+1),\ldots,x_t(N)=x_1(t-1))$ at time t. Thus $S_N(N,1)=\{R_1,R_2,\ldots,R_N,R_1,\ldots\}$. Now we consider the stability of the k replica solution

$$S_N(N,k)=\{\langle R_1,\ldots,R_1\rangle_k,\langle R_2,\ldots,R_2\rangle_k,\ldots\langle R_N,\ldots,R_N\rangle_k\}\ .$$

The stability will be determined by the Jacobian

$$J=J_N\cdots J_2J_1\ . \tag{39}$$

Now looking at the fact that the traveling-wave solution will look like a fixed point in a frame of reference moving with the same velocity, one can infer that the different Jacobian matrices should be related by a unitary transformation. The relevant transformation π which is a $kN\times kN$ matrix is obvious. It should take the value of the variable at site i to the site $i+1$ cyclically. We have

$$\pi=\text{circ}(0,1,0,\ldots,0) \tag{40}$$

and the Jacobian matrix at time t is given by

$$\begin{aligned}J_t&=\pi J_{t-1}\pi^{-1}\\&=\pi^{t-1}J_1(\pi^{-1})^{t-1}\ ,\end{aligned} \tag{41}$$

where J_t is a $kN\times kN$ matrix given by

$$J_t=\begin{bmatrix} h_0f_0'(x_t(t)) & h_1f_1'(x_t(t+1)) & \cdots & 0 & h_{-1}f_{-1}'(x_t(t-1)) \\ h_{-1}f_{-1}'(x_t(t)) & h_0f_0'(x_t(t+1)) & & 0 & 0 \\ \vdots & & & & \vdots \\ h_1f_1'(x_t(t)) & 0 & \cdots & h_{-1}f_{-1}'(x_t(t-2)) & h_0f_0'(x_t(t-1)) \end{bmatrix}. \tag{42}$$

Using Eq. (41) and the property $\pi^{kN-1}=\pi^{-1}$ we get

$$J_{kN}\cdots J_2J_1=[\pi^{-1}J_1]^{kN}\ . \tag{43}$$

We also have [Eq. (39)]

$$J_{kN}\cdots J_2J_1=[J_N,\ldots,J_1]^k=J^k\ ,$$

where we have used the time periodicity $J_{rN+i}=J_i, r=0,1,2,\ldots,$ and $i=1,2,\ldots,N$. Hence

$$J=[\pi^{-1}J_1]^N\ . \tag{44}$$

Thus eigenvalues of $\pi^{-1}J_1$ are enough to infer about the stability of the traveling-wave solution. In fact, they are Nth roots of the eigenvalues of J. The matrix π is unitary and real and hence $\pi^{-1}=\pi^T$. Also, the fact to be noted is that π^{-1} can be block diagonalized by the same transformation as for J_1 and so one can block diagonalize both, take the product, and find the eigenvalues. Thus for a pattern $R_1=(x_1(1),x_1(2),\ldots,x_1(N))$ repeated k times, we need to consider the eigenvalues of $N\times N$ matrices $M(\theta)$ which are given by

$$M(\theta)=\begin{bmatrix} 0 & 0 & \cdots & 0 & e^{i\theta} \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}\begin{bmatrix} h_0f_0'(x_1(1)) & h_1f_1'(x_1(2)) & \cdots & h_{-1}f_{-1}'(x_1(N))e^{i\theta} \\ h_{-1}f_{-1}'(x_1(1)) & h_0f_0'(x_1(2)) & \cdots & 0 \\ \vdots & & & \vdots \\ h_1f_1'(x_1(1))e^{-i\theta} & 0 & \cdots & h_0f_0'(x_1(N)) \end{bmatrix}, \tag{45}$$

where $\theta=0,2\pi/k,\ldots,(k-1)2\pi/k$.

For the traveling-wave moving with velocity greater than 1 (say p), π will have to be replaced by π^p, which can again be block diagonalized in a similar manner.

Let us illustrate this procedure with the help of oscillating period-two solution considered in Eq. (33) [example (b) of the preceding section]. The stability of this solution can be studied using the matrix $\pi^{-1}J_1$ and the corresponding eigenvalue equation is

$$\lambda^2(2\epsilon-1)-2\lambda\epsilon-32\epsilon^2+28\epsilon-4=0\ . \tag{46}$$

We note that this equation is the same as Eq. (38) and the conditions for stability are as discussed before. The stability of the k replica solution $S_2(2,k)$ can be obtained by considering the eigenvalues of the k matrices $M(\theta)$ given by [Eq. (45)]

$$\begin{bmatrix} 0 & e^{i\theta} \\ 1 & 0 \end{bmatrix}\begin{bmatrix} (1-\epsilon)f'(x_t(1)) & \frac{\epsilon}{2}(1+e^{i\theta})f'(x_t(2)) \\ \frac{\epsilon}{2}(1+e^{-i\theta})f'(x_t(1)) & (1-\epsilon)f'(x_t(2)) \end{bmatrix}. \tag{47}$$

The results that we get using this matrix match with the one obtained while discussing this example in the preceding section.

V. HIGHER-DIMENSIONAL CASE

Let us consider a two-dimensional coupled-map lattice model with periodic boundary conditions. This is an evolution on the two-dimensional lattice L_{N_x,N_y} which is a $N_x \times N_y$ lattice. Let the evolution be given by the map

$$x_{n+1}(i,j)=h_0 f_0(x_n(i,j))+h_{x+}f_{x+}(x_n(i+1,j))+h_{x-}f_{x-}(x_n(i-1,j))+h_{y+}f_{y+}(x_n(i,j+1))+h_{y-}f_{y-}(x_n(i,j-1)) , \quad (48)$$

where $1 \le i \le N_x$ and $1 \le j \le N_y$. The functions f_0, f_{x+}, f_{x-}, f_{y+}, f_{y-} are some functions which describe evolution in an otherwise isolated space. The parameters h_0, h_{x+}, h_{y+}, h_{x-}, h_{y-} represent coupling strength and are chosen so that $x_{t+1}(i)$ lies in the same phase space and are assumed to be positive. We have associated variable x at time t to each point of the two-dimensional lattice L_{N_x,N_y}. As in the one-dimensional case we impose cyclic boundary conditions. In the one-dimensional case the lattice points form a ring of maps whereas here they form a torus. Let

$$R_t=[(x_t(1,1),x_t(2,1),\ldots,x_t(N_x,1)),(x_t(1,2),\ldots x_t(N_x,2)),\ldots,(x_t(1,N_y),\ldots,x_t(N_x,N_y))] \quad (49)$$

denote the state of the lattice at time t. Let $S_\tau(N_x,1;N_y,1)$ denote solution of Eq. (48) with temporal periodicity τ, i.e.,

$$S_\tau(N_x,1;N_y,1)=\{R_1,R_2,\ldots,R_\tau,R_1,\ldots\} .$$

Now arguing on the lines of arguments in the one-dimensional case, one can see that this solution repeated p times in the direction of the first index and q times in the direction of second index, i.e.,

$$S_\tau(N_x,p;N_y,q)=\{\langle[R_1,\ldots,R_1]_p\rangle_q , \langle[R_2,\ldots,R_2]_p\rangle_q , \vdots \langle[R_\tau,\ldots,R_\tau]_p\rangle_q\} ,$$

is a solution for the two-dimensional lattice L_{pN_x,qN_y} with temporal periodicity τ. Here $\langle[R_t,\ldots,R_t]_p\rangle_q$ represents a state R_t [Eq. (49)] repeated p times in the x direction and q times in the y direction.

We again pose the same question as in the one-dimensional case. We address the problem of what can be stated about the stability properties of $S_\tau(N_x,p;N_y,q)$ from the analysis of stability matrix of $S_\tau(N_x,1;N_y,1)$.

A. Homogeneous case

We begin with the case $N_x=1$, $N_y=1$, i.e., a homogeneous solution. We will suppress the indices N_x, N_y, i.e., $S_\tau(N_x,p;N_y,q)\equiv S_\tau(p,q)$. We will also suppress the indices i and j, i.e., $x_t(i,j)\equiv x_t$. Let us consider the stability of the periodic solution with period τ,

$$S_\tau(1,1)=\{R_1,\ldots,R_\tau,R_1,\ldots\}=\{x_1,\ldots,x_\tau,x_1,\ldots\} \quad (50)$$

for the lattice $L_{1,1}$. This is a stable solution provided

$$|f'(x_1)f'(x_2)\cdots f'(x_\tau)|<1 , \quad (51)$$

where

$$f(x)=h_0 f_0(x)+h_{x+}f_{x+}(x)+h_{x-}f_{x-}(x)+h_{y+}f_{y+}(x)+h_{y-}f_{y-}(x) .$$

For the homogeneous solution $S_\tau(p,q)$, the stability condition is that modulus of all eigenvalues of the $pq \times pq$ stability matrix $J=J_\tau \cdots J_2 J_1$ have magnitude less than one. Now we define Jacobian matrix J_t as

$$(J_t)_{m,n}=\frac{\partial x_t(k,l)}{\partial x_{t-1}(i,j)} , \quad (52)$$

where

$$m=(l-1)p+k$$

and

$$n=(j-1)p+i .$$

One can see that J_t is a $pq \times pq$ Jacobian matrix given by

$$J_t=b\ \mathrm{circ}(A_0,A_{y+},0_p,\ldots,0_p,A_{y-}) \quad (53)$$

where A_0, A_{y+}, A_{y-} are $p \times p$ matrices given by

$$A_0=\mathrm{circ}(h_0 f_0'(x_t),h_{x+}f_{x+}'(x_t),0,\ldots,0,h_{x-}f_{x-}'(x_t)) ,$$
$$A_{y+}=\mathrm{circ}(h_{y+}f_{y+}'(x_t),0,0,\ldots,0) ,$$
$$A_{y-}=\mathrm{circ}(h_{y-}f_{y-}'(x_t),0,0,\ldots,0) ,$$

and 0_p is a square matrix of order p with all elements equal to 0. Note that A_{y+} and A_{y-} are diagonal matrices and can also be written as

$$A_{y+}=h_{y+}f_{y+}'(x_t)I_p ,$$
$$A_{y-}=h_{y-}f_{y-}'(x_t)I_p ,$$

where I_p is an unit matrix of order p.

Note that J_t is a block circulant matrix with circulant blocks. We can block diagonalize J_t in matrices $\gamma_t(s)$, $s=1,2,\ldots,q$ where

$$\gamma_t(s)=A_0+\omega_s A_{y+}+\omega_s^{-1}A_{y-}$$
$$=\text{circ}(h_0f_0'(x_t)+h_{y+}f_{y+}'(x_t)\omega_s+h_{y-}f_{y-}'(x_t)\omega_s^{-1},h_{x+}f_{x+}'(x_t),0,\ldots,0,h_{x-}f_{x-}'(x_t)) \quad (54)$$

where

$$\omega_s=e^{i(2\pi(s-1)/q)} . \quad (55)$$

$\gamma_t(s)$ can be further diagonalized to give the eigenvalues $\lambda_t(r,s)$, $r=1,2,\ldots,p$ where

$$\lambda_t(r,s)=h_0f_0'(x_t)+h_{y+}f_{y+}'(x_t)\omega_s+h_{y-}f_{y-}'(x_t)\omega_s^{-1}+h_{x+}f_{x+}'(x_t)\omega_r+h_{x-}f_{x-}'(x_t)\omega_r^{-1} , \quad (56)$$

where

$$\omega_r=e^{i(2\pi(r-1)/p)} . \quad (57)$$

Let us define the following vectors:

$$\begin{aligned}\mathbf{F}_{t+}&=(h_{x+}f_{x+}(x_t),h_{y+}f_{y+}(x_t)) ,\\ \mathbf{F}_{t-}&=(h_{x-}f_{x-}(x_t),h_{y-}f_{y-}(x_t)) ,\\ \mathbf{\Omega}_+(r,s)&=(\omega_r,\omega_s) ,\\ \mathbf{\Omega}_-(r,s)&=(\omega_r^{-1},\omega_s^{-1}) .\end{aligned} \quad (58)$$

Thus Eq. (56) can be written as

$$\lambda_t(r,s)=h_0f_0'(x_t)+\mathbf{F}_{t+}\cdot\mathbf{\Omega}_+(r,s)+\mathbf{F}_{t-}\cdot\mathbf{\Omega}_-(r,s) . \quad (59)$$

Equation (59) suggests that it is possible to extend the result to higher dimensions. For example, in three dimensions, the same formulas hold except that each of the vectors will have three components.

The pq eigenvalues of the total Jacobian matrix are given by

$$\lambda(r,s)=\prod_{t=1}^{\tau}\lambda_t(r,s) , \quad (60)$$

where $r=1,\ldots,p$ and $s=0,1,\ldots,q$.

Let us consider the special case

$$f_0=f_{x-}=f_{x+}=f_{y-}=f_{y+}=f . \quad (61)$$

Now the eigenvalues of the Jacobian for this case are given by

$$\lambda(r,s)=\prod_{t=1}^{\tau}(h_0+h_{y+}\omega_s+h_{y-}\omega_s^{-1}+h_{x+}\omega_\tau+h_{x-}\omega_r^{-1})f'(x_t) . \quad (62)$$

By triangle inequality and the fact that couplings are positive we get

$$|\lambda(r,s)|\le|\lambda(1,1)| . \quad (63)$$

However, $\lambda(1,1)$ is the eigenvalue for $S_\tau(1,1)$. The condition for its stability is given in Eq. (51). Thus in this case, stability of the homogeneous solution $S_\tau(p,q)$ is guaranteed by the stability of the single-point solution $S_\tau(1,1)$ for the same parameters of the map. We have seen that a similar result holds good in one dimension. Thus the above statement appears to hold irrespective of the dimensionality of the lattice.

The second example is that considered by Oppo and Kapral [13]. They consider the maps

$$\begin{aligned}&h_0f_0(x)=f(x)-4\gamma x ,\\ &h_{x+}f_{x+}(x)=h_{x-}f_{x-}(x)=h_{y+}f_{y+}(x)=h_{y-}f_{y-}(x)=\gamma x .\end{aligned} \quad (64)$$

Let $p=q=k$. Using Eq. (62) for the fixed point and periodic point, we obtain the condition for $\lambda=\pm1$, i.e., the condition for marginal stability. Our results coincide with those of Ref. [13]. For the spatially homogeneous periodic point, using Eq. (62), N^2 eigenvalues $\lambda(k_1,k_2)$, ($k_1=1,2,\ldots,k$; $k_2=1,2,\ldots,k$) are given by

$$\begin{aligned}\lambda(k_1,k_2)&=\prod_{t=1}^{\tau}[f'(x_t)-4\gamma+\gamma(\omega_{k_1}+\omega_{k_1}^{-1}+\omega_{k_2}+\omega_{k_2}^{-1})]\\ &=\prod_{t=1}^{\tau}\{f'(x_t)+4\gamma(\cos[\pi(k_1+k_2)/k]\cos[2\pi(k_1-k_2)/k]-1)\} .\end{aligned} \quad (65)$$

The condition for marginal stability is that at least one of these eigenvalues is of magnitude unity and no eigenvalue is having magnitude greater than unity. The criterion coincides with Eqs. (2.6) and (2.7) of Ref. [13].

B. Case of higher spatial period

Now let us consider the case of higher spatial periodicity. As one would guess from the previous discussions, the Jacobian in this case should turn out to be the block circulant matrix with block circulant blocks and one should be able to block diagonalize them. This expectation is true indeed. We will discuss this case in detail now.

The problem is what can be stated about stability properties of $S_\tau(N_x,p;N_y,q)$ from the analysis of stability analysis of stability matrices of $S_\tau(N_x,1;N_y,1)$. To make the analysis easy we number the sites in a particular way.

Let us associate a function $g(i,j)$ with the site (i,j) defined by

$$g(i,j)=\left[\frac{j}{N_y}\right](pN_xN_y)+\left[\frac{i}{N_x}\right](N_xN_y)+[(j \bmod N_y)-1]N_x+(i \bmod N_x)\ , \tag{66}$$

where [ ] denotes the integer part. One can see that this function associates a unique number between 1 and N_xN_ypq with each point on the lattice. Now we define Jacobian matrix J_t as

$$(J_t)_{m,n}=\frac{\partial x_t(k,l)}{\partial x_{t-1}(i,j)}\ , \tag{67}$$

where

$$m=g(k,l)$$

and

$$n=g(i,j)\ .$$

Using Eq. (67) the Jacobian matrix looks like the following:

$$J_t=b\ \mathrm{circ}(A_0, A_{y+}, 0_{N_xN_yp}, \ldots, 0_{N_xN_yp}, A_{y-}) \tag{68}$$

is a block circulant matrix of order q with blocks being $N_xN_yp\times N_xN_yp$ matrices, where

$$A_0=b\ \mathrm{circ}(B_0, B_{x+}, 0_{N_xN_y}, \ldots, 0_{N_xN_y}, B_{x-})\ ,$$

$$A_{y+}=b\ \mathrm{circ}(B_{y+}, 0_{N_xN_y}, \ldots, 0_{N_xN_y})\ ,$$

$$A_{y-}=b\ \mathrm{circ}(B_{y-}, 0_{N_xN_y}, \ldots, 0_{N_xN_y})\ .$$

Here A_0, A_{y+}, A_{y-} are block circulant matrices of order p with blocks being matrices of order $N_x\times N_y$. Matrices A_{y+}, A_{y-} are themselves block-diagonal matrices and can be written in form of direct product with identity matrix,

$$A_{y+}=B_{y+}\otimes I_p\ ,$$

$$A_{y-}=B_{y-}\otimes I_p\ .$$

While indexing the site (i,j) by using the function $g(i,j)$, we have effectively divided the lattice in pq identical blocks. If one scans the lattice sites corresponding to the ascending order defined by the function g, the value of the variable x associated with the site is periodic with period N_xN_y. Any ith element in this order is equivalent to $k(N_xN_y)+i$th element. Thus one can view the lattice as comprising of pq identical blocks. The nonzero elements in the matrix B_0 describe connections between the sites within a block while the nonzero elements in the matrices B_{x+}, B_{x-}, B_{y+}, B_{y-} represent the connections between different blocks.

The only nonzero matrix elements of B_{x+}, B_{x-}, B_{y+}, B_{y-} are

$$(B_{x+})_{m,n}=X_+(x_t(1,j)),\quad m=jN_x,\ n=(j-1)N_x+1\ ;$$

$$(B_{x-})_{m,n}=X_-(x_t(N_x,j)),\quad m=(j-1)N_x+1,\ n=jN_x\ ;$$

$$(B_{y+})_{m,n}=Y_+(x_t(i,1)),\quad m=(N_y-1)N_x+i,\ n=i\ ;$$

$$(B_{y-})_{m,n}=Y_-(x_t(i,N_y)),\quad m=i,\ n=(N_y-1)N_x+i\ .$$

where $j=1,\ldots,N_y$ and $i=1,\ldots,N_x$. The remaining elements are zero. We have also used the following simplifying notation:

$$W(z)=h_0f_0'(z)\ ,$$

$$X_+(z)=h_{x+}f_{x+}'(z)\ ,$$

$$X_-(z)=h_{x-}f_{x-}'(z)\ ,$$

$$Y_+(z)=h_{y+}f_{y+}'(z)\ ,$$

$$Y_-(z)=h_{y-}f_{y-}'(z)\ .$$

Matrix B_0 is a bit more complicated. It is almost like the Jacobian for the case $p=q=1$ except for the fact that the elements coming in due to periodic boundary conditions are absent. The nonzero elements of B_0 are given by

$$(B_0)_{m,m}=W(x_t(i,j))\ ,$$

$$(B_0)_{m,m+1}=X_+(x_t(i+1,j)),\quad i<N_x\ ,$$

$$(B_0)_{m,m-1}=X_-(x_t(i-1,j)),\quad i>1\ ,$$

$$(B_0)_{m,m+N_x}=Y_+(x_t(i,j+1)),\quad j<N_y\ ,$$

$$(B_0)_{m,m-N_x}=Y_-(x_t(i,j-1)),\quad j>1\ ,$$

where $m=(j-1)N_x+i$, $i=1,\ldots,N_x$, $j=1,\ldots,N_y$.

Matrix J_t can be block diagonalized into blocks of order N_xN_yp and the blocks are given by

$$\Gamma_t(s)=A_0+A_{y+}\omega_s+A_{y-}\omega_s^{-1}$$

$$=b\ \mathrm{circ}(B_0+B_{y+}\omega_s+B_{y-}\omega_s^{-1}, B_{x+}, 0_{N_XN_Yp}, \ldots, 0_{N_XN_Yp}, B_{x-})\ , \tag{69}$$

where $s=1,2,\ldots,q$. The matrices $\Gamma_t(s)$ can be further block diagonalized as [see Eqs. (55) and (57)]

$$M_t(r,s)=B_0+B_{y+}\omega_s+B_{y-}\omega_s^{-1}+B_{x+}\omega_r+B_{x-}\omega_r^{-1} . \quad (70)$$

Let us define the following vectors. The components of these vectors are $N_xN_y\times N_xN_y$ matrices

$$\begin{aligned} \mathbf{F}_{t+}&=(B_{x+},B_{y+}) ,\\ \mathbf{F}_{t-}&=(B_{x-},B_{y-}) ,\\ \mathbf{\Omega}_+(r,s)&=(\omega_r I_{N_xN_y},\omega_s I_{N_xN_y}) ,\\ \mathbf{\Omega}_-(r,s)&=(\omega_r^{-1} I_{N_xN_y},\omega_s^{-1} I_{N_xN_y}) . \end{aligned} \quad (71)$$

This allows us to write $M_t(r,s)$ in a vector form,

$$M_t(r,s)=B_0+\mathbf{F}_{t+}\cdot\mathbf{\Omega}_+(r,s)+\mathbf{F}_{t-}\cdot\mathbf{\Omega}_-(r,s) . \quad (72)$$

Thus the final matrices appear in the form

$$M(r,s)=\prod_{t=1}^{\tau}[B_0+\mathbf{F}_{t+}\cdot\mathbf{\Omega}_+(r,s)+\mathbf{F}_{t-}\cdot\mathbf{\Omega}_-(r,s)] . \quad (73)$$

Thus the job of diagonalizing $N_xN_ypq\times N_xN_ypq$ matrix is reduced to diagonalization of pq matrices of order N_xN_y.

VI. DISCUSSION

We have discussed the conditions that ensure the stability of spatially and temporally periodic orbits. In addition, our analysis also leads to the following important conclusion about unstable periodic orbits. As noted in a comment after Eq. (21), the matrix j appears as a block of the matrix D. Hence, a solution built out of the replicas of unstable periodic orbits will also be unstable. Enlargement of phase space and the effect of couplings cannot stabilize an unstable replica solution. The unstable periodic orbits are dense on the chaotic attractor. They are supposed to form the backbone of the dynamics on the attractor. One can calculate properties like invariant density, Lyapunov exponent knowing the periodic orbits [21,22]. One can even predict the time series using them [23]. Our formalism will be useful if one tries to use unstable periodic orbits to analyze the spatially extended systems. It is clear that the replica solutions can be used to construct a hierarchy of unstable periodic orbits based on the orbits for building blocks. This may help in the organization of spatio-temporal chaos on the lines of arguments in Ref. [21].

So far as spatially and temporally periodic orbits are concerned, we have shown that the stability of spatially and temporally periodic orbits can be analyzed in terms of smaller ones made up of building blocks of spatial periodicity. We find that for the homogeneous solution no further conditions are imposed if $f_0=f_1=f_{-1}$ and the stable solution for a single point remains stable on the enlargement of phase space and the introduction of couplings. However, solutions with larger wavelengths require additional conditions for stability. These conditions depend on the stability matrices for the building block of spatial periodicity and the roots of unity. The stability conditions undergo an additional simplification in the case of a traveling wave solution. We have discussed this briefly.

We have also discussed the two-dimensional extension of our formalism. From the convenient form in which the equations can be set, it is obvious that the generalization to higher dimensions is also possible.

If one tries to analyze the problems similar to the ones analyzed here, in oscillator arrays this procedure can be easily used to simplify the computation. Even if the model involves more than nearest-neighbor interactions, such as next-nearest-neighbor interaction or global coupling, the procedure still remains useful with minor modifications. Thus the scope of our formalism is fairly general and can be used to analyze a variety of physical problems.

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