Soliton on a disordered lattice

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(Received 2 September 1992)

A stochastic version of the lattice nonlinear Schrödinger equation, allowing treatment by means of the inverse-scattering technique and having an exact one-soliton solution, is introduced. It is shown that such a model is a useful tool for investigation of a wide class of nonlinear lattices affected by spatiotemporally random forces. A number of the most important statistical characteristics of soliton dynamics governed by such models can be evaluated without any assumption about the smallness of random perturbations. The problem is studied in detail in two limiting cases: small and large intensities of fluctuations of a stochastic term in the integrable equation.

PACS number(s): 05.45.+b, 03.20.+i

In a recent paper Scharf and Bishop [1] reported a new exactly integrable version of a "perturbed" lattice nonlinear Schrödinger equation, which is a combination of the Ablowitz-Ladik model [2] and a linear potential. The integrability of the continuum limit of that equation has been known [3, 4]. Moreover, as has been shown by Besieris [5], a one-soliton solution of a nonlinear evolution equation, in which coefficients of the linear potential are time-dependent functions, can also be found. Simultaneously in Ref. [5] another, more general, version of the stochastic nonlinear Schrödinger equation has been introduced. In the last case a stochastic term was described by a Gaussian field, being δ-function-correlated in time and having a squared correlator in space. The known solution of the first equation, a time dependent coefficient which also was assumed to be a Gaussian δ-function-correlated process, together with the observation of the equivalence of statistical properties of the two randomly perturbed evolution equations, allowed Besieris to solve exactly (in statistical meaning) the nonlinear Schrödinger equation affected by spatiotemporally disorder. To the best of the author's knowledge the method of the effective equation (as we shall call the approach of Ref. [5]) being successfully applied to the mentioned particular case up to now did not find extension for other nonlinear models [6]. In the present report we show a possibility of one more application of that technique, which in fact demonstrates wide perspectives of the method. We shall deal with discrete evolution equations. The work should be considered within the framework of general investigations of randomly perturbed discrete systems, which have created recent interest [7] and in some sense have been studied much less than their continuum analogs [8].

First of all we introduce a lattice model, slightly generalized in comparison with that of Scharf and Bishop. It reads

$$i(d\psi_n/dt) = -(1+|\psi_n|^2)(\psi_{n-1}+\psi_{n+1}) + a(t)n\psi_n. \quad (1)$$

Here $a(t)$ is a time dependent function (cf. Ref. [1]) and a term of type $b(t)\psi_n$ is assumed to be taken into account by corresponding phase renormalization. Using a U-V pair from Ref. [1], and doing direct calculations, one can make sure that Eq. (1) can also be included into the scheme of the inverse scattering technique. The integrability of the continnum limit of (1) has been stated by Balakrishnan [4].

Knowing a one-soliton solution of Eq. (1) at $a(t)$ =const [1], it is not difficult to find a corresponding solution of Eq. (1) for an arbitrary function $a(t)$. It is given by

$$\psi_n(t) = \sinh\beta \,\mathrm{sech}\{\beta[n-x(t)]\}\, \exp\{-i[\phi(t)+n\alpha(t)]\}, \quad (2)$$

where β is a constant and functions $\phi(t)$, $x(t)$, and $\alpha(t)$ are defined by

$$d\phi(t)/dt = -2\cosh\beta\,\cos\alpha(t), \quad (3)$$

$$dx(t)/dt = -(2/\beta)\sinh\beta\,\sin\alpha(t), \quad (4)$$

$$d\alpha(t)/dt = a(t). \quad (5)$$

We shall consider a case of a random function $a(t)$ being a Gaussian process with zero mean value: $\langle a(t)\rangle = 0$, dispersion σ^2, and zero correlation radius

$$\langle a(t)a(t')\rangle = 2\sigma^2\delta(t-t') \quad (6)$$

(throughout the paper we use angular brackets for averaging over all statistical realizations of respective random processes).

Let us also introduce a more general stochastic nonlinear lattice equation

$$i(dv_n/dt) = -(1+|v_n|^2)(v_{n-1}+v_{n+1}) + f_n(t)v_n, \quad (7)$$

where a Gaussian random function $f_n(t)$ is defined by statistical characteristics as follows:

$$\langle f_n(t)\rangle = 0, \quad \langle f_n(t)f_m(t')\rangle = 2D_{n,m}\delta(t-t'). \quad (8)$$

Following Ref. [5] the main subject of the further investigation will be the correlation function $\langle V_N(\mathbf{n},t)\rangle$, where $V_N(\mathbf{n},t)$ is given by

$$V_N(\mathbf{n},t) = \prod_{p=1}^{N/2} v^*_{n_{2p}}(t)v_{n_{2p-1}}(t) \quad (9)$$

with $\mathbf{n} = (n_1, ..., n_N)$, N being assumed to be even, and the asterisk denoting complex conjugation. Also we introduce a quantity

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$$\Psi_N(\mathbf{n},t)=\prod_{p=1}^{N/2}\psi^*_{n_{2p}}(t)\psi_{n_{2p-1}}(t). \qquad (10)$$

Two equations for V_N and Ψ_N follow directly from (7),(1) and definitions (9),(10)

$$i\frac{d\langle V_N\rangle}{dt}=\sum_{q=1}^{N}(-1)^q(\langle V_N^{(n_q+1)}\rangle+\langle V_{N+2}^{(n_q+1)}\rangle+\langle V_N^{(n_q-1)}\rangle+\langle V_{N+2}^{(n_q-1)}\rangle)-\sum_{q=1}^{N}(-1)^q\langle f_{n_q}(t)V_N\rangle, \qquad (11)$$

$$i\frac{d\langle \Psi_N\rangle}{dt}=\sum_{q=1}^{N}(-1)^q(\langle \Psi_N^{(n_q+1)}\rangle+\langle \Psi_{N+2}^{(n_q+1)}\rangle+\langle \Psi_N^{(n_q-1)}\rangle+\langle \Psi_{N+2}^{(n_q-1)}\rangle)-\sum_{q=1}^{N}(-1)^q n_q\langle a(t)\Psi_N\rangle. \qquad (12)$$

Here the index $(n_q\pm1)$ means that a subindex n_p in the products (9) and (10) is replaced by $n_q\pm1$ for each $p=q$, and

$$V_{N+2}^{(n_q\pm1)}=v^*_{n_q}v_{n_q}V_N^{(n_q\pm1)} \qquad (13)$$

($\Psi_N^{(n_q\pm1)}$ is similar).

In fact (11) and (12) are infinite systems of bounded equations for correlators of $\langle V_N(\mathbf{n},t)\rangle$ and $\langle\Psi_N(\mathbf{n},t)\rangle$ types (note that values with the subindex $N+2$ belong to the mentioned class). In order to close them, i.e., to transform the last term on the right-hand side, we employ the Novikov theorem [9], taking into account statistical properties of the random processes. This yields

$$\langle f_{n_q}(t)V_N\rangle=i\sum_{p=1}^{N}(-1)^pD_{n_p,n_q}\langle V_N\rangle, \qquad (14)$$

$$n_q\langle a(t)\Psi_N\rangle=i\sigma^2\sum_{p=1}^{N}(-1)^pn_pn_q\langle\Psi_N\rangle. \qquad (15)$$

Let us suppose now that both Eqs. (1) and (7) are subject to the same initial conditions, i.e.,

$$\Psi_N(\mathbf{n},t=0)=V_N(\mathbf{n},t=0) \qquad (16)$$

for all N and $\mathbf{n}$. Then one can conclude that

$$\langle\Psi_N(\mathbf{n},t)\rangle=\langle V_N(\mathbf{n},t)\rangle \qquad (17)$$

at any moment of time, if

$$\sum_{q=1}^{N}\sum_{p=1}^{N}(-1)^q(-1)^pD_{n_p,n_q}\langle V_N\rangle=\sigma^2\sum_{q=1}^{N}\sum_{p=1}^{N}(-1)^q(-1)^pn_pn_q\langle V_N\rangle. \qquad (18)$$

Indeed in that case both systems (11),(14) and (12),(15) are identical. Thus under the condition (18) corresponding statistical characteristics of Eq. (7) can be found through calculation of characteristics of Eq. (1), for which the exact one-soliton solution is known. In this sense Eq. (1) can be considered as an *effective equation* associated with Eq. (7).

By analogy with Ref. [5] one can verify that (18) is satisfied identically in a particular case as follows:

$$D_{n,m}=D_0-\tfrac{1}{2}\sigma^2(n-m)^2. \qquad (19)$$

However, this correlator has an "unphysical" region, when $(n-m)^2>2D_0/\sigma^2$, since it takes negative values. Nevertheless, it can be used. After insertion $D''(0)$ (the primes here mean the derivative with respect to the argument) instead of σ^2, Eq. (19) can be considered as the first terms of the Taylor expansion of a correlation function $D_{n,m}\equiv D(n-m)$ of a rather general type [note that due to summation a term proportional to the first power of $(n-m)$ will give zero contribution]. Thus, the requirement (18) is satisfied approximately for sufficiently close points n_p and n_q and for a differentiable function $D(n-m)$, depending on n and m through their difference. It allows one to construct a perturbation theory for statistical characteristics of a soliton of Eq. (7), using another small parameter, rather than amplitude of the random perturbation. In particular, (18) is exactly valid for momenta of the soliton intensity (i.e., when all n_p are equal).

Let us now discuss an applicability region of the expansion of $D_{n,m}$. The main problem is that the discreteness of the evolution equation (7) raises restrictions (in this sense even the statement about equality of intensity momenta of both lattices is not exact). In order to explain this we recall Eqs. (11) and (12) and suppose that a quantity of main interest is a momentum, say, in a point $n_q=n_*$. The equations for $\langle V_N\rangle$ and $\langle\Psi_N\rangle$ will contain terms with $n_*\pm1$ and higher ones given in $N+2$ points. Therefore in order to calculate the momentum of order N one has to obtain all even momenta having orders greater than N. But each equation for a higher momentum and for correlators containing $n_q=n_*\pm2$ gives larger "deviation" of points n_q from n_*. For example, momenta with $N+2l$ (l being integer) will be related to correlators in points $n_*\pm2l$. Thus the treatment of Eq. (19) as an expansion fails with N.

However, this difficulty can be avoided by taking into account the following arguments. We are interested in dynamics of a soliton, initially localized in the region of order of β^{-1} in the neighborhood of $x(0)$ [see requirement (16) and representation (2)]. This means that initially both sides of Eq. (18) are exponentially small at

$$|n_p-n_q|>\beta^{-1}. \qquad (20)$$

Thus, only for the region $|n_p-n_q|<\beta^{-1}$ does one have to demand validity of the expansion (19). It gives immediately a requirement

$$\mu_0^2=\beta^{-2}D''(0)\ll1. \qquad (21)$$

Besides the case of weak fluctuations it is satisfied for strongly localized pulses, having large β, and (or) smooth spatial perturbations. As is clear μ_0 [$\sim D(0)/r\beta$; r being a spatial correlation radius] is a small parameter of the problem under consideration and the expansion error is of order of μ_0^3.

In fact, applicability of the method is wider, since another limit, $\beta \ll 1$, can also be treated. As is clear, in that case a soliton width is much greater than distance between neighbor lattice sites, and one deals with the "quasicontinuum" limit for which

$$\mu_1^2(\sim \max\{\beta^2; D(0)r^{-2}\})$$
$$= \left| \frac{\langle V_N^{(n_q+1)}\rangle + \langle V_N^{(n_q-1)}\rangle - 2\langle V_N\rangle}{\langle V_N^{(n_q+1)}\rangle + \langle V_N^{(n_q-1)}\rangle} \right| \ll 1. \quad (22)$$

Then as usual the sum $\langle V_N^{(n_q+1)}\rangle + \langle V_N^{(n_q-1)}\rangle$ can be replaced by $\partial^2 V_N/\partial n_q^2 + 2\langle V_N\rangle$. All terms in Eqs. (11) and (12) for any N will be defined on the same set of points $\mathbf{n}$. Thus the perturbation theory for studying Eq. (7) can be constructed with respect to the small parameter μ_1, as well. However, in the last case one has to take into account that terms proportional to higher derivatives in the expansion of $D(n-m)$ yield a perturbation as well (in contrast to the previous case it is not taken into account by only the small parameter μ_1). But now its role is different. It results in a restriction to points on which the correlators $\langle V_N\rangle$ and $\langle \Psi_N\rangle$ are equal approximately: correlators in points far from each other cannot be considered. In general the method allows one to study correlators given on a set of points, satisfying the condition $(n_p - n_q)^2 r^{-2} \ll 1$. This means in particular that smooth random perturbations having large correlation radii allow one to describe larger diversity of the correlation functions $\langle V_N\rangle$. But what is most important is that neither case (21) nor (22) requires smallness of stochastic term (what typically is a background of most of the preceding approaches [8]).

Thus soliton dynamics of both Eqs. (1) and (7) subject to the supposition introduced may be described by the system of the stochastic equations (3)–(5). Despite the system essentially differing from that obtained for the continuous nonlinear Schrödinger equation (see Ref. [5]) and being more complicated, some results can be obtained. In particular, introducing a three-point probability density $P(\phi, x, \alpha, t)$, which describes distribution of $\phi(t)$, $x(t)$, and $\alpha(t)$ at any moment of time, in the usual manner [10]:

$$P(\phi, x, \alpha, t) = \langle \delta(\phi(t) - \phi)\delta(x(t) - x)\delta(\alpha(t) - \alpha)\rangle \quad (23)$$

[$\delta(\,)$ being Dirac's δ function], one derives the Fokker-Planck equation

$$\frac{\partial P}{\partial t} = 2\cosh\beta\cos\alpha\frac{\partial P}{\partial\phi} + \frac{2}{\beta}\sinh\beta\sin\alpha\frac{\partial P}{\partial x} + \sigma^2\frac{\partial^2 P}{\partial\alpha^2}. \quad (24)$$

Its solution has to satisfy the initial condition

$$P(\phi, x, \alpha, 0) = \delta(\phi(0) - \phi_0)\delta(x(0) - x_0)\delta(\alpha), \quad (25)$$

where ϕ_0 and x_0 are the initial phase and position of a soliton. Equation (24) can be treated in an evident form in limiting cases.

Let $\sigma^2 \ll 1$, which corresponds to the large correlation radius of $D_{n,m}$. The quantity α in a region giving the main contribution is small and one can expand $\cos\alpha$ and $\sin\alpha$. Then the leading order of the probability density takes the form

$$P = \frac{\sqrt{3}}{2\pi t^2\sigma^2}\delta(\phi_0 + 2t\cosh\beta)\exp\left(-\frac{3}{\sigma^2 t^2}\left[\frac{t^2\alpha^2}{3} + \frac{\beta^2(x-x_0)^2}{4\sinh^2\beta} - \frac{\alpha\beta t(x-x_0)}{2\sinh\beta}\right]\right). \quad (26)$$

Thus in this case the phase $\phi(t)$ does not undergo fluctuations.

The averaged soliton intensity $\langle|\psi_n(t)|^2\rangle$ and its fluctuations are important characteristics of soliton dynamics. They depend only on the distribution of x [see Eq. (2)]. From (26) one finds a corresponding one-point distribution

$$P_1(x,t) = \int\int d\phi\, d\alpha\, P(\phi, x, \alpha, t) = (1/2\sigma)\sqrt{3/\pi t^3}\exp[-3\beta^2(x-x_0)^2/2^4\sinh^2\sigma^2 t^3]. \quad (27)$$

A remarkable fact is that now intensity momenta of the lattice soliton display the same stochastic dynamics as that of a soliton in the continuum limit [5] (see also [8]).

Another case allowing rather complete investigation is the limit of large σ^2. Let us examine in this limit the mean value $\langle I(n-x)\rangle$ where $I(n-x)$ is an arbitrary finite localized function on x (as is clear, intensity momenta are such functions). It can be expressed through the Fourier transform $\tilde{I}(\kappa)$

$$\langle I(n-x)\rangle = \frac{1}{\sqrt{2\pi}}\int d\kappa \tilde{I}(\kappa)e^{-i\kappa n}\langle e^{i\kappa x}\rangle, \quad (28)$$

where $\tilde{I}(\kappa)$ is a regular function defined by initial conditions. Thus it is sufficient to study a mean value $\langle\exp(i\kappa x)\rangle$, which depends on time. To this end we multiply both sides of Eq. (24) by $(2\pi)^{-1/2}\exp(-i\kappa x + im\alpha)$ (m being integer) and integrate over ϕ, x, and α. Then introducing designations $\lambda = \frac{\sinh\beta}{\beta\sigma^2}$ and $\tau = \sigma^2 t$ one gets the result in a form of a differential-difference equation

$$\partial\Pi_m/\partial\tau = \lambda\kappa(\Pi_{m-1} - \Pi_{m+1}) - m^2\Pi_m, \quad (29)$$

where

$$\Pi_m = \Pi_m(\kappa, \tau)$$
$$= \frac{1}{\sqrt{2\pi}}\int dx e^{i\kappa x}\int d\alpha e^{i\alpha m}\int d\phi P(\phi, x, \alpha, t). \quad (30)$$

As is clear, now $\langle\exp(i\kappa x)\rangle = \Pi_0(\kappa, \tau)$ and

$$\langle I(n-x)\rangle = \int d\kappa \tilde{I}(\kappa)e^{-i\kappa n}\Pi_0(\kappa, \tau). \quad (31)$$

Equation (29) has to be considered subject to the initial condition

$$\Pi_m(\kappa, 0) = (1/\sqrt{2\pi})e^{i\kappa x_0} \quad (32)$$

[it follows from (25) and the definition (30)].

The differential-difference equation (29) has a small parameter λ ($\lambda \to 0$ at constant β and $\sigma \to \infty$). The characteristic value of κ is defined as the inverse width of the localization of $I(n-x)$, i.e., $\kappa \le \beta$ [see Eqs. (2) and (31)]. Therefore one can look for a solution of Eq. (29) at $|n| \ge 1$ using expansion with respect to the small parameter λ:

$$\Pi_m = \Pi_m^{(0)} + \lambda \Pi_m^{(1)} + \cdots \tag{33}$$

[note that such representation is not valid for Π_0 since at $m = 0$ Eq. (29) does not contain the small parameter in an evident form]. Correspondingly $\Pi_m^{(0)}$ are considered subject to the initial condition (32) and $\Pi_m^{(j)} = 0$ at $t = 0$ and $j \ge 1$.

Insertion of (33) into (29) yields in zero order

$$\Pi_m^{(0)} = (1/\sqrt{2\pi}) \exp(-i\kappa x_0 - m^2\tau) \tag{34}$$

for $|m| \ge 1$. Since this result implies $\Pi_{-1}^{(0)} - \Pi_1^{(0)} = 0$, in order to find Π_0 one needs the terms $\Pi_{\pm 1}^{(1)}$. They are expressed through Π_0

$$\Pi_{\pm 1}^{(1)} = \pm \lambda\kappa e^{-\tau} \int_0^\tau dt (\Pi_2(\kappa, 0) - \Pi_0) e^t. \tag{35}$$

Substituting the representation of $\Pi_{\pm 1}^{(1)}$ into Eq. (29) and setting $m = 0$, one obtains the integral equation for Π_0,

$$\frac{\partial \Pi_0}{\partial \tau} = 2(\lambda\kappa)^2 e^{-\tau} \int_0^\tau dt (\Pi_2(\kappa, 0) - \Pi_0) e^t. \tag{36}$$

It is not difficult to rewrite it in the differential form

$$(\partial^2 \tilde{\Pi}/\partial \tau^2) - \omega^2 \tilde{\Pi} = 2(\lambda\kappa)^2 \Pi_2(\kappa, 0) e^{\tau/2}, \tag{37}$$

where we introduce designations as follows: $\tilde{\Pi} = e^{\tau/2}\Pi_0$ and $\omega^2 = \frac{1}{4} - 2(\lambda\kappa)^2$. Equation (37) is solved trivially under initial conditions: $\tilde{\Pi}(0) = (2\pi)^{-1/2} \exp(i\kappa x_0)$ and $\partial\tilde{\Pi}/\partial\tau = \tilde{\Pi}(0)/2$, which result from (32) and (36). Finally one finds

$$\Pi_0(\kappa, \tau) = \frac{1}{\sqrt{2\pi}} e^{i\kappa x_0} \left(1 - \frac{\omega - 1}{2\omega} \sinh(\omega\tau) e^{-\tau/2}\right). \tag{38}$$

Since we are interested in the case $\sigma^2 \gg 1$, in a region giving the main contribution to the integral (31) $\tau \gg 1$ ($t \gg \sigma^{-2}$) and expression (38) is simplified,

$$\Pi_0(\kappa, \tau) \approx (1/\sqrt{2\pi})\ e^{i\kappa x_0} \{1 - \tfrac{1}{4}[1 + 8(\lambda\kappa)^2] \times \exp(2\sinh^2 \beta\kappa^2 t/\beta^2\sigma^2)\}. \tag{39}$$

Thus we come to essentially different dynamics of a soliton in comparison with the case $\sigma^2 \ll 1$, which can be treated as a discreteness effect. Afterwards narrow transition region ($t < t_1 \approx \sigma^{-2}$) behavior of averaged quantities $\langle I(n-x)\rangle$ is stabilized. The momentum takes the approximate value $\frac{3}{4} I(n - x_0)$ and maintains it till $t_2 \approx \sigma^2/2\sinh^2\beta$. The phase of a soliton undergoes strong fluctuations. All changes of $\langle I(n-x)\rangle$ in the time domain (t_1, t_2) are of the order of $\sigma^{-2} t \sinh^2\beta$. At $t \gg t_2$ the expression (38) reverts to the initial form, but this region is outside the time interval when the expansion (33) is valid.

Though the example just considered is interesting by itself in that it demonstrates the distinction between dynamics of discrete and continuous solitary waves, it requires some stipulations from the viewpoint of the effective equation method since the condition (21) is not better satisfied. To this end we consider the substantially discrete limit $\beta \gg 1$. Then Eq. (20) says that in the leading order of the expansion of D_{n_p, n_q} one has to set $n_p = n_q$, as far as these are discrete parameters. An error of the approximation is given by items corresponding to $n_p = n_q \pm 1$ in the sum (15) and being of order of $D''(0)\exp(-2\beta)$. Demanding these terms to be much less than 1 and bearing in mind that $D''(0)$ is associated with σ^2 one states that the effective equation method is applicable for the case $\exp(2\beta) \gg \sigma^2 \gg \exp(\beta)/\beta$, which is compatible with all approximations of the last limiting case studied above. These arguments allow one to conclude also that the restriction (21) is rather strong and can be weakened at $\beta \gg 1$.

To conclude, we have found qualitatively different behavior of a lattice soliton on the integrable disordered lattice in cases of small and large intensities of external force fluctuations. In the last case discreteness strongly manifests itself. Such a model can serve as an effective equation for studying the Ablowitz-Ladik lattice under random perturbations being δ-function-correlated in time and rather slowly varying in space. The technique used does not require smallness of fluctuations and is based on statistical equivalence of two different lattices. This application of the *effective equation method*, originally proposed in Ref. [5], demonstrates possibilities for its further extension to other systems (discrete or continuous) describing soliton dynamics affected by spatiotemporally disorder. Some of the extensions may be related to multisoliton problems, especially to interaction of solitons in disordered lattices, to the use of new integrable systems as effective equations, to other random lattices associated with integrable ones, etc.

The author is grateful to I. M. Besieris whose comments stimulated this work and F.G. Bass for some preliminary discussions. The work has been supported by the Universidad Complutense de Madrid.

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[1] R. Scharf and A.R. Bishop, Phys. Rev. A **43**, 6535 (1991).
[2] M.J. Ablowitz and G.F. Ladik, J. Math. Phys. **16**, 598 (1985).
[3] H.H. Chen and C.S. Liu, Phys. Rev. Lett. **37**, 693 (1976).
[4] R. Balakrishnan, Phys. Rev. A **32**, 1144 (1985).
[5] I. M. Besieris, in *Nonlinear Electromagnetics*, edited by P.L.E. Uslenghi (Academic, New York, 1980).
[6] I. M. Besieris (private communication).
[7] *Disorder and Nonlinearity*, edited by A.R. Bishop, D.K. Campbell, and S. Pnevmatikos (Springer-Verlag, Berlin, 1989); *Nonlinearity with Disorder*, edited by F.Kh. Abdullaev, A.R. Bishop, and S. Pnevmatikos (Springer-Verlag, Berlin, 1992).
[8] F.G. Bass *et al.*, Phys. Rep. **157**, 63 (1988).
[9] E.A. Novikov, Zh. Eksp. Teor. Fiz. **47**, 1919 (1964) [Sov. Phys. JETP **20**, 1290 (1965)].
[10] V.I. Klyatskin, *Stochastic Equations and Waves in Randomly Inhomogeneous Media* (Nauka, Moscow, 1980).