Pattern selection at high nonlinearity in a diffusively coupled logistic lattice

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A different pattern-selection regime is shown to exist in regions of high nonlinearity in the diffusively coupled logistic lattice with nearest-neighbor interaction. For this regime, the phase diagram of several prominent attractors is determined. The results indicate that the spatiotemporal chaos in some of the regions so far believed to be areas of fully developed spatiotemporal chaos is of a transient nature.

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Coupled map lattices (CML's) have been studied extensively in recent years as a model for spatiotemporal chaos [1–6]. This is not only because they can be applied to a wide range of areas, such as turbulence, chemical reactions, biology, etc., but also because they exhibit an extremely rich phenomenology, which includes kink dynamics, suppression of chaos, intermittency, and spatiotemporal chaos.

One of the simplest and most intensively investigated CML is the diffusively coupled logistic lattice with nearest-neighbor interaction. In one dimension, the case considered exclusively here, this can be written as

$$x_{n+1}(i)=F\left[(1-\epsilon)x_n(i)+\frac{\epsilon}{2}[x_n(i+1)+x_n(i-1)]\right], \tag{1}$$

where n is the discrete time, i the lattice site, ϵ the lattice coupling constant, and F the logistic map, which is given by

$$F(x_n)=x_{n+1}=1-\alpha x_n^2, \tag{2}$$

with the nonlinearity α a constant. Periodic boundary conditions are used throughout.

One of the interesting phenomena exhibited by the diffusively coupled logistic lattice is pattern selection with the suppression of chaos. When patterns are selected, the entire system is in a very regular state and is not only temporally but also spatially periodic. This is well known to occur in regions of medium nonlinearity ($\alpha\simeq 1.55-1.75$), i.e., regions where the single logistic map is chaotic (except for windows) if ϵ is not too small.

A nice overview of the regions in which pattern selection occurs is, for example, given in Kaneko's phase diagram in Ref. [2]. One can view this regime as a kind of equilibrium between the tendency of the chaotic motion of the local element to make the system inhomogeneous and the tendency of the diffusion to do the opposite. Spatiotemporal chaos may occur if the coupling is too weak or the nonlinearity too strong for the equilibrium to be established.

Kaneko's phase diagram, however, only covers values of the coupling constant of up to $\epsilon=0.4$. If the above picture of an equilibrium has any validity, *a priori* one cannot exclude the possibility that the pattern-selection regime extends, at least somewhat, further into the direction of high nonlinearity for values of $\epsilon>0.4$.

The main topic of the present Brief Report is to investigate how far "somewhat" may be. Clarifying this is not only important in its own right but also regarding its consequences for the analysis of spatiotemporal chaos. The phase diagram in [2] indicates, for example, that fully developed spatiotemporal chaos (FDSTC) governs when the nonlinearity is greater than that of the period-3 window for $\epsilon\gtrsim 0.2$. Although the phases for $\epsilon>0.4$ are neither indicated nor directly mentioned in the text, the diagram suggests somewhat that regions of high nonlinearity and large coupling are spatiotemporally chaotic. Various authors have determined Lyapunov spectra between $\alpha=1.8$ and $\alpha=1.9$ for $\epsilon=0.4$, for example, to illustrate how these spectra appear for regions of spatiotemporal chaos. It is shown below, however, that a periodic attractor exists in this area. Accordingly, when starting from random initial conditions, as is customary, there is no guarantee that the obtained spectrum is accurate, however likely, without making certain that for the chosen random initial conditions the transient time is sufficiently long.

As it turns out, regions of high nonlinearity and large coupling are not spatiotemporally chaotic at all. Figure 1 shows an attractor for $\alpha=1.9$ and $\epsilon=0.85$. Random initial conditions are used and the system size is $N=60$. The periodicity of the attractor is four. If one defines the number of domains n_d as the number of times that the line connecting the amplitudes of the lattice sites at a certain time crosses the fixed point of the logistic map $[x^*=(\sqrt{1+4\alpha}-1)/2\alpha]$ in the positive direction, it can clearly be seen that the attractor has seven domains. This means that there is some frustration, i.e., the ratio of the system size and the number of domains is noninteger, and indicates thus that the attractor must be quite stable. This can also be confirmed by adding dynamical noise. In the case of Fig. 1, the addition of 6% noise was necessary to destroy the attractor.

In order to check whether pattern selection is generic for the above parameter values, the distribution of the transient times when starting from 100 000 random initial conditions was determined. The result is displayed in Fig. 2. The longest transient time was 471 136 steps and

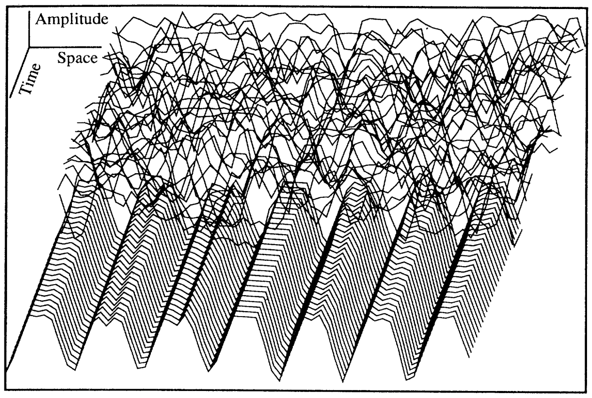

FIG. 1. Attractor at high nonlinearity. The system size is $N=60$; $\alpha=1.9$ and $\epsilon=0.85$. The figure shows the situation just after the system has fallen onto the attractor. Every 1000th step was plotted.

the shortest a mere 192 steps. The distribution has a peak for the 2×10^3 bin and an exponential tail. Since all the initial conditions reached a period-4 attractor, it seems justifiable to conclude that pattern selection is indeed the prevalent final state for these parameter values. The possibility that other attractors with very small basins of attraction exist cannot, of course, be excluded. The above result, for example, should not drastically change for parameter values in a small neighborhood of those above, and since the chaotic region of the single logistic map is densely interspersed with windows, one could choose a value of α close to 1.9 so that the single logistic map is in a periodic window. In that case, the homogeneous attractor does exist; its basin of attraction, however, is so small that it is extremely unlikely to be found when starting with random initial conditions.

Next, it will be interesting to see in which regions of parameter space a nontrivial period-4 attractor exists. This could be done in the same way as is usual in constructing a basin of attraction, but instead of gradually changing the initial conditions one would have to change the parameters. There are, however, some severe drawbacks with this method. For example, if several attractors coexist, one might have to trace a large number of trajectories (starting from random initial conditions) for every α-ϵ pair to obtain a good picture of the most prevalent attractors. Also, for some parameter values, the average transient time might be impractically long for a detailed numerical analysis. Therefore the following, more efficient, method was used.

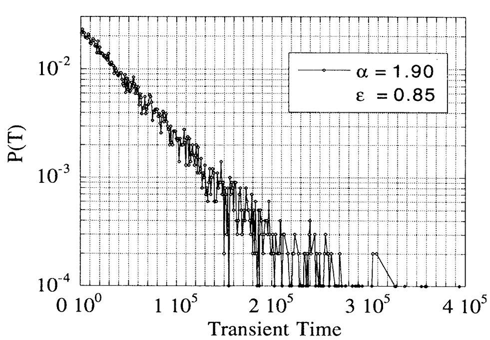

FIG. 2. Distribution of transient times before the system reaches a period-4 attractor at $\alpha=1.9$ and $\epsilon=0.85$. The system size was $N=60$ and the bin size was 1000.

Given a certain number of domains, a pair of α and ϵ was first determined where the wanted attractor would generally be selected in an acceptable time. Then the obtained attractors were used as a starting point for a parameter sweep. Initially, only ϵ was swept in the positive direction, until a value was reached where a subsequent sweep in the α direction would not destroy the attractor in the entire range of α investigated. In the second step, α was swept, and in the final step ϵ, first in the positive and then in the negative direction, until the attractor was destroyed. Every time an attractor was destroyed, a new sweep would start again from the starting attractor and the procedure repeated. Only in the case of the $n_d=11$ attractor did an extra sweep step have to be inserted because there was no value for ϵ that could be used as the starting point for the sweep in the α direction that would cover the entire range.

The advantage of this method is that it is able to show where the wanted attractors exist in a computationally efficient way; its drawback is that no conclusions can be drawn about the sizes of the basins of attraction or about the average transient times. It should, therefore, be viewed not as a replacement but as a complement to the one mentioned first.

After every step of the sweep, the system was allowed to adjust to the new parameters for 60 time steps. The next 40 steps were used for the periodicity check, where the average difference of ten successive periods was not allowed to be more than 2%. The step size in the α direction was 0.01, and the step size in the ϵ direction 0.005. In order to make certain that only regions where the attractors are (fairly) stable were included, 1% dynamical noise was added throughout the sweeping process.

The resulting phase diagram for attractors that have from 6 to 11 domains and periodicity 4 is given in Fig. 3. The large stars indicate the starting attractors, and the numbers the number of domains n_d in each region. As can clearly be seen, these attractors cover almost all of the parameter space under consideration, and most of the boundaries are virtually linear in ϵ. In the case of ten domains, the upper boundary suddenly seems to jump to $\epsilon=1.0$ around $\alpha\simeq1.85$. This stems from a rather smooth transition from a period-4 to a period-2 attractor with the same number of domains. The reason that this smooth transition can only be observed for $n_d=10$ is the match between system size and the number of domains, which leads to extra stability of the domain structure. Without a smooth transition, the $n_d=11$ attractor shows a jump at around $\alpha=1.7$ from $\epsilon\simeq0.35$ to $\epsilon\simeq0.45$. Why only this attractor shows such a jump is not clear at present.

The dashed box roughly indicates one of the areas of FDSTC in the phase diagram of Ref. [2]. Since nontrivial

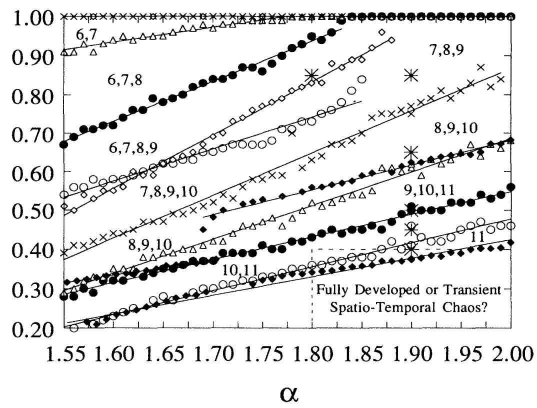

FIG. 3. Phase diagram for the period-4 attractor with the numbers of domains ranging from 6 to 11. The system size is $N=60$. The large stars indicate the starting points of the parameter sweeps. The dashed box corresponds to one of the regions of FDSTC in [2]. The numbers represent the number of domains n_d. The upper and lower boundaries are indicated with the following markers, respectively: (◊) $n_d=6$, (×) $n_d=7$, (△) $n_d=8$, (●) $n_d=9$, (○) $n_d=10$, (♦) $n_d=11$.

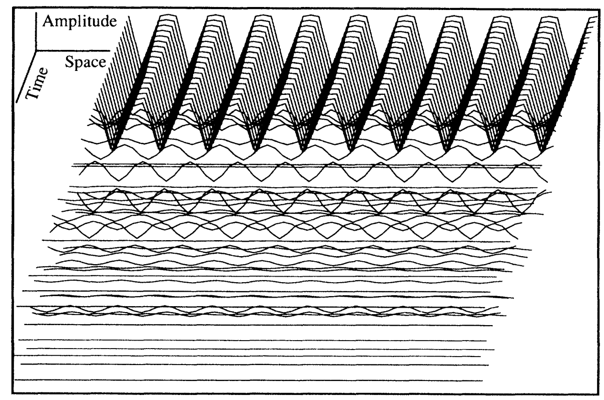

FIG. 4. Transition from the nonhomogeneous to the homogeneous attractor under a parameter sweep. Every fourth time step is plotted. With every time step, ϵ is increased by 1.0×10^{-5}, and the transition occurs at $\epsilon\simeq0.73$.

attractors exist in the upper left corner, it seems reasonable to assume that, at least in this region, FDSTC is a transient phenomenon. One of the starting attractors ($n_d=11$, $\alpha=1.90$, $\epsilon=0.4$) even lies clearly in the FDSTC region, albeit at the border of the plot. Although the transient time of this attractor can be considered as rather long (the average of 75 runs was 1.48×10^7 while the shortest was 110 464), it certainly is not astronomical, nor is it outrageous when compared to recent simulations. Grassberger and Schreiber, for example, have determined the density of defects in a CML using 10^8 iterates [3].

The transient times seem to increase with decreasing ϵ. Up to the maximum nonlinearity $\alpha=2.0$, however, a zigzag regime exists for $\epsilon\simeq0.15$. Hence there must be a value of ϵ between 0.4 and 0.15 where the transient time is at a maximum. If this maximum is not astronomically large, the only region in which "true" FDSTC may occur will be the one of very small ϵ, eventually sustained by supertransients [7].

Although for higher values of ϵ, as stated above, numerical evidence suggests that the basin of attraction of an eventually coexisting chaotic attractor is quite small if it exists, this need not necessarily be true for smaller values of ϵ.

The domains in all the starting attractors were, in principle, stationary in time, i.e., the domain walls did not move. In the medium nonlinearity regime, traveling waves were recently discovered for large ϵ [5]. These traveling waves also occur almost everywhere in the high nonlinearity regime. This is no surprise, since, in principle, stationary and traveling waves are composed of the same elements, the four phases of the basic waveform. Their only difference lies in the order of the elements. Owing to the movement of the domain walls, however, traveling waves are less stable and destroyed sooner when being swept.

The fact that a stable, homogeneous attractor exists in periodic windows makes a transition from a periodic, spatially inhomogeneous attractor to the homogeneous one possible in principle. Simply extrapolating the results of [7] would imply that, for larger values of the coupling, the homogeneous attractor, in practical terms, can never be reached. There is, however, a way to achieve this in a reasonably short time. In general, when starting with random initial conditions and $\epsilon\simeq0.35$, a period-4 attractor with ten domains is reached within a few hundred thousand steps for $\alpha\simeq1.755$, i.e., the single logistic map is in the period-3 window. After the attractor is reached, ϵ can be swept in the positive direction. As can be seen from the phase diagram, at $\epsilon\simeq0.7$, the $n_d=10$ attractor ceases to exist. At this point, the transition to the period-3 attractor may occur, as is illustrated in Fig. 4. Of course, the same scenario also works if one does not wait for the system to fall onto the attractor at the starting value of ϵ but sweeps slowly enough. So far, this transition was only observed in the period-3 window; it may be expected, though, that it will also occur under similar circumstances in other windows. Furthermore, this transition seems to be extremely sensitive to noise.

In conclusion, a different pattern-selection regime at high nonlinearity has been reported. The phase diagram constructed shows that this regime covers a large part of the parameter space from medium to high nonlinearity. The results do not contradict, but supplement and exceed, previous findings, and there may be important consequences for the understanding of spatiotemporal chaos in a diffusively coupled logistic lattice.

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