Length-scale competition for the one-dimensional nonlinear Schrödinger equation with spatially periodic potentials

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The effect of spatially periodic on-site potentials on solitons of the (1+1)-dimensional nonlinear Schrödinger equation depends not only on the amplitude of the perturbation but also on two length ratios—between the wavelength of the potential and either the spatial width of the soliton or its phase-modulation wavelength. For solitons which are slowly moving or at rest, only the first ratio is important: Solitons which are narrow compared to the wavelength of the potential move like particles in an effective potential; wide solitons move like "renormalized" particles; and for competing length scales, solitons are easily destabilized by the perturbation and break up into smaller localized excitations and radiation. For initially rapidly moving solitions in short-wavelength potentials, a different resonance effect is described which can inhibit soliton propagation. Away from this resonance the solitons can move in a "dressed" form radiating very slowly. Furthermore, it is shown numerically that two such "dressed" solitons reemerge after collision, essentially unchanged. We also comment on the structural stability of completely integrable dynamics under spatially periodic perturbations. Finally, we remark on the relevance of our results for the dynamical behavior of large polarons and bipolarons in disordered ionic lattices.

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I. INTRODUCTION

Adding spatial disorder to completely integrable nonlinear dynamics such as the nonlinear Schrödinger (NLS) equation or the sine-Gordon (sG) equation leads to a variety of novel effects having practical relevance [1], e.g., in materials science and optics. Competition of the length scales introduced by the perturbation and by the nonlinearity is one example we have described recently in the case of the (1+1)-dimensional sG equation [2]. If these length scales are very different from each other the perturbed dynamics can support solitonlike or breatherlike excitations. The particlelike motion of these excitations can be described by a collective variable approach [3–5]. Even within this "effective-particle" approximation, interesting effects, including sG-breather breakup [6] or NLS-soliton chaos [7], can be understood.

On the other hand, if the length scales are comparable, coherent excitations break up or dissipate into radiation even for relatively small strengths of the perturbation. This behavior has already been reported in the case of the sG equation [2].

In the present article we argue that breakup of coherent nonlinear excitations through length-scale competition is a general feature of soliton-bearing nonlinear dynamics with stationary perturbations. We illustrate that for the (1+1)-dimensional NLS equation perturbed parametrically by a spatially periodic potential.

The soliton solutions of the NLS equation possess two characteristic length scales (related to the amplitude and to the phase of the soliton) leading to two possible situations of length-scale competition in the presence of a spatially periodic perturbation. We address these competitions in detail below.

The present article has the following structure. Following this introduction we define and motivate the model in Sec. II. In Sec. III we investigate the "effective-particle" limit when the spatial width of a soliton is much smaller than the wavelength of the perturbing potential. In Sec. IV we turn to the opposite limit of "renormalized particles" when the width of the soliton is much larger than the wavelength of the potential and the solitons acquire a "dressing." We also address the case of potentials with several characteristic wave lengths. In Sec. V we show how the phase modulation of a moving soliton can give rise to a situation of competing length scales leading to a breakup of the soliton for certain velocities. In Sec. VI we investigate a second case of competing length scales when the width of the soliton and the wavelength of the potential are comparable. The conclusion and discussion of related topics are given in Sec. VII.

All numerical results in this article we obtained upon numerically integrating the integrable spatial discretization of the NLS equation, given by Ablowitz and Ladik [8] (see also [5]), in the continuum limit using a Runge-Kutta-Verner fifth- and sixth-order method. To damp away unwanted excitations and radiation, the spatial domain was divided into a central half supporting the soliton (being at rest or moving), and two quarters at the boundary where the system was damped with a damping constant increasing smoothly from zero towards the boundaries.

II. THE MODEL

The $(1+1)$-dimensional NLS equation and the behavior of its solutions under a variety of perturbations are well investigated [9]. This body of work can be characterized by the kind of perturbation added to the NLS equation: (i) driving, periodic in space and time, with inclusion of damping; (ii) isolated impurities; (iii) spatially periodic stationary potentials; and (iv) spatially random stationary potentials. The initial conditions chosen depend on the application in question. One can either look for modulational instability starting from a homogeneous initial state, investigate transmission properties using plane wave solutions entering the nonlinear disordered region from a linear region without disorder, or start with soliton solutions of the unperturbed NLS equation and monitor their behavior under the perturbation.

We are interested in the behavior of coherent excitations under stationary, spatially periodic perturbations. This is the simplest model of disorder in nonlinear materials having a wide range of applications. Away from the perturbative regime knowledge about exact solutions is very limited. Therefore the usual strategy is to start with exact solutions of the unperturbed NLS equation and observe numerically whether and how they adjust to the perturbation. After different scenarios are identified, perturbation expansions starting from nontrivial perturbed solutions might give some insight into the mechanisms underlying the different scenarios.

One numerical tool is especially useful in this respect: the inverse scattering transformation (IST) to soliton and radiation coordinates of the unperturbed NLS equation for the numerically integrated perturbed problem. This allows monitoring of how the soliton and radiation contents of the perturbed system change in time under the influence of the perturbation (for a recent example see [10]).

Here we investigate a perturbed NLS equation of the following form:

$$i\psi_t+\psi_{xx}+2\psi|\psi|^2=\epsilon\psi\cos(kx) \ . \tag{1}$$

The perturbation introduces a length scale $\lambda=2\pi/k$ which breaks the length-scale invariance of the unperturbed NLS equation ($\epsilon=0$). Rescaling spatial and temporal coordinates and the amplitude of the field allows us to choose the wavelength of the potential as the unit of length. Therefore only length-scale ratios involving the period of the potential will be relevant variables.

Neglecting the cubic term in Eq. (1) leads to the corresponding linear problem which is known to have time-periodic Bloch-wave solutions in terms of Mathieu functions with eigenfrequency $\omega(\kappa)$. The linear spectrum has gaps at $\kappa=k$. At this k value the solutions have the character of standing waves.

For vanishing perturbation ($\epsilon=0$) the traveling nonlinear wave solutions of Eq. (1) are proportional to the elliptic functions $\mathrm{cn}(u)$ or $\mathrm{dn}(u)$ with $u\propto(x-vt-x_0)$. In addition to these extended wavelike solutions, localized soliton solutions are possible,

$$\psi(x,t)=2i\eta\frac{e^{i\dot{q}x/2-i\Phi}}{\cosh[2\eta(x-q)]} \ , \tag{2}$$

with the soliton position $q(t)=-4\xi t+q_0$ and its phase $\Phi(t)=4(\xi^2-\eta^2)t+\Phi_0$, where $\xi+i\eta$ is the complex pole of the analytically continued transmission coefficient corresponding to a single soliton (see for example [11]).

For small values of ϵ we expect solutions to exist which resemble the unperturbed solitons with spatial modulations due to the perturbation. That this picture is essentially correct, apart from certain resonances, will be shown in the following sections.

III. THE "EFFECTIVE-PARTICLE" LIMIT

If the width of a soliton is much smaller than the period of the perturbing potential ($\eta\gg k$) then the soliton is only influenced by local properties of the potential, e.g., its gradient, its curvature, and so on. If we take into account only the gradient of the perturbing potential $V(x)$ then the resulting NLS equation is still completely integrable [12],

$$i\psi_t+\psi_{xx}+2\psi|\psi|^2=\epsilon\psi(V_0+V_1x) \ . \tag{3}$$

In particular it allows for soliton solutions of the form given in Eq. (2) with the position q and the phase Φ fulfilling the following equations of motion:

$$\ddot{q}=-2\epsilon V_1, \quad \dot{\Phi}=\tfrac{1}{4}\dot{q}^2-4\eta^2+\epsilon V_0 \ . \tag{4}$$

If the potential is nonlinear then we make a Taylor expansion around the instantaneous position of the soliton. We keep only terms linear in $(x-q)$, as higher-order terms will be proportional to higher powers of the length-scale ratio k/η, which we assumed to be a small quantity in this section. This leads to

$$\begin{aligned}&\ddot{q}=-2\epsilon V'(q) \ ,\\&\dot{\Phi}=\tfrac{1}{4}\dot{q}^2=4\eta^2+\epsilon[V(q)-qV'(q)] \ .\end{aligned} \tag{5}$$

Obviously the position of the soliton does not depend on its phase. Therefore we will drop the phase $\Phi(t)$ in our considerations.

For a general perturbing potential the NLS equation will no longer be completely integrable. Nevertheless, the dynamics still possesses two integrals of motion, namely the energy E and the "norm" N,

$$\begin{aligned}&N=\int_{-\infty}^{+\infty}|\psi|^2dx \ ,\\&E=\int_{-\infty}^{+\infty}[|\psi_x|^2-|\psi|^4+\epsilon|\psi|^2V(x)] \ .\end{aligned} \tag{6}$$

We can refine our previous considerations, which led to an equation of motion for the soliton position q, by using the collective coordinate ansatz (2) for the solution and demanding the constancy of energy and norm,

$$N=4\eta, \quad E=\eta(\dot{q}^2-\tfrac{16}{3}\eta^2)+V_{\text{eff}}(q) \ , \tag{7}$$

where the effective potential is given by

$$V_{\text{eff}}(q)=4\eta^2\epsilon\int_{-\infty}^{+\infty}\mathrm{sech}^2[2\eta(x-q)]V(x) \ . \tag{8}$$

This leads directly to the following conditions:

$$\dot{\eta}=0, \quad \ddot{q}=-\frac{1}{2\eta}V'_{\mathrm{eff}}(q) . \tag{9}$$

In particular we find for $V(x)=\cos(kx)$,

$$V_{\mathrm{eff}}(q)=\frac{\epsilon k \pi}{\sinh(k\pi/4\eta)}\cos(kq) . \tag{10}$$

From this we derive the equation of motion for the soliton position,

$$\ddot{q}=\frac{\epsilon k^2 \pi}{2\eta \sinh(k\pi/4\eta)}\sin(kq) . \tag{11}$$

Comparing this expression with the equation of motion for q derived earlier [see Eq. (5)], we observe the influence of the length-scale ratio k/η. If the soliton is very narrow compared to the wavelength of the potential

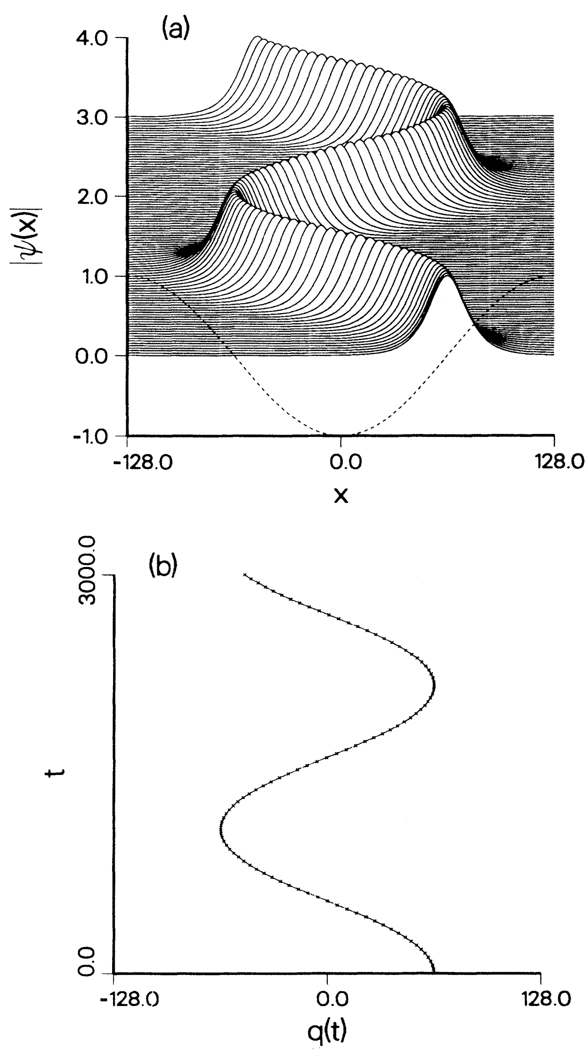

FIG. 1. The particle limit. Parameters: $\epsilon=-0.01$, $\lambda=256$. Initial values: $\dot{q}=0$, $\eta=0.05$. Integration time: $T=3000$. (a) The behavior of the full NLS equation (1) (dashed line: perturbing cosine potential). (b) The position of the NLS soliton (solid line) compared with the position of the effective-particle dynamics (11) (crosses).

($k/\eta \ll 1$) then we recover the previous result. Numerical evidence shows that Eq. (11) is still useful even for $k/\eta \approx \frac{1}{2}$, when the approximation (5) is no longer sufficient. Figure 1 gives an example for a soliton moving in a long-wavelength cosine potential and compares its motion with the prediction of the effective particle dynamics. Note the excellent agreement.

The opposite limit ($k/\eta \gg 1$) is also notable. Although the soliton ansatz (2) is no longer justified the effective potential $V_{\mathrm{eff}}(q)$ given in (10) is exponentially small in the length-scale ratio. Indeed, as we will see in the next section, when the coherent excitation covers many periods of the perturbing potential the influence of the potential on the center-of-mass motion of the excitation is exponentially small in the length-scale ratio.

So far we have neglected changes of the soliton shape and radiative effects completely. Shape changes become important in the following sections and will be discussed there. For sufficiently small perturbations radiative effects can be made arbitrarily small even on the long time scale which is needed to reveal the effects of the potential on the center-of-mass motion of the soliton. The tool of inverse scattering perturbation theory [13,14] can be used to calculate radiative effects perturbatively. Work is in progress to combine this well-established technique with the adiabatic "effective-particle" approximation discussed above.

Two cases are accessible to analytical treatment: When the soliton emits radiation without interacting with it (see for example [15]), or when the soliton traps the radiation in the form of a shape modulation of sufficiently simple form. We address the second case in the next two sections.

Numerical IST gives information about the soliton and radiation content in the system. Alone it does not allow us, however, to distinguish free radiation from virtual radiation trapped as shape modes in the case of a nonintegrably perturbed dynamics. IST together with direct inspection of the field $\psi(x,t)$ is necessary for a complete analysis in such cases.

IV. THE "RENORMALIZED PARTICLE" LIMIT

Now we turn to the opposite limit ($\eta \ll k$) when the width of the soliton is much larger than the period or the characteristic length scale of the perturbing potential. Taking the soliton given in Eq. (2) as an initial condition, we observe that the effective potential it experiences initially in the presence of a perturbation $\cos(kx)$ is given by Eq. (10). This effective potential is exponentially small for $k/\eta \gg 1$. The initial energy of the soliton depends only very weakly on the perturbing potential because the soliton "smooths over" the short-wavelength perturbation. Nevertheless, the potential will affect the coherent excitation locally and change its shape in the course of time. Numerical evidence shows that the soliton is influenced by the perturbation on two different time scales. An initial adjustment happens nearly instantaneously leading to a burst of radiation emitted by the soliton. An immediate effect of this is that the soliton shape shows a modulation with the wavelength of the perturbation. Then, on a much longer time scale, the soliton continues to radiate with much smaller power compared to the initial burst.

As the two length scales involved are very different the dynamical equations can be solved approximately using a separation of length scales. We separate the "dressed soliton" solution of Eq. (1) into two parts,

$$\psi(x,t)=\Psi(x,t)[1+\chi(kx)] \ . \tag{12}$$

The first ("slow") part Ψ turns out to be the unperturbed soliton (2) fulfilling the homogeneous NLS equation ($\epsilon=0$). The second ("fast") part gives the effect of the perturbation $\epsilon\cos(kx)$ on the short length scale.

If we put the ansatz (12) into Eq. (1) and average over the fast variables we find for the slow part of the dynamics, up to $O(\epsilon)$,

$$i\Psi_t+\Psi_{xx}+2\Psi|\Psi|^2=0 \ , \tag{13}$$

which is the unperturbed NLS equation with the soliton solution given in Eq. (2). The fast part of the dynamics fulfills, up to $O(\epsilon)$,

$$\chi_{xx}\Psi+2\chi_x\Psi_x+2(\chi+\bar{\chi})\Psi|\Psi|^2=\Psi\cos(kx) \ , \tag{14}$$

where $\bar{\chi}$ denotes the complex conjugate of χ. In this equation we treat $\Psi(x,t)$ as an adiabatic perturbation in the spatial variable leading to χ of the form [upon neglecting terms of $O(k^{-4})$]

$$\chi=-\frac{\cos(kx)}{k^2-\dot{q}^2}+\frac{i\dot{q}\,\sin(kx)}{k(k^2-\dot{q}^2)} \ . \tag{15}$$

This gives [up to $O(\epsilon)$]

$$|\psi(x,t)|^2=4\eta^2\,\mathrm{sech}^2[2\eta(x-q)]\left[1-\frac{2\epsilon\cos(kx)}{k^2-\dot{q}^2}\right] \ . \tag{16}$$

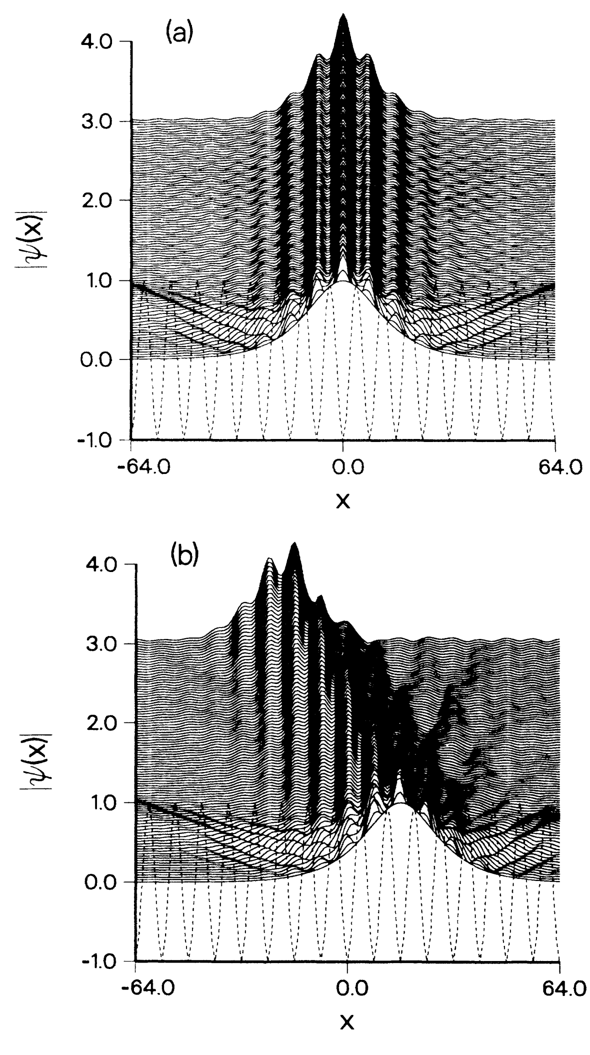

FIG. 2. The renormalized particle limit. Parameters: $\epsilon=-0.1$, $\lambda=8$. Initial value: $\eta=0.05$. Integration time: $T=200$. (a) Soliton at rest: $\dot{q}=0$. (b) Slowly moving soliton: $\dot{q}=-0.2$.

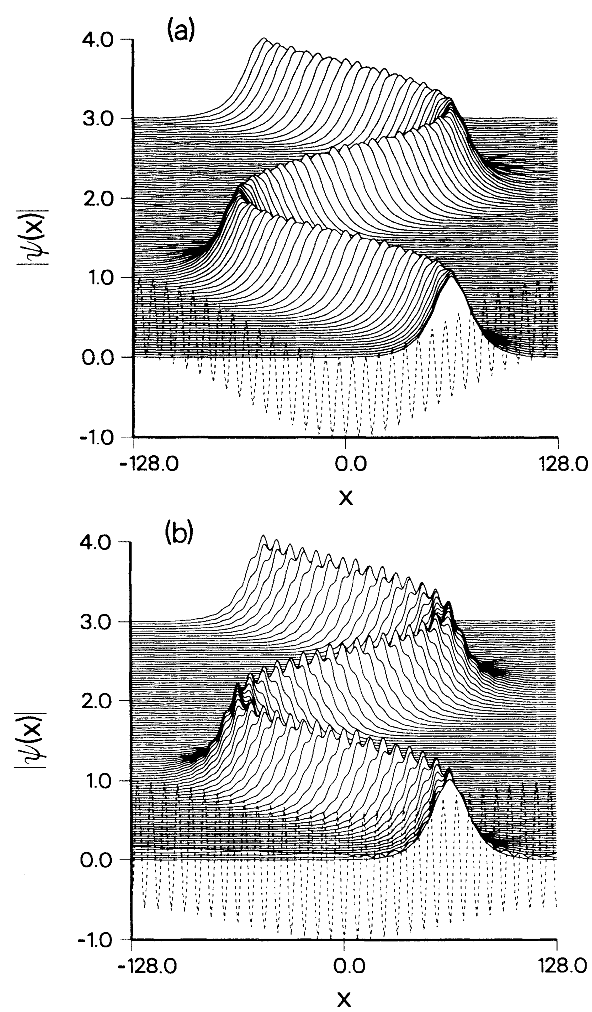

FIG. 3. The renormalized particle limit. Parameter: $\epsilon=-0.01$. Initial values: $\eta=0.05$, $\dot{q}=0$. Potential of the form $V(x)=\cos(2\pi x/\lambda_1)+r\cos(2\pi x/\lambda_2)$ with $\lambda_1=256$ and $\lambda_2=8$. Integration time: $T=3000$. (a) $r=1$. (b) $r=4$. Compare with Fig. 1.

The result (16) shows that for slow solitons ($|\dot{q}| < k$) the spatial modulations of $|\psi|$ are out of phase with the perturbing cosine potential by π: local maxima of the shape modulations of the soliton fall into minima of the potential. This is what one would expect for the case of a soliton at rest: excitation and weight of the field will move away from the maxima of the potential and condense around its minima. Equation (16) predicts that a similar result holds for sufficiently slow solitons as well. Figure 2 gives examples for solitonlike excitations at rest or slowly moving in a cosine potential, which are in agreement with our theoretical prediction. Note that $|\dot{q}| < k$ is fulfilled in both cases.

In Sec. V we will also address the large velocity case and the transition region between small and large velocity, giving rise to an unexpected additional length-scale competition which destabilizes coherent excitations.

If the soliton with shape parameter η moves in a potential with several widely differing characteristic length scales, like $V(x)=\cos(k_1 x)+r\cos(k_2 x)$ with $k_1 \ll \eta \ll k_2$, then the soliton experiences the two Fourier components of the potential very differently and independently. The k_1 component alone gives rise to a long-wavelength modulation in which the soliton will move like a particle. The k_2 component alone on the other hand will result in a small-wavelength modulation, which the soliton will average over and move in like a dressed particle. Both modulations taken together will lead to a dressed particle moving in an effective long-wavelength potential. Figure 3 gives an illustration of this behavior. Note that the dressed soliton is accelerated in the effective potential but its maximum velocity is sufficiently small to guarantee $|\dot{q}| < k_2$. The long-wavelength component of the potential in Fig. 3 has the same amplitude as in Fig. 1 leading to a center-of-mass motion for the dressed soliton which is nearly indistinguishable from the bare soliton motion in Fig. 1. The short-wavelength component is averaged over even if it

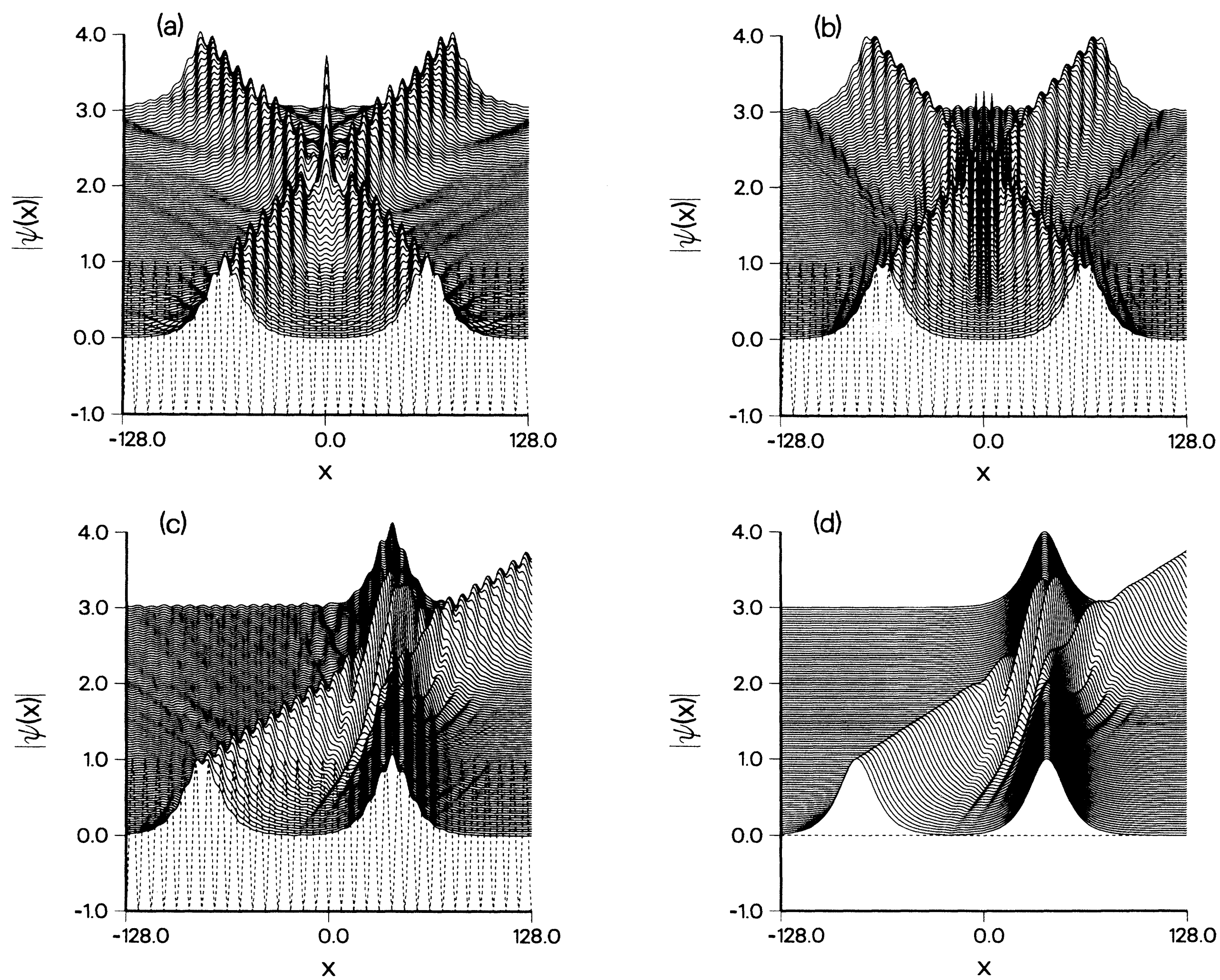

FIG. 4. Collision of two dressed solitons. Parameters: $\epsilon=-0.05$, $\lambda=8$. Soliton parameters: $\eta_1=\eta_2=0.05$. (a) $\dot{q}_1=-\dot{q}_2=-0.2$, $T=700$. (b) $\dot{q}_1=-\dot{q}_2=-1.2$, $T=120$. (c) $\dot{q}_1=0$, $\dot{q}_2=1.2$, $T=200$. (d) The integrable case ($\epsilon=0$) with $\dot{q}_1=0$, $\dot{q}_2=1.2$, $T=200$ for comparison.

has an amplitude which is much larger than the amplitude of the long-wavelength component.

Another particlelike aspect of the dressed solitons is worth mentioning. Two dressed solitons moving in a perturbing short-wavelength potential can collide and reemerge after collision essentially unchanged like unperturbed solitons of the NLS equation. This is illustrated in Fig. 4 on which we will comment later.

The shape modulation of the dressed solitons can be analyzed in terms of radiation with wavelength $\lambda = 2\pi/k$ being trapped on the soliton. To understand why this excitation is trapped on the soliton, assume it would be able to propagate away from it. If the amplitude of the radiation is small then we can neglect the nonlinear term in the perturbed NLS equation. The radiation has a wavelength coinciding with the wavelength of the potential. In this case only standing-wave solutions exist for the linear Schrödinger equation and free propagation is not possible. Excitations of this wavelength, which occur naturally as shape modes of solitons in spatially periodic potentials, are therefore trapped on the solitons and are forced to move with them.

V. THE PHASE–LENGTH-SCALE COMPETITION

In Sec. IV, where we discussed the renormalized particle limit, we restricted the velocity of the soliton to sufficiently small values. Now we make more precise what "sufficiently small" means. Equation (15), giving the shape modulation of the soliton in the presence of a short-wavelength perturbation, is valid not only for $|\dot{q}| \ll k$, but as long as $|\dot{q}^2 - k^2| \gg \eta$. Now we turn to the case $|\dot{q}| > k$ and find from Eq. (16) that the spatial modulation of $|\psi|$ is *in phase* with the perturbing potential.

Equations (15) and (16) already indicated where the crossover to the large velocity behavior occurs, namely at $|\dot{q}| = k$. Here the sign of the shape modulation changes. In addition at this point the perturbative result suffers

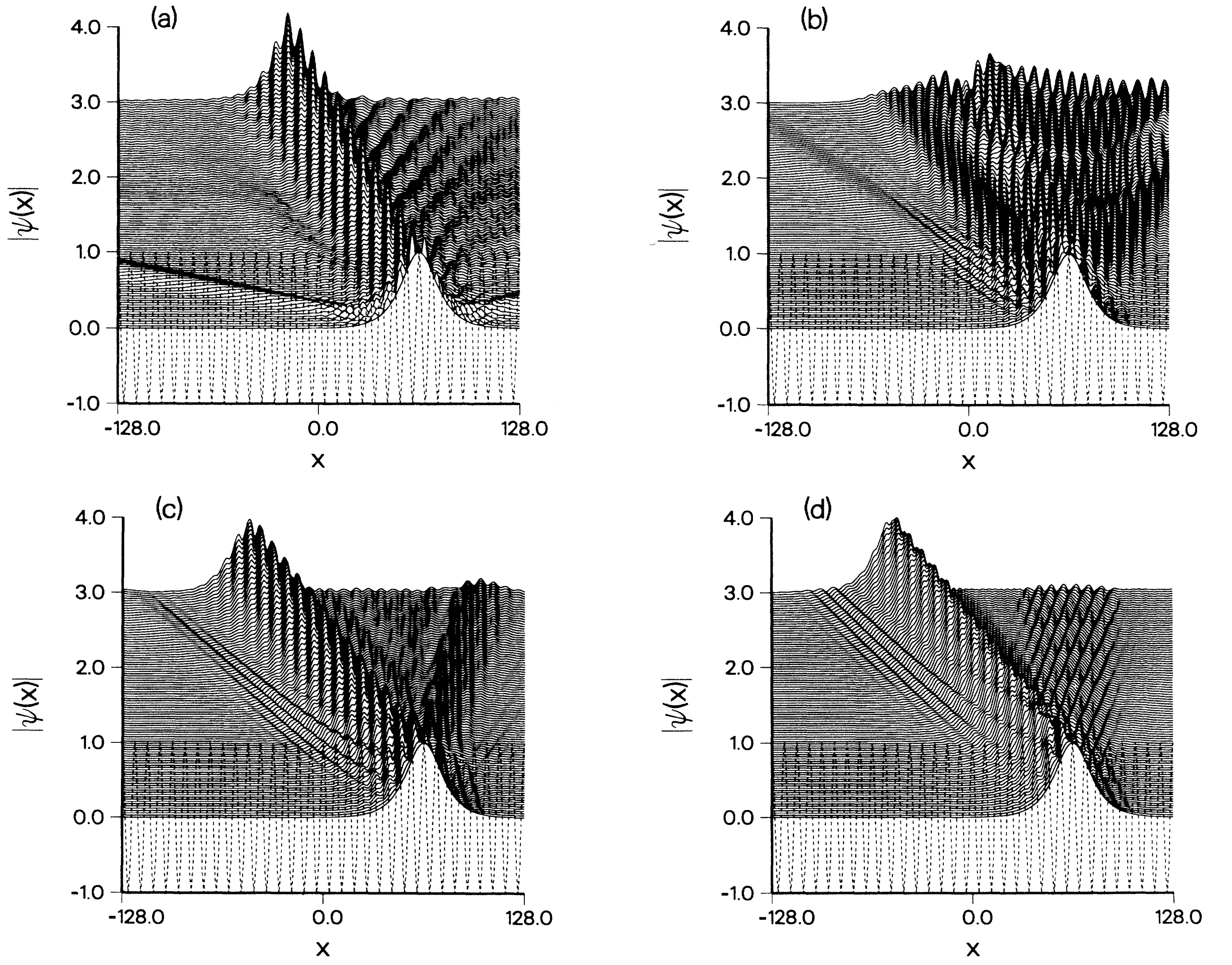

FIG. 5. The phase resonance $|\dot{q}| \approx k$. Parameters: $\epsilon = 0.1$, $\lambda = 8$ (or $k \approx 0.785$), $\eta = 0.05$. (a) $\dot{q} = -0.2$, $T = 500$. (b) $\dot{q} = -0.8$ (or $|\dot{q}| \approx k$), $T = 125$. (c) $\dot{q} = -1.2$, $T = 100$. (d) $\dot{q} = -1.6$ (or $|\dot{q}| \approx 2k$), $T = 80$.

from a singularity which signals the breakdown of the perturbation expansion. The competition between the wavelength of the perturbation and the wavelength of the phase modulation leads to a resonance which breaks up coherent excitations already for very small amplitude of the perturbation. This is illustrated in Fig. 5 together with the small and large velocity dressed soliton cases. Exactly at the resonance ($|\dot{q}|=k$) something interesting happens. The soliton breaks up into a standing wave train with $\lambda=k$ which switches back and forth between being in phase and out of phase with the potential.

Staying well away from this resonance, either above ($|\dot{q}|>k$) or below ($|\dot{q}|<k$), guarantees stable propagation of coherent excitation. These dressed solitons are quite robust and behave like bare solitons in many respects, for example upon collision. Coming back to Fig. 4 we see this in the case of two slow solitons ($|\dot{q}_1|,|\dot{q}_2|<k$), two fast solitons ($|\dot{q}_1|,|\dot{q}_2|>k$), as well as for the case of a fast soliton colliding with a slow soliton ($|\dot{q}_1|<k<|\dot{q}_2|$). Comparing the latter case with the collision of two bare soliton solutions of the unperturbed NLS equation reveals striking similarities (spatial shifts of the solitons resulting from the collision, strong modulations during the collision, etc.).

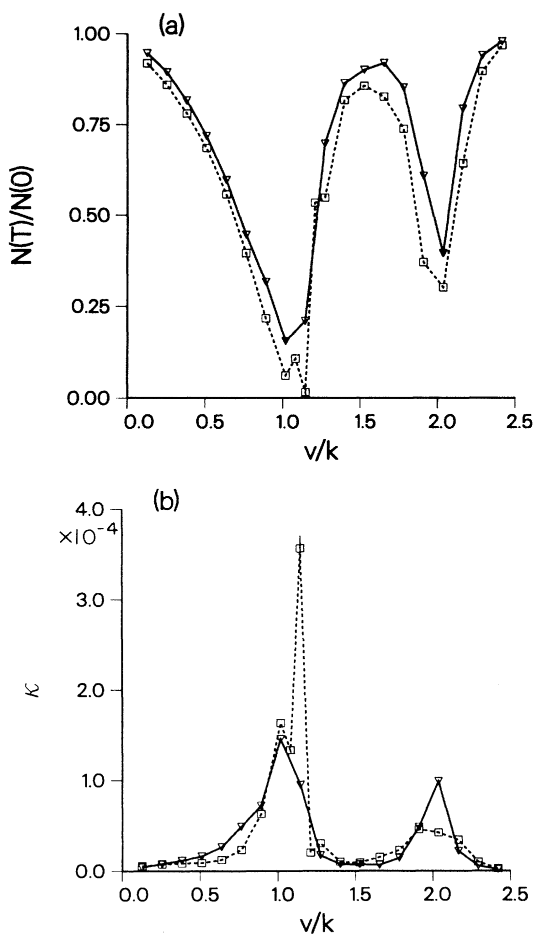

FIG. 6. Phase resonance and soliton stability. Parameters: $\epsilon=-0.1$, $\lambda=8$, $\eta=0.05$ (dashed line and squares); $\epsilon=-0.025$, $\lambda=16$, $\eta=0.025$ (solid line and triangles). Integration time $T=10\,000$. (a) Norm ratio $N(T)/N(0)$. (b) Exponential decay rate κ of the norm.

We collect our results in Fig. 6 for fixed k and variable soliton velocity $v=\dot{q}$ showing the norm N of the excitation after 10 000 time steps and the exponential decay rate κ of the norm (after damping away radiation and other excitations traveling with velocities different from the soliton velocity), both giving information about the stability of the excitation. As can be seen from Fig. 6 the solitons are most unstable for $|\dot{q}|\approx k$ and $|\dot{q}|\approx 2k$; but only in the former case is a resonance observed as described in Eq. (15). The difference between these two cases is also illustrated in Fig. 5. For $|\dot{q}|\approx 2k$ the soliton decays by spreading very slowly without apparent resonant behavior.

VI. THE AMPLITUDE–LENGTH-SCALE COMPETITION

If the width of the coherent excitation is comparable to the period of the perturbing potential ($\eta\approx k$) then neither the particle picture nor the renormalized particle picture can describe the outcome appropriately. That both approximation schemes fail has a deeper reason. The competition of length scales leads to complicated behavior in space and time which involves many degrees of freedom. There no longer exists a small number of collective degrees of freedom which allows a description of the essential features of the dynamics. As all approximation techniques fail badly we have to resort to numerical techniques to determine the parameter range in which length-scale competition leads to breakup of coherent excitations.

In Fig. 7 we illustrate the length-scale competition in question for a soliton with initial velocity $|\dot{q}|$ well below the resonance at k for potentials with widely different k values but fixed amplitudes. Upon varying the wavelength of the potential from $\lambda \ll 1/\eta$ through $\lambda\approx 1/\eta$ to $\lambda \gg 1/\eta$, we observe smoothing over the potential, breakup of the soliton and trapping in the potential, and particlelike motion in the long-wavelength potential, respectively. For the long-wavelength case see Fig. 1, where the potential has the same amplitude as in Fig. 7. As our numerical studies indicate for stronger perturbations the most robust excitation is the dressed soliton ($\lambda \ll 1/\eta$), whereas the effective particle soliton ($\lambda \gg 1/\eta$) very quickly begins to show dispersion induced by the curvature of the potential for the relatively large perturbation amplitude used in this case.

We collect our numerical results for solitons with initial velocity $|\dot{q}|$ well above the resonance at k in Fig. 8. It shows the effect of cosine potentials with a wide range of wavelengths on a fast moving soliton for large times. As long as the initial condition is well separated from the parameter region of competing length scales ($\eta\approx k$) the soliton can propagate in particlelike fashion for relatively large perturbations.

If the perturbation has several characteristic wavelengths, then only the ones comparable to the width of

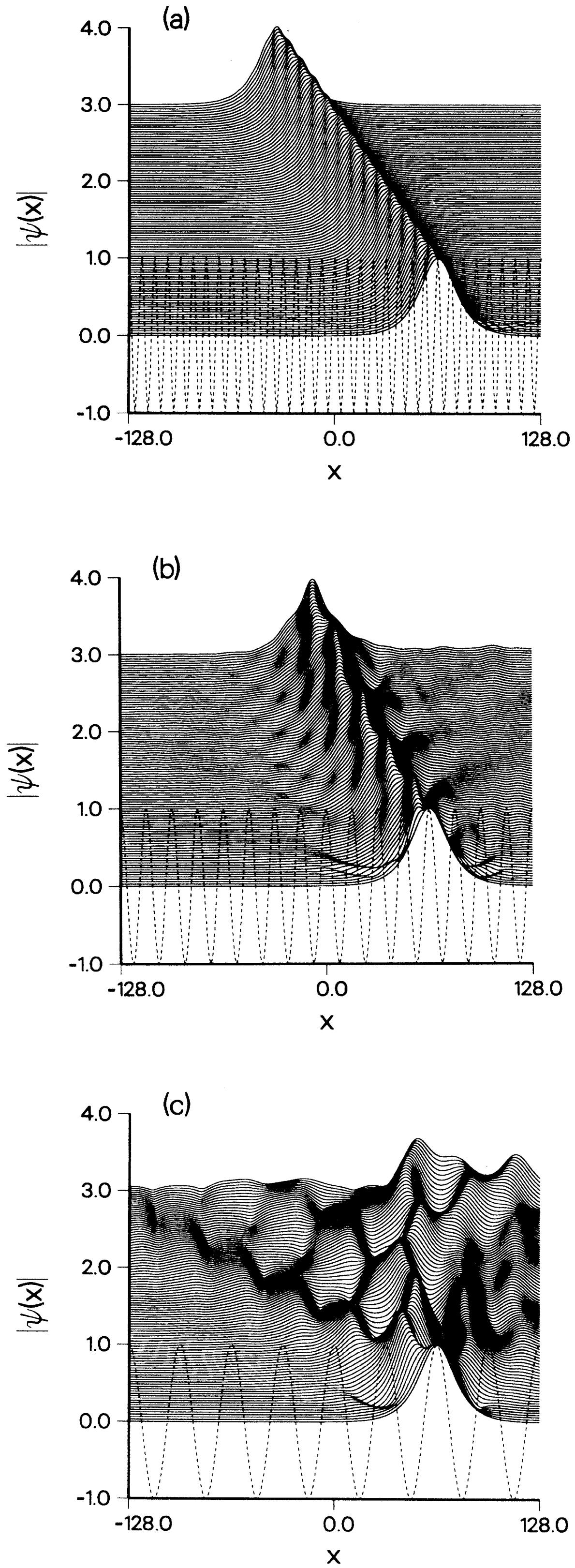

FIG. 7. The shape resonance $\eta \approx k$. Parameters: $\epsilon=0.01$, $\eta=0.05$, $\dot{q}=-0.04$, $T=2500$. (a) $\lambda=8$. (b) $\lambda=16$. (c) $\lambda=32$. For $\lambda=256$ see Fig. 1.

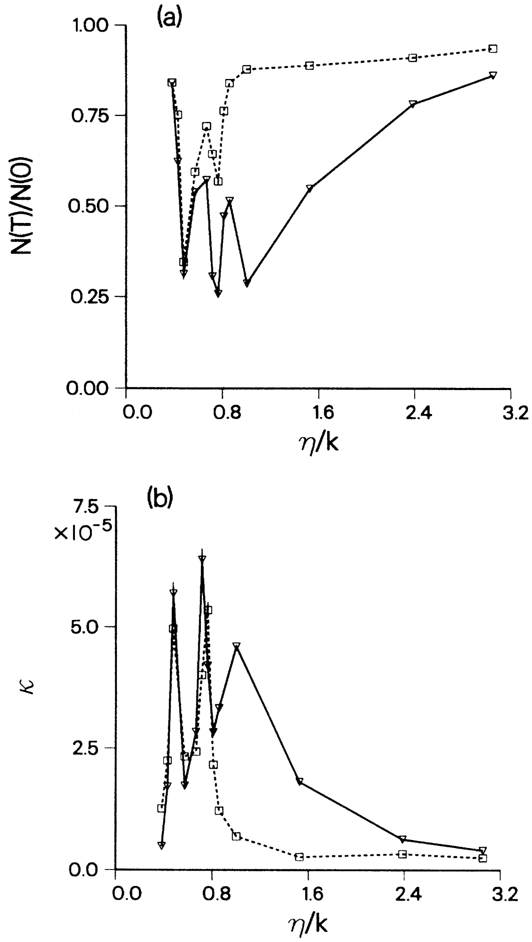

FIG. 8. Shape resonance and soliton stability. Parameters: $\dot{q}=1.2$, $\eta=0.05$, $T=10\,000$. $\epsilon=0.1$ (dashed line and squares); $\epsilon=0.2$ (solid line and triangles). (a) Norm ratio $N(T)/N(0)$. (b) Exponential decay rate κ of the norm.

the soliton are important and need to have small amplitude if they are not to destroy the particlelike behavior of the soliton.

VII. CONCLUSION

In summary, we have seen that the perturbed NLS equation can support solitonlike excitations as long as their two characteristic length scales are very different from the typical length scale of the perturbing potential. These excitations have either the shape of unperturbed solitons and move like particles in a long-wavelength effective potential, or their shape is strongly modulated by a short-wavelength potential and they move like renormalized particles.

We identified two different situations of competing length scales when the simple picture leading to a collective variable description fails: phase–length-scale competition and amplitude–length-scale competition. In both cases a large number of degrees of freedom is involved in the dynamics and a simple collective variable description seems no longer feasible. In these cases a small perturbation already can lead to a breakup of the coherent excitation, generating radiation as well as trapped smaller excitations.

Multicolor perturbing potentials of sufficiently small amplitude and with small or vanishing Fourier components at the wavelengths leading to a length-scale competition allow for long-lived solitonlike coherent excitations. The dynamics of these dressed solitons can be described by a collective variable approach similar to the one given above.

The nearly elastic collision of two dressed solitons shows that, apart from situations of competing length scales, those features of the NLS equation which are usually connected to its complete integrability are relatively stable under a large class of spatially periodic or quasiperiodic perturbations which break the complete integrability. This "structural stability" of the NLS equation explains its relevance for many physical systems (e.g., optical fibers, plasmas) where perturbation of the completely integrable NLS equation is not necessarily small. It is *not* the complete integrability of the NLS equation which seems to be important for this structural stability. We have, for example, demonstrated [7] that already a cosine perturbation such as in Eq. (1) can lead to soliton chaos in the two-soliton long-wavelength case, emphasizing the nonintegrability under this perturbation. Nevertheless, a collective variable description is still possible in this case.

The relevant feature for the structural stability of the NLS equation seems to be the property of the perturbed NLS equation (1) that for certain classes of initial conditions a collective variable description is still possible. This in turn leads to a dynamics governed by a few degrees of freedom which slave all the other degrees of freedom. In general this dynamics will not be integrable [7]. The stability under the parametric perturbations we have studied is more robust than in comparable studies of additive perturbations, where additional degrees of freedom are more easily excited [16].

Our results indicate that, for a given correlation length of the disorder present in a nonlinear system, spatially extended coherent excitations are much more stable than excitation with a width comparable to the correlation length. Correspondingly, in the case of a lattice we expect that nonlinear excitations extending over many lattice spacings are more stable than excitations extending only over a few lattice sites. The former can more effectively average out impurities and modulations on the scale of a few lattice spacings than the latter.

In the case of polarons the charge of an electron added to an ionic crystal leads to a lattice distortion giving rise to an effective (nonlinear) self-interaction of the excess charge. Polarons (and bipolarons for pairs of electrons) are the resulting coherent nonlinear excitations [17]. They can extend over a few lattice spacings distorting the lattice strongly (small polaron) or over many lattice spacings with a milder lattice distortion (large polaron). In disordered ionic lattices large polarons are therefore expected to smooth out the disorder and propagate long distances before they are scattered. In contrast, small polarons are much more strongly affected by disorder. They are expected to be scattered more often and to be trapped even by local impurities.

Finally we want to indicate two directions for future research. First, one should extend the class of perturbations to spatially random disorder. A collective variable approach combined with inverse-scattering perturbation theory should give information about the radiation generated in these potentials. Second, one should apply the collective variable approach to higher-dimensional perturbed-field equations. The behavior of single excitations is expected to be much richer in two or more spatial dimensions. Furthermore there is the possibility for having a larger class of qualitatively different excitations. Work on these questions is in progress.

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