Transition operators in electromagnetic-wave diffraction theory. II. Applications to optics

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This paper is the second of a series. The first [G. E. Hahne, Phys. Rev. A **45**, 7526 (1992)] developed a general theory of the transition operator approach to diffraction of time-harmonic electromagnetic waves from fixed obstacles, such that the response of the obstacle, denoted by Ω, to an impinging electromagnetic signal with wave number k_0 was simulated by nonlocal, homogeneous Leontovich (i.e., impedance) boundary conditions on the obstacle's surface, which surface is called $\partial\Omega$. Moreover, the exterior region, called Ω^{ex}, was presumed to be unbounded empty space, and has an electromagnetic response that can be expressed in terms of the so-called radiation impedance operator, denoted $\breve{Z}_{k_0}^{+}$; $\breve{Z}_{k_0}^{+}$ is a certain invertible, linear functional operator that maps the space of complex tangent-vector fields on $\partial\Omega$ into itself. The matching of the limiting tangential electric and magnetic fields on $\partial\Omega$ yielded a functional-operator expression for the transition operator and thereby a formal reduction to quadratures of the entire direct-scattering problem. This paper is intended to serve as an illustration and elaboration of the formalism presented in the first paper. After a brief recapitulation of the theory, the following topics are dealt with. First, an infinite sum in terms of vector spherical harmonics is obtained for $\breve{Z}_{k_0}^{+}$ for $\partial\Omega$ a sphere. Second, a quasiplanar approximation, based directly on the exact result for planar $\partial\Omega$, is obtained for $\breve{Z}_{k_0}^{+}$ when $|k_0|$ is large compared to the local surface curvatures of $\partial\Omega$ and attention can be restricted to small neighborhoods on $\partial\Omega$; when augmented by the method of stationary phase, this approximation is shown to lead to the familiar physical optics method for smooth, convex surfaces. Third, the latter method, supported by further interventions of the method of stationary phase, is applied in a well-established manner to secure the results of geometrical optics for the complete Green's function for the time-harmonic Maxwell field in the presence of a smooth-surfaced, convex, perfectly electrically conducting obstacle. One feature of the latter computations is that the original source currents from which the free-space Maxwell Green's function is constructed are presumed to generate ordinary outgoing ("causal") electromagnetic waves for electric current sources, and purely ingoing ("anticausal") waves for magnetic current sources (i.e., sinks) of electromagnetic radiation. Fourth, a formal construction is derived for mapping the Leontovich boundary conditions on an inner surface into a set of Leontovich boundary conditions on an outer surface, in the circumstance that a layer comprised of a material medium, which has possibly nonuniform and nonsymmetric tensors representing its constitutive properties, fills the domain between the surfaces; the construction presumes that a Green's function is available for the corresponding Maxwell equations in the medium when the medium is extended appropriately to fill all space. The main body of the paper concludes with a brief discussion of directions of possible future work and applications. The first appendix shows how to apply the method of stationary phase in the present context, in particular for the mixed case that the original radiation source is anticausal and the response currents generated in the obstacle are causal sources of radiation. A second appendix develops an acoustic (Helmholtz equation) analog to the fourth topic mentioned above, and exhibits an interplay of the theory with symplectic transformations in the case that the principle of reciprocity holds; it is noted that a symplectic connection also exists in the electromagnetic case when the propagation medium's constitutive properties are such that reciprocity holds.

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I. INTRODUCTION

The subject of this paper is an elaboration of a previously established theory—cf. Ref. [1]—of the diffraction of time-harmonic electromagnetic waves from obstacles. In Ref. [1], a transition (T) operator formalism was proposed for the diffraction of electromagnetic waves from an impenetrable obstacle of general geometrical shape, with imposed surface boundary conditions (SBC's) of general homogeneous, nonlocal, linear type enforced on the joint electric and magnetic field at the obstacle's boundary; the latter are called impedance, or (generalized) Leontovich, SBC's. In this paper we consider applications and an extension of the theory of Ref. [1].

The applications comprise first, a spherical harmonic expansion of the so-called radiation impedance operator, as defined in Ref. [1], for a spherical surface, and second, a reconsideration of familiar short-wavelength approximations from the new standpoint. The latter includes a derivation of the so-called physical optics method on the basis of a quasiplanar approximation to the radiation impedance operator, augmented by the method of stationary phase; it includes further a (re-)derivation of the geometrical optics approximation for the complete Green's function for the electromagnetic field in the presence of a smooth- and convex-surfaced perfectly electrically conducting obstacle.

The extension of the theory of Ref. [1] that is obtained here has as its principal objective the (one way) mapping of Leontovich boundary conditions on a surface into an analogous simulation on the exterior surface of a new obstacle, which is comprised of the original obstacle augmented by a material layer with specified geometry and (linear) constitutive properties. The construction of the mapping requires that a Green's function be available, at least in principle, for the medium that fills the region between the two surfaces when it is extended in some suitable manner to fill all space; accordingly, numerical applications will normally be feasible only when this medium is electromagnetically homogeneous and isotropic, as in Ref. [1], Appendix C. In this manner, an obstacle that is both materially and geometrically complex can be simulated in the respect of its electromagnetic scattering by Leontovich boundary conditions on a geometrically simpler circumscribing surface, as a sphere. The mapping is one way in the sense that the method derived here does not permit the mapping of Leontovich boundary conditions from an exterior surface to an interior one.

Two previous papers [2,3] were dedicated to the establishment of a T operator theory of diffraction of time-harmonic acoustic (scalar) waves from impenetrable obstacles, such that general homogeneous, nonlocal SBC's of the Robin (that is, impedance) type are satisfied by the acoustic signal at the obstacle's bounding surface. The first [2] of these papers treated the general theory, while the second [3] contained an investigation of certain important aspects of the theory from the new viewpoint, that is, of the acoustic radiation impedance operator and of the geometrical acoustics limit for special obstacle geometries and simple SBC's. The theory of Refs. [2] and [3] is extended herein in Appendix B: a method of propagating certain types of solutions of the Helmholtz equation from surface to surface, and a connection with projection operators and symplectic transformations on the joint linear space of surface values and normal derivatives, are established. There proves to be a structural analogy between the acoustic and the electromagnetic theory; the principal differences between electromagnetic-wave diffraction theory as it is treated here, and the acoustic theory presented in Refs. [2] and [3] and in Appendix B, derive from the vector character of the electric and the magnetic fields. Other differences are (i) that in Secs. IV, V, and VI, which deal with short-wavelength approximations, the hypothesis is made that true physical electric and magnetic currents give rise to, respectively, outgoing-wave and ingoing-wave solutions to Maxwell's equations, and (ii) that in Sec. VII, which deals with electromagnetic-wave propagation in a medium, reciprocity-violating constitutive properties are introduced.

The remainder of this paper is organized as follows. In Sec. II, we shall recapitulate briefly parts of the general theory of Ref. [1]: the kinematics of a diffraction problem for electromagnetic waves, SBC's for perfect electrical conductors (called E-type obstacles and SBC's), free-space and complete Green's functions, the radiation impedance operator $\breve{Z}_{k_0}^{+}$, and the T operator associated with an E-type obstacle. In Sec. III, we shall obtain and discuss an infinite series expansion, in terms of vector spherical harmonics, of $\breve{Z}_{k_0}^{+}$ for the case of a spherical boundary. In Sec. IV, we shall propose a "quasiplanar" approximation for $\breve{Z}_{k_0}^{+}$, which seems particularly suited to applications for which the electromagnetic wavelength is small compared to an obstacle's local curvature radii. In Sec. V, we shall verify that the quasiplanar approximation for $\breve{Z}_{k_0}^{+}$ will, with the aid of the method of stationary phase, yield the familiar short-wavelength approximation known as "physical optics" as a consequence. In Sec. VI, we shall apply the results of Secs. IV and V, again with the help of the method of stationary phase, to obtain an approximation for the complete Green's function in an E-type diffraction problem in the extreme short-wavelength (i.e., geometrical optics) realm; the results will be a mixture of the familiar and unfamiliar due to our presumption of conventional and unconventional forms of radiation from electric and magnetic current sources, respectively. In Sec. VII we shall derive, with the aid of a suitable Green's function, a mapping of given Leontovich boundary conditions on an interior surface into Leontovich boundary conditions on an exterior surface, where a physically complex linear medium may fill the domain between the surfaces. Section VIII concludes the main part of the paper with a brief discussion of possible directions for further work.

The paper is augmented by two appendices. In Appendix A, we derive the rules for stationary phase integrals (and implicitly geometrical optics involving reflection from a surface) both in the "elliptic" and "hyperbolic" cases that the specular points are the stationary points of the sum and difference function, respectively, of distances from a point on the surface to a source point and to a field point. In Appendix B, we shall extend the theory of Refs. [2] and [3] to develop a theory of propagation of initial values (including both function values and their normal derivatives) of the Helmholtz equation from surface to surface, and show how this theory is intertwined with that of certain projection operators and symplectic transformations, in the linear space of boundary values. It is argued in Sec. VII that an analogous symplectic structure exists for the time-harmonic Maxwell field when the principle of reciprocity obtains.

II. SKETCH OF FORMAL DIFFRACTION THEORY

In this section we shall give a brief recapitulation of those results of Ref. [1], specialized in the respects of geometry and boundary conditions, that are needed for our purposes here.

We denote Euclidean three-dimensional space by $\mathcal{E}^3$, established a fixed Cartesian coordinate system in it, and denote a point by a three-vector as $\mathbf{r}$, $\mathbf{r}_1$, etc.; the corresponding volume measures are denoted d^3r and d^3r_1, respectively. Specializing from Ref. [1], Sec. II, we let Ω be the open set occupied by the obstacle, let Ω^{ex} be the open set comprising the unbounded exterior region in which the electromagnetic field propagates, and let $\partial\Omega$ be the surface, presumed smooth, that is the common boundary of Ω^{ex} and Ω. Smoothness means that the outward unit normal vector and the surface curvature matrix are continuous everywhere in $\partial\Omega$. Points in the subset $\partial\Omega$ are denoted by three-vectors with a first subscript ∂, as $\mathbf{r}_\partial$ or $\mathbf{r}_{\partial 1}$, and the corresponding element of surface area measure is called dA_∂ or $dA_{\partial 1}$, respectively. The unit outward (pointing toward Ω^{ex}) normal vector to $\partial\Omega$ at $\mathbf{r}_{\partial 1}$ is denoted $\hat{\mathbf{n}}(\mathbf{r}_{\partial 1})$. These notations will need to be generalized for the computations of Sec. VII and Appendix B, where two or more surfaces are involved.

The electromagnetic fields and sources are presumed to have the (unexpressed) time dependence $\exp(-ik_0ct)$, where c is the speed of light in a vacuum, and the wave number k_0 can be any nonzero real number. The electromagnetic field is described by the direct sum of two three-vector fields, the electric field $\mathbf{E}_{k_0}(\mathbf{r})$, and the magnetic field $c\mathbf{B}_{k_0}(\mathbf{r})$, where we use *Système International* units, and the c is for dimensional consistency. An electromagnetic field is denoted $(\mathbf{E}_{k_0}(\mathbf{r});c\mathbf{B}_{k_0}(\mathbf{r}))^\tau$, where the τ means transpose, that is, the field at each $\mathbf{r}$ is a column vector with six entries.

In the geometrical optics application of Sec. VI, the obstacle occupying Ω is taken to be a perfect electrical conductor, so that electrical currents flow only within $\partial\Omega$, and no magnetic currents are induced in $\Omega\cup\partial\Omega$. The region Ω^{ex} is taken to be a vacuum. Accordingly, the exterior limiting tangential component of the electric field, denoted $-\hat{\mathbf{n}}(\mathbf{r}_\partial)\times[\hat{\mathbf{n}}(\mathbf{r}_\partial)\times\mathbf{E}_{k_0}(\mathbf{r}_\partial\pm)]$, is forced to be the zero vector; the extra $+$ (or $-$) in the argument implies that a limit from points in Ω^{ex} (or Ω) is to be taken. These physical SBC's will be called E-type boundary conditions; the E may appear in a superscript, as in the Green's function and the T operator. In acoustic-wave diffraction, the limiting cases of "sound-hard" (Neumann) and of "sound-soft" (Dirichlet) SBC's are associated with differently structured T operators in the degree of singularity encountered—see Ref. [2], Eqs. (59) and (60), and Ref. [3], Eqs. (5) and (6). For electromagnetic-wave diffraction, the dual case of a perfect magnetic conductor does not offer anything substantially new from a mathematical viewpoint, so that we shall not consider this case of hypothetical boundary conditions.

We denote the space of sufficiently well-behaved complex tangent-vector fields on $\partial\Omega$ by $\mathcal{V}^{\partial\Omega}$. Linear operators that map this space into itself will have a superimposed "breve" accent, as $\breve{C}$; these operators will be construed to annihilate the normal components of a general three-vector field on $\partial\Omega$. The identity operator in $\mathcal{V}^{\partial\Omega}$ is called $\breve{I}_\partial$. We let $\breve{X}_\partial$ be the operator having local Cartesian components $\epsilon_{jlk}\hat{\mathbf{n}}_l(\mathbf{r}_\partial)$ (ϵ_{jlk} is the Levi-Civita symbol, and the summation convention is operative on Cartesian indices); note that

$$\breve{X}_\partial^2=-\breve{I}_\partial \ . \tag{1}$$

If $\mathbf{E}_1\in\mathcal{V}^{\partial\Omega}$ and $\mathbf{E}_2\in\mathcal{V}^{\partial\Omega}$, we define the bilinear inner product

$$(\mathbf{E}_1;\mathbf{E}_2)_{\partial\Omega}\equiv\int_{\partial\Omega}\mathbf{E}_1(\mathbf{r}_\partial)\cdot\mathbf{E}_2(\mathbf{r}_\partial)dA_\partial \ , \tag{2}$$

and the $\mathbf{E}_1,\mathbf{E}_2$ bilinear matrix element of an operator $\breve{C}$ as $(\mathbf{E}_1;\breve{C}\mathbf{E}_2)_{\partial\Omega}$. The adjoint of $\breve{C}$ in this context is called its transpose, and is defined as that operator $\breve{C}^\tau$ for which

$$(\mathbf{E}_2;\breve{C}^\tau\mathbf{E}_1)_{\partial\Omega}=(\mathbf{E}_1;\breve{C}\mathbf{E}_2)_{\partial\Omega} \tag{3}$$

for all choices of pairs of fields $\mathbf{E}_1\in\mathcal{V}^{\partial\Omega}$ and $\mathbf{E}_2\in\mathcal{V}^{\partial\Omega}$. An operator that equals its transpose is called symmetric.

By virtue of the uniqueness and existence theorems for Maxwell fields in Ω^{ex} satisfying asymptotic conditions of outgoing-wave type (cf. Ref. [4], Theorems 4.18 and 4.27), each tangential electric-field distribution $\breve{I}_\partial\mathbf{E}_{k_0}(\mathbf{r}_\partial+)$ is uniquely associated with a tangential magnetic-field distribution $\breve{I}_\partial c\mathbf{B}_{k_0}(\mathbf{r}_\partial+)$ on $\partial\Omega$; these fields are the tangential projections of the exterior limiting values of a particular solution to Maxwell's equations of outgoing-wave type. By definition, then, there exists an invertible operator $\breve{Z}_{k_0}^+$, called the radiation impedance operator, such that [cf. Ref. [1], Eq. (61)]

$$\breve{I}_\partial\mathbf{E}_{k_0}(\mathbf{r}_{\partial+})=-(\breve{Z}_{k_0}^+\breve{X}_\partial c\mathbf{B}_{k_0})(\mathbf{r}_\partial+) \ , \tag{4}$$

where the $-\breve{X}_\partial$ is inserted for convenience elsewhere. From the duality symmetry of Maxwell's equations in empty space [Ref. [5], Eq. (6.151)], it follows that

$$(\breve{Z}_{k_0}^+)^{-1}=-\breve{X}_\partial\breve{Z}_{k_0}^+\breve{X}_\partial \ . \tag{5}$$

It is easy to show that $\breve{Z}_{k_0}^+$ is symmetric [Ref. [1], Eq. (64)]:

$$(\breve{Z}_{k_0}^+)^\tau=\breve{Z}_{k_0}^+ \ . \tag{6}$$

We denote electromagnetic Green's functions briefly by a symbol $\Gamma_{k_0}^{X+\pm}$, and in detail as entities with components $\Gamma_{k_0,\alpha\beta,jk}^{X+\pm}(\mathbf{r}_1;\mathbf{r}_2)$. The physical meaning of the various superscripts, subscripts, and arguments follows the notation of Ref. [1], Sec. III, and is as follows. The X can be M, E, or 0 in the present context: the M denotes the Green's function when all space is filled by a material medium, the properties of which will be specified in context; E stands for the complete Green's function when a perfectly electrically conducting obstacle is present; and 0 stands for the free-space Green's function. The second superscript $+$ means that electrical sources are presumed always to be associated with outgoing-wave solutions to Maxwell's equations. The third superscript can be $+$ or $-$, depending on whether magnetic sources are presumed to couple to outgoing-wave solutions or to ingoing-wave solutions, respectively, to Maxwell's equations; we emphasize that only true magnetic sources are presumed to be of the latter type, while any fictitious magnetic sources arising from the modeling of the obstacle—presumed to be made up of ordinary electrical matter—are taken to be of the former type. Concerning the subscripts and arguments, k_0c is the angular frequency of oscillation of the sources and fields, the source is an electric ($\beta=e$) or magnetic ($\beta=m$) current element (δ function) pointing in direction $k=x$, y, or z, and located at $\mathbf{r}_2$; the Green's function then gives the electric- ($\alpha=e$) and magnetic- ($\alpha=m$) field components in directions $j=x$, y, and z, at the field point $\mathbf{r}_1$.

Note that each of the four sub-blocks of a Green's function, extracted by fixing both α and β indices, can be regarded as an ordinary dyadic operator that maps the space of vector fields in $\mathcal{E}^3$ into itself linearly—cf. Ref. [1], Eq. (34); for the free-space Green's function these are expressible in terms of conventional Green's dyadics, as in Ref. [1], Eqs. (A2) and (A3). We recapitulate these results. We take

$$\Gamma_{k_0}^{0+\pm}=\begin{bmatrix}(\Gamma_{k_0}^{0+\pm})_{ee} & (\Gamma_{k_0}^{0+\pm})_{em}\\ (\Gamma_{k_0}^{0+\pm})_{me} & (\Gamma_{k_0}^{0+\pm})_{mm}\end{bmatrix} =\begin{bmatrix}G_{k_0,ee}^{0+} & G_{k_0,em}^{0\pm}\\ G_{k_0,me}^{0+} & G_{k_0,mm}^{0\pm}\end{bmatrix}, \tag{7}$$

where the four constituent dyadics are constructed as follows: We have the causal (anticausal) scalar Green's function

$$G_{k_0}^{0\pm}(\mathbf{r}_1;\mathbf{r}_2)=-(4\pi|\mathbf{r}_1-\mathbf{r}_2|)^{-1}\exp(\pm ik_0|\mathbf{r}_1-\mathbf{r}_2|)\ , \tag{8}$$

where the $+$ ($-$) superscript or sign corresponds to outgoing (ingoing) radiation. The Green's dyadics of the right-hand side (rhs) of Eq. (7) are now

$$\begin{aligned}G_{k_0,ee,jk}^{0\pm}(\mathbf{r}_1;\mathbf{r}_2)&=G_{k_0,mm,jk}^{0\pm}(\mathbf{r}_1;\mathbf{r}_2)\\ &=(k_0\delta_{jk}+k_0^{-1}\nabla_{1j}\nabla_{1k})G_{k_0}^{0\pm}(\mathbf{r}_1;\mathbf{r}_2)\ ,\end{aligned} \tag{9}$$

$$\begin{aligned}G_{k_0,me,jk}^{0\pm}(\mathbf{r}_1;\mathbf{r}_2)&=-G_{k_0,em,jk}^{0\pm}(\mathbf{r}_1;\mathbf{r}_2)\\ &=-i(\nabla_1\times)_{jk}G_{k_0}^{0\pm}(\mathbf{r}_1;\mathbf{r}_2)\ ,\end{aligned} \tag{10}$$

where the differentiations are to be carried out after integration over $\mathbf{r}_2$ when necessary to avoid singularities.

The T operator $\mathcal{T}_{k_0}^{E+}$ associated with a perfectly conducting obstacle is defined implicitly as that operator that makes the following equation [Ref.[1], Eqs. (38), (39)] hold:

$$\Gamma_{k_0}^{E+-}=\Gamma_{k_0}^{0+-}+\Gamma_{k_0}^{0++}\mathcal{T}_{k_0}^{E+}\Gamma_{k_0}^{0+-}\ , \tag{11}$$

where the second summand on the rhs is an operator product. The first summand on the rhs of Eq. (11) represents the signal in the absence of an obstacle, i.e., when $\mathcal{T}_{k_0}^{E+}$ is the zero operator. The second summand comprises the complete scattered wave, where the cause and effect sequence, but not necessarily the temporal sequence, reads from right to left: In response to the impinging field given by $\Gamma_{k_0}^{0+-}$, the $\mathcal{T}_{k_0}^{E+}$ operator creates, linearly but nonlocally, currents on $\Omega\cup\partial\Omega$; the operation of $\Gamma_{k_0}^{0++}$ on these currents then gives rise to an additional electromagnetic field that exactly cancels the impinging field within Ω and has the character of an outgoing-wave scattered field in Ω^{ex}, such that the total exterior field satisfies the given SBC's.

For any open set $\Delta\subset\mathcal{E}^3$ we define the unit step function Θ_Δ as follows:

$$\Theta_\Delta(\mathbf{r})\equiv\begin{cases}1 & \text{if } \mathbf{r}\in\Delta\\ 0 & \text{otherwise}\ .\end{cases} \tag{12}$$

We shall now give the results for $\Gamma_{k_0}^{E+-}$ derived in Ref. [1], Eqs. (97) and (98), following applications of Eqs. (94) and (95) of Ref. [1]:

$$\begin{aligned}\Gamma_{k_0,\alpha\beta,jk}^{E+-}(\mathbf{r}_1;\mathbf{r}_2)=&[1-(\tfrac{1}{2})\Theta_\Omega(\mathbf{r}_1)]G_{k_0,\alpha\beta,jk}^{0\pm}(\mathbf{r}_1;\mathbf{r}_2)-(\tfrac{1}{2})G_{k_0,\alpha\beta,jk}^{0+}(\mathbf{r}_1;\mathbf{r}_2)\Theta_\Omega(\mathbf{r}_2)\\ &+(i/2)[G_{k_0,\alpha e}^{0+}(\check{X}_\partial\check{Z}_{k_0}^{+}\check{X}_\partial G_{k_0,e\beta}^{0\pm}-\check{X}_\partial G_{k_0,m\beta}^{0\pm})]_{jk}(\mathbf{r}_1;\mathbf{r}_2)\\ &+(i/2)[(G_{k_0,\alpha e}^{0+}\check{X}_\partial\check{Z}_{k_0}^{+}\check{X}_\partial-G_{k_0,\alpha m}^{0+}\check{X}_\partial)G_{k_0,e\beta}^{0\pm}]_{jk}(\mathbf{r}_1;\mathbf{r}_2)\ .\end{aligned} \tag{13}$$

The choices of plus and minus in the superscripts on the rhs of Eq. (13) correspond to $\beta=e$ and $\beta=m$, respectively. The implied operator products on the rhs of Eq. (13) entail restriction of the inner arguments of the Green's functions to $\partial\Omega$, summation over inner Cartesian indices, and integration of all the inner arguments over $\partial\Omega$. The form Eq. (13) and its constituent expressions will be estimated in the short-wavelength approximations of Secs. V and VI.

III. VECTOR SPHERICAL HARMONIC EXPANSION OF $\check{Z}_{k_0}^{+}$ FOR SPHERICAL $\partial\Omega$

In this section we shall obtain an infinite expansion in terms of a complete orthonormal set of vector spherical harmonics of the operator $\check{Z}_{k_0}^{+}$, for the geometrical case that $\partial\Omega$ is a sphere of radius a centered at the origin, which sphere we call $S^2(a)$. Note that in this section orthogonality and normalization of states and operator adjoints (called Hermitian conjugates) are defined in terms of the sesquilinear inner product familiar from quantum mechanics: that is, if $\mathbf{E}_1(\mathbf{r}_\partial)$ and $\mathbf{E}_2(\mathbf{r}_\partial)$ are two (not necessarily tangential) vector fields on $\partial\Omega$, we have the Hermitian inner product

$$\langle\mathbf{E}_1|\mathbf{E}_2\rangle=\int_{\partial\Omega}[\mathbf{E}_1(\mathbf{r}_\partial)]^*\cdot\mathbf{E}_2(\mathbf{r}_\partial)dA_\partial\ . \tag{14}$$

We shall follow and augment the constructions in Jackson (Ref. [5], p. 746) of vector spherical harmonics, but make certain changes of notation: For example, we shall use J,M rather than l,m as the indices for multipole order and orientation, as the former symbols are a notation that is more in keeping with the tradition in quantum mechanics (cf. Ref. [6], Chap. 5.9, or Ref. [7], Chap. 2) for total angular momentum $\mathbf{J}=\mathbf{S}+\mathbf{L}$ and z projection $J_z=S_z+L_z$ (with quantum numbers J and M, respectively) in the kinematics of vector addition of spin $\mathbf{S}$ (with quantum numbers $s=1$, and $m_s=\pm 1,0$ for vector fields) plus orbital angular momentum $\mathbf{L}$ (with quantum numbers l,m).

Let us consider the space of complex three-vector fields defined on the unit sphere $S^2(1)$. Each such field, denoted $\mathbf{E}(\hat{\mathbf{r}})$ say, will have components $E_j(\hat{\mathbf{r}})$, where $j=x,y,z$ refers to fixed Cartesian axes and $\hat{\mathbf{r}}\in S^2(1)$. A complete orthonormal set of such vector fields can be constructed explicitly. In fact, let $Y_{lm}(\hat{\mathbf{r}})$, $l=0,1,2,\ldots$, $m=-l,-l+1,\ldots,+l$ be the spherical harmonics as defined in Ref. [6], Eq. (2.5.5), and let $\mathbf{S}$, $\mathbf{L}$, and $\mathbf{J}$ be the vector operators with Cartesian components

$$(S_j)_{kl}=i\epsilon_{kjl}, \quad (L_j)_{kl}=\delta_{kl}\epsilon_{jpq}r_p\frac{1}{i}\frac{\partial}{\partial r_q}, \qquad (J_j)_{kl}=(S_j)_{kl}+(L_j)_{kl} ; \tag{15}$$

the operator $\mathbf{L}$ can be expressed in terms of $\hat{\mathbf{r}}$ coordinates alone—cf. Ref. [6], Eq. (2.1.4). The desired set of vector fields on $S^2(1)$ will be called $\mathbf{X}^{pJM}(\hat{\mathbf{r}})$, $p=1,2,3$, and is obtained by the following construction (Ref. [5], p. 746, as modified; see also Ref. [7], Chap. 2):

$$X_j^{1JM}(\hat{\mathbf{r}})\equiv[J(J+1)]^{-1/2}L_jY_{JM}(\hat{\mathbf{r}}) , \tag{16}$$

$$X_j^{2JM}(\hat{\mathbf{r}})\equiv[\hat{\mathbf{r}}\times\mathbf{X}^{1JM}(\hat{\mathbf{r}})]_j , \tag{17}$$

$$X_j^{3JM}(\hat{\mathbf{r}})\equiv\hat{\mathbf{r}}_jY_{JM}(\hat{\mathbf{r}}) . \tag{18}$$

In Eqs. (16) and (17) $J=1,2,3,\ldots$, in Eq. (18) $J=0,1,2,\ldots$, and in Eqs. (16)–(18) $M=-J,-J+1,\ldots,J-1,J$. The sets $\mathbf{X}^{1JM}(\hat{\mathbf{r}})$ and $\mathbf{X}^{2JM}(\hat{\mathbf{r}})$ together are complete within the space of tangent-vector fields to $S^2(1)$.

We remark that the vector harmonics of Eqs. (16)–(18) can be regarded as coordinate representatives of certain state vectors, which states may be called $|pJM\rangle$ in Dirac notation; that is,

$$\langle j\hat{\mathbf{r}}|pJM\rangle=X_j^{pJM}(\hat{\mathbf{r}}) . \tag{19}$$

A particular unitary transformation among the three states $|pJM\rangle$, $p=1,2,3$ (there is only one state with $J=0$) yields the set of states that are simultaneous eigenstates of three compatible (that is, mutually commuting) Hermitian operators: (i) the so-called r-helicity operator [8] $\omega\equiv\hat{\mathbf{r}}\cdot\mathbf{J}=\hat{\mathbf{r}}\cdot\mathbf{S}=i(\hat{\mathbf{r}}\times)$, (ii) the operator $\mathbf{J}\cdot\mathbf{J}$, and (iii) the operator J_z. In fact, if we call the latter states $|\omega JM\rangle$, where the eigenvalue ω can take the values $+1,0,-1$ for $J>0$, and 0 for $J=0$, it can be verified that

$$|\omega JM\rangle=\sum_{p=1}^{3}|pJM\rangle\langle p\|\omega\rangle_J , \tag{20}$$

where for $J>0$ we have the J-independent matrix

$$\langle p\|\omega\rangle_J=\begin{array}{c} \\ p=1 \\ p=2 \\ p=3 \end{array}\begin{array}{c} \begin{array}{ccc} \omega=+1 & \omega=0 & \omega=-1 \end{array} \\ \begin{bmatrix} -1\sqrt{2} & 0 & 1/\sqrt{2} \\ -i/\sqrt{2} & 0 & -i/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix} \end{array} . \tag{21}$$

For $J=0$ the transformation matrix is 1×1; $\langle 3\|0\rangle_0$ is the number $+1$. An analogous set of vector harmonics can be constructed on the unit sphere in wave-vector ($\mathbf{k}$) space; there is a corresponding set of states that are eigenstates of (i) the k-helicity operator $\lambda\equiv\hat{\mathbf{k}}\cdot\mathbf{J}=\hat{\mathbf{k}}\cdot\mathbf{S}=i(\hat{\mathbf{k}}\times)$ (called p helicity in quantum mechanics), (ii) the operator $\mathbf{J}\cdot\mathbf{J}$, and (iii) the operator J_z. The values $\lambda=+1$, 0, and -1 correspond, respectively, to left circular, longitudinal, and right circular polarization of the vector field—cf. Ref. [5], p. 274, and Ref. [9], p. 28. Note that the position-coordinate representative of the Hermitian operator $k\lambda=\mathbf{k}\cdot\mathbf{J}=\mathbf{k}\cdot\mathbf{S}$ is the curl operator $\nabla\times$ within the space of three-vector fields on $\mathcal{E}^3$, where Fourier transformation is taken as the unitary transformation between $\mathbf{r}$ space and $\mathbf{k}$ space, with an obvious definition for the sesquilinear inner product of two vector fields on $\mathcal{E}^3$.

Now let $f(r)$ be a continuously differentiable function except possibly at $r=0$. Then away from the origin of coordinates we have the gradient, curl, and divergence formulas [derivations omitted—related formulas are in Ref. [6], Eqs. (5.9.17)–(5.9.23)]

$$\nabla[f(r)Y_{JM}(\hat{\mathbf{r}})]=-\frac{i}{r}f(r)[J(J+1)]^{1/2}\mathbf{X}^{2JM}(\hat{\mathbf{r}}) +\frac{d}{dr}[f(r)]\mathbf{X}^{3JM}(\hat{\mathbf{r}}) , \tag{22}$$

$$\nabla\times[f(r)\mathbf{X}^{1JM}(\hat{\mathbf{r}})]=\frac{1}{r}\frac{d}{dr}[rf(r)]\mathbf{X}^{2JM}(\hat{\mathbf{r}}) +\frac{i}{r}f(r)[J(J+1)]^{1/2}\mathbf{X}^{3JM}(\hat{\mathbf{r}}) , \tag{23}$$

$$\nabla\times[f(r)\mathbf{X}^{2JM}(\hat{\mathbf{r}})]=-\frac{1}{r}\frac{d}{dr}[rf(r)]\mathbf{X}^{1JM}(\hat{\mathbf{r}}) , \tag{24}$$

$$\nabla\times[f(r)\mathbf{X}^{3JM}(\hat{\mathbf{r}})]=-\frac{i}{r}f(r)[J(J+1)]^{1/2}\mathbf{X}^{1JM}(\hat{\mathbf{r}}) , \tag{25}$$

$$\nabla\cdot[f(r)\mathbf{X}^{1JM}(\hat{\mathbf{r}})]=0 , \tag{26}$$

$$\nabla\cdot[f(r)\mathbf{X}^{2JM}(\hat{\mathbf{r}})]=-[J(J+1)^{1/2}(i/r)f(r)Y_{JM}(\hat{\mathbf{r}}) , \tag{27}$$

$$\nabla\cdot[f(r)\mathbf{X}^{3JM}(\hat{\mathbf{r}})]=\frac{1}{r^2}\frac{d}{dr}[r^2f(r)]Y_{JM}(\hat{\mathbf{r}}) . \tag{28}$$

The time-harmonic Maxwell equations in empty space are [Ref. [4], Eq. (4.2)], with $k_0\neq 0$,

$$\mathbf{E}_{k_0}(\mathbf{r})=-(ik_0)^{-1}(\nabla\times c\mathbf{B}_{k_0})(\mathbf{r}) , \tag{29}$$

$$c\mathbf{B}_{k_0}(\mathbf{r})=(ik_0)^{-1}(\nabla\times\mathbf{E}_{k_0})(\mathbf{r}) . \tag{30}$$

Using Eqs. (23)–(30) and the properties of the spherical Hankel functions $h_J^{(1)}(k_0r)$ (cf. Ref. [10], p. 437), we can establish that for each $J \geq 1$ and $|M| \leq J$ the electromagnetic field distribution $[\mathbf{E}^{+mJM}_{k_0}(\mathbf{r}); c\mathbf{B}^{+mJM}_{k_0}(\mathbf{r})]^\tau$, where

$$\mathbf{E}^{+mJM}_{k_0}(\mathbf{r}) = h_J^{(1)}(k_0 r)\mathbf{X}^{1JM}(\hat{\mathbf{r}}) , \tag{31}$$

$$c\mathbf{B}^{+mJM}_{k_0}(\mathbf{r}) = -\frac{i}{k_0 r}\frac{d}{dr}[r h_J^{(1)}(k_0(r)]\mathbf{X}^{2JM}(\hat{\mathbf{r}}) + \frac{1}{k_0 r} h_J^{(1)}(k_0 r)[J(J+1)]^{1/2}\mathbf{X}^{3JM}(\mathbf{r}) \tag{32}$$

satisfies Maxwell's equations in the domain $\mathcal{E}^3 - \{\mathbf{0}\}$, and satisfies the Silver-Müller radiation conditions of Ref. [4], Theorem 4.4. The superscripts $+m$ stand for outgoing-wave magnetic multipole (this is, transverse electric) fields, and J,M for the multipole order 2^J and orientation. The corresponding electric multipole (i.e., transverse magnetic) fields of outgoing-wave type are obtained as the dual solutions to Maxwell's equations:

$$[\mathbf{E}^{+eJM}_{k_0}(\mathbf{r}); c\mathbf{B}^{+eJM}_{k_0}(\mathbf{r})]^\tau = [c\mathbf{B}^{+mJM}_{k_0}(\mathbf{r}); -\mathbf{E}^{+mJM}_{k_0}(\mathbf{r})]^\tau . \tag{33}$$

A general solution to Eqs. (29) and (30) of outgoing-wave type has the multipole expansion

$$\mathbf{E}^{+}_{k_0}(\mathbf{r}) = \sum_{J,M} \{ A_m^{JM}\mathbf{E}^{+mJM}_{k_0}(\mathbf{r}) + A_e^{JM}\mathbf{E}^{+eJM}_{k_0}(\mathbf{r})\} , \tag{34}$$

$$c\mathbf{B}^{+}_{k_0}(\mathbf{r}) = \sum_{J,M} \{ A_m^{JM} c\mathbf{B}^{+mJM}_{k_0}(\mathbf{r}) + A_e^{JM} c\mathbf{B}^{+eJM}_{k_0}(\mathbf{r})\} . \tag{35}$$

We can find the corresponding tangential fields on $S^2(a)$ by substituting $r=a$ in Eqs. (34) and (35) and dropping all contributions from $\mathbf{X}^{3JM}(\hat{\mathbf{r}})$. Let us define the logarithmic derivative $\xi_J^{(1)}(z)$ for the Riccati-Hankel functions (Ref. [7], Chap. 2.2.1)

$$\xi_J^{(1)}(z) \equiv [z h_J^{(1)}(z)]^{-1}(d/dz)[z h_J^{(1)}(z)] . \tag{36}$$

It follows now from the definition Eq. (4) that the radiation impedance operator for $\partial\Omega = S^2(a)$ has the following expansion for its kernel:

$$\langle j'\hat{\mathbf{r}}'|\breve{Z}^{+}_{k_0}|j\hat{\mathbf{r}}\rangle \equiv \breve{Z}^{+}_{k_0,j'j}(\hat{\mathbf{r}}';\hat{\mathbf{r}}) = a^{-2}\sum_{J>0,M}\{X_{j'}^{1JM}(\hat{\mathbf{r}}')[-i\xi_J^{(1)}(k_0 a)]^{-1}[X_j^{1JM}(\hat{\mathbf{r}})]^* + X_{j'}^{2JM}(\hat{\mathbf{r}}')[-i\xi_J^{(1)}(k_0 a)][X_j^{2JM}(\hat{\mathbf{r}})]^*\} . \tag{37}$$

We have asymptotically for large J and fixed z,

$$\xi_J^{(1)}(z) \approx -\frac{J}{z} + \frac{z}{2J-1} . \tag{38}$$

The expansion Eq. (37) for $\breve{Z}^{+}_{k_0}$ divides into two suboperators involving either $\mathbf{X}^{1JM}$ or $\mathbf{X}^{2JM}$ exclusively, where the latter contributes the dominant singular terms to the generalized (in the sense of containing differential operations) kernel of $\breve{Z}^{+}_{k_0}$. In fact, let $\mathbf{\Phi}(\hat{\mathbf{r}})$ and $\mathbf{\Psi}(\hat{\mathbf{r}})$ be continuously differentiable tangent-vector fields on $S^2(a)$. Then we define the derived scalar fields

$$\phi_1(\hat{\mathbf{r}}) \equiv \mathbf{L}\cdot\mathbf{\Phi}(\hat{\mathbf{r}}), \quad \phi_2(\hat{\mathbf{r}}) \equiv (\hat{\mathbf{r}}\times\mathbf{L})\cdot\mathbf{\Phi}(\hat{\mathbf{r}}) ; \tag{39}$$

and similarly for ψ_1 and ψ_2 in terms of $\mathbf{\Psi}$. Following integrations by parts, we now have from Eqs. (37), (16), and (17)

$$\langle\mathbf{\Phi}|\breve{Z}^{+}_{k_0}|\mathbf{\Psi}\rangle = a^2\int_{S^2(1)} d^2\mathbf{r}'\int_{S^2(1)} d^2\hat{\mathbf{r}}\{[\phi_1(\hat{\mathbf{r}}')]^* K_1(\hat{\mathbf{r}}';\hat{\mathbf{r}})\psi_1(\hat{\mathbf{r}}) + [\phi_2(\hat{\mathbf{r}}')]^* K_2(\hat{\mathbf{r}}';\hat{\mathbf{r}})\psi_2(\hat{\mathbf{r}})\} , \tag{40}$$

where we have used the definitions

$$K_1(\hat{\mathbf{r}}';\hat{\mathbf{r}}) \equiv \sum_{J>0,M}[-iJ(J+1)\xi_J(k_0 a)]^{-1} \times Y_{JM}(\hat{\mathbf{r}}')[Y_{JM}(\hat{\mathbf{r}})]^* , \tag{41}$$

$$K_2(\hat{\mathbf{r}}';\hat{\mathbf{r}}) \equiv \sum_{J>0,M}[J(J+1)]^{-1}[-i\xi_J(k_0 a)] \times Y_{JM}(\hat{\mathbf{r}}')[Y_{JM}(\hat{\mathbf{r}})]^* . \tag{42}$$

In view of Eq. (38) and Ref. [3], Eqs. (A3)–(A7), $K_1(\hat{\mathbf{r}}',\hat{\mathbf{r}})$ is a continuous kernel, while the dominant singular term in K_2 is

$$K_2(\hat{\mathbf{r}}';\hat{\mathbf{r}}) \approx i[2\pi k_0 a|\hat{\mathbf{r}}'-\hat{\mathbf{r}}|]^{-1} , \tag{43}$$

with a correction term of order $\ln|\hat{\mathbf{r}}'-\hat{\mathbf{r}}|$.

We conclude our discussion of vector spherical harmonic expansions by noting the feasibility of obtaining an expansion in these terms of the free-space Green's function $\Gamma^{0+\pm}_{k_0,\alpha\beta,jk}(\mathbf{r}_1;\mathbf{r}_2)$. Beyond the first step, the computation amounts to an application of Dirac's transformation theory, and goes as follows. First, the Green's function matrix is obtained in $\mathbf{k}$ space; this computation is algebraic, since the matrix is local in $\mathbf{k}$. Second, the dyadic constituents, which all commute with the operators corresponding to the "quantum numbers" λ, J, M, are transformed into diagonal form in the corresponding basis of states. Third, a suitable specialization of Ref. [8], Eq. (24) is applied to express the Green's function in an r-helicity basis; this transformation entails the computation of a number of distinct Fourier-Bessel transforms. Finally, Eqs. (16)–(21) are applied to express the Green's function in conventional Cartesian components and coordinates in $\mathbf{r}$ space. Both the computation and the results, which include generalized functions, are lengthy [11], and will not be reproduced here. The problem of expanding the electromagnetic Green's dyadics in vector spherical harmonics has been the subject of a number of investigations—see Refs. [12] and [13], and other papers cited therein—and a recent book [14] discusses the singularity structure of these dyadics.

IV. QUASIPLANAR APPROXIMATION FOR $\breve{Z}_{k_0}^{+}$

The "primitive" operators $\breve{M}_{k_0}$ and $\breve{N}_{k_0}$ are defined in Ref. [4], Eqs. (2.82) and (2.85), respectively—see also Ref. [1], Eqs. (A4) and (A5); these operators effect a linear mapping of the space $\mathcal{V}^{\partial\Omega}$ into itself. According to Ref. [1], Eq. (68), we have

$$\breve{Z}_{k_0}^{+} = -(i/k_0)\breve{X}_\partial \breve{N}_{k_0}\breve{X}_\partial(\breve{I}_\partial + \breve{M}_{k_0})^{-1} . \tag{44}$$

Moreover, if $\partial\Omega$ is a plane, it is easy to verify that $\breve{M}_{k_0}$ is the zero operator in $\mathcal{V}^{\partial\Omega}$,

$$\breve{M}_{k_0} = \breve{0}_\partial , \tag{45}$$

so that the result

$$\breve{Z}_{k_0}^{+} = -(i/k_0)\breve{X}_\partial \breve{N}_{k_0}\breve{X}_\partial \tag{46}$$

is exact for planar $\partial\Omega$. If instead $\partial\Omega$ is smooth and convex, it is plausible on physical grounds that locally, that is, over distances $|\mathbf{r}'_\partial - \mathbf{r}_\partial|$ that are small compared to the local curvature radii of $\partial\Omega$, the correction terms to the kernel $\breve{Z}_{k_0,j'j}^{+}(\mathbf{r}'_\partial;\mathbf{r}_\partial)$ arising from the higher-order approximations in

$$(\breve{I}_\partial + \breve{M}_{k_0})^{-1} \approx \breve{I}_\partial - \breve{M}_{k_0} + \cdots \tag{47}$$

can be neglected. Therefore we propose the quasiplanar approximation to the radiation impedance operator for such a $\partial\Omega$ to be ${}^0\breve{Z}_{k_0}^{+}$, where

$$\breve{Z}_{k_0}^{+} \approx {}^0\breve{Z}_{k_0}^{+} \equiv -(i/k_0)\breve{X}_\partial \breve{N}_{k_0}\breve{X}_\partial . \tag{48}$$

A correction to ${}^0\breve{Z}_{k_0}^{+}$ derived by applying the linear term in $\breve{M}_{k_0}$ on the rhs of Eq. (47) to Eq. (44) can be computed with methods similar to those applied in Ref. [3], Sec. V, but we shall not do so here.

V. DERIVATION OF THE PHYSICAL OPTICS APPROXIMATION

In this section we shall apply the approximation Eq. (48) and the method of stationary phase to obtain a short-wavelength approximation to terms that appear in parentheses in the last two summands on the rhs of Eq. (13). These results are an alternative statement of the so-called physical optics approximation (Ref. [15], p. 29), which approximation is postulated on physical grounds as an extrapolation to gently curved, convex $\partial\Omega$ of the results of the exactly solvable problem of reflection from a perfectly conducting plane.

We will work out estimates for the expressions

$$\mathcal{R}_{k_0,\beta,lk}^{\pm}(\mathbf{r}_{\partial 3};\mathbf{r}_2) \equiv (\breve{X}_\partial\, {}^0\breve{Z}_{k_0}^{+}\breve{X}_\partial G_{k_0,e\beta}^{0\pm} - \breve{X}_\partial G_{k_0,m\beta}^{0\pm})_{lk}(\mathbf{r}_{\partial 3};\mathbf{r}_2) . \tag{49}$$

The reciprocity property for free-space Green's functions of purely outgoing-wave type [Ref. [1], Eq. (51)] and the symmetry of $\breve{Z}_{k_0}^{+}$ [Eq. (6)] imply that

$$(G_{k_0,\alpha e}^{0+}\breve{X}_\partial\, {}^0\breve{Z}_{k_0}^{+}\breve{X}_\partial - G_{k_0,\alpha m}^{0+}\breve{X}_\partial)_{jl}(\mathbf{r}_1;\mathbf{r}_{\partial 3}) = (-1)^{\delta_{\alpha m}}\mathcal{R}_{k_0,\alpha,lj}^{+}(\mathbf{r}_{\partial 3};\mathbf{r}_1) . \tag{50}$$

Explicit formulas for the scalar Green's functions and dyadic constituents of the Green's function for the free-space Maxwell field are given in Eqs. (8)–(10).

It is convenient to use the following abbreviated symbols for vectors, intervals, and unit vectors:

$$\mathbf{r}_{1,2} = -\mathbf{r}_{2,1} = \mathbf{r}_1 - \mathbf{r}_2 = r_{1,2}\hat{\mathbf{r}}_{1,2} , \tag{51}$$

$$\mathbf{r}_{\partial 1,2} = -\mathbf{r}_{2,\partial 1} = \mathbf{r}_{\partial 1} - \mathbf{r}_2 = r_{\partial 1,2}\hat{\mathbf{r}}_{\partial 1,2} . \tag{52}$$

We continue to follow the summation convention for Cartesian indices $j,k,l,p,q,\ldots$. We carry out the indicated differentiations in Eqs. (9) and (10); insofar as we will obtain an asymptotic approximation as $|k_0| \to \infty$, we can drop all derivative terms that do not bring down a factor k_0 from the exponent:

$$G_{k_0,ee,jk}^{0\pm}(\mathbf{r}_1;\mathbf{r}_2) = G_{k_0,mm,jk}^{0\pm}(\mathbf{r}_1;\mathbf{r}_2) \approx -k_0(4\pi r_{1,2})^{-1}\exp(\pm ik_0 r_{1,2}) \times[\delta_{jk} - (\hat{\mathbf{r}}_{1,2})_j(\hat{\mathbf{r}}_{1,2})_k] , \tag{53}$$

$$G_{k_0,me,jk}^{0\pm}(\mathbf{r}_1;\mathbf{r}_2) = -G_{k_0,em,jk}^{0\pm}(\mathbf{r}_1;\mathbf{r}_2) \approx \mp k_0(4\pi r_{1,2})^{-1}\exp(\pm ik_0 r_{1,2}) \times \epsilon_{jlk}(\hat{\mathbf{r}}_{1,2})_l . \tag{54}$$

Moreover, in view of Eqs. (46), (1), and Ref. [4], Eq. (2.85) [see also Ref. [1], Eq. (A5)], we have

$$(\breve{X}_\partial\, {}^0\breve{Z}_{k_0}^{+}\breve{X}_\partial)_{jl}(\mathbf{r}_{\partial 3};\mathbf{r}_{\partial 4}) = -(i/k_0)(\breve{N}_{k_0})_{jl}(\mathbf{r}_{\partial 3};\mathbf{r}_{\partial 4}) \tag{55}$$

$$= -\frac{i}{2\pi k_0}\lim_{\mathbf{r}_3\to\mathbf{r}_{\partial 3}}(\breve{X}_\partial)_{jp}(\mathbf{r}_{\partial 3})[\nabla_3\times(\nabla_3\times)]_{pq}\frac{\exp(ik_0 r_{3,\partial 4})}{r_{3,\partial 4}}(\breve{X}_\partial)_{ql}(\mathbf{r}_{\partial 4}) , \tag{56}$$

where the limit is to be taken after an integral over $\mathbf{r}_{\partial 4}$ is carried out; according to Ref. [4], p. 64, the same result is obtained whether $\mathbf{r}_3$ approaches $\mathbf{r}_{\partial 3}$ from Ω^{ex} or from Ω. The first summand on the rhs of Eq. (49) reduces to

$$\frac{i}{8\pi^2}\lim_{\mathbf{r}_3\to\mathbf{r}_{\partial 3}}(\check{X}_\partial)_{lp}(\mathbf{r}_{\partial 3})[\nabla_3\times(\nabla_3\times)]_{pq}\int_{\partial\Omega}dA_{\partial 4}(r_{3,\partial 4}r_{\partial 4,2})^{-1}$$
$$\times\exp[ik_0(r_{3,\partial 4}\pm r_{\partial 4,2})](\check{X}_\partial)_{qs}(\mathbf{r}_{\partial 4})\times\begin{cases}[\delta_{sk}-(\hat{\mathbf{r}}_{\partial 4,2})_s(\hat{\mathbf{r}}_{\partial 4,2})_k] & \text{if } \beta=e\\ [\mp\epsilon_{stk}(\hat{\mathbf{r}}_{\partial 4,2})_t] & \text{if } \beta=m\ .\end{cases} \tag{57}$$

We evaluate the integral in Eq. (57) by the method of stationary phase—see Appendix A. We obtain an estimate for this integral as a sum over contributions from points of stationary phase. Our approximation leads to significant and incorrect contributions from stationary phase points remote from $\mathbf{r}_{\partial 3}$, since Eq. (48) does not account for the rapid decrease in magnitude of the exact kernel $(\check{Z}^+_{k_0})_{jk}(\mathbf{r}_{\partial 3};\mathbf{r}_{\partial 4})$ as its arguments $\mathbf{r}_{\partial 3}$ and $\mathbf{r}_{\partial 4}$ become remote from one another on a convex $\partial\Omega$. We shall ignore these other contributions, and keep the contribution from the single stationary phase point $\mathbf{r}_{\partial\chi a}$ for which $|\mathbf{r}_{\partial 3}-\mathbf{r}_{\partial\chi a}|\to 0$ as $\mathbf{r}_3\to\mathbf{r}_{\partial 3}$; only the contribution, which proves to be finite, from this point of stationary phase yields a local contribution to the integral, in which circumstance we expect that the approximation Eq. (48) will be valid. Close to the limit $\mathbf{r}_3\to\mathbf{r}_{\partial 3}$, the corresponding matrix of Eq. (A8) is dominated by a single term:

$$\Lambda_{\chi a\zeta\eta}\approx[2r_{3,\partial\chi a}]^{-1}[\delta_{\zeta\eta}-(\hat{\mathbf{r}}_{3,\partial\chi a}\cdot\hat{\mathbf{t}}_{a\chi\zeta})(\hat{\mathbf{r}}_{3,\partial\chi a}\cdot\hat{\mathbf{t}}_{a\chi\eta})]\ . \tag{58}$$

We find for this circumstance that in the two possible geometrical arrangements (that is, $\mathbf{r}_2$ and $\mathbf{r}_3$ are on the same or on opposite sides of the plane tangent to $\partial\Omega$ at $\mathbf{r}_{\partial a}$) and for both elliptic and hyperbolic cases, we have

$$\begin{aligned}&\sigma_{\chi a1}=\sigma_{\chi a2}=+1\\ &|\det(\Lambda_{\chi a\zeta\eta})|^{-1/2}\approx 2r_{3,\partial\chi a}|\hat{\mathbf{n}}_{\partial\chi a}\cdot\hat{\mathbf{r}}_{3,\partial\chi a}|^{-1}\ .\end{aligned} \tag{59}$$

After applying Eqs. (58), (59), and (A11) to Eq. (57), and effecting the limit $\mathbf{r}_3\to\mathbf{r}_{\partial 3}$ in all but one of the factors, we must evaluate the limit

$$\check{X}_{\partial,lp}(\mathbf{r}_{\partial 3})\lim_{\mathbf{r}_3\to\mathbf{r}_{\partial 3}}[\delta_{pq}-(\hat{\mathbf{r}}_{3,\partial\chi a})_p(\hat{\mathbf{r}}_{3,\partial\chi a})_q]\check{X}_{\partial,qs}(\mathbf{r}_{\partial 3})\ . \tag{60}$$

For the cases that $\mathbf{r}_2,\mathbf{r}_{\partial\chi a},\mathbf{r}_3$ lie on a straight line we have $\hat{\mathbf{r}}_{3,\partial\chi a}=\mp\hat{\mathbf{r}}_{2,\partial 3}$, while if the three points lie on a broken line, we have

$$\hat{\mathbf{r}}_{3,\partial\chi a}=\mp\{\hat{\mathbf{r}}_{2,\partial 3}-2\hat{\mathbf{n}}(\mathbf{r}_{\partial 3})[\hat{\mathbf{n}}(\mathbf{r}_{\partial 3})\cdot\hat{\mathbf{r}}_{2,\partial 3}]\}\ . \tag{61}$$

In all cases, therefore, the expression in Eq. (60) reduces to

$$\check{X}_{\partial,lp}(\mathbf{r}_{\partial 3})[\delta_{pq}-(\hat{\mathbf{r}}_{2,\partial 3})_p(\hat{\mathbf{r}}_{2,\partial 3})_q]\check{X}_{\partial,qs}(\mathbf{r}_{\partial 3})\ . \tag{62}$$

We must now apply Eq. (62) to Eq. (57) and carry out the indicated sums—we shall not display the details of the computation—over Cartesian indices in the limit $\mathbf{r}_3\to\mathbf{r}_{\partial 3}$, where the limit is taken after the integral is calculated by the method of stationary phase. We find the following estimate, which is asymptotic for $|k_0|$ very large, for Eq. (57):

$$\frac{k_0}{4\pi}\frac{\exp(\pm ik_0r_{2,\partial 3})}{r_{2,\partial 3}}\frac{\hat{\mathbf{n}}(\mathbf{r}_{\partial 3})\cdot\hat{\mathbf{r}}_{2,\partial 3}}{|\hat{\mathbf{n}}(\mathbf{r}_{\partial 3})\cdot\hat{\mathbf{r}}_{2,\partial 3}|}\check{X}_{\partial,lq}(\mathbf{r}_{\partial 3})$$
$$\times\begin{cases}[-\epsilon_{qrk}(\hat{\mathbf{r}}_{2,\partial 3})_r] & \text{if } \beta=e\ ,\\ \{\pm[\delta_{qk}-(\hat{\mathbf{r}}_{2,\partial 3})_q(\hat{\mathbf{r}}_{2,\partial 3})_k]\} & \text{if } \beta=m\ .\end{cases} \tag{63}$$

We can now assemble results and, with the aid of Eqs. (53) and (54), infer an estimate for the rhs of Eq. (49) and the lhs of Eq. (50):

$$\mathcal{R}^{\pm}_{k_0,\beta,lk}(\mathbf{r}_{\partial 3};\mathbf{r}_2)\approx-(\check{X}_\partial G^{0\pm}_{k_0,m\beta})_{lk}(\mathbf{r}_{\partial 3};\mathbf{r}_2)\times\left[1\pm\frac{\hat{\mathbf{n}}(\mathbf{r}_{\partial 3})\cdot\hat{\mathbf{r}}_{2,\partial 3}}{|\hat{\mathbf{n}}(\mathbf{r}_{\partial 3})\cdot\hat{\mathbf{r}}_{2,\partial 3}|}\right]\ , \tag{64}$$

$$(-1)^{\delta_{\alpha m}}\mathcal{R}^{+}_{k_0,\alpha,lj}(\mathbf{r}_{\partial 3};\mathbf{r}_1)\approx-(G^{0+}_{k_0,\alpha m}\check{X}_\partial)_{jl}(\mathbf{r}_1;\mathbf{r}_{\partial 3})\times\left[1+\frac{\hat{\mathbf{n}}(\mathbf{r}_{\partial 3})\cdot\hat{\mathbf{r}}_{1,\partial 3}}{|\hat{\mathbf{n}}(\mathbf{r}_{\partial 3})\cdot\hat{\mathbf{r}}_{1,\partial 3}|}\right]\ , \tag{65}$$

where we used the reciprocity property [Ref. [1], Eq. (51)] for the causal free-space Green's function in deriving Eq. (65).

Equations (64) and (65) generalize the physical optics approximation [Ref. [15], Chap. I.2.13.4, Eq. (I.126)]. Note in Eq. (64) that if $\mathbf{r}_2\in\Omega^{\mathrm{ex}}$, the geometrically illuminated side and shadowed side of $\partial\Omega$ are interchanged in moving from the elliptic case (upper sign) to the hyperbolic case (lower sign), as is expected according to whether the signal diverges from, or converges to, the source point $\mathbf{r}_2$. Note also that if $\mathbf{r}_2\in\Omega$, Eq. (64) gives zero in the elliptic case, consistent with the exact result Eq. (4) for an outgoing electromagnetic field, the sources of which lie entirely in $\Omega\cup\partial\Omega$.

VI. GEOMETRICAL OPTICS LIMIT

In this section we shall obtain the geometrical optics (that is, asymptotic for extreme short wavelength) approximation for the complete Green's function of Eq. (13). In Sec. V we obtained estimates for the quantities within parentheses in the last two summands of Eq. (13). We shall again use the method of stationary phase to estimate the remaining implied integrals, and assemble the separate results; in particular, we shall verify approximately what might be called the extinction properties, whether exact (for the signal within the obstacle) or asymptotic (for the signal in the geometrical shadow region with respect to the radiation source or sink and the obstacle). Many of the results obtained in this section could be obtained by ray optics methods, as described in, say, Ref. [16].

According to Eqs. (13), (49), (50), (64), and (65), we need to estimate the following integrals:

$$-(i/2)\int_{\partial\Omega} dA_{\partial 3} G^{0+}_{k_0,\alpha e,jl}(\mathbf{r}_1;\mathbf{r}_{\partial 3})\left[1\pm\frac{\hat{\mathbf{n}}(\mathbf{r}_{\partial 3})\cdot\hat{\mathbf{r}}_{2,\partial 3}}{[\hat{\mathbf{n}}(\mathbf{r}_{\partial 3})\cdot\hat{\mathbf{r}}_{2,\partial 3}|}\right](\check{X}_{\partial} G^{0\pm}_{k_0,m\beta})_{lk}(\mathbf{r}_{\partial 3};\mathbf{r}_2) \ , \tag{66}$$

$$-(i/2)\int_{\partial\Omega} dA_{\partial 3}(G^{0+}_{k_0,\alpha m}\check{X}_{\partial})_{jl}(\mathbf{r}_1;\mathbf{r}_{\partial 3})\left[1+\frac{\hat{\mathbf{n}}(\mathbf{r}_{\partial 3})\cdot\hat{\mathbf{r}}_{1,\partial 3}}{|\hat{\mathbf{n}}(\mathbf{r}_{\partial 3})\cdot\hat{\mathbf{r}}_{1,\partial 3}|}\right] G^{0\pm}_{k_0,e\beta,lk}(\mathbf{r}_{\partial 3};\mathbf{r}_2) \ , \tag{67}$$

where the upper signs correspond to electric current sources of radiation ($\beta=e$, which corresponds to $\chi=\epsilon$ in Appendix A) and the lower signs to magnetic current sinks of radiation ($\beta=m$, which corresponds to $\chi=v$ in Appendix A).

We approximate the Green's dyadics in Eqs. (66) and (67) by Eqs. (53) and (54), and apply the method of stationary phase as described in Appendix A. Two categories of stationary phase points $\mathbf{r}_{\partial\chi a}$ emerge, according to whether $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_{\partial\chi a}$ (i) do, or (ii) do not lie on a common straight line. In the straight-line case (i) the stationary phase estimates take a simple form, as they do not depend on the local curvature of $\partial\Omega$ at $\mathbf{r}_{\partial\chi a}$—the coefficient of the curvature matrix in Eq. (A8) is zero. We shall sketch the evaluation of the stationary phase estimates for this category, both in the elliptic and hyperbolic cases.

If $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_{\partial\chi a}$ lie on a straight line, we define

$$\xi_{\chi a}=(r_{1,\partial\chi a}\pm r_{2,\partial\chi a})/|r_{1,\partial\chi a}\pm r_{2,\partial\chi a}| \ , \tag{68}$$

where we continue the sign convention as above for $\chi=\epsilon,v$. We infer from Eq. (A8) that in the notation of Eq. (A10),

$$\sigma_{\chi a1}=\sigma_{\chi a2}=\pm\xi_{\chi a}k_0/|k_0| \ , \tag{69}$$

$$|\Lambda_{\chi a}|^{-1/2}=2r_{1,\partial\chi a}r_{2,\partial\chi a}/|(r_{2,\partial\chi a}\pm r_{1,\partial\chi a})[\hat{\mathbf{n}}(\mathbf{r}_{\partial\chi a})\cdot\hat{\mathbf{r}}_{1,\partial\chi a}]| \ , \tag{70}$$

$$r_{1,\partial\chi a}\pm r_{2,\partial\chi a}=\xi_{\chi a}r_{1,2} \ . \tag{71}$$

Further simplifications of type (i) contributions to Eqs. (66) and (67) are straightforward according to the rules developed in Appendix A, but careful attention to algebraic signs is needed. We combine these results and incorporate them into an estimate for the rhs of Eq. (13):

$$\begin{aligned}\Gamma^{E+-}_{k_0,\alpha\beta,jk}(\mathbf{r}_1;\mathbf{r}_2)\approx{}&[1-(\tfrac{1}{2})\Theta_\Omega(\mathbf{r}_1)]G^{0\pm}_{k_0,\alpha\beta,jk}(\mathbf{r}_1;\mathbf{r}_2)-(\tfrac{1}{2})G^{0+}_{k_0,\alpha\beta,jk}(\mathbf{r}_1;\mathbf{r}_2)\Theta_\Omega(\mathbf{r}_2)\\&-(-1)^{\delta_{\beta m}}(\tfrac{1}{2})\sum_{(\mathrm{i}),a}\xi_{\chi a}G^{0\xi_{\chi a}}_{k_0,\alpha\beta,jk}(\mathbf{r}_1;\mathbf{r}_2)+\sum_{(\mathrm{ii}),b}[\text{type-(ii) contributions}] \ .\end{aligned} \tag{72}$$

We shall not make use of the contributions from type-(ii) points of stationary phase beyond noting whether or not they are zero, and hence we do not give explicit formulas for them; these contributions will vanish in circumstances determined by physical optics, that is, according to the vanishing at a point of stationary phase of the factors in square brackets in the integrands of Eqs. (66) and (67).

We remark that in Eq. (72), for $\beta=m$, a straight-line contribution can appear to be of retarded type ($\xi_{va}=+1$) or advanced type ($\xi_{va}=-1$), according to whether $\mathbf{r}_2$ or $\mathbf{r}_1$ is closer to $\mathbf{r}_{\partial va}$, respectively. This result can be understood in physical terms: as the original wave converges on the electromagnetic sink at $\mathbf{r}_2$, an outgoing scattered wave is generated by the obstacle in the vicinity of (in particular) $\mathbf{r}_{\partial va}$, which secondary wave propagates to $\mathbf{r}_1$. Depending on whether $\mathbf{r}_1$ is farther from or closer to $\mathbf{r}_{\partial va}$ than $\mathbf{r}_2$, the secondary wave reaches $\mathbf{r}_1$ later or sooner, respectively, than the original wave reaches $\mathbf{r}_2$, hence the result. We can now verify the several extinction properties in the geometrical optics approximation.

We first consider the case of electric sources, that is, $\beta=e$, $\chi=\epsilon$. Because of the factors in brackets in Eqs. (66) and (67), rays that bounce from the inside of $\partial\Omega$ do not contribute to Eq. (72); type-(ii) rays of this character only are present in the geometrical cases that one or both $\mathbf{r}_1$ and $\mathbf{r}_2$ are in Ω, or that $\mathbf{r}_1$ and $\mathbf{r}_2$ are both in Ω^{ex} but are mutually invisible. In these same geometrical cases, zero, one, or two stationary phase points of type (i) will be present, and it is straightforward to verify that the various terms on the rhs of Eq. (72) will cancel in all these geometrical cases. If $\mathbf{r}_1$ and $\mathbf{r}_2$ are both in Ω^{ex} and are mutually visible, Eq. (72) shows that the signal at $\mathbf{r}_1$ will be the superposition of a direct wave, that is, the contribution from $G^{0+}_{k_0,\alpha e}$, plus a scattered wave resulting from a nonzero type-(ii) contribution from a ray that bounces from a specular point on $\partial\Omega$ that is illuminated from $\mathbf{r}_2$ and visible from $\mathbf{r}_1$. The stationary phase point on the invisible side of $\partial\Omega$ does not contribute.

Next we consider the case $\beta=m$, which by our hypothesis entails taking $\chi=v$ in Appendix A and the lower signs in Eqs. (66), (67), and (72). First, if both $\mathbf{r}_1$ and $\mathbf{r}_2$ are in Ω, there are no type-(ii) points of stationary phase, and two of type (i), one of which has $\xi_{va}=+1$, the other $\xi_{va'}=-1$. The rhs of Eq. (72) is therefore zero.

Continuing with $\beta=m$, we will make use of the following geometrical property, which is implied by Eq. (A6): Let $\mathbf{r}_{\partial vb}$ be a type-(ii) stationary phase point, and consider the plane determined by $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_{\partial vb}$. That tangent direction to $\partial\Omega$ at $\mathbf{r}_{\partial vb}$ which lies in this plane is the angular bisector of the lines determined by the vectors $\hat{\mathbf{r}}_{1,\partial vb}$ and $\hat{\mathbf{r}}_{2,\partial bv}$. Accordingly, if $\mathbf{r}_2\in\Omega^{\mathrm{ex}}$, and if either $\mathbf{r}_1\in\Omega$, or $\mathbf{r}_1$ is in Ω^{ex} but is in the shadow zone of the obstacle with respect to the sink at $\mathbf{r}_2$, there will be no contributions from type-(ii) stationary phase points, since the factor in brackets in Eq. (66) and that in (67) will each be zero for such points. For the latter geometrical cases as well, there will be one or two type-(i) stationary phase points, both with $\xi_{va}=-1$; accordingly, the rhs of Eq. (72) again adds up to zero.

If $\mathbf{r}_2\in\Omega$ and $\mathbf{r}_1\in\Omega^{\mathrm{ex}}$, the term $-(\frac{1}{2})G^{0+}_{k_0,\alpha m}$ on the rhs of Eq. (72) will be cancelled by a type-(i) contribution, and to the primary free-space signal $G^{0-}_{k_0,\alpha m}$ at $\mathbf{r}_1$ will be superimposed a scattered wave from type-(ii) contributions. In physical terms, the incoming spherical wave converging on $\mathbf{r}_2$ propagates freely until it impinges on $\partial\Omega$, whereupon secondary currents in $\partial\Omega\cup\Omega$ generate a signal in Ω, which cancels the free-space field in Ω. The extinction property holds only in an incomplete sense in this case.

Finally, if $\mathbf{r}_2\in\Omega^{\mathrm{ex}}$, and if $\mathbf{r}_1\in\Omega^{\mathrm{ex}}$ and is not in the shadow of the obstacle with respect to the radiation sink at $\mathbf{r}_2$, both the primary free-space signal and a secondary scattered wave, which arises from type-(ii) contributions, will be present at the field point $\mathbf{r}_1$. Moreover, if $\mathbf{r}_1$ and $\mathbf{r}_2$ are such that the extended line through them intercepts $\partial\Omega$ (twice, since we exclude tangential contact), the net signal at $\mathbf{r}_1$, aside from type-(ii) contributions, will be

$$G^{-}_{k_0,\alpha m}(\mathbf{r}_1;\mathbf{r}_2)+G^{+}_{k_0,\alpha m}(\mathbf{r}_1;\mathbf{r}_2)\ , \tag{73}$$

as there are two straight-line contributions, both with $\xi_{va}=+1$. In physical terms, the original incoming wave induces currents on $\partial\Omega$, which currents generate a secondary signal that, in the geometrical optics approximation, extinguishes the incoming wave in the conelike shadow between $\partial\Omega$ and the apex $\mathbf{r}_2$. This secondary signal is focused to a real image at $\mathbf{r}_2$, in fact precisely to a point focus in the geometrical optics limit. (Note that if the source and obstacle are both causal, the secondary signal that extinguishes the outgoing wave in the geometrical shadow has only a virtual image at the original source.) This secondary signal continues to propagate beyond $\mathbf{r}_2$ into the other branch of the cone formed by $\partial\Omega$ and $\mathbf{r}_2$, and (we can infer) is present at times later than the time that the original signal is absorbed at $\mathbf{r}_2$; that is, it approximates a retarded signal generated at $\mathbf{r}_2$ within this branch of the cone. In this qualified sense it is possible for a radiation sink to give rise to secondary signals that in a restricted domain closely resemble a retarded source of radiation.

VII. OBSTACLE PLUS A LAYER

A. Outline

In this section we return to the general theory of Ref. [1] and arbitrary $k_0\neq 0$, and analyze scattering from the following setup: Suppose that we have found an operator pair, as in Ref. [1], Eq. (35), that simulates the exterior electromagnetic response of a given obstacle Ω_a by generalized Leontovich boundary conditions on its surface $\partial\Omega_a\equiv\Sigma_a$. We augment the obstacle by a layer, say $\mathcal{D}_{ab}\subset\Omega_a^{\mathrm{ex}}$, and define $\Omega_a\cup\Sigma_a\cup\mathcal{D}_{ab}\equiv\Omega_b$. The layer $\mathcal{D}_{ab}$ has inner boundary Σ_a and outer boundary Σ_b; we denote this relationship of the surfaces by $\Sigma_b \rhd \Sigma_a$. The layer $\mathcal{D}_{ab}$ is to be occupied by a medium, called M, with given, and possibly nonuniform and nonsymmetric, tensor constitutive parameters $\epsilon_{jk}(\mathbf{r}),\mu_{jk}(\mathbf{r}),\sigma_{jk}(\mathbf{r})$; we note, but do not denote, the fact that each of these can depend on k_0, subject to the limitations imposed by causality in the time domain. We shall, using the methods of Appendix B, find an operator expression for Leontovich boundary conditions on Σ_b that simulate the scattering engendered by the combined obstacle within Ω_b as far as the region exterior to Σ_b is concerned. A requirement of the argument is the availability of a Green's function, of outgoing-wave type, when the medium within $\mathcal{D}_{ab}$ is imagined to be extended in some physically appropriate but otherwise arbitrary fashion so as to fill all space $\mathcal{E}^3$; although exceptional cases might be found, explicit closed-form analytic expressions for such Green's functions will normally be available only if the constitutive properties of the medium are uniform and isotropic—cf. Ref. [1], Appendix C. As a mathematical statement, however, the following considerations apply whenever a suitable Green's function exists in principle. Appendix B is intended to be an introduction to the material in this section. The analogs obtained here for the electromagnetic case are more general than those in Appendix B, as the argument is carried through (except in Sec. VII G) for media which can be lossy and for which the constitutive properties and corresponding Green's functions need not satisfy the principle of reciprocity.

The results derived in this section may be of use in several circumstances, including the following: first, in the analysis of the electromagnetic effects of an optical coating, or second, as one of a sequence of steps in which a physical obstacle is approximated by an onionlike structure of layers $\mathcal{D}_{ab},\mathcal{D}_{bc},\mathcal{D}_{cd},\ldots,$ such that each layer is made up of homogeneous, isotropic substance, or third, if Σ_b is an artificial boundary inserted in the medium, for which extended medium—as empty space—the Green's function is available, it will be the case that the derived Leontovich boundary conditions on Σ_b will give rise to the same scattered field outside of Σ_b as do the original boundary conditions on Σ_a. If in the second case, the onion is more or less flattened, so that the interfaces between successive layers are noncrossing deformed planes, the methods to be applied here can be used to obtain a (formally exact) treatment of the propagation of the Maxwell field in an optical system laid out along an axis, and thereby give a precise antecedent of the approximate "ray optics" and "wave optics" treatments of such propagation, as described in, say, Refs. [17], [18], or [19]. In the third case, the result stated—see Eq. (126)—apparently entails for its application the equivalent of determining the radiation impedance operator for the geometrically complex inner surface, which requirement, unless it can be circumvented or mitigated, may make it impractical to transform the boundary problem from a complex boundary to a simple one.

B. Field equations and Green's functions

Maxwell's equations in a material medium are [Ref. [19], Eq. (1.30), or Ref. [20], pp. 14–23]

$$-\frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}+\nabla\times c\mathbf{B}=\mu_0 c\left[\mathbf{J}_c+\nabla\times\mathbf{M}+\frac{\partial \mathbf{P}}{\partial t}+\mathbf{J}_e\right], \tag{74}$$

$$-\nabla\times\mathbf{E}-\frac{1}{c}\frac{\partial c\mathbf{B}}{\partial t}=\mu_0 c\mathbf{J}_m . \tag{75}$$

The rhs of Eq. (74) contains the vector sum of the conduction current $\mathbf{J}_c$, the Amperian current $\nabla\times\mathbf{M}$, the polarization current $\partial\mathbf{P}/\partial t$, and the electric source current $\mathbf{J}_e$; the rhs of Eq. (75) contains the magnetic source current $\mathbf{J}_m$. We presume the time dependence $\exp(-ik_0ct)$ everywhere, multiply both sides of Eqs. (74) and (75) by $-i$, presume the forms

$$(J_c)_j(\mathbf{r})=\sigma_{jk}(\mathbf{r})E_k(\mathbf{r}) , \tag{76}$$

$$M_j(\mathbf{r})=[\mu_0^{-1}\delta_{jk}-(\mu^{-1})_{jk}(\mathbf{r})]B_k(\mathbf{r}) , \tag{77}$$

$$P_j(\mathbf{r})=[\epsilon_{jk}(\mathbf{r})-\epsilon_0\delta_{jk}]E_k(\mathbf{r}) , \tag{78}$$

and introduce a new magnetic-field variable $\mathbf{F}(\mathbf{r})$ and new constitutive parameters $\kappa_{jk}(\mathbf{r})$ and $\lambda_{jk}(\mathbf{r})$:

$$F_j(\mathbf{r})\equiv\mu_0(\mu^{-1})_{jk}cB_k(\mathbf{r}) , \tag{79}$$

$$\kappa_{jk}(\mathbf{r})\equiv k_0\epsilon_0^{-1}\epsilon_{jk}(\mathbf{r})+i\mu_0c\sigma_{jk}(\mathbf{r}) , \tag{80}$$

$$\lambda_{jk}(\mathbf{r})\equiv k_0\mu_0^{-1}\mu_{jk}(\mathbf{r}) . \tag{81}$$

Maxwell's equations (74) and (75) now read in matrix operator form

$$\begin{bmatrix}\kappa_{jk} & -i(\nabla\times)_{jk}\\ i(\nabla\times)_{jk} & \lambda_{jk}\end{bmatrix}\begin{bmatrix}E_k^M\\ F_k^M\end{bmatrix}=\begin{bmatrix}-i\mu_0cJ_{e,j}\\ -i\mu_0cJ_{m,j}\end{bmatrix}, \tag{82}$$

where the superscript distinguishes fields in the medium M. We remark that spatially local, linear constitutive relations of a more general character than those of Eqs. (76)–(78) are physically realizable, and have been the subjects of theoretical studies of electromagnetic-wave propagation—cf. the treatments of so-called bianisotropic media in Refs. [21]–[23]—but we shall not attempt here to develop a theory at that level of generality.

We note that Eq. (82) implies that the tangential components of $\mathbf{E}(\mathbf{r})$ and $\mathbf{F}(\mathbf{r})$, and the normal components of $\kappa\mathbf{E}$ and of $\lambda\mathbf{F}$ are continuous across a surface of discontinuity of the constitutive tensors κ and λ in the usual circumstance that there are no magnetic or electric source currents or charges embedded in the surface of discontinuity. The jump conditions at the surface of an idealized object of infinite electrical conductivity are treated as an exception to the latter circumstance.

Corresponding to Eq. (82) we introduce a physically distinct medium called M', the constitutive parameters of which are the transposes of those for the medium M; Maxwell's equations for the fields in M' are

$$\begin{bmatrix}(\kappa^{\tau})_{jk} & -i(\nabla\times)_{jk}\\ i(\nabla\times)_{jk} & (\lambda^{\tau})_{jk}\end{bmatrix}\begin{bmatrix}E_k^{M'}\\ F_k^{M'}\end{bmatrix}=\begin{bmatrix}-i\mu_0cJ_{e,j}\\ -i\mu_0cJ_{m,j}\end{bmatrix} . \tag{83}$$

The introduction of the second, "adjoint" system of field equations follows a pattern suggested by Morse and Feshbach (Ref. [24], Part I, Chap. 7.5). We shall make use of these adjoint fields and the associated Green's functions in what follows.

We introduce two completely causal Green's functions $\Gamma^M_{\alpha\beta,jk}(\mathbf{r}_1;\mathbf{r}_2)$ and $\Gamma^{M'}_{\alpha\beta,jk}(\mathbf{r}_1;\mathbf{r}_2)$, which generalize those in Eq. (7) and Ref. [1], Eq. (C5), and which represent the propagation of electromagnetic waves in the material medium M and M', respectively. (We have suppressed the superscripts $++$ and the subscript k_0 in the interest of notational simplicity.) These satisfy the differential equations

$$\kappa_{pq}(\mathbf{r}_3)\Gamma^M_{e\alpha,qj}(\mathbf{r}_3;\mathbf{r}_1)-i(\nabla_3\times)_{pq}\Gamma^M_{m\alpha,qj}(\mathbf{r}_3;\mathbf{r}_1)=\delta_{pj}\delta_{e\alpha}\delta^3(\mathbf{r}_3-\mathbf{r}_1) , \tag{84}$$

$$i(\nabla_3\times)_{pq}\Gamma^M_{e\alpha,qj}(\mathbf{r}_3;\mathbf{r}_1)+\lambda_{pq}(\mathbf{r}_3)\Gamma^M_{m\alpha,qj}(\mathbf{r}_3;\mathbf{r}_1)=\delta_{pj}\delta_{m\alpha}\delta^3(\mathbf{r}_3-\mathbf{r}_1) , \tag{85}$$

$$(\kappa^{\tau})_{qp}(\mathbf{r}_3)\Gamma^{M'}_{e\beta,pk}(\mathbf{r}_3;\mathbf{r}_2)-i(\nabla_3\times)_{qp}\Gamma^{M'}_{m\beta,pk}(\mathbf{r}_3;\mathbf{r}_2)=\delta_{qk}\delta_{e\beta}\delta^3(\mathbf{r}_3-\mathbf{r}_2) , \tag{86}$$

$$i(\nabla_3\times)_{qp}\Gamma^{M'}_{e\beta,pk}(\mathbf{r}_3;\mathbf{r}_2)+(\lambda^{\tau})_{qp}(\mathbf{r}_3)\Gamma^{M'}_{m\beta,pk}(\mathbf{r}_3;\mathbf{r}_2)=\delta_{qk}\delta_{m\beta}\delta^3(\mathbf{r}_3-\mathbf{r}_2) . \tag{87}$$

We want both Green's functions to satisfy outgoing-wave asymptotic conditions at large distances: we assume that the tensors σ, ϵ, and μ are asymptotically uniform, isotropic, and positive definite (σ can be positive or zero), so that

$$\begin{aligned}\kappa_{jk}(R\hat{\mathbf{R}})&\underset{R\to\infty}{\sim}\kappa_\infty\delta_{jk}+O(1/R^2) ,\\ \lambda_{jk}(R\hat{\mathbf{R}})&\underset{R\to\infty}{\sim}\lambda_\infty\delta_{jk}+O(1/R^2) ,\end{aligned} \tag{88}$$

and we define the complex wave number ω_∞ to be

$$\omega_\infty=[\kappa_\infty\lambda_\infty]^{1/2} , \tag{89}$$

such that $\mathrm{Im}(\omega_\infty)\geq 0$ for both $k_0>0$ and $k_0<0$. The electric and magnetic fields emanating from each type and orientation of source satisfy the asymptotic conditions (compare Ref. [4], Corollary 4.9)

$$\mathbf{E}(R\hat{\mathbf{R}}) \underset{R\to\infty}{\sim} \mathbf{E}^{(0)}(\hat{\mathbf{R}})\exp(i\omega_\infty R)/R + O(1/R^2) ,$$
$$\mathbf{F}(R\hat{\mathbf{R}}) \underset{R\to\infty}{\sim} \mathbf{F}^{(0)}(\hat{\mathbf{R}})\exp(i\omega_\infty R)/R + O(1/R^2) ,$$
$$\mathbf{E}^{(0)}(\hat{\mathbf{R}}) = -(\omega_\infty/\kappa_\infty)\hat{\mathbf{R}}\times\mathbf{F}^{(0)}(\hat{\mathbf{R}}) ,$$
$$\mathbf{F}^{(0)}(\hat{\mathbf{R}}) = (\omega_\infty/\lambda_\infty)\hat{\mathbf{R}}\times\mathbf{E}^{(0)}(\hat{\mathbf{R}}) , \tag{90}$$

where $\mathbf{E}$ and $\mathbf{F}$ stand for any coupled pair of electric and magnetic fields emanating from the same bounded source, whether in the presence of the medium M or M'. We take it for granted that the tensors σ, ϵ, and μ are such that both Green's functions exist and are unique, provided that Eqs. (84)–(90) are satisfied.

We shall obtain a relation between the two Green's functions. Let us multiply both sides of Eqs. (84), (85), (86), and (87) by $\Gamma^{M'}_{e\beta,pk}(\mathbf{r}_3;\mathbf{r}_2)$, $-\Gamma^{M'}_{m\beta,pk}(\mathbf{r}_3;\mathbf{r}_2)$, $-\Gamma^{M}_{e\alpha,qj}(\mathbf{r}_3;\mathbf{r}_1)$, and $\Gamma^{M}_{m\alpha,qj}(\mathbf{r}_3;\mathbf{r}_1)$, respectively, then sum over repeated Cartesian indices and sum corresponding sides of the resulting four equations. The terms involving κ and λ cancel, and the remainder of the lhs forms a perfect divergence. Exchanging the two sides of the equation, we find

$$\delta_{\alpha e}\Gamma^{M'}_{\alpha\beta,jk}(\mathbf{r}_1;\mathbf{r}_2)\delta^3(\mathbf{r}_3-\mathbf{r}_1) - \delta_{\alpha m}\Gamma^{M'}_{\alpha\beta,jk}(\mathbf{r}_1;\mathbf{r}_2)\delta^3(\mathbf{r}_3-\mathbf{r}_1) - \delta_{\beta e}\Gamma^{M}_{\beta\alpha,jk}(\mathbf{r}_2;\mathbf{r}_1)\delta^3(\mathbf{r}_3-\mathbf{r}_2) + \delta_{\beta m}\Gamma^{M}_{\beta\alpha,jk}(\mathbf{r}_2;\mathbf{r}_1)\delta^3(\mathbf{r}_3-\mathbf{r}_2)$$
$$= -i\frac{\partial}{\partial x_{3r}}[\Gamma^{M'}_{e\beta,pk}(\mathbf{r}_3;\mathbf{r}_2)\epsilon_{prq}\Gamma^{M}_{m\alpha,qj}(\mathbf{r}_3;\mathbf{r}_1) + \Gamma^{M'}_{m\beta,pk}(\mathbf{r}_3;\mathbf{r}_2)\epsilon_{prq}\Gamma^{M}_{e\alpha,qj}(\mathbf{r}_3;\mathbf{r}_1)] ; \tag{91}$$

note that the summation convention applies only for repeated Cartesian indices. We integrate both sides of Eq. (91) over the interior of a large sphere $S^2(R)$ centered at the coordinate origin, apply the divergence theorem to the rhs, and let $R\to\infty$; the surface integral tends to zero as a result of Eq. (90). We infer that

$$\Gamma^{M'}_{\alpha\beta,jk}(\mathbf{r}_1;\mathbf{r}_2) = \pm\Gamma^{M}_{\beta\alpha,kj}(\mathbf{r}_2;\mathbf{r}_1) = \pm(\Gamma^{M}_{\beta\alpha})^{\tau}_{jk}(\mathbf{r}_1;\mathbf{r}_2) , \tag{92}$$

where the upper sign (lower sign) holds if $\alpha=\beta$ ($\alpha\neq\beta$), and the transpose notation follows Ref. [1], Eq. (11). Let us define the operator Π—cf. Ref. [1], Eq. (43)—with matrix elements

$$\Pi_{\alpha\beta,jk}(\mathbf{r}_1;\mathbf{r}_2) \equiv (\delta_{\alpha e}\delta_{e\beta} - \delta_{\alpha m}\delta_{m\beta})\delta_{jk}\delta^3(\mathbf{r}_1-\mathbf{r}_2) ; \tag{93}$$

then Eq. (92) can be expressed as

$$\Gamma^{M'} = \Pi(\Gamma^{M})^{\tau}\Pi . \tag{94}$$

The Green's function $\Gamma^{M'}$ will be termed the adjoint to Γ^{M}. If the constitutive tensors κ and λ are everywhere symmetric, then Γ^{M} is self-adjoint, or, in other words, satisfies the principle of reciprocity—compare Ref. [1], Eq. (44).

C. Representation theorems for the fields

We define a six-component electromagnetic field $\Phi^{M}_{\alpha,j}(\mathbf{r})$ as follows [the symbol $\Phi^{M}_{\alpha,j}(\mathbf{r})$ is also used in the sequel]:

$$\Phi^{M}_{\alpha,j}(\mathbf{r}) = \begin{cases} E^{M}_j(\mathbf{r}) & \text{if } \alpha=e \\ F^{M}_j(\mathbf{r}) & \text{if } \alpha=m . \end{cases} \tag{95}$$

We want Φ^{M} to be a solution of Maxwell's source-free equations in an open subdomain $\mathcal{S}\subset\mathcal{E}^3$, which is to be specified, and which is occupied by the medium M:

$$\kappa_{pq}(\mathbf{r}_3)\Phi^{M}_{e,q}(\mathbf{r}_3) - i(\nabla_3\times)_{pq}\Phi^{M}_{m,q}(\mathbf{r}_3) = 0 , \tag{96}$$

$$i(\nabla_3\times)_{pq}\Phi^{M}_{e,q}(\mathbf{r}_3) + \lambda_{pq}(\mathbf{r}_3)\Phi^{M}_{m,q}(\mathbf{r}_3) = 0 . \tag{97}$$

We can combine Eqs. (86), (87), (96), and (97) in a very similar way to the steps that led to Eq. (91), again with the result that the terms involving κ and λ cancel and a perfect divergence appears on the lhs; we find that

$$\delta_{\alpha e}\Phi^{M}_{m,j}(\mathbf{r}_1)\delta^3(\mathbf{r}_3-\mathbf{r}_1) - \delta_{\alpha m}\Phi^{M}_{e,j}(\mathbf{r}_1)\delta^3(\mathbf{r}_3-\mathbf{r}_1)$$
$$= i\frac{\partial}{\partial x_{3r}}[\Gamma^{M'}_{m\alpha,pj}(\mathbf{r}_3;\mathbf{r}_1)\epsilon_{prq}\Phi^{M}_{e,q}(\mathbf{r}_3) + \Gamma^{M'}_{e\alpha,pj}(\mathbf{r}_3;\mathbf{r}_1)\epsilon_{prq}\Phi^{M}_{m,q}(\mathbf{r}_3)] . \tag{98}$$

Now let Σ_a, Σ_b, and $\mathcal{D}_{ab}$ be as in Sec. VII A. We consider two classes of solutions to Eqs. (96) and (97) and corresponding domain $\mathcal{S}$: (i) $\Phi^{M0}_{\alpha,j}(\mathbf{r})$ is a regular solution in the bounded domain Ω_b, which is the complete interior of the surface Σ_b, and (ii) $\Phi^{M+}_{\alpha,j}(\mathbf{r})$ is an outgoing-wave solution in the domain Ω^{ex}_a, which is the complete exterior of the surface Σ_a. In case (i), we integrate both sides of Eq. (98) over Ω_b, and apply the divergence theorem to the rhs; in case (ii), we integrate both sides of Eq. (98) over the domain between Σ_a and a large sphere $S^2(R)$ centered at the origin of coordinates, apply the divergence theorem, and let $R\to\infty$, whereupon the surface integral over $S^2(R)$ tends to zero. As in Eq. (1), we define the linear operators $\check{X}_\Sigma$ and $\check{I}_\Sigma$, which map a general vector field defined on the surface Σ into a tangent-vector field on Σ by restriction and tangential projection

$$\check{X}_\Sigma(\mathbf{r}_\Sigma) \equiv \hat{\mathbf{n}}_\Sigma(\mathbf{r}_\Sigma)\times$$
$$\check{I}_\Sigma \equiv -(\check{X}_\Sigma)^2 , \tag{99}$$

where $\hat{\mathbf{n}}(\mathbf{r}_\Sigma)$ is the outward normal to Σ at $\mathbf{r}_\Sigma\in\Sigma$. (If the vector field is originally defined and continuous in a neighborhood of Σ, we consider that the operator has the additional function of first restricting the field to its values on Σ; successive application of these operations is therefore well defined.) Also, we multiply both sides of each equation by the operator Π defined in Eq. (93). The outcomes of these manipulations are the following integral formulas for Φ^{M0} and Φ^{M+} in terms of their limiting tangential components on Σ_b and Σ_a, respectively, and the Green's function Γ^{M}:

$$\Phi^{M0}_{\alpha,j}(\mathbf{r}_1)\Theta_{\Omega_b}(\mathbf{r}_1)=+i\int_{\Sigma_b} dA_{\Sigma_b}[(\Gamma^M\Pi)_{\alpha e,jp}(\mathbf{r}_1;\mathbf{r}_{\Sigma_b})(\check{X}_{\Sigma_b})_{pq}(\mathbf{r}_{\Sigma_b})\Phi^{M0}_{m,q}(\mathbf{r}_{\Sigma_b})+(\Gamma^M\Pi)_{\alpha m,jp}(\mathbf{r}_1;\mathbf{r}_{\Sigma_b})(\check{X}_{\Sigma_b})_{pq}(\mathbf{r}_{\Sigma_b})\Phi^{M0}_{e,q}(\mathbf{r}_{\Sigma_b})] , \tag{100}$$

$$\Phi^{M+}_{\alpha,j}(\mathbf{r}_1)\Theta_{\Omega_a^{\mathrm{ex}}}(\mathbf{r}_1)=-i\int_{\Sigma_a} dA_{\Sigma_a}[(\Gamma^M\Pi)_{\alpha e,jp}(\mathbf{r}_1;\mathbf{r}_{\Sigma_a})(\check{X}_{\Sigma_a})_{pq}(\mathbf{r}_{\Sigma_a})\Phi^{M+}_{m,q}(\mathbf{r}_{\Sigma_a}) + (\Gamma^M\Pi)_{\alpha m,jp}(\mathbf{r}_1;\mathbf{r}_{\Sigma_a})(\check{X}_{\Sigma_a})_{pq}(\mathbf{r}_{\Sigma_a})\Phi^{M+}_{e,q}(\mathbf{r}_{\Sigma_a})] . \tag{101}$$

We note that the above formulas give zero for propagation in the "wrong" direction, that is to say, outward for Φ^{M0} and inward for Φ^{M+}; in effect, we have in each case used the field values on a surface to define source current distributions on that surface that give rise to zero electromagnetic fields in the exterior of Σ_b in Eq. (100) and in the interior of Σ_a in Eq. (101). The present method gives no information on an "analytic" extrapolation of a regular electromagnetic field outward from a surface, or of an outgoing field inward from a surface.

We can define a six-component field $\Phi^{M'}_{\alpha,j}(\mathbf{r})$ in terms of the fields $E^{M'}_j(\mathbf{r})$ and $F^{M'}_j(\mathbf{r})$ as in Eq. (95), and presume that the fields satisfy the source-free version of Eq. (83). We combine the latter equations with Eqs. (84) and (85) in a similar manner to the steps that led to Eq. (98). Using Eq. (94), we can infer the representation theorems for the adjoint fields in terms of their tangential values on a surface and the adjoint Green's function; we shall not display these results, as they are simply a repetition of Eqs. (100) and (101) with M' substituted for M everywhere.

D. Propagators and projection operators

Let us now restrict the argument $\mathbf{r}_1$ to Σ_a in Eq. (100) and to Σ_b in Eq. (101), and extract tangential components of the vectors on both sides. It is expedient to define and employ certain limiting tangential components $\tilde{\Phi}^Q_{\Sigma,\alpha,j}(\mathbf{r}_\Sigma)$, of an electromagnetic field on a surface Σ, where $Q=M$ or $Q=M'$:

$$\tilde{\Phi}^Q_{\Sigma,\alpha,j}(\mathbf{r}_\Sigma)\equiv\begin{cases}(\check{I}_\Sigma\Phi^Q_e)_j(\mathbf{r}_\Sigma) & \text{if } \alpha=e\\ -(\check{X}_\Sigma\Phi^Q_m)_j(\mathbf{r}_\Sigma) & \text{if } \alpha=m .\end{cases} \tag{102}$$

We now define two operators $\mathcal{P}^{Q0}_{\Sigma_a,\Sigma_b}$ and $\mathcal{P}^{Q+}_{\Sigma_b,\Sigma_a}$, which will be called an inward and an outward propagator, respectively, in terms of two-by-two matrices of dyadics as in Eq. (7), as follows, again with $Q=M$ or M':

$$\mathcal{P}^{Q0}_{\Sigma_a,\Sigma_b}\equiv\begin{bmatrix} -i\check{I}_{\Sigma_a}(\Gamma^Q)_{em}\check{X}_{\Sigma_b} & -i\check{I}_{\Sigma_a}(\Gamma^Q)_{ee}\check{I}_{\Sigma_b}\\ +i\check{X}_{\Sigma_a}(\Gamma^Q)_{mm}\check{X}_{\Sigma_b} & +i\check{X}_{\Sigma_a}(\Gamma^Q)_{me}\check{I}_{\Sigma_b}\end{bmatrix} , \tag{103}$$

$$\mathcal{P}^{Q+}_{\Sigma_b,\Sigma_a}=\begin{bmatrix} +i\check{I}_{\Sigma_b}(\Gamma^Q)_{em}\check{X}_{\Sigma_a} & +i\check{I}_{\Sigma_b}(\Gamma^Q)_{ee}\check{I}_{\Sigma_a}\\ -i\check{X}_{\Sigma_b}(\Gamma^Q)_{mm}\check{X}_{\Sigma_a} & +i\check{X}_{\Sigma_b}(\Gamma^Q)_{me}\check{I}_{\Sigma_a}\end{bmatrix} . \tag{104}$$

In terms of the limiting values of Eq. (102) and the propagators of Eqs. (103) and (104), Eqs. (100) and (101) and their adjoint equivalents specialize to

$$\tilde{\Phi}^{Q0}_{\Sigma_a}=\mathcal{P}^{Q0}_{\Sigma_a,\Sigma_b}\tilde{\Phi}^{Q0}_{\Sigma_b} , \tag{105}$$

$$\tilde{\Phi}^{Q+}_{\Sigma_b}=\mathcal{P}^{Q+}_{\Sigma_b,\Sigma_a}\tilde{\Phi}^{Q+}_{\Sigma_a} . \tag{106}$$

What are by now standard manipulations of Eqs. (91) and (94) can be made to yield the following results:

$$\mathcal{P}^{Q0}_{\Sigma_a,\Sigma_b}\mathcal{P}^{Q0}_{\Sigma_b,\Sigma_c}=\mathcal{P}^{Q0}_{\Sigma_a,\Sigma_c} \quad \text{if } \Sigma_c\rhd\Sigma_b\rhd\Sigma_a \tag{107}$$

$$\mathcal{P}^{Q+}_{\Sigma_c,\Sigma_b}\mathcal{P}^{Q+}_{\Sigma_b,\Sigma_a}=\mathcal{P}^{Q+}_{\Sigma_c,\Sigma_a} \quad \text{if } \Sigma_c\rhd\Sigma_b\rhd\Sigma_a \tag{108}$$

$$\mathcal{P}^{Q+}_{\Sigma_c,\Sigma_a}\mathcal{P}^{Q0}_{\Sigma_a,\Sigma_b}=0 \quad \text{if } \Sigma_c\rhd\Sigma_a \text{ and } \Sigma_b\rhd\Sigma_a \tag{109}$$

$$\mathcal{P}^{Q0}_{\Sigma_a,\Sigma_c}\mathcal{P}^{Q+}_{\Sigma_c,\Sigma_b}=0 \quad \text{if } \Sigma_c\rhd\Sigma_a \text{ and } \Sigma_c\rhd\Sigma_b . \tag{110}$$

We let the layer $\mathcal{D}_{ab}$ have uniform thickness, in some suitable sense, and let the thickness tend to zero, which process we denote correspondingly by $\lim_{\Sigma_b\searrow\Sigma_a}$. It can be inferred from Eqs. (107)–(110) that in this limit the inward and outward propagators tend to projection operators:

$$\lim_{\Sigma_b\searrow\Sigma_a}\mathcal{P}^{Q+}_{\Sigma_b,\Sigma_a}=\mathring{P}^{Q+}_{\Sigma_a} , \tag{111}$$

$$\lim_{\Sigma_b\searrow\Sigma_a}\mathcal{P}^{Q0}_{\Sigma_a,\Sigma_b}=\mathring{I}_{\Sigma_a}-\mathring{P}^{Q+}_{\Sigma_a} , \tag{112}$$

where $Q=M$ or M'. The "overcircle" accent means an operator in the linear space formed from the direct sum of two tangent-vector fields on Σ_a, as in Eq. (102), and $\mathring{I}_{\Sigma_a}$ is the identity operator in this space—compare Ref. [1], in the final paragraph of Sec. II and Eq. (74). The operator $\mathring{P}^{M+}_{\Sigma_a}$ has as its unit space all tangential fields $\tilde{\Phi}^{M+}_{\Sigma_a}$ derived from outgoing-wave solutions to Eqs. (96) and (97) in Ω_a^{ex}, and as its null space all tangential fields $\tilde{\Phi}^{M0}_{\Sigma_a}$ derived from regular solutions (in Ω_a) to Eqs. (96) and (97):

$$\mathring{P}^{M+}_{\Sigma_a}\tilde{\Phi}^{M+}_{\Sigma_a}=\tilde{\Phi}^{M+}_{\Sigma_a} , \tag{113}$$

$$\mathring{P}^{M+}_{\Sigma_a}\tilde{\Phi}^{M0}_{\Sigma_a}=0 . \tag{114}$$

In terms of the tangential fields of type $\tilde{\Phi}^{Q0}_{\Sigma_a}$ and $\tilde{\Phi}^{Q+}_{\Sigma_a}$, the Leontovich boundary conditions for an interior regular solution [cf. Ref. [1], Eq. (35)] and the definition of the radiation impedance operator [cf. Ref. [1], Eq. (61)] in the medium $Q=M$ or $Q=M'$ now take the form

$$\breve{A}^{Q}_{\Sigma_a}\tilde{\Phi}^{Q0}_{\Sigma_a,e}+\breve{C}^{Q}_{\Sigma_a}\tilde{\Phi}^{Q0}_{\Sigma_a,m}=0\ , \tag{115}$$

$$\tilde{\Phi}^{Q+}_{\Sigma_a,e}=\breve{Z}^{Q}_{\Sigma_a}\tilde{\Phi}^{Q+}_{\Sigma_a,m}\ , \tag{116}$$

where both $\breve{Z}^{Q}_{\Sigma_a}$ and $\breve{A}^{Q}_{\Sigma_a}\breve{Z}^{Q}_{\Sigma_a}+\breve{C}^{Q}_{\Sigma_a}$ must be invertible, but the reciprocity conditions that $\breve{Z}^{Q}_{\Sigma_a}$ and $\breve{A}^{Q}_{\Sigma_a}(\breve{C}^{Q}_{\Sigma_a})^{\tau}$ be symmetric [cf. Ref. [1], Eqs. (55) and (64)] are normally invalid in the present, more general, circumstances—see Sec. VII F for further discussion on this matter. We infer from Eqs. (113)–(116) that the projection operator must have the following matrix of dyadics:

$$\mathring{P}^{Q+}_{\Sigma_a}=\begin{bmatrix}\breve{Z}^{Q}_{\Sigma_a}(\breve{A}^{Q}_{\Sigma_a}\breve{Z}^{Q}_{\Sigma_a}+\breve{C}^{Q}_{\Sigma_a})^{-1}\breve{A}^{Q}_{\Sigma_a} & \breve{Z}^{Q}_{\Sigma_a}(\breve{A}^{Q}_{\Sigma_a}\breve{Z}^{Q}_{\Sigma_a}+\breve{C}^{Q}_{\Sigma_a})^{-1}\breve{C}^{Q}_{\Sigma_a}\\ (\breve{A}^{Q}_{\Sigma_a}\breve{Z}^{Q}_{\Sigma_a}+\breve{C}^{Q}_{\Sigma_a})^{-1}\breve{A}^{Q}_{\Sigma_a} & (\breve{A}^{Q}_{\Sigma_a}\breve{Z}^{Q}_{\Sigma_a}+\breve{C}^{Q}_{\Sigma_a})^{-1}\breve{C}^{Q}_{\Sigma_a}\end{bmatrix}, \tag{117}$$

which does not depend on the choice among equivalent operator pairs in Eq. (115) representing the Leontovich boundary conditions, and which generalizes Ref. [1], Eq. (C24). We could, in fact, use the "normalized" operator pairs taken directly from the first or second row of operators on the rhs of Eq. (117) to state the Leontovich boundary conditions Eq. (115).

We have now reached a point at which, to the degree of generality provided by the present context, we can prove the unproved assertion in the paragraph following that containing Eq. (37) in Ref. [1], Sec. III; that is, that any physically plausible obstacle with a linear response to electromagnetic signals can be simulated in the respect of its scattering by an equivalence class of Leontovich boundary conditions Eq. (115). Let us replace the medium M in the exterior of Σ_a by empty space, without changing the composition of the inside. Then the linear space of regular solutions of Maxwell's equations inside of Σ_a does not change, that is, Eq. (115) must continue to hold for the total electromagnetic signal (impinging wave plus scattered, or response, wave) with no change in the operators; but the response of the exterior region is now characterized by the vacuum value $\breve{Z}^{+}_{\Sigma_a,k_0}$ for the radiation impedance operator. We construct a new projection operator $\mathring{P}^{M\downarrow+}_{\Sigma_a}$ from the modified ingredients, where the superscript $\downarrow$ means that M only occupies the interior Ω_a of Σ_a, while the exterior Ω^{ex}_a is now a vacuum:

$$\mathring{P}^{Q+}_{\Sigma_a}=\begin{bmatrix}\breve{Z}^{Q}_{\Sigma_a}(\breve{A}^{Q}_{\Sigma_a}\breve{Z}^{Q}_{\Sigma_a}+\breve{C}^{Q}_{\Sigma_a})^{-1}\breve{A}^{Q}_{\Sigma_a} & \breve{Z}^{Q}_{\Sigma_a}(\breve{A}^{Q}_{\Sigma_a}\breve{Z}^{Q}_{\Sigma_a}+\breve{C}^{Q}_{\Sigma_a})^{-1}\breve{C}^{Q}_{\Sigma_a}\\ (\breve{A}^{Q}_{\Sigma_a}\breve{Z}^{Q}_{\Sigma_a}+\breve{C}^{Q}_{\Sigma_a})^{-1}\breve{A}^{Q}_{\Sigma_a} & (\breve{A}^{Q}_{\Sigma_a}\breve{Z}^{Q}_{\Sigma_a}+\breve{C}^{Q}_{\Sigma_a})^{-1}\breve{C}^{Q}_{\Sigma_a}\end{bmatrix}, \tag{118}$$

In view of the fact that

$$\mathring{P}^{M\downarrow+}_{\Sigma_a}\mathring{P}^{M+}_{\Sigma_a}=\mathring{P}^{M\downarrow+}_{\Sigma_a},\quad \mathring{P}^{M+}_{\Sigma_a}\mathring{P}^{M\downarrow+}_{\Sigma_a}=\mathring{P}^{M+}_{\Sigma_a}\ , \tag{119}$$

both projection operators have the same null space, but generally have distinct unit spaces—compare Eq. (B13), below. When we recall that the tangential components of **E** and **F** will be continuous across Σ_a even if the exterior medium is empty space, we infer that either row of the projection operator $\mathring{P}^{M+}_{\Sigma_a}$, which was derived on the assumption that the medium M fills all space, nevertheless provides an operator pair that fulfills the requirements for a simulation of the electromagnetic scattering of the obstacle embedded in otherwise empty space. The limitation of the generality of this construction corresponds to the limitation of the generality of the constitutive relations Eqs. (76)–(78).

E. Solution to the layer problem

We can now obtain a solution to the problem posed in the first paragraph of this section. We suppose initially that the material medium M fills all space; that part of it contained in Ω_a is removed and replaced by an obstacle Ξ of unspecified internal properties, but which is simulated on its boundary Σ_a in the respect of its scattering properties by Leontovich boundary conditions of the type of Eq. (115), with a nontrivial operator pair $\breve{A}^{\Xi}_{\Sigma_a},\breve{C}^{\Xi}_{\Sigma_a}$, for which the dependence on k_0 is implicit. Since the exterior region is still occupied by the medium M, the radiation impedance operator for Σ_a remains the same, that is, $\breve{Z}^{M}_{\Sigma_a}$. Accordingly, we have the associated projection operator $\mathring{P}^{\Xi+}_{\Sigma_a}$:

$$\mathring{P}^{\Xi+}_{\Sigma_a}=\begin{bmatrix}\breve{Z}^{M}_{\Sigma_a}(\breve{A}^{\Xi}_{\Sigma_a}\breve{Z}^{M}_{\Sigma_a}+\breve{C}^{\Xi}_{\Sigma_a})^{-1}\breve{A}^{\Xi}_{\Sigma_a} & \breve{Z}^{M}_{\Sigma_a}(\breve{A}^{\Xi}_{\Sigma_a}\breve{Z}^{M}_{\Sigma_a}+\breve{C}^{\Xi}_{\Sigma_a})^{-1}\breve{C}^{\Xi}_{\Sigma_a}\\ (\breve{A}^{\Xi}_{\Sigma_a}\breve{Z}^{M}_{\Sigma_a}+\breve{C}^{\Xi}_{\Sigma_a})^{-1}\breve{A}^{\Xi}_{\Sigma_a} & (\breve{A}^{\Xi}_{\Sigma_a}\breve{Z}^{M}_{\Sigma_a}+\breve{C}^{\Xi}_{\Sigma_a})^{-1}\breve{C}^{\Xi}_{\Sigma_a}\end{bmatrix}. \tag{120}$$

This projection operator has the same unit space as that of Eq. (117), that is, any set $\tilde{\Phi}_{\Sigma_a}^{M+}$ of limiting tangential values of an outgoing-wave solution of Eqs. (96) and (97) in the region exterior to Σ_a. The new projection operator has a null space that differs from the old in the following manner: let a generic $\Phi^{Mt}(\mathbf{r})$, where the t superscript stands for total, be a solution of Eqs. (96) and (97) for all $\mathbf{r}\in\Omega_a^{\text{ex}}$, where

$$\Phi^{Mt}(\mathbf{r})=\Phi^{M0}(\mathbf{r})+\Phi^{M+}(\mathbf{r}) \ , \tag{121}$$

and such that it satisfies the given Leontovich boundary conditions on the surface Σ_a of the obstacle Ξ:

$$\check{A}_{\Sigma_a}^{\Xi}\tilde{\Phi}_{\Sigma_a,e}^{Mt}+\check{C}_{\Sigma_a}^{\Xi}\tilde{\Phi}_{\Sigma_a,m}^{Mt}=0 \ . \tag{122}$$

In Eq. (120), Φ^{M0} is the specified impinging wave that would have been the complete solution if the obstacle Ξ had not replaced the medium M within Σ_a, and Φ^{M+} is the additional scattered wave that is uniquely determined by the impinging wave and the boundary conditions Eq. (122). The determination can be made as follows. Since we have

$$\mathring{P}_{\Sigma_a}^{\Xi+}\tilde{\Phi}_{\Sigma_a}^{Mt}=0 \ , \tag{123}$$

$$\mathring{P}_{\Sigma_a}^{\Xi+}\tilde{\Phi}_{\Sigma_a}^{M+}=\tilde{\Phi}_{\Sigma_a}^{M+} \ , \tag{124}$$

we infer from Eqs. (121)–(124) that

$$\tilde{\Phi}_{\Sigma_a}^{M+}=-\mathring{P}_{\Sigma_a}^{\Xi+}\tilde{\Phi}_{\Sigma_a}^{M0} \ . \tag{125}$$

We can now apply Eq. (101) to compute $\Phi^{M+}(\mathbf{r})$ everywhere in the exterior region Ω_a^{ex}, and, in particular, on Σ_b via Eq. (106).

It is now straightforward to compute a projection operator $\mathring{P}_{\Sigma_b}^{\Xi\cup\mathcal{D}_{ab}+}$ that engenders the same scattered wave as that derived from Eqs. (125) and (101), but which operates on the values $\tilde{\Phi}_{\Sigma_b}^{M0}$ of the impinging wave restricted to tangential components on Σ_b. We state the result and then prove it:

$$\mathring{P}_{\Sigma_b}^{\Xi\cup\mathcal{D}_{ab}+}=\mathring{P}_{\Sigma_b}^{M+}+\mathcal{P}_{\Sigma_b,\Sigma_a}^{M+}\mathring{P}_{\Sigma_a}^{\Xi+}\mathcal{P}_{\Sigma_a,\Sigma_b}^{M0} \ . \tag{126}$$

In view of Eqs. (105) and (106), and the results corresponding to Eqs. (113) and (114) on Σ_b, we infer that

$$\begin{aligned}\mathring{P}_{\Sigma_b}^{\Xi\cup\mathcal{D}_{ab}+}\tilde{\Phi}_{\Sigma_b}^{Mt}&=(\mathring{P}_{\Sigma_b}^{M+}+\mathcal{P}_{\Sigma_b,\Sigma_a}^{M+}\mathring{P}_{\Sigma_a}^{\Xi+}\mathcal{P}_{\Sigma_a,\Sigma_b}^{M0})(\tilde{\Phi}_{\Sigma_b}^{M0}+\tilde{\Phi}_{\Sigma_b}^{M+})\\&=\tilde{\Phi}_{\Sigma_b}^{M+}+\mathcal{P}_{\Sigma_b,\Sigma_a}^{M+}\mathring{P}_{\Sigma_a}^{\Xi+}\tilde{\Phi}_{\Sigma_a}^{M0}\\&=\tilde{\Phi}_{\Sigma_b}^{M+}-\mathcal{P}_{\Sigma_b,\Sigma_a}^{M+}\tilde{\Phi}_{\Sigma_a}^{M+}\\&=\tilde{\Phi}_{\Sigma_b}^{M+}-\tilde{\Phi}_{\Sigma_b}^{M+}=0 \ .\end{aligned} \tag{127}$$

We can also show, using Eqs. (107)–(110) and their limiting forms [cf. Eqs. (111) and (112)], that the operator $\mathring{P}_{\Sigma_b}^{\Xi\cup\mathcal{D}_{ab}+}$ specified in Eq. (126) is idempotent. The operator of Eq. (126) therefore provides the sought-for result: for if we now replace the medium M in the region exterior to Σ_b by empty space, the argument made in connection with Eq. (119) shows that either of its rows provides a suitable operator pair to specify Leontovich boundary conditions on Σ_b that simulate the combined obstacle Ξ in Ω_a plus a surrounding layer of M filling the region $\mathcal{D}_{ab}$ between Σ_a and Σ_b. The difficulties connected with applications of Eq. (126) have been mentioned in Sec. VII A; more insight can be gleaned from scrutiny of the analogous theory for the Helmholtz equation in Appendix B, where Eq. (126) has a counterpart in Eq. (B49).

F. Introduction of the "symplectic" form $\mathring{J}_\Sigma$

In this subsection we shall establish further analogies between the formalism in the electromagnetic case and that of the acoustic case treated in Appendix B. We shall continue to assume that the constitutive tensor properties given by σ, ϵ, and μ are nonsymmetric, so that the results obtained here will be of a more general character than those in Appendix B. For a generic bounding surface Σ we now define the electromagnetic analog $\mathring{J}_\Sigma$ to the symplectic [25] form J_Σ^S of Eq. (B2):

$$(\mathring{J}_\Sigma)_{\alpha\beta,jk}(\mathbf{r}_{\Sigma,1};\mathbf{r}_{\Sigma,2})\equiv(\delta_{\alpha e}\delta_{m\beta}-\delta_{\alpha m}\delta_{e\beta})(\check{I}_\Sigma)_{jk}(\mathbf{r}_{\Sigma,1};\mathbf{r}_{\Sigma,2}) \ . \tag{128}$$

Straightforward computations with Eqs. (103), (104), (92), and (128) lead to the following results, the analogs of Eq. (B36):

$$\begin{aligned}\mathcal{P}_{\Sigma_a,\Sigma_b}^{M'0}&=\mathring{J}_{\Sigma_a}(\mathcal{P}_{\Sigma_b,\Sigma_a}^{M+})^{\tau}(\mathring{J}_{\Sigma_b})^{-1} \ ,\\\mathcal{P}_{\Sigma_b,\Sigma_a}^{M'+}&=\mathring{J}_{\Sigma_b}(\mathcal{P}_{\Sigma_a,\Sigma_b}^{M0})^{\tau}(\mathring{J}_{\Sigma_a})^{-1} \ .\end{aligned} \tag{129}$$

We can take $\lim_{\Sigma_b\searrow\Sigma_a}$ of either line of Eq. (129), and apply Eqs. (111) and (112) with the result

$$\mathring{I}_{\Sigma_a}-\mathring{P}_{\Sigma_a}^{M'+}=\mathring{J}_{\Sigma_a}(\mathring{P}_{\Sigma_a}^{M+})^{\tau}(\mathring{J}_{\Sigma_a})^{-1} \ . \tag{130}$$

We note in this connection that the last equality in Ref. [1], Eq. (79), is incorrect: The operator $\mathring{X}$ defined in Ref. [1], Eq. (78), is a symmetric operator in a place that a skew-symmetric operator, defined with both constituents $\check{X}_\partial$ having the same sign, is needed to make the last part of Eq. (79) of Ref. [1] valid.

We define the skew-symmetric, "Wronskian" inner product of two kinematically allowable electromagnetic fields $\Phi(\mathbf{r})$ and $\Psi(\mathbf{r})$ (these need not be solutions to any Maxwell equations) with respect to the surface Σ to be, using Eqs. (102) and (128),

$$\begin{aligned}&\mathcal{W}_\Sigma(\Phi;\Psi)\\&\quad\equiv\int_\Sigma dA_{\Sigma,1}\int_\Sigma dA_{\Sigma,2}\sum_{\alpha,\beta}(\tilde{\Phi}_\Sigma)_{\alpha,j}(\mathbf{r}_{\Sigma,1})\\&\qquad\times(\mathring{J}_\Sigma)_{\alpha\beta,jk}(\mathbf{r}_{\Sigma,1};\mathbf{r}_{\Sigma,2})\\&\qquad\times(\tilde{\Psi}_\Sigma)_{\beta,k}(\mathbf{r}_{\Sigma,2}) \ .\end{aligned} \tag{131}$$

We let Ψ^M satisfy Eqs. (96) and (97), and let $\Phi^{M'}$ be a solution of the corresponding adjoint equations, in an open domain including $\Sigma_a\cup\mathcal{D}_{ab}\cup\Sigma_b$. Familiar manipulations of the differential equations now lead to the result that the Wronskian of $\Phi^{M'}$ and Ψ^M does not depend upon the surface on which it is computed:

$$\mathcal{W}_{\Sigma_b}(\Phi^{M'};\Psi^M)=\mathcal{W}_{\Sigma_a}(\Phi^{M'};\Psi^M) . \tag{132}$$

Evidently the Wronskian of $\Phi^{M'}$ and Ψ^M is invariant under the choice of surface in a class of surfaces; each surface in a class must be equivalent to every other by the following requirement: there should exist a chain of (say, N) intermediate surfaces, $\Sigma_a \equiv \Sigma_1, \Sigma_2, \ldots, \Sigma_N \equiv \Sigma_b$, such that adjacent pairs in the chain form, in either order, the inner and outer boundary of a domain such as $\mathcal{D}_{i,i+1}$, and such that Eqs. (96) and (97) are satisfied by Ψ^M, and the adjoint equations by $\Phi^{M'}$, in an open set covering each domain and its boundary.

In particular, if $\Phi^{M'0}$ and Ψ^{M0} are regular solutions in the medium M' and M, respectively, in the entire geometrical region inside of Σ, we can show that their Wronskian vanishes:

$$\mathcal{W}_{\Sigma}(\Phi^{M'0};\Psi^{M0})=0 . \tag{133}$$

Similarly, if $\Phi^{M'+}$ and Ψ^{M+} are respective solutions of outgoing-wave type in the exterior of Σ, we can show that

$$\mathcal{W}_{\Sigma}(\Phi^{M'+};\Psi^{M+})=0 . \tag{134}$$

We expect, on the other hand, that if both $\Phi^{M'+}$ and Ψ^{M0} are nontrivial, then their Wronskian on Σ can be nonzero, and similarly for the Wronskian of $\Phi^{M'0}$ and Ψ^{M+}.

If Eq. (134) is written out in terms of its electric- and magnetic-field constituents, and since the (say) tangential magnetic fields on Σ can be chosen arbitrarily for an outgoing wave, we infer that the radiation impedance operator $\breve{Z}^{M'}_{\Sigma}$ is the transpose of that when the exterior of Σ is filled with the medium M:

$$\breve{Z}^{M'}_{\Sigma}=(\breve{Z}^{M}_{\Sigma})^{\tau} . \tag{135}$$

Moreover, using the form Eq. (117) for the projection operators $\mathring{P}^{M+}_{\Sigma}$ and $\mathring{P}^{M'+}_{\Sigma}$ as well as Eq. (130) we can show for any representative pairs of operators $\breve{A}^{M}_{\Sigma},\breve{C}^{M}_{\Sigma}$ and $\breve{A}^{M'}_{\Sigma},\breve{C}^{M'}_{\Sigma}$ that another adjointness condition is satisfied:

$$\breve{A}^{M'}_{\Sigma}(\breve{C}^{M}_{\Sigma})^{\tau}=\breve{C}^{M'}_{\Sigma}(\breve{A}^{M}_{\Sigma})^{\tau} . \tag{136}$$

Equations (135) and (136) characterize the relationship between the exterior and, respectively, interior simulations of the medium M and the adjoint medium M' in terms of operators on the dividing surface Σ. When $M'=M$, we recover the reciprocity conditions generalizing Ref. [1], Eqs. (64) and (55).

Generalizing Eq. (B16), we can now find adjoint sets of invertible transformations that reduce to symplectic transformations when the reciprocity conditions are satisfied. We define, for $Q=M$ or M',

$$\mathring{T}^{Q}_{\Sigma}\equiv\begin{bmatrix}(\breve{A}^{Q}_{\Sigma}\breve{Z}^{Q}_{\Sigma}+\breve{C}^{Q}_{\Sigma})^{-1}\breve{A}^{Q}_{\Sigma} & (\breve{A}^{Q}_{\Sigma}\breve{Z}^{Q}_{\Sigma}+\breve{C}^{Q}_{\Sigma})^{-1}\breve{C}^{Q}_{\Sigma}\\ -\breve{I}_{\Sigma} & \breve{Z}^{Q}_{\Sigma}\end{bmatrix} . \tag{137}$$

It follows from Eqs. (128), (135), and (136) that

$$(\mathring{T}^{M}_{\Sigma})^{-1}=\mathring{J}_{\Sigma}(\mathring{T}^{M'}_{\Sigma})^{\tau}(\mathring{J}_{\Sigma})^{-1} . \tag{138}$$

Furthermore, if $\mathring{P}^{C}_{\Sigma}$ is the "canonical" projector

$$\mathring{P}^{C}_{\Sigma}\equiv\begin{bmatrix}\breve{I}_{\Sigma} & \breve{0}_{\Sigma}\\ \breve{O}_{\Sigma} & \breve{0}_{\Sigma}\end{bmatrix} , \tag{139}$$

then we have

$$\mathring{P}^{M+}_{\Sigma}=(\mathring{T}^{M}_{\Sigma})^{-1}\mathring{P}^{C}_{\Sigma}\mathring{T}^{M}_{\Sigma} , \tag{140}$$

analogous to Eq. (B20).

We note that Eqs. (139) and (140) can be generalized in a way that the symplectic transformations achieve a block diagonalization of the propagators. It is an immediate consequence of Eqs. (115), (116), and (138), that for any regular solution $\Phi^{M0}(\mathbf{r})$ and any outgoing-wave solution $\Phi^{M+}(\mathbf{r})$, we have

$$\mathring{T}^{M}_{\Sigma}\tilde{\Phi}^{M0}_{\Sigma}=\begin{bmatrix}\mathbf{0}\\ (\breve{I}_{\Sigma}\mathbf{U})(\mathbf{r}_{\Sigma})\end{bmatrix} , \tag{141}$$

$$\mathring{T}^{M}_{\Sigma}\tilde{\Phi}^{M+}_{\Sigma}=\begin{bmatrix}(\breve{I}_{\Sigma}\mathbf{V})(\mathbf{r}_{\Sigma})\\ \mathbf{0}\end{bmatrix} , \tag{142}$$

where the nonzero matrix entries can be any sufficiently well-behaved tangent-vector fields on Σ. It follows from Eqs. (140), (141), (105), (106), and (129), now, that there is an operator $S^{M+}_{\Sigma_b,\Sigma_a}$, which propagates tangent vectors (as opposed to a pair of such vectors) from an inner surface Σ_a to an outer one Σ_b through the medium M, such that

$$\mathring{T}^{M}_{\Sigma_b}\mathcal{P}^{M+}_{\Sigma_b,\Sigma_a}[\mathring{T}^{M}_{\Sigma_a}]^{-1}=\begin{bmatrix}S^{M+}_{\Sigma_b,\Sigma_a} & 0\\ 0 & 0\end{bmatrix} , \tag{143}$$

$$\mathring{T}^{M'}_{\Sigma_a}\mathcal{P}^{M'0}_{\Sigma_a,\Sigma_b}[\mathring{T}^{M'}_{\Sigma_b}]^{-1}=\begin{bmatrix}0 & 0\\ 0 & (S^{M+}_{\Sigma_b,\Sigma_a})^{\tau}\end{bmatrix} . \tag{144}$$

Evidently we have from Eq. (140) that

$$\lim_{\Sigma_b\searrow\Sigma_a} S^{M+}_{\Sigma_b,\Sigma_a}=\breve{I}_{\Sigma_a} . \tag{145}$$

G. Analog to Schrödinger theory

We conclude the section by addressing the question of establishing an electromagnetic analog to the considerations near the end of Appendix B. Suppose that the principle of reciprocity is satisfied and that the medium $M=M'$ is lossless, that is, $\sigma=0$, and ϵ and μ, and hence κ and λ, are all symmetric, real tensors everywhere. Given a suitable one-parameter family of surfaces $\Sigma(a)$, and restricting oneself to the subspace of purely outgoing-wave solutions to (the now self-adjoint) Eqs. (96) and (97), does there exist a surface-independent, sesquilinear inner product law for any pair of solutions that defines a positive-definite norm? Electromagnetic energy is now conserved, so that a surface integral of the normal component of the Poynting vector is a plausible candidate for the desired positive-definite norm of an outgoing wave, and the construct generalizes to yield a sesquilinear product law for pairs of outgoing waves. We note that with the given restrictions on κ and λ, if $\Phi^{M+}(\mathbf{r})$ is an outgoing-wave solution to Eqs. (96) and (97), then the time-reversed field $[\Pi\Phi^{M+}]^*(\mathbf{r})$ is also a solution, of ingoing-wave type. We define

$$\langle \Phi_1^{M+}|R_{\Sigma(a)}|\Phi_2^{M+}\rangle \equiv (\tfrac{1}{2})\mathcal{W}_{\Sigma(a)}([\Pi\Phi_1^{M+}]^*;\Phi_2^{M+}) \ . \tag{146}$$

It is straightforward to show that Eq. (146) reduces to

$$\langle \Phi_1^{M+}|R_{\Sigma(a)}|\Phi_2^{M+}\rangle = \int_{\Sigma(a)} dA_{\Sigma(a)}\mathrm{Re}\{\hat{\mathbf{n}}_{\Sigma(a)}\cdot[\mathbf{E}_1^{M+}\times(\mathbf{F}_2^{M+})^*]|_{\Sigma_a}\} \tag{147}$$

$$=([\check{X}_{\Sigma(a)}\mathbf{F}_1^{M+}]^*, \tfrac{1}{2}[\check{Z}_{\Sigma(a)}^{M+}+(\check{Z}_{\Sigma(a)}^{M+})^\dagger]\check{X}_{\Sigma(a)}\mathbf{F}_2^{M+})_{\Sigma(a)} \ , \tag{148}$$

where we used the inner product notation of Eqs. (2) and (3). We proved in Ref. [1], Appendix B, that the vacuum radiation impedance operator $\check{Z}^{+}_{\Sigma(a),k_0}$ has a positive-definite Hermitian part; it is plausible that the same result holds in the present, more general circumstances. Accordingly, the sesquilinear inner product law Eq. (146) defines a so-called pre-Hilbert space [26] on the complex linear space of tangential electromagnetic fields associated with outgoing-wave solutions; moreover, the inner product thereby obtained between a pair of solutions does not depend on which of the surfaces in the family is used to evaluate it. Although this matter has not been investigated, it is plausible that a Schrödinger-like theory can be established for the propagation of time-harmonic electromagnetic waves along a one-parameter family of surfaces, where the parameter plays a role analogous to that of the time in time-dependent nonrelativistic quantum mechanics. In turn, the "classical," or short-wavelength, limit of such a theory could yield a Hamiltonian form of geometrical optics. The resulting theory would presumably be more general than that derived in Ref. [19], Chap. III, where the one-parameter family of surfaces employed is taken to be the family of wave-front surfaces of a step-function electromagnetic source.

The above construct would also be more general than those that depend on the separability of Maxwell's equations or the Helmholtz equation, for which one of the three families of coordinate surfaces serves as the set along which Maxwell's equations are propagated; the propagator $\mathcal{P}^{M+}_{\Sigma_b,\Sigma_a}$ from Eq. (104) serves as the analog of the time evolution operator $\exp[-iH(t_b-t_a)/\hbar]$ of Schrödinger theory, where H is a time-independent Hamiltonian. In the present case the "Hamiltonian" is not given, but could be computed from the limiting derivative of the propagator, and will generally depend on the time-analogous parameter a that labels surfaces in the family.

VIII. DISCUSSION

Several lines of investigation are suggested by the material presented herein, in the forms either of (re-)derivation of previously established or accepted results, or of exploration and application of the methods proposed here with the hope of facilitating the treatment of a range of previously refractory problems.

First, it is clearly of interest, and appears to be mathematically feasible, to obtain a better [than Eq. (48)] short-wavelength approximation to the radiation impedance operator for a sphere immersed in a uniform isotropic medium by approximately summing the series (or their generalizations for a medium other than empty space) of Eq. (37), or Eqs. (41) and (42), when $|k_0a|\gg 1$. Similar to the sums in Ref. [3], Eqs. (A1) and (A2), the Sommerfeld-Watson transformation (Ref. [15], Chap. 10; Ref. [27]) is a good candidate for effecting this approximation. This result in turn may be applied to give an alternative derivation of that subspecies of the "geometric(al) theory of diffraction" [28] that concerns diffraction from smooth-surfaced, convex, perfectly conducting obstacles, and the theory of so-called creeping waves. We note in the present connection that the rhs of Eq. (37) is made up of two parts that annihilate one another, that have distinct singularity structure for nearby points on the sphere, and that exchange roles when the inverse operator is computed according to Eq. (5). It is an open question if such a decomposition exists for a more general surface, whether or not it has the topology of a sphere.

Another set of canonical problems (Ref. [28], Part II) that lie at the foundation of the geometrical theory of diffraction is that of diffraction from a perfectly conducting wedge with an aperture angle from just above zero to 2π (in the latter case the obstacle is a half plane). From the point of view of the transition operator approach to diffraction, we need to find a suitable approximation to the radiation impedance operator for a wedge-shaped surface in order to reduce Eq. (13) to quadratures. Integral representations are available—cf. Ref. [10], Chap. 6.5, and references given therein—for the Hertz vectors, and hence the complete electromagnetic Green's function, associated with arbitrarily oriented electric and magnetic dipoles in the presence of a perfectly conducting wedge. For a perfect conductor with boundary Σ, the projection operator of Eq. (117) reduces to

$$\mathring{P}^{\Sigma+}=\begin{bmatrix} \check{I}_\Sigma & 0 \\ (\check{Z}^{+}_{\Sigma,k_0})^{-1} & 0 \end{bmatrix} , \tag{149}$$

which operator is obtainable as a limit of the associated propagator of Eq. (104), where the role of $\Gamma^{M+}_{k_0}$ is now played by the complete Green's function. Perhaps asymptotic estimates for the radiation impedance operator can be inferred from the integral representations. We note that the nonlocality of the operator means that, considered as a two-point kernel, the radiation impedance operator couples points on distinct faces, as well as on the same face, of the wedge. These results may in their turn suggest *ad hoc* extensions that plausibly represent short-wavelength approximations for the radiation impedance operator for curved wedges, and provide an alternative, and arguably more rigorous, derivation of a range of diffraction phenomena now treatable by the geometrical theory of diffraction.

Other old problems that may yield more readily to the methods presented here are those of inferring the projection operator Eq. (118) in connection with a half space filled with a good, but not perfect, conductor—say the earth, locally flat; Eq. (125) could then be used to infer the surface values of the scattered wave for a radiating dipole above the earth, neglecting surface curvature. More generally, it appears to be feasible to obtain Leontovich boundary conditions for the case that the obstacle is a half space filled with a homogeneous but possibly anisotropic medium. One could then infer the projection operator Eq. (120) in the two-dimensional wave-vector space associated with the planar interface, and thereby the surface values of the scattered wave generated by any chosen impinging wave. These considerations would generalize those of Ref. [29], Chap. IX, and afford a means of describing the external response of anisotropic absorbing crystals, as those described in Ref. [9], Chap. 14.6.

A class of problems for which the methods developed here may facilitate treatment is that in which a perfectly conducting object has the highly nonconvex geometry that the obstacle surrounds a cavity that is connected to the outside by a relatively small aperture. A special case was considered in Ref. [1], Appendix C; the treatment there depended on the partial separability of Maxwell's equations in cylindrical geometries, and on the case that the aperture was taken to be a plane section of the cylindrical cavity. Such partly analytical methods will generally not be available: for example, if the obstacle is a thin spherical shell with a small aperture removed. The latter problem could nevertheless be approached by first dividing it into exterior and interior problems with respect to the complete sphere. We have discussed the exterior problem, which is solved in principle by the radiation impedance operator of Sec. III. The interior problem may be best handled numerically: we need a numerically complete set of regular interior solutions to Maxwell's equations for a given k_0 such that the tangential electric field is zero everywhere except on the aperture, which set of solutions would provide a nontrivial relationship of the form of Eq. (122) between the tangential electric and magnetic fields on the aperture. As in Ref. [1], Eq. (C28), the latter results provide a pair of Leontovich operators that simulate the complete obstacle plus cavity inside the sphere, and the problem reduces to one of the general type treated in Ref. [1]. As the class of examples of Ref. [1], Appendix C, shows, the operators involved in this process are not necessarily representable as continuous kernels, so that the numerical process sketched out will likely encounter difficulties in practice. A theoretical approach that may facilitate the treatment of this class of problems is an analog to Wigner's $\mathcal{R}$-matrix theory, as sketched at the end of Appendix B.

We conclude with remarks on a comment by Cho (Ref. [30], final paragraph in Chap. 9.8). Cho observes that conventional integral equation methods for electromagnetic scattering from perfect conductors are based on integral equations that, notwithstanding their successes both in theory and application, fail to have unique solutions at frequencies equal to interior cavity eigenfrequencies. Cho considers that a principal roadblock to progress along these lines is the lack of suitable integral representations for the scattered fields such that these new representations yield integral equations free of the stated defect. The theory presented here and in Ref. [1] in effect addresses this problem, but goes about it in a nonstraightforward way: In the integral equation treatment, the integral operators are known, but suffer from a defect that make their use problematic. In the approach embodied in Eq. (149), or more generally in Eq. (118), the operators that solve the problem can be proved formally to exist whenever $\mathrm{Im}(k_0)=0$ and $k_0\neq 0$, in particular when k_0 is a cavity eigenfrequency; these operators are not "off-the-shelf" entities, however, and the present theory redirects attention to the determination of suitable approximations to the projection operators that map the boundary data inferred from a given impinging wave directly into the sought-for boundary values from which the scattered wave can be constructed.

APPENDIX A: METHOD OF STATIONARY PHASE

We wish to evaluate an integral of the type

$$\int_{\partial\Omega} dA_{\partial 3} f(\mathbf{r}_{\partial 3}) \exp[ik_0 d_\chi(\mathbf{r}_1;\mathbf{r}_{\partial 3};\mathbf{r}_2)] \ , \tag{A1}$$

where $\chi=\epsilon$ corresponds to the elliptic case and $\chi=v$ to the hyperbolic case:

$$d_\chi(\mathbf{r}_1;\mathbf{r}_{\partial 3};\mathbf{r}_2) \equiv \begin{cases} r_{1,\partial 3}+r_{\partial 3,2} & \text{if } \chi=\epsilon \\ r_{1,\partial 3}-r_{\partial 3,2} & \text{if } \chi=v \ . \end{cases} \tag{A2}$$

If $|k_0|$ is sufficiently large, we can obtain an asymptotic estimate for this integral by the method of stationary phase (Ref. [9], Appendix III.3). We need to obtain the set of stationary points of the function $d_\epsilon(\mathbf{r}_1;\mathbf{r}_{\partial 3};\mathbf{r}_2)$ or $d_v(\mathbf{r}_1;\mathbf{r}_{\partial 3};\mathbf{r}_2)$ for $\mathbf{r}_1$ and $\mathbf{r}_2$ fixed, as $\mathbf{r}_{\partial 3}$ ranges over $\partial\Omega$.

The points of stationary phase can be characterized in the following geometrical terms. We first consider the elliptic case. Let Δ be a length parameter with $\Delta>|\mathbf{r}_1-\mathbf{r}_2|$. For each such Δ the locus of points $\mathbf{r}_3$ such that

$$|\mathbf{r}_1-\mathbf{r}_3|+|\mathbf{r}_3-\mathbf{r}_2|=\Delta \tag{A3}$$

is a prolate ellipsoid of rotation with $\mathbf{r}_1$ and $\mathbf{r}_2$ as its foci. Any point of tangency of one of this family of ellipsoids with $\partial\Omega$, or any point of intersection [31] of the *interior* of the straight-line segment between $\mathbf{r}_1$ and $\mathbf{r}_2$ with $\partial\Omega$, is a point of stationary $d_\epsilon(\mathbf{r}_1;\mathbf{r}_{\partial 3};\mathbf{r}_2)$. In the hyperbolic case, let Δ be a parameter such that $-|\mathbf{r}_1-\mathbf{r}_2|<\Delta<|\mathbf{r}_1-\mathbf{r}_2|$. The locus of points $\mathbf{r}_3$ such that

$$|\mathbf{r}_1-\mathbf{r}_3|-|\mathbf{r}_3-\mathbf{r}_2|=\Delta \tag{A4}$$

is a one-sheeted hyperboloid of rotation that has $\mathbf{r}_1$ and $\mathbf{r}_2$ as its foci; any point of tangency of one of these hyperboloids with $\partial\Omega$, or any point of intersection [31] of $\partial\Omega$ with the extended straight line connecting $\mathbf{r}_1$ and $\mathbf{r}_2$ and lying *exterior* to the closed line segment between $\mathbf{r}_1$ and $\mathbf{r}_2$, is a point of stationary $d_v(\mathbf{r}_1;\mathbf{r}_{\partial 3};\mathbf{r}_2)$.

We shall denote the stationary points of d_ϵ (d_v) by $\mathbf{r}_{\partial\epsilon a}$ ($\mathbf{r}_{\partial v a}$), with $a=1,2,\ldots$. We define

$$\hat{\mathbf{n}}(\mathbf{r}_{\partial\chi a})\equiv\hat{\mathbf{n}}_{\chi a} \ , \tag{A5}$$

and suppose that $\hat{\mathbf{t}}_{\chi a\zeta}$, $\zeta=1,2$ are orthogonal vectors tangent to $\partial\Omega$ at $\mathbf{r}_{\partial\chi a}$. We use the notation of Eqs. (51) and (52); then

$$\begin{aligned}&\hat{\mathbf{r}}_{1,\partial\chi a}\cdot\hat{\mathbf{t}}_{\chi a\zeta}=\mp\hat{\mathbf{r}}_{2,\partial\chi a}\cdot\hat{\mathbf{t}}_{\chi a\zeta}\quad\text{for }\zeta=1,2\\&|\hat{\mathbf{r}}_{1,\partial\chi a}\cdot\hat{\mathbf{n}}_{\chi a}|=|\hat{\mathbf{r}}_{2,\partial\chi a}\cdot\hat{\mathbf{n}}_{\chi a}|\ ,\end{aligned}\tag{A6}$$

where the upper sign (lower sign) corresponds to $\chi=\epsilon$ ($\chi=v$).

As in Ref. [3], Sec. III, we introduce local coordinates (u^1,u^2) in a neighborhood of $\mathbf{r}_{\partial 3}=\mathbf{r}_{\partial\chi a}$ such that

$$\begin{aligned}\mathbf{r}_{\partial 3}&=\mathbf{R}_{\chi a}(u^1,u^2)\\&=\mathbf{r}_{\partial\chi a}+\sum_{\zeta=1}^{2}u^\zeta\hat{\mathbf{t}}_{\chi a\zeta}-\tfrac{1}{2}\sum_{\zeta,\eta=1}^{2}K_{\chi a\zeta\eta}u^\zeta u^\eta\hat{\mathbf{n}}_{\chi a}\\&\quad+O^3(u^1,u^2)\ ,\end{aligned}\tag{A7}$$

where $O^3(u^1,u^2)$ is a neglected term of third order in u^1,u^2; the 2×2 matrix $K_{\chi a\zeta\eta}$ is the curvature matrix of $\partial\Omega$ at $\mathbf{r}_{\partial\chi a}$ in the chosen coordinates, where positive curvatures correspond to convex $\partial\Omega$. We define the 2×2 matrix $\Lambda_{\chi a\zeta\eta}$ associated with a stationary phase point $\mathbf{r}_{\partial\chi a}$ as follows:

$$\begin{aligned}\Lambda_{\chi a\zeta\eta}\equiv(\tfrac{1}{2})\{&(\hat{\mathbf{r}}_{1,\partial\chi a}\pm\hat{\mathbf{r}}_{2,\partial\chi a})\cdot\hat{\mathbf{n}}_{\chi a}K_{\chi a\zeta\eta}\\&+(1/r_{1,\partial\chi a})[\delta_{\zeta\eta}-(\hat{\mathbf{r}}_{1,\partial\chi a}\cdot\hat{\mathbf{t}}_{\chi a\zeta})(\hat{\mathbf{r}}_{1,\partial\chi a}\cdot\hat{\mathbf{t}}_{\chi a\eta})]\\&\pm(1/r_{2,\partial\chi a})[\delta_{\zeta\eta}-(\hat{\mathbf{r}}_{2,\partial\chi a}\cdot\hat{\mathbf{t}}_{\chi a\zeta})(\hat{\mathbf{r}}_{2,\partial\chi a}\cdot\hat{\mathbf{t}}_{\chi a\eta})]\}\ ,\end{aligned}\tag{A8}$$

where the upper signs (lower signs) belong to the case $\chi=\epsilon$ ($\chi=v$). With Eqs. (A7) and (A8) we can expand the distance function of Eq. (A2) in powers of u^1,u^2:

$$\begin{aligned}d_\chi(\mathbf{r}_1;\mathbf{r}_{\partial 3};\mathbf{r}_2)=d_\chi(\mathbf{r}_1;\mathbf{r}_{\partial\chi a};\mathbf{r}_2)+\sum_{\zeta,\eta=1}^{2}\Lambda_{\chi a\zeta\eta}u^\zeta u^\eta\\+O^3(u^1,u^2)\ .\end{aligned}\tag{A9}$$

We presume that the eigenvalues of the $\Lambda_{\chi a\zeta\eta}$ matrix are $\lambda_{\chi a\zeta}$, with $\zeta=1,2$. Both eigenvalues are taken to be nonzero in order that the stationary phase approximation to be used here is valid. We take

$$\begin{aligned}&\det(\Lambda_{\chi a\zeta\eta})\equiv\Lambda_{\chi a}=\lambda_{\chi a1}\lambda_{\chi a2}\neq 0\ ,\\&\sigma_{\chi a\zeta}\equiv k_0\lambda_{\chi a\zeta}/|k_0\lambda_{\chi a\zeta}|\quad\text{for }\zeta=1,2\ .\end{aligned}\tag{A10}$$

Then the stationary phase approximation for the integral of Eq. (A1) is

$$\begin{aligned}\sum_a f(\mathbf{r}_{\partial\chi a})(\pi/|k_0|)|\Lambda_{\chi a}|^{-1/2}\\\times\exp[ik_0d_\chi(\mathbf{r}_1;\mathbf{r}_{\partial\chi a};\mathbf{r}_2)+i(\pi/4)(\sigma_{\chi a1}+\sigma_{\chi a2})]\ .\end{aligned}\tag{A11}$$

We conclude this appendix by noting that there is a relationship between the topology of $\partial\Omega$ and the properties of a phase function, as that in the exponent in Eq. (A1), at its collection of stationary points. We fix $\mathbf{r}_1$ and $\mathbf{r}_2$, order the stationary points $\mathbf{r}_{\partial\chi a}$ of $k_0d_\chi(\mathbf{r}_1;\mathbf{r}_{\partial 3};\mathbf{r}_2)$ so that increasing $a=1,2,\ldots$, corresponds to increasing $k_0d_\chi(\mathbf{r}_1;\mathbf{r}_{\partial\chi a};\mathbf{r}_2)$, and assume that the $k_0^2\Lambda_{\chi a}$ are nonzero for all $a=1,2,\ldots$. Then $k_0d_\chi(\mathbf{r}_1;\mathbf{r}_{\partial 3};\mathbf{r}_2)$ can be considered to be a so-called *Morse function* (named for the mathematician Marston Morse) for $\partial\Omega$, that is, a smooth mapping of $\partial\Omega$ into the reals for which all stationary points (called *critical points* in this context) are nondegenerate. The pair of algebraic signs $\sigma_{\chi a1},\sigma_{\chi a2}$ can be called the *signature* of the critical point, and the number of -1's (be it 0, 1, or 2) among the $\sigma_{\chi a\zeta}$ is called the *index* of the critical point. Then *Morse theory* asserts that the manifold $\partial\Omega$ can be reconstructed up to homotopy equivalence from the ordered sequence of indices of its critical points—see Ref. [32], Part I. This property can be used as a partial verification that all the points of stationary $k_0d_\chi(\mathbf{r}_1;\mathbf{r}_{\partial 3};\mathbf{r}_2)$, as $\mathbf{r}_{\partial 3}$ ranges over $\partial\Omega$, have been collected, and the signatures computed properly.

APPENDIX B: HELMHOLTZ PROJECTION OPERATORS, SYMPLECTIC TRANSFORMATIONS, AND PROPAGATORS

In this appendix we shall establish a theoretical structure analogous to that of Sec. VII, but for the case of time-harmonic acoustic-wave scattering. We choose to append this material, as the argument for scalar fields is more transparent than that manufactured for the vector fields of the electromagnetic case. The theory in this appendix depends primarily on that of Refs. [2] and [3] and can be read independently of the electromagnetic theory in the remainder of the paper. We shall find that projection operators, symplectic geometry (in function space), and propagators play a corresponding role in both cases. As in Sec. VII, a principal result is a formula that permits impedance boundary conditions on an inner surface, which conditions simulate an interior obstacle's acoustic scattering properties, to be transformed into impedance boundary conditions on a circumscribing surface. We note that analogous methods may be of use in scattering theory for time-harmonic elastic waves and for the time-independent Schrödinger equation.

As discussed in, for example, Ref. [33], Vol. II, pp. 228–231, the Cauchy problem for an elliptic partial differential equation (PDE) is ill posed in the sense of Hadamard, although the initial surface and initial values are taken to be analytic: that is, elaborate strategems are needed to estimate the behavior of the solution function away from a small neighborhood of the initial surface, and the solution is unstable to general small variations of the initial values. This extrapolation process may be compared to analytic continuation in the complex plane; the results that we shall obtain are in some respects reminiscent of the Laurent expansion in the theory of analytic functions of a complex variable. (See Ref. [34] for a discussion of, and references to papers on, the solution of a range of ill-posed problems.) The methods that will be developed here and in Sec. VII circumvent the instabilities of the Cauchy problem, but are dependent on the availability of a Green's function; to this extent the methods will ordinarily be practicable only for propagation of solutions in domains within which the acoustic or electromagnetic properties are uniform, although the mathematical structure is valid for more general situations. Only a partial solution to the Cauchy problem for the Helmholtz equation is achieved: the linear space of boundary values (taken in a suitable sense) is decomposed by a projection operator into a direct sum of two subspaces such that one subspace can be propagated in a characteristic direction away from the initial surface, while the complementary subspace can be propagated in the opposite direction. The method affords no direct information on the propagation of either subspace in the respective "wrong" direction.

Let $\mathcal{F}^{\Sigma}$ be the space of complex-valued functions defined in Σ; we call the two-component direct sum space $\mathcal{F}^{\Sigma\oplus\Sigma}\equiv\mathcal{F}^{\Sigma}\oplus\mathcal{F}^{\Sigma}$. Let $\Upsilon(\mathbf{r})$ be a complex-valued function that is defined on, and continuously differentiable in, a neighborhood of a closed surface Σ in $\mathcal{E}^3$. From the space of functions of type Υ we construct a linear mapping into the space $\mathcal{F}^{\Sigma\oplus\Sigma}$ by the following operations. We associate with each Υ the entity $\tilde{\Upsilon}_{\Sigma}\in\mathcal{F}^{\Sigma\oplus\Sigma}$, which comprises the two-component function of limiting values and normal derivatives of Υ on Σ:

$$\tilde{\Upsilon}_{\Sigma}(\mathbf{r}_{\Sigma})\equiv\begin{bmatrix}\Upsilon(\mathbf{r}_{\Sigma})\\ \dfrac{\partial\Upsilon}{\partial n_{\Sigma}}(\mathbf{r}_{\Sigma})\end{bmatrix}, \tag{B1}$$

where $\mathbf{r}_{\Sigma}\in\Sigma$, and $\partial/\partial n_{\Sigma}$ stands for the gradient of a function in the outward normal direction to Σ. The linear mapping defined by Eq. (B1) will be an explicit or implicit part of a number of the constructions below.

We define $J^{S}_{\Sigma\oplus\Sigma}$ to be a skew-symmetric operator that maps $\mathcal{F}^{\Sigma\oplus\Sigma}$ into itself linearly, in terms of a two-by-two matrix of operators in $\mathcal{F}^{\Sigma}$, as follows (the S stands for complex-valued scalar functions):

$$J^{S}_{\Sigma\oplus\Sigma}\equiv\begin{bmatrix}0^{S}_{\Sigma} & I^{S}_{\Sigma}\\ -I^{S}_{\Sigma} & 0^{S}_{\Sigma}\end{bmatrix}, \tag{B2}$$

where 0^{S}_{Σ} and I^{S}_{Σ} are the zero operator and the unit operator, respectively, on $\mathcal{F}^{\Sigma}$. The operator $J^{S}_{\Sigma\oplus\Sigma}$ establishes a nondegenerate Wronskian bilinear inner product $W_{\Sigma}(\Upsilon^1;\Upsilon^2)$ between pairs of functions $\Upsilon^1(\mathbf{r})$ and $\Upsilon^2(\mathbf{r})$,

$$W_{\Sigma}(\Upsilon^1;\Upsilon^2)\equiv(\tilde{\Upsilon}^{1}_{\Sigma};J^{S}_{\Sigma\oplus\Sigma}\tilde{\Upsilon}^{2}_{\Sigma})_{\Sigma\oplus\Sigma} \tag{B3}$$

$$=-W_{\Sigma}(\Upsilon^2;\Upsilon^1), \tag{B4}$$

in an obvious notation. The Wronskian $W_{\Sigma}(\Upsilon^1;\Upsilon^2)$ is evaluated by restricting the functions and their gradients to their limiting values and limiting normal components, respectively, on a particular choice of surface Σ, which can be any surface within the domain of definition and good behavior of both functions. The value of the Wronskian normally depends on the choice of Σ.

Let us now consider two surfaces Σ_a and Σ_b in $\mathcal{E}^3$, such that Σ_a is entirely inside of, or at most touching, Σ_b; we denote the domain between the surfaces as $\mathcal{D}_{ab}\subset\mathcal{E}^3$. (If both Σ_a and Σ_b are unbounded, what is "inside" or "outside" is a matter of choice.) Let $\Upsilon^1(\mathbf{r})$ and $\Upsilon^2(\mathbf{r})$ be two complex-valued solutions of the scalar Helmholtz equation in an open domain that covers $\Sigma_a\cup\mathcal{D}_{ab}\cup\Sigma_b$:

$$(\nabla^2+k_0^2)\Upsilon^{\alpha}(\mathbf{r})=0 \quad \text{for } \alpha=1,2\ . \tag{B5}$$

Familiar manipulations—see Ref. [24], p. 804, Eq. (7.2.2)—of Eq. (B5) lead to the result

$$W_{\Sigma_a}(\Upsilon^1;\Upsilon^2)=W_{\Sigma_b}(\Upsilon^1;\Upsilon^2)\ . \tag{B6}$$

Equation (B6) establishes an invariance principle for the chosen symplectic inner product of two solutions to the scalar Helmholtz equation: for solution pairs of Eq. (B5), the Wronskian is independent of the choice of Σ, at least when the choices are subjected to the aforementioned geometrical constraints. In fact, if we generalize the notion of equivalence of a pair of surfaces to be contingent on the existence of a chain of domains as described following Eq. (132), and such that both solutions satisfy Eq. (B5) in open sets covering the closures of all the domains, then the Wronskian is independent of the choice of surface in an equivalence class.

If Eq. (B5) is satisfied in the complete exterior of Σ_a, and both Υ^1 and Υ^2 are of outgoing-wave type, we keep Σ_a fixed and let Σ_b tend to infinity; the rhs of Eq. (B6) then tends to zero. If Eq. (B5) is satisfied in the entire interior of Σ_b, with both functions regular, we keep Σ_b fixed and shrink Σ_a to a point; the lhs of Eq. (B6) then tends to zero. (If Σ_a is unbounded, the latter assertion applies to pairs of functions that represent waves that are outgoing toward "minus infinity.") We expect that if Υ^1, say, satisfies Eq. (B5) and is not identically zero and regular within Σ_b, and if Υ^2 satisfies Eq. (B5) and is not identically zero and of outgoing-wave type outside of Σ_a, then their Wronskian can be nonzero.

Hence, given the limitations on the behavior of Υ^1 and Υ^2 with respect to Σ and $\mathcal{E}^3$, we have identified two linear subspaces of $\mathcal{F}^{\Sigma\oplus\Sigma}$ that (we could plausibly show) are disjoint except for the zero vector and whose direct sum spans $\mathcal{F}^{\Sigma\oplus\Sigma}$. This decomposition can be termed a symplectic decomposition of $\mathcal{F}^{\Sigma\oplus\Sigma}$: that is to say, the symplectic inner product of pairs of elements is always zero when both belong to the same subspace, while if the elements are both nontrivial and belong to different subspaces, their symplectic inner product can be nonzero. If the surface Σ is compact it may be possible to obtain a complete symplectic coordinate system in the sense of Ref. [25] in the space $\mathcal{F}^{\Sigma\oplus\Sigma}$, that is to say a countably infinite-dimensional analog to the coordinate system that exists in the finite-dimensional case whenever the coefficient field is of characteristic zero (cf. Ref. [25], p. 166); we shall not attempt this construction here.

We consider a geometrical configuration as described in Ref. [2], Sec. II, and assume that free-space acoustic propagation prevails everywhere in $\mathcal{E}^3$; note that the interior of $\partial\Omega$ need not be connected, but the exterior must be connected and unbounded. We fix the surface $\Sigma\equiv\partial\Omega$, and have recourse to the theory of Ref. [2], Secs. III A and III C, that there is a projection operator P^{F+}_{Σ} (the superscripts F and $+$ signify free-space solutions and the outgoing-wave case, respectively), where

$$P_{\Sigma}^{F+}=\begin{bmatrix}\frac{1}{2}(I_{\Sigma}^{S}+V_{\Sigma,k_0}) & -\frac{1}{2}U_{\Sigma,k_0}\\ \frac{1}{2}W_{\Sigma,k_0} & \frac{1}{2}(I_{\Sigma}^{S}-V_{\Sigma,k_0}^{\tau})\end{bmatrix};\tag{B7}$$

the operators U_{Σ,k_0}, V_{Σ,k_0}, V^{τ}_{Σ,k_0}, and W_{Σ,k_0} are defined in Ref. [2], Eqs. (18)–(21), respectively.

The operator P_{Σ}^{F+} is the similarity transform of the operator P defined in Ref. [2], Eq. (42) by the operator X of Ref. [2], Eq. (43); the operator P is defined with respect to singlet and doublet layers on Σ, while P_{Σ}^{F+} operates directly on the linear space of limiting function values and normal derivatives as defined above. The operator P_{Σ}^{F+} has as its unit space the limits on Σ of the values and derivatives of outgoing-wave solutions in the exterior of Σ, while the null space of P_{Σ}^{F+} comprises the limiting values and derivatives of solutions to Eq. (B5) that are regular within Σ.

Next we suppose that the interior of Σ contains an unspecified, and possibly complex, substance and structure that responds linearly to acoustic signals. Acoustic-wave functions that are regular in the interior of Σ are called $\Upsilon^{R}(\mathbf{r})$ (the superscript R stands for Robin—cf. Ref. [2]), and continue into free-space solutions in the exterior region, whose exterior limiting values and derivatives on Σ satisfy the following boundary conditions of generalized impedance, or Robin, type on Σ:

$$(A_{\Sigma,k_0}\Upsilon^{R})(\mathbf{r}_{\Sigma})+\left[B_{\Sigma,k_0}\frac{\partial\Upsilon^{R}}{\partial n_{\Sigma}}\right](\mathbf{r}_{\Sigma})=0\ ,\tag{B8}$$

where, in contrast to Ref. [2], we make the surface dependence of the operators explicit. The pair of operators $A_{\Sigma,k_0},B_{\Sigma,k_0}$ are representatives of an equivalence class with respect to simultaneous left multiplication of both by any invertible operator Y_{Σ} of the type that maps $\mathcal{F}^{\Sigma}$ into itself; this class of operator pairs is supposed to have a structure such that if, and only if, an acoustic-wave function satisfies Eq. (B8), it is necessarily of the type that extends to a regular solution in the interior of Σ. We denote the acoustic radiation impedance operator for the surface Σ by $\check{Z}_{\Sigma,k_0}$—note the superimposed "haček" accent, as distinct from a "breve" accent in the electromagnetic case; we note that $\check{Z}_{\Sigma,k_0}$ is invertible and symmetric—cf. Ref. [2], Eqs. (28) and (30). Wave functions of outgoing-wave type comprise those solutions $\Upsilon^{+}(\mathbf{r})$ of Eq. (B5) in the exterior of Σ that satisfy Sommerfeld's radiation condition—cf. Ref. [4], Eq. (3.7)–at infinity, and whose limiting values on Σ therefore satisfy

$$\Upsilon^{+}(\mathbf{r}_{\Sigma})-\left[\check{Z}_{\Sigma,k_0}\frac{\partial\Upsilon^{+}}{\partial n_{\Sigma}}\right](\mathbf{r}_{\Sigma})=0\ .\tag{B9}$$

We assume that the three operators are such that a unique left and right inverse operator and its adjoint

$$(A_{\Sigma,k_0}\check{Z}_{\Sigma,k_0}+B_{\Sigma,k_0})^{-1},\quad [\check{Z}_{\Sigma,k_0}(A_{\Sigma,k_0})^{\tau}+(B_{\Sigma,k_0})^{\tau}]^{-1}\tag{B10}$$

exist—cf. Ref. [2], Appendix.

We can now construct a projection operator that generalizes Eq. (B7) to the case that the interior of Σ is an acoustically complex entity that is simulated by Robin boundary conditions Eq. (B8) on Σ, with the exterior being free space:

$$P_{\Sigma}^{R+}(A_{\Sigma,k_0},B_{\Sigma,k_0},\check{Z}_{\Sigma,k_0})\equiv\begin{bmatrix}\check{Z}_{\Sigma,k_0}(A_{\Sigma,k_0}\check{Z}_{\Sigma,k_0}+B_{\Sigma,k_0})^{-1}A_{\Sigma,k_0} & \check{Z}_{\Sigma,k_0}(A_{\Sigma,k_0}\check{Z}_{\Sigma,k_0}+B_{\Sigma,k_0})^{-1}B_{\Sigma,k_0}\\ (A_{\Sigma,k_0}\check{Z}_{\Sigma,k_0}+B_{\Sigma,k_0})^{-1}A_{\Sigma,k_0} & (A_{\Sigma,k_0}\check{Z}_{\Sigma,k_0}+B_{\Sigma,k_0})^{-1}B_{\Sigma,k_0}\end{bmatrix}.\tag{B11}$$

The unit space and the null space of the operator $P_{\Sigma}^{R+}(A_{\Sigma,k_0},B_{\Sigma,k_0},\check{Z}_{\Sigma,k_0})$ comprise the vectors $\tilde{\Upsilon}_{\Sigma}(\mathbf{r}_{\Sigma})$ of Eq. (B1) derived, respectively, from outgoing-wave solutions $\Upsilon^{+}(\mathbf{r})$ in the unbounded exterior of Σ, and regular solutions $\Upsilon^{R}(\mathbf{r})$ in the acoustically complex interior of Σ.

In the latter connection, we note that the projection operators have certain algebraic properties. We fix Σ and k_0, and dispense with these subscripts and the haček accent on the arguments. Then we have

$$P_{\Sigma}^{R+}(A_1,B_1,Z)P_{\Sigma}^{R+}(A_2,B_2,Z)=P_{\Sigma}^{R+}(A_2,B_2,Z)\ ,\tag{B12}$$

$$P_{\Sigma}^{R+}(A,B,Z_1)P_{\Sigma}^{R+}(A,B,Z_2)=P_{\Sigma}^{R+}(A,B,Z_1)\ .\tag{B13}$$

The result Eq. (B12) and the corresponding result with (A_1,B_1) and (A_2,B_2) interchanged imply that the operators $P_{\Sigma}^{R+}(A_1,B_1,Z)$ and $P_{\Sigma}^{R+}(A_2,B_2,Z)$ have the same unit space, while Eq. (B13) and a corresponding equation with Z_1 and Z_2 interchanged imply that $P_{\Sigma}^{R+}(A,B,Z_1)$ and $P_{\Sigma}^{R+}(A,B,Z_2)$ have the same null space. These mathematical results reflect the physical circumstances that, in the case of Eq. (B12) we are dealing with two scattering problems with different obstacles but the same exterior environment, while in the case of Eq. (B13) the two scattering problems dealt with have acoustically identical obstacles embedded in distinct external fluid environments.

A canonical (C) form P_{Σ}^{C} for projection operators for the space of vectors as that in Eq. (B1) is defined to be

$$P_{\Sigma}^{C}\equiv\begin{bmatrix}I_{\Sigma}^{S} & 0_{\Sigma}^{S}\\ 0_{\Sigma}^{S} & 0_{\Sigma}^{S}\end{bmatrix}.\tag{B14}$$

It is natural to ask whether the projection operator of Eq. (B11) is equivalent to that of Eq. (B14) via a similarity transformation. At least if the reciprocity criterion

$$A_{\Sigma,k_0}(B_{\Sigma,k_0})^{\tau}=B_{\Sigma,k_0}(A_{\Sigma,k_0})^{\tau}\tag{B15}$$

[Ref. [2], Eq. (A8)] is satisfied, the answer is affirmative;

indeed the transformation, which is suggested by the results of Ref. [1], Sec. IV, and by an analogous computation in the acoustic case, is equivalent to a symplectic transformation in $\mathcal{F}^{\Sigma\oplus\Sigma}$. A linear operator T_Σ that maps $\mathcal{F}^{\Sigma\oplus\Sigma}$ into itself is said to be symplectic if

$$(T_\Sigma)^\tau J^S_{\Sigma\oplus\Sigma} T_\Sigma = J^S_{\Sigma\oplus\Sigma} \ . \tag{B16}$$

We define

$$T^R_\Sigma(A_{\Sigma,k_0},B_{\Sigma,k_0},\check{Z}_{\Sigma,k_0}) = \begin{bmatrix} A_{\Sigma,k_0} & B_{\Sigma,k_0} \\ -[\check{Z}_{\Sigma,k_0}(A_{\Sigma,k_0})^\tau+(B_{\Sigma,k_0})^\tau]^{-1} & [\check{Z}_{\Sigma,k_0}(A_{\Sigma,k_0})^\tau+(B_{\Sigma,k_0})^\tau]^{-1}\check{Z}_{\Sigma,k_0} \end{bmatrix} . \tag{B17}$$

We can, using Eq. (B15), easily verify that the inverse operator is

$$[T^R_\Sigma(A_{\Sigma,k_0},B_{\Sigma,k_0},\check{Z}_{\Sigma,k_0})]^{-1} = J^S_{\Sigma\oplus\Sigma}[T^R_\Sigma(A_{\Sigma,k_0},B_{\Sigma,k_0},\check{Z}_{\Sigma,k_0})]^\tau (J^S_{\Sigma\oplus\Sigma})^{-1} \tag{B18}$$

$$= \begin{bmatrix} \check{Z}_{\Sigma,k_0}(A_{\Sigma,k_0}\check{Z}_{\Sigma,k_0}+B_{\Sigma,k_0})^{-1} & -(B_{\Sigma,k_0})^\tau \\ (A_{\Sigma,k_0}\check{Z}_{\Sigma,k_0}+B_{\Sigma,k_0})^{-1} & (A_{\Sigma,k_0})^\tau \end{bmatrix} ; \tag{B19}$$

moreover,

$$P^{R+}_\Sigma(A_{\Sigma,k_0},B_{\Sigma,k_0},\check{Z}_{\Sigma,k_0}) = [T^R_\Sigma(A_{\Sigma,k_0},B_{\Sigma,k_0},\check{Z}_{\Sigma,k_0})]^{-1} \times P^C_\Sigma T^R_\Sigma(A_{\Sigma,k_0},B_{\Sigma,k_0},\check{Z}_{\Sigma,k_0}) \ . \tag{B20}$$

We note that, even given the requirement that T^R_Σ is symplectic, it is not uniquely determined by the requirement that Eq. (B20) be satisfied, as left multiplication of T^R_Σ by a block-diagonal symplectic operator of the form

$$\Lambda_{\Sigma\oplus\Sigma}[Y_\Sigma] \equiv \begin{bmatrix} Y_\Sigma & 0^S_\Sigma \\ 0^S_\Sigma & (Y_\Sigma)^{-\tau} \end{bmatrix} , \tag{B21}$$

where Y_Σ is any invertible linear operator in $\mathcal{F}^\Sigma$, does not change the outcome Eq. (B20). In particular, we define

$$T^{R'}_\Sigma(A_{\Sigma,k_0},B_{\Sigma,k_0},\check{Z}_{\Sigma,k_0}) \equiv \Lambda_{\Sigma\oplus\Sigma}[(A_{\Sigma,k_0}\check{Z}_{\Sigma,k_0}+B_{\Sigma,k_0})^{-1}] \times T^R_\Sigma(A_{\Sigma,k_0},B_{\Sigma,k_0},\check{Z}_{\Sigma,k_0}) \ . \tag{B22}$$

The latter form of the symplectic transformation is independent of the choice of representatives $A_{\Sigma,k_0},B_{\Sigma,k_0}$ for the Robin boundary conditions Eq. (B8).

The symplectic transformations defined by Eqs. (B17) or (B22) have a certain group-theoretical property, depending on whether (i) the exterior environment (i.e., $\check{Z}_{\Sigma,k_0}$) is kept fixed and the interior (i.e., A_{Σ,k_0} and B_{Σ,k_0}) is permitted to vary, or (ii) the fluid occupying the exterior of Σ is permitted to vary in a way that is consistent with the existence of a varying $\check{Z}_{\Sigma,k_0}$ operator, while the acoustic response of the interior region is kept fixed. We note first that the operators $K_{\Sigma,r}(X_\Sigma)$ and $K_{\Sigma,l}(X_\Sigma)$, where

$$K_{\Sigma,r}(X_\Sigma) \equiv \begin{bmatrix} I^S_\Sigma & X_\Sigma \\ 0_\Sigma & I^S_\Sigma \end{bmatrix} , \tag{B23}$$

$$K_{\Sigma,l}(X_\Sigma) \equiv \begin{bmatrix} I^S_\Sigma & 0^S_\Sigma \\ X_\Sigma & I^S_\Sigma \end{bmatrix} , \tag{B24}$$

are each symplectic if X_Σ is any symmetric operator that maps $\mathcal{F}^\Sigma$ into itself, and each type belongs to an Abelian subgroup of the full group of symplectic transformations on $\mathcal{F}^{\Sigma\oplus\Sigma}$:

$$K_{\Sigma,a}(X_\Sigma)K_{\Sigma,a}(Y_\Sigma) = K_{\Sigma,a}(X_\Sigma+Y_\Sigma) , \tag{B25}$$

where $a=l$ or $a=r$. Now we drop the subscripts Σ,k_0 and accents on the arguments, and consider operators $T^{R'}_\Sigma(A,B,Z)$ and $T^R_\Sigma(A,B,Z)$. For case (i) we find, with the aid of some algebra, that

$$T^{R'}_\Sigma(A_1,B_1,Z)[T^{R'}_\Sigma(A_2,B_2,Z)]^{-1} = K_{\Sigma,r}[-(A_1Z+B_1)^{-1}A_1] \times K_{\Sigma,r}[(A_2Z+B_2)^{-1}A_2] \ . \tag{B26}$$

For case (ii), we find that

$$T^R_\Sigma(A,B,Z_1)[T^R_\Sigma(A,B,Z_2)]^{-1} = K_{\Sigma,l}[(Z_1A^\tau+B^\tau)^{-1}(Z_1-Z_2)(AZ_2+B)^{-1}] \ . \tag{B27}$$

The argument of the rhs of Eq. (B27) decomposes in a manner analogous to the rhs of Eq. (B26) if either A or B is invertible:

$$(Z_1A^\tau+B^\tau)^{-1}(Z_1-Z_2)(AZ_2+B)^{-1} = \begin{cases} -(AZ_1A^\tau+AB^\tau)^{-1}+(AZ_2A^\tau+BA^\tau)^{-1} & \text{if } A^{-1} \text{ exists} \\ (BA^\tau+BZ_1^{-1}B^\tau)^{-1}-(AB^\tau+BZ_2^{-1}B^\tau)^{-1} & \text{if } B^{-1} \text{ exists .} \end{cases} \tag{B28}$$

I have not been able to achieve an additive decomposition comparable to the rhs of Eq. (B26) in the general case that neither A nor B is invertible, always assuming that the current Z^{-1} and $(AZ+B)^{-1}$ exist, and that Z and AB^{τ} are symmetric.

The property Eq. (B26) means that the class of operators $T_{\Sigma}^{R\prime}(A,B,Z)$ for fixed Z all belong to a common right (sometimes called left) coset of a certain Abelian subgroup of the symplectic group on $\mathcal{F}^{\Sigma\oplus\Sigma}$. (We follow the definition of the right cosets in Ref. [35], pp. 60–61.) Similarly, Eq. (B27) implies that the class $T_{\Sigma}^{R}(A,B,Z)$ for fixed A and B belong to a common right coset of another Abelian subgroup of the symplectic group on $\mathcal{F}^{\Sigma\oplus\Sigma}$. These results can be understood partly on the grounds that an operator of the form $K_{\Sigma,r}(X)$ leaves the unit space of P_{Σ}^{C} invariant, while a $K_{\Sigma,l}(X)$ leaves the null space of P_{Σ}^{C} invariant, corresponding to holding fixed the acoustic properties of the exterior and interior region, respectively, to the surface Σ. These mathematical results may simplify applications of the theory when acoustic scattering in a family of geometrically invariable systems with variable interior acoustic properties is studied.

Now let us formulate the qualified notion of propagating the boundary values of a solution of the Helmholtz equation between surfaces. We consider two surfaces Σ_a and Σ_b in $\mathcal{E}^3$, such that Σ_a lies within Σ_b with a well-defined domain $\mathcal{D}_{ab}$ between them, and assume free-space propagation of sound waves everywhere. Let $G_{k_0}^{+}(\mathbf{r}_1;\mathbf{r}_2)$ be the Green's function of outgoing-wave type, which satisfies the differential equation

$$(\nabla_2^2+k_0^2)G_{k_0}^{+}(\mathbf{r}_1;\mathbf{r}_2)=\delta^3(\mathbf{r}_1-\mathbf{r}_2)\ , \tag{B29}$$

and has the analytic expression

$$G_{k_0}^{+}(\mathbf{r}_1;\mathbf{r}_2)=-(4\pi|\mathbf{r}_1-\mathbf{r}_2|)^{-1}\exp(ik_0|\mathbf{r}_1-\mathbf{r}_2|)\ . \tag{30}$$

Suppose that $\Upsilon^0(\mathbf{r})$ is regular within Σ_b and that $\Upsilon^{+}(\mathbf{r})$ is regular outside of Σ_a and is of outgoing-wave type. Then manipulations of Eqs. (B5), (B29), and (B30) along the lines of Ref. [24], following Eq. (7.2.6) on p. 805, and the use of the notation of Eq. (B1), lead to the following results:

$$\tilde{\Upsilon}_{\Sigma_a}^{0}(\mathbf{r}_{\Sigma_a})=\int_{\Sigma_b}\mathcal{P}_{\Sigma_a,\Sigma_b}^{F0}(\mathbf{r}_{\Sigma_a};\mathbf{r}_{\Sigma_b})\tilde{\Upsilon}_{\Sigma_b}^{0}(\mathbf{r}_{\Sigma_b})dA_{\Sigma_b}\ , \tag{B31}$$

$$\tilde{\Upsilon}_{\Sigma_b}^{+}(\mathbf{r}_{\Sigma_b})=\int_{\Sigma_a}\mathcal{P}_{\Sigma_b,\Sigma_a}^{F+}(\mathbf{r}_{\Sigma_b};\mathbf{r}_{\Sigma_a})\tilde{\Upsilon}_{\Sigma_a}^{+}(\mathbf{r}_{\Sigma_a})dA_{\Sigma_a}\ , \tag{B32}$$

where we have used the definitions

$$\mathcal{P}_{\Sigma_a,\Sigma_b}^{F0}(\mathbf{r}_{\Sigma_a};\mathbf{r}_{\Sigma_b})\equiv\begin{bmatrix}\dfrac{\partial G_{k_0}^{+}}{\partial n_{\Sigma_b}}(\mathbf{r}_{\Sigma_a};\mathbf{r}_{\Sigma_b}) & -G_{k_0}^{+}(\mathbf{r}_{\Sigma_a};\mathbf{r}_{\Sigma_b})\\ \dfrac{\partial^2 G_{k_0}^{+}}{\partial n_{\Sigma_a}\partial n_{\Sigma_b}}(\mathbf{r}_{\Sigma_a};\mathbf{r}_{\Sigma_b}) & -\dfrac{\partial G_{k_0}^{+}}{\partial n_{\Sigma_a}}(\mathbf{r}_{\Sigma_a};\mathbf{r}_{\Sigma_b})\end{bmatrix}, \tag{B33}$$

$$\mathcal{P}_{\Sigma_b,\Sigma_a}^{F+}(\mathbf{r}_{\Sigma_b};\mathbf{r}_{\Sigma_a})\equiv\begin{bmatrix}-\dfrac{\partial G_{k_0}^{+}}{\partial n_{\Sigma_a}}(\mathbf{r}_{\Sigma_b};\mathbf{r}_{\Sigma_a}) & G_{k_0}^{+}(\mathbf{r}_{\Sigma_b};\mathbf{r}_{\Sigma_a})\\ -\dfrac{\partial^2 G_{k_0}^{+}}{\partial n_{\Sigma_b}\partial n_{\Sigma_a}}(\mathbf{r}_{\Sigma_b};\mathbf{r}_{\Sigma_a}) & \dfrac{\partial G_{k_0}^{+}}{\partial n_{\Sigma_b}}(\mathbf{r}_{\Sigma_b};\mathbf{r}_{\Sigma_b})\end{bmatrix}. \tag{B34}$$

We call the entities

$$\mathcal{P}_{\Sigma_a,\Sigma_b}^{F0}(\mathbf{r}_{\Sigma_a};\mathbf{r}_{\Sigma_b}),\quad \mathcal{P}_{\Sigma_b,\Sigma_a}^{F+}(\mathbf{r}_{\Sigma_b};\mathbf{r}_{\Sigma_a}) \tag{B35}$$

the free-space inward and outward propagators for scalar waves, respectively. Since Eq. (B6) holds for arbitrary pairs of solutions of Eq. (B5), straightforward manipulations lead to the result

$$\mathcal{P}_{\Sigma_a,\Sigma_b}^{F0}=J_{\Sigma_a\oplus\Sigma_a}^{S}(\mathcal{P}_{\Sigma_b,\Sigma_a}^{F+})^{\tau}(J_{\Sigma_b\oplus\Sigma_b}^{S})^{-1}\ . \tag{B36}$$

We can infer from Ref. [2], Eqs. (18)–(21), that as the surface Σ_b shrinks down uniformly to coincide with Σ_a, the limiting behavior of the propagators is, referring to Eq. (B7),

$$\lim_{\Sigma_b\searrow\Sigma_a}\mathcal{P}_{\Sigma_b,\Sigma_a}^{F+}=P_{\Sigma_a}^{F+} \tag{B37}$$

and

$$\lim_{\Sigma_b\searrow\Sigma_a}\mathcal{P}_{\Sigma_a,\Sigma_b}^{F0}=I_{\Sigma_a\oplus\Sigma_a}^{S}-P_{\Sigma_a}^{F+}\ ,$$
$$=J_{\Sigma_a\oplus\Sigma_a}^{S}(P_{\Sigma_a}^{F+})^{\tau}(J_{\Sigma_a\oplus\Sigma_a}^{S})^{-1}\ , \tag{B38}$$

where the notation $\searrow$ means that the limit is taken as Σ_b maintains a given distance from Σ_a along the family of normals to Σ_a, which distance is then allowed to tend to zero uniformly over Σ_a. Therefore, Eq. (B36) generalizes Ref. [2], Eq. (44). Moreover, Eqs. (B36) and (B37) and the discussion following Eq. (B7) show that the propagator $\mathcal{P}_{\Sigma_a,\Sigma_b}^{F0}$ annihilates boundary functions $\tilde{\Upsilon}^{+}$ that belong to outgoing-wave solutions, and propagates the regular solutions of the Helmholtz equation only from an outer surface Σ_b to an interior surface, while the propagator $\mathcal{P}_{\Sigma_b,\Sigma_a}^{F+}$ annihilates boundary functions $\tilde{\Upsilon}^{0}$ that belong to solutions that are regular in the interior, and propagates outgoing-wave solutions correctly from an interior to an exterior surface. The method therefore achieves an incomplete solution of the Cauchy problem for the elliptic PDE Eq. (B5): the linear space of boundary values, of the type of Eq. (B1), is divided into two subspaces such that an explicit construction can be given for the propagation of each subspace in a stable (and mutually opposing) direction. The method in effect achieves a nonlocal decomposition of the elliptic PDE into two parabolic problems analogous to the heat equation, which two have oppositely directed "time" senses. (The latter notion can be made analytically explicit with Fourier transform techniques if the surfaces Σ_a and Σ_b are parallel planes.) The above construction yields zero, and hence zero information, for attempted propagation in the wrong, or unstable, direction for each subspace. Singularities, which can be construed as inhomogeneous terms on the rhs of Eq. (B5), can exist in the contrary direction for a given type of solution, and extrapolation of boundary values to the vicinity of such singularities is a difficult task, which is not encompassed by the methods established here.

We infer from Ref. [2], Eqs. (36), (34), and (32), that we can recover the free-space projector of Eq. (B7) from that of Eq. (B11) by a choice of arguments:

$$P_{\Sigma}^{F+}=P_{\Sigma}^{R+}(\tfrac{1}{2}W_{\Sigma,k_0},\tfrac{1}{2}(I_{\Sigma}^{S}-V_{\Sigma,k_0}^{\tau}),\check{Z}_{\Sigma,k_0}) . \quad \text{(B39)}$$

We can specialize the symplectic operator of Eq. (B17) correspondingly:

$$\begin{aligned} T_{\Sigma}^{F} &\equiv T_{\Sigma}^{R}(\tfrac{1}{2}W_{\Sigma,k_0},\tfrac{1}{2}(I_{\Sigma}^{S}-V_{\Sigma,k_0}^{\tau}),\check{Z}_{\Sigma,k_0}) \\ &= \begin{bmatrix} \tfrac{1}{2}W_{\Sigma,k_0} & \tfrac{1}{2}(I_{\Sigma}^{S}-V_{\Sigma,k_0}^{\tau}) \\ -I_{\Sigma}^{S} & \check{Z}_{\Sigma,k_0} \end{bmatrix} . \end{aligned} \quad \text{(B40)}$$

The symplectic operators for Σ_a and Σ_b applied to the propagators of Eqs. (B33) and (B34) afford a generalization of Eq. (B20). That is, there exists a k_0-dependent operator $Q_{\Sigma_b,\Sigma_a}^{F}$, which propagates one-component scalar functions from Σ_a to Σ_b, such that

$$\begin{aligned} T_{\Sigma_a}^{F}\mathcal{P}_{\Sigma_a,\Sigma_b}^{F0}(T_{\Sigma_b}^{F})^{-1} &= \begin{bmatrix} 0^S & 0^S \\ 0^S & (Q_{\Sigma_b,\Sigma_a}^{F})^{\tau} \end{bmatrix} , \\ T_{\Sigma_b}^{F}\mathcal{P}_{\Sigma_b,\Sigma_a}^{F+}(T_{\Sigma_a}^{S})^{-1} &= \begin{bmatrix} Q_{\Sigma_b,\Sigma_a}^{F} & 0^S \\ 0^S & 0^S \end{bmatrix} , \end{aligned} \quad \text{(B41)}$$

where 0^S is the zero propagator for one-component scalar functions. As Σ_b shrinks down to Σ_a, we must recover Eq. (B20) from Eq. (B41), that is,

$$\lim_{\Sigma_b \searrow \Sigma_a} Q_{\Sigma_b,\Sigma_a}^{F}=I_{\Sigma_a}^{S} . \quad \text{(B42)}$$

A computation shows that if Σ_a and Σ_b are concentric spheres of radius a and b, respectively, we have

$$\begin{aligned} Q_{\Sigma_b,\Sigma_a}^{F}(b\hat{\mathbf{r}}_b;a\hat{\mathbf{r}}_a) &= a^{-2}\sum_{l,m} Y_{l,m}(\hat{\mathbf{r}}_b)[Y_{l,m}(\hat{\mathbf{r}}_a)]^* h_l^{(1)\prime}(k_0 b) \\ &\quad \times [h_l^{(1)\prime}(k_0 a)]^{-1} , \end{aligned} \quad \text{(B43)}$$

where the notation for spherical harmonics and spherical Hankel functions follows that of Ref. [3], Appendix.

We are now in a position to deal with the problem set in the first paragraph of this appendix, the reformulation of a Robin boundary-value problem from an interior surface Σ_a to an exterior surface Σ_b, where the fluid medium occupying the domain between the two surfaces is here taken to be free space. Let $P_{\Sigma_a}^{R+}$ denote the projector, as in Eq. (B11), that simulates the scattering obstacle interior to Σ_a. We want to find a corresponding projector $P_{\Sigma_b}^{R+}$ that generates the same scattered wave from a given initial free-space wave as the former projector; then we shall be able, as far as the domain exterior to Σ_b is concerned, to simulate the original obstacle by Robin boundary conditions on Σ_b.

In the region exterior to Σ_a let $\Upsilon^R(\mathbf{r})$, $\Upsilon^0(\mathbf{r})$, and $\Upsilon^+(\mathbf{r})$, where

$$\Upsilon^R(\mathbf{r})=\Upsilon^0(\mathbf{r})+\Upsilon^+(\mathbf{r}) , \quad \text{(B44)}$$

be the total signal, impinging free-space signal, and scattered wave, respectively. Equations (B1), (B8), and (B9) imply that

$$P_{\Sigma_a}^{R+}\tilde{\Upsilon}_{\Sigma_a}^{R}=0 , \quad \text{(B45)}$$

$$P_{\Sigma_a}^{R+}\tilde{\Upsilon}_{\Sigma_a}^{+}=\tilde{\Upsilon}_{\Sigma_a}^{+} ; \quad \text{(B46)}$$

hence the inner boundary values of the scattered wave are

$$\tilde{\Upsilon}_{\Sigma_a}^{+}=-P_{\Sigma_a}^{R+}\tilde{\Upsilon}_{\Sigma_a}^{0} . \quad \text{(B47)}$$

Using the properties Eqs. (B31) and (B32) of the inward and outward propagators, we can easily infer from Eq. (B47) that

$$\tilde{\Upsilon}_{\Sigma_b}^{+}=-\mathcal{P}_{\Sigma_b,\Sigma_a}^{F+}P_{\Sigma_a}^{R+}\mathcal{P}_{\Sigma_a,\Sigma_b}^{F0}\tilde{\Upsilon}_{\Sigma_b}^{0} . \quad \text{(B48)}$$

We have now obtained the values of the scattered wave on Σ_b. Comparison of Eq. (B48) with Eq. (B47) suggests the following guess for the definition of the corresponding Robin projector Σ_b:

$$P_{\Sigma_b}^{R+}\equiv P_{\Sigma_b}^{F+}+\mathcal{P}_{\Sigma_b,\Sigma_a}^{F+}P_{\Sigma_a}^{R+}\mathcal{P}_{\Sigma_a,\Sigma_b}^{F0} , \quad \text{(B49)}$$

where the $P_{\Sigma_b}^{F+}$ is the free-space projector of Eq. (B7). Since

$$\begin{aligned} &\mathcal{P}_{\Sigma_a,\Sigma_b}^{F0}P_{\Sigma_b}^{F+}=0, \quad \mathcal{P}_{\Sigma_a,\Sigma_b}^{F0}\mathcal{P}_{\Sigma_b,\Sigma_a}^{F+}=0 , \\ &P_{\Sigma_b}^{F+}\mathcal{P}_{\Sigma_b,\Sigma_a}^{F+}=\mathcal{P}_{\Sigma_b,\Sigma_a}^{F+}, \quad \mathcal{P}_{\Sigma_b,\Sigma_a}^{F+}P_{\Sigma_a}^{F+}=\mathcal{P}_{\Sigma_b,\Sigma_a}^{F+} , \end{aligned} \quad \text{(B50)}$$

the rhs of Eq. (B49) is idempotent, and acts as the unit operator on any outgoing free-space wave. We need to show that $P_{\Sigma_a}^{R+}$ annihilates $\tilde{\Upsilon}_{\Sigma_b}^{R}$:

$$\begin{aligned} P_{\Sigma_b}^{R+}\tilde{\Upsilon}_{\Sigma_b}^{R} &= (P_{\Sigma_b}^{F+}+\mathcal{P}_{\Sigma_b,\Sigma_a}^{F+}P_{\Sigma_a}^{R+}\mathcal{P}_{\Sigma_a,\Sigma_b}^{F0})(\tilde{\Upsilon}_{\Sigma_b}^{0}+\tilde{\Upsilon}_{\Sigma_b}^{+}) \\ &= \tilde{\Upsilon}_{\Sigma_b}^{+}+\mathcal{P}_{\Sigma_b,\Sigma_a}^{F+}P_{\Sigma_a}^{R+}\tilde{\Upsilon}_{\Sigma_a}^{0} \\ &= \tilde{\Upsilon}_{\Sigma_b}^{+}-\mathcal{P}_{\Sigma_b,\Sigma_a}^{F+}\tilde{\Upsilon}_{\Sigma_a}^{+} \\ &= \tilde{\Upsilon}_{\Sigma_b}^{+}-\tilde{\Upsilon}_{\Sigma_b}^{+}=0 , \end{aligned} \quad \text{(B51)}$$

where we used Eqs. (B31), (B32), and (B47).

In deriving Eq. (B49), we made use of the circumstance that Eq. (B32) yields the values $\tilde{\Upsilon}_{\Sigma_b}^{+}$ from the values $\tilde{\Upsilon}_{\Sigma_a}^{+}$ of the scattered wave. It follows that these methods do not afford a contribution to the inverse scattering problem, that is, do not permit a set of Robin boundary conditions on an outer surface to be replaced by an equivalent (in an obvious sense) set of Robin boundary conditions on an interior surface.

Let us examine the result Eq. (B49) more closely when it is applied to a common set of problems. That is, suppose that Σ_a is an approximately planar surface, as a slightly rough surface or an acoustic diffraction grating.

We suppose further that on Σ_a either homogeneous Neumann (N) boundary conditions (that is, $A_{\Sigma_a,k_0}=0^S_{\Sigma_a}$ and $B_{\Sigma_a,k_0}=I^S_{\Sigma_a}$) or homogeneous Dirichlet (D) boundary conditions (that is, $A_{\Sigma_a,k_0}=I^S_{\Sigma_a}$ and $B_{\Sigma_a,k_0}=0^S_{\Sigma_a}$) are satisfied in Eq. (B8). We want to obtain the surface values and derivatives $\tilde{\Upsilon}^+_{\Sigma_b}$ of the scattered wave on a planar surface Σ_b that lies entirely above Σ_a; we consider Σ_b to be the plane of observation. We note that the corresponding projection operators on Σ_a are

$$P^{N+}_{\Sigma_a}=\begin{bmatrix} 0^S_{\Sigma_a} & \check{Z}_{\Sigma_a,k_0} \\ 0^S_{\Sigma_a} & I^S_{\Sigma_a} \end{bmatrix}, \tag{B52}$$

$$P^{D+}_{\Sigma_a}=\begin{bmatrix} I^S_{\Sigma_a} & 0^S_{\Sigma_a} \\ (\check{Z}_{\Sigma_a,k_0})^{-1} & 0^S_{\Sigma_a} \end{bmatrix}. \tag{B53}$$

Moreover, we infer from Eq. (B7) and Ref. [2], Eqs. (32) and (34), that these can be reexpressed as follows:

$$P^{N+}_{\Sigma_a}=P^{F+}_{\Sigma_a}\begin{bmatrix} 0^S_{\Sigma_a} & 0^S_{\Sigma_a} \\ 0^S_{\Sigma_a} & 2(I^S_{\Sigma_a}-V^{\tau}_{\Sigma_a,k_0})^{-1} \end{bmatrix}, \tag{B54}$$

$$P^{D+}_{\Sigma_a}=P^{F+}_{\Sigma_a}\begin{bmatrix} 2(I^S_{\Sigma_a}+V_{\Sigma_a,k_0})^{-1} & 0^S_{\Sigma_a} \\ 0^S_{\Sigma_a} & 0^S_{\Sigma_a} \end{bmatrix}. \tag{B55}$$

Let $\Upsilon^0(\mathbf{r})$ be the impinging wave; then $P^{F+}_{\Sigma_b}$ annihilates $\tilde{\Upsilon}^0_{\Sigma_b}$ and $\mathcal{P}^{F0}_{\Sigma_a,\Sigma_b}$ transforms it to $\tilde{\Upsilon}^0_{\Sigma_a}$, so that the rhs of Eq. (B48) reduces to $-\mathcal{P}^{F+}_{\Sigma_b,\Sigma_a}P^{X+}_{\Sigma_a}\tilde{\Upsilon}^0_{\Sigma_a}$, where $X=N$ or $X=D$. Since Σ_a is approximately planar the operators $V^{\tau}_{\Sigma_a,k_0}$, V_{Σ_a,k_0} are small, as argued in Ref. [3], Sec. II, so that the derived formulas are well adapted to the use of perturbation theory. Note that the conventional physical optics approximation entails taking the source density on $\partial\Omega$ for the scattered field to be

$$\begin{bmatrix} 2\Upsilon^0(\mathbf{r}_{\Sigma_a}) \\ 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 \\ 2\dfrac{\partial\Upsilon^0}{\partial n_{\Sigma_a}} \end{bmatrix} \tag{B56}$$

for the N case and the D case, respectively, while Eqs. (B54) and (B55) entail taking the corresponding densities to be

$$\begin{bmatrix} 0 \\ -2\dfrac{\partial\Upsilon^0}{\partial n_{\Sigma_a}} \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} -2\Upsilon^0(\mathbf{r}_{\Sigma_a}) \\ 0 \end{bmatrix}. \tag{B57}$$

Inasmuch as the differences of the alternative sets of boundary values are, both in the N case and in the D case, the boundary values of a regular free-space solution (and hence the difference functions are annihilated by $P^{F+}_{\Sigma_a}$), either approximate choice of boundary values will yield the same scattered wave in corresponding cases, apart from numerical errors in a practical computation. The present method has therefore achieved an expression for obtaining perturbative corrections to the physical optics approximation, which approach is sometimes used to treat the scattering of waves from rough surfaces—see, for example, Ref. [36], and references cited therein. If a nonperturbative approach is required, one can have recourse to Eqs. (B54) and (B55) if the surface is unbounded in a way that there are no cavity eigenfrequencies, or to Eqs. (B52) and (B53) for general geometries.

An item of unfinished business in the theory presented thus far is the question of how the elaboration of the theory of the Helmholtz equation described in this appendix can be viewed as leading, in the extreme short-wavelength limit, to the Hamiltonian version of geometrical optics presented in Refs. [17] and [18]. Evidently an analog to the Schrödinger equation, which is known to lead to classical Hamiltonian mechanics in the short-(deBroglie) wavelength limit, should be inherent in the formalism. The following considerations suggest that such an analog exists.

Let us imagine a family of nonintersecting surfaces $\Sigma(a)$ that depend on a real parameter a, such that the surfaces $\Sigma(a+\delta a)$ and $\Sigma(a)$ have a normal separation that is linear in δa, and that is uniformly bounded above and below, as $\delta a\to 0$. It should, therefore, be meaningful to compute derivatives of wave functions and operators (such as the propagation operators) with respect to the upper or lower limits. We restrict ourselves to outgoing-wave solutions to Eq. (B5). If Υ^{1+} and Υ^{2+} are two such solutions, we know that their Wronskian is independent of the choice of surface in the family, and in fact is zero, according to the remarks following Eq. (B6). A slight modification of this computation leads to a useful result: We note that the complex conjugate $(\Upsilon^+)^*$ of an outgoing-wave solution of Eq. (B5) is also a solution, and in fact is an ingoing-wave solution. Moreover, as the examples of plane and spherical waves show, the Wronskian of such a solution with its complex conjugate is nonzero. Let us adopt the following definition, therefore, for the sesquilinear inner product of two outgoing-wave solutions $\Upsilon^{1+}(\mathbf{r})$ and $\Upsilon^{2+}(\mathbf{r})$ with respect to a surface $\Sigma(a)$:

$$\langle\Upsilon^{1+}|\rho_{\Sigma(a)}|\Upsilon^{2+}\rangle\equiv-(ik_0/2)W_{\Sigma(a)}([\Upsilon^{1+}]^*;\Upsilon^{2+}). \tag{B58}$$

The inner product is independent of a, as noted, and is defined in Dirac notation as the matrix element of a certain "weighting" operator $\rho_{\Sigma(a)}$. Let us evaluate the rhs of Eq. (B58) by Eq. (B9). We find the result

$$\langle\Upsilon^{1+}|\rho_{\Sigma(a)}|\Upsilon^{2+}\rangle=\int_{\Sigma(a)}dA_{\Sigma(a),1}\int_{\Sigma(a)}dA_{\Sigma(a),2}\left[\left[\frac{\partial\Upsilon^{1+}}{\partial n_{\Sigma(a)}}(\mathbf{r}_{\Sigma(a),1})\right]^*\left[\frac{ik_0}{2}\right]\right.$$
$$\left.\times[\check{Z}^+_{k_0}-(\check{Z}^+_{k_0})^\dagger](\mathbf{r}_{\Sigma(a),1};\mathbf{r}_{\Sigma(a),2})\frac{\partial\Upsilon^{2+}}{\partial n_{\Sigma(a)}}(\mathbf{r}_{\Sigma(a),2})\right]. \tag{B59}$$

It follows from Ref. [2], Appendix, that the Hermitian operator

$$(ik_0/2)[\check{Z}_{k_0}^{+}-(\check{Z}_{k_0}^{+})^{\dagger}] \tag{B60}$$

is positive definite for $k_0 \neq 0$, a result connected physically with the positive outward acoustic energy flux associated with any nonzero outgoing wave. Moreover, a computation shows that our definition of the inner product entails

$$\langle \Upsilon^{2+}|\rho_{\Sigma(a)}|\Upsilon^{1+}\rangle=[\langle \Upsilon^{1+}|\rho_{\Sigma(a)}|\Upsilon^{2+}\rangle]^* \ . \tag{B61}$$

The mathematical system formed by adjoining the inner product Eq. (B58) to the linear space of outgoing-wave solutions to the Helmholtz equation therefore satisfies the axioms of a so-called pre-Hilbert space—see, for example, Ref. [26], Chap. II. (A pre-Hilbert space is a Hilbert space if it is complete, i.e., every Cauchy sequence within the space converges to a limit in the space.) We have therefore defined a conserved (with respect to the parameter a) sesquilinear inner product that is similar to a quantum-mechanical transition amplitude in Schrödinger theory, where time is the parameter. This result suggests that it may be possible to establish an extended analogy of the wave optics of outgoing waves with nonrelativistic quantum mechanics, such that position and momentum operators, etc., can be defined in favorable geometries. In turn, the short-wavelength limit of this hypothetical theory will plausibly yield the Hamiltonian form of geometrical optics. This possibility remains to be investigated both in the acoustic case and in the similar electromagnetic case, as remarked in Sec. VII G.

We note finally four bodies of published work on acoustic, electromagnetic, or quantum scattering theory that were overlooked or incompletely cited in Refs. [1]–[3], and that have common elements with the scattering theory proposed in Refs. [1]–[3] and elaborated in the present work. The first group of papers are those concerned with the so-called delta boundary operator (DBO) method—see Refs. [37]–[42]. The method was first proposed for treating the scattering of electromagnetic waves impinging at right angles to perfectly conducting cylindrical obstacles, such that the component of the magnetic-field vector parallel to the cylinder axis is zero—this state of affairs is not changed in the scattering from a perfect conductor, and the wave is called a transverse magnetic (TM) wave in waveguide nomenclature (some authors, as in Ref. [37], call it a TE wave). The problem reduces to a D-type boundary-value problem for the two-dimensional Helmholtz equation, and hence is appropriately discussed here rather than in Sec. VII. We call the cylinder C, which is the surface of the conductor, let z be the axial direction, let $\boldsymbol{\rho}=(x,y)$ be the position vector in an orthogonal plane section, and let s be an arc-length parameter along a section of C, measured from any fixed initial point on C, such that $\boldsymbol{\rho}_C(s)$ is the parametrized position of the cylinder. Maystre [37] considers an outgoing-wave solution $U^{+}(\boldsymbol{\rho},s_0)$ to the Helmholtz equation (we have changed Maystre's notation in part)

$$(\nabla_\rho^2+k_0^2)U^{+}(\boldsymbol{\rho},s_0)=0 \ , \tag{B62}$$

such that

$$\lim_{\rho\to\rho_C(s)} U^{+}(\boldsymbol{\rho},s_0)=-\delta(s-s_0) \ , \tag{B63}$$

where s_0 is the position of a δ source on the boundary. The limiting normal derivative of this solution is the DBO

$$\psi_C(s,s_0)\equiv\frac{\partial U^{+}}{\partial n_C(s)}(\boldsymbol{\rho}_C(s),s_0) \ , \tag{B64}$$

where $\hat{\mathbf{n}}_C(s)$ is the outward normal to C at s. Evidently the DBO $\psi_C(s,s_0)$ is the two-dimensional analog of minus the inverse radiation impedance operator, that is, of $-(\check{Z}_{k_0})^{-1}$, for the surface C. Analytical expressions for $\psi_C(s,s_0)$ have been obtained [cf. Ref. [37], Eqs. (29) and (42)] for C a plane or circular cylinder, which are the two-dimensional analogs of Ref. [3], Eqs. (54) and (A2), respectively. Different approaches to treating the singularity in the DBO are proposed in Refs. [37] and [3]. An empirical fit to the difference between the DBO for a plane and that for a circular cylinder was presented in Ref. [37], Eq. (89). The DBO method was extended to enable the treatment of scattering from dielectrics and imperfect conductors with cylindrical boundaries, and of TE as well as TM impinging waves, in Ref. [41]. Approximation schemes that are based on the results for planes and circular cylinders were proposed for the DBO for more general cylinders, including rough surfaces and gratings, and numerical studies were carried out, with generally favorable results, in Refs. [38–40,42].

The second group of papers is concerned with the numerical treatment of the propagation of pulsed signals in seismological work, for which either the acoustic or elastic wave equation are the modeling theories. We cite three papers, that is, Refs. [43]–[45], among a body of related work: Ref. [43] deals with the two-dimensional problem of the scattering of sound waves from a right-angled wedge, and Refs. [44] and [45] are concerned with establishing nonreflecting boundary conditions for the scalar wave equation on the edge of a grid, in Ref. [44] by a condition that in effect approximates the time-domain radiation impedance operator, and by an apparently completely different method in Ref. [45].

The third group of papers comprises those that are concerned with the electromagnetic version of Waterman's method, also called the T-matrix method, the null-field method, or the extended boundary-condition method. The relationship between Waterman's method and the transition operator approach was discussed for the case of acoustic-wave scattering in Ref. [2], Sec. III D. Waterman also proposed [46,47], and others have further discussed or applied—cf. Refs. [48–51] or the citation in Ref. [7], Note 4.1.2—a method of treating medium-to-long-wavelength electromagnetic-wave scattering from nonspherical obstacles based on truncated expansions, as Eqs. (34) and (35), of the electromagnetic field in terms of the multipole fields. Waterman's method can be obtained in the case of electromagnetic-wave scattering from a perfectly conducting and approximately spherically shaped obstacle in a manner analogous to the treatment of the acoustic case presented in Ref. [4], Theorem 3.45. That is, in Eq. (134) let $M'=M$ be empty space, let Ψ_n^{M+}, $n=1,2,\ldots,N$, be a suitable set of $N<\infty$ outgoing-wave multipole electromagnetic fields [as in Eqs. (31)–(33), with the origin inside the obstacle], and let $\Phi^{M'+}$ be the scattered wave, of which the tangential electric component on the surface is known and the tangential magnetic component is to be determined. As in Ref. [4], Eqs. (3.92) and (3.93), Eq. (134) now establishes a moment problem for the unknown tangential magnetic field of the scattered wave on the surface of the obstacle. The numerical solution of this moment problem for a sufficiently large class of electric fields on the surface amounts to the determination of an approximate inverse radiation impedance operator—cf. Eqs. (4) and (5). We note also that Waterman [52] proposed a third version of the method for the scattering of elastic waves from nonuniformities in solids.

The fourth group of papers concerns the so-called $\mathcal{R}$-matrix method, the founding of which is due to Wigner and Eisenbud and Duke [53–55], and which has been used extensively in the theory of nuclear reactions [56–58] and other applications of quantum-mechanical collision theory [59–65]. In the present context, the $\mathcal{R}$-matrix method could be realized along the following lines. The obstacle, presumed bounded and penetrable, is enclosed by a $\partial\Omega$ that is a sphere $S^2(a)$ of radius a, such that the exterior region is free space. Wave functions for all r, and in particular for $r=a$, are expanded in spherical harmonics (Helmholtz case) or vector spherical harmonics (Maxwell case), so that the linear operators defined elsewhere herein, which map fields defined on $S^2(a)$ into other such fields, are now realized as matrices, albeit of countably infinite dimensionality. The $\mathcal{R}$ matrix is defined as the matrix of the operator that, in the Helmholtz case, maps the normal derivatives on $S^2(a)$ of the regular interior wave functions into the corresponding wave-function values on $S^2(a)$; in the Maxwell case, an analogous entity could be defined (with $k_0\neq 0$) as the matrix of the operator that maps the tangential magnetic field into the tangential electric field, where both tangential fields are derived from regular interior solutions to Maxwell's equations. Hence, the $\mathcal{R}$ matrix for the regular interior solutions is a counterpart to the radiation impedance operator for the exterior solutions of outgoing-wave type; unlike the latter entity, the $\mathcal{R}$ matrix can be singular. In the operator notation of Eqs. (B8) and (122), we have

$$\mathcal{R}=\begin{cases} -(A_{S^2(a),k_0})^{-1}B_{S^2(a),k_0} & \text{(Helmholtz case)} \\ -(\breve{A}^{\Xi}_{S^2(a)})^{-1}\breve{C}^{\Xi}_{S^2(a)} & \text{(Maxwell case)} .\end{cases} \tag{B65}$$

Once the $\mathcal{R}$ matrix is determined, the S matrix can be determined by a solution matching across $S^2(a)$. Although the $\mathcal{R}$ matrix, taken as a function of k_0, has a singularity whenever the null space of the operator $A_{S^2(a),k_0}$ (or $\breve{A}^{\Xi}_{S^2(a)}$) is nontrivial, the derived S matrix is well behaved.

The standard $\mathcal{R}$-matrix method is equipped with an approach to solving the interior problem in a manner that yields an estimate for the $\mathcal{R}$ matrix: Corresponding to the conservation of probability and the unitarity of the S matrix in quantum scattering processes, we presume that the scattering medium is lossless, and that a convenient set of artificial homogeneous boundary conditions can be chosen on $S^2(a)$ such that a self-adjoint (in the Hermitian sense) eigenfunction or eigenvalue problem is established in the interior of $S^2(a)$. Remarkably, a complete set of eigenfunctions and eigenvalues for this problem yields a formula for the $\mathcal{R}$ matrix, despite the fact that the individual eigenfunctions by construction satisfy a different set of boundary conditions on $S^2(a)$. This method was originally established to treat the cases of—in general, many—scattering resonances that occur in nuclear reactions as the energy (frequency) is varied, and a suitable variant may find applications to classical wave scattering where analogous resonance phenomena occur, as scattering from an impenetrable obstacle with a cavity, as described in Sec. VIII.

As advocated by Breit (Ref. [57], Sec. 29), the essential character of the $\mathcal{R}$-matrix method derives not from a particular scheme chosen to treat the interior problem, but from its property as a representation of the action of the interior region in determining the exterior properties of the wave function. We can take this interpretation to comprehend the possibility of lossy structures in the interior region, which are more characteristic of classical than of quantum-mechanical scattering problems. In this sense the impedance boundary-condition approach for simulating an obstacle that was introduced in Refs. [1,2] is an alternative version of $\mathcal{R}$-matrix theory, restricted in the number of dimensions of the position space to three or fewer, but generalized in the respect of the geometry of the surface that divides the interior and exterior regions.

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