

Extreme fractal dimension at periodicity cascades in parameter spacesCarlos E. P. Abreu 

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In the parameter spaces of nonlinear dynamical systems, we investigate the boundaries between periodicity and chaos and unveil the existence of fractal sets characterized by a singular fractal dimension that deviates greatly from the fractal sets in their vicinity. This extreme fractal dimension stands out from the typical value previously considered universal for these parameter boundaries. We show that such singular fractal sets dwell along parameter curves, called extreme curves, that intersect periodicity cascades at their centers of stability across all scales of parameter spaces. The results reported here are generally demonstrated for the class of one-dimensional maps with at least two control parameters. Generalizations to other classes of systems are possible.

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Introduction. In nonlinear systems, order and chaos are two profoundly contrasting dynamics, yet they often intricately intertwine within the system's parameter spaces. Although the domains of parameter sets corresponding to chaotic attractors are generally not continuous, they are dense enough to present a positive Lebesgue measure [1]. Arbitrarily close to these sets there are continuous periodic windows with stable periodic behavior [2]. Consequently, the parameters that lead to chaotic attractors form *fat fractal* sets, and chaotic dynamics can be replaced by stable periodic behavior through an arbitrarily small variation of the system parameters [3]. Moreover, since periodic windows occur across various scales of the parameter space, they give rise to periodicity *cascades*, an infinite set of self-similar periodic windows, densely distributed in the

parameter space. Along cascades, the distribution of periodic windows can be governed by scaling rules related to their size [4], period [5], and other topological measures associated with their periodic orbits [6,7].

In the literature, considerable efforts have been dedicated to characterizing the complexity arising from the self-similarity of periodic windows within the context of one-dimensional maps, where only a single bifurcation parameter is available. Grebogi *et al.* developed an approach to determine the “exterior dimension” [8] of these sets through an estimation of the so-called “uncertainty exponent” [9]. They estimated this exponent for the periodic windows of the quadratic map, finding it to be $\alpha = 0.413(5)$. Additionally, Farmer proposed an alternative approach to also characterize the intertwined structure of periodicity and chaos, finding scaling exponents to be $\beta = 0.45(4)$ for the quadratic and sine maps [10]. He conjectured that this exponent could be

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universal among one-dimensional maps up to a certain order of their maxima. Subsequently, Hunt *et al.* [3] theoretically and numerically estimated the uncertainty exponent solely for “large” chaotic attractors of the quadratic map, i.e., excluding the “small” chaotic attractor appearing via the Feigenbaum scenario. This approach yields a different value for the uncertainty exponent, $\gamma = 0.51(3)$. Such a discrepancy between γ and α has been addressed by Joglekar *et al.*, who demonstrated a relationship between these exponents and conjectured that both values are universal for one-dimensional maps with a quadratic maximum [11].

In planar parameter spaces, where two bifurcation parameters are available, cascades of self-similar periodic windows can manifest in two main ways: (i) aligned towards specific directions in two-parameter spaces, giving rise to periodicity cascades that accumulate in a parameter region corresponding to periodic behavior [7,12,13], and (ii) periodicity hubs, characterized by infinitely many spiral-shaped sequences emanating from a single point in the parameter space, corresponding to a homoclinic bifurcation [14–16]. Such a nontrivial organization of periodic windows has been observed in two-parameter spaces of several classes of dynamical systems, both in computational [17–25] and in laboratory experiments [26–28]. In this context, Medeiros *et al.* estimated the uncertainty exponent α for three different continuous-time systems exhibiting periodicity cascades in their planar parameter spaces [29]. They found that the values of α for all systems fell within the interval $\alpha = 0.40(4)$, supporting the conjectured universality of this exponent even in another class of dynamical systems [29].

Here, we present evidence challenging the longstanding belief in this universality. To this end, we consider the class of one-dimensional maps generically governed by the following equation,

$$x_{n+1} = f(x_n, \mathbf{a}), \quad (1)$$

where f is sufficiently smooth and at least bimodal, the variable $x \in \mathbb{R}$ represents the states of the system, and the vector $\mathbf{a} \in \mathbb{R}^N$ accounts for the $N \geq 2$ control parameters. In these high-dimensional parameter spaces, we first specify parameter curves, called extreme curves, that intersect periodicity cascades at their centers of stability. Subsequently, we estimate the uncertainty exponents along the extreme curves and obtain $\alpha \sim 0.23$. Interestingly, we find that the transition from these values to those previously reported in the literature occurs abruptly in the vicinity of the extreme curves. Moreover, by considering the uncertainty exponent as an approximation of the exterior codimension, we obtain the singular fractal dimension of the parameter sets residing at the boundaries of periodicity cascades and chaos, which we term the extreme fractal dimension.

Extreme curves. Since our proposed extreme fractal sets exist in the multidimensional parameter space of mappings given by Eq. (1), we now specify the location of such fractal objects. In general, starting from an initial condition x_0 , the successive iterates of Eq. (1) produce trajectories that, depending on the parameters \mathbf{a} , eventually approach an asymptotic solution such as a fixed point, a periodic orbit, or a chaotic attractor. However, even before converging, such trajectories may include sequences of critical points

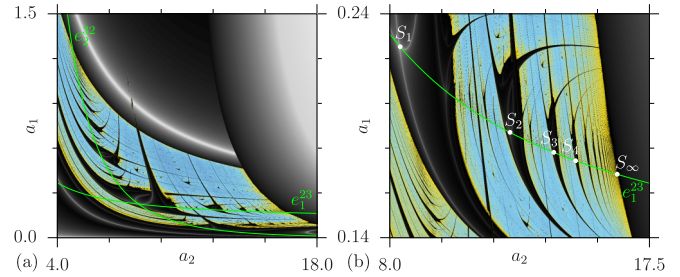


FIG. 1. (a) Periodicity cascade in the parameter plane Γ along the green extreme curves e_1^{23} and e_2^{32} . Dark shading represents periodic dynamics ($\lambda < 0$), while yellow-blue shadings indicate chaotic behavior ($\lambda > 0$). (b) Emphasis on e_1^{23} crossing the cascade through all superstability locus, sequence $(S_i)_{i=1}^\infty$.

$\{x_1^*, x_2^*, \dots, x_i^*, \dots\}$, each satisfying $f'(x_i^*) = 0$, which are associated to extreme points (local maxima or minima) of the mapping in Eq. (1). Naturally, any pair of critical points x_i^* and x_j^* may be connected by k successive iterates of the mapping, i.e., $x_j^* = f^k(x_i^*)$, where $k \in \mathbb{N}$, f^0 is the identity, and $f \equiv f^1$. Such trajectories of length k connecting critical points in the state space of the mapping are referred to as k -extreme orbits and their corresponding parameters $e_k^{ij} = \{\mathbf{a} \in \mathbb{R}^N | x_j^* = f^k(x_i^*, \mathbf{a})\}$ constitute codimensional one sets known as extreme curves in planar parameter spaces [30]. The extreme curves host the extreme and singular fractal sets of interest in this Letter.

Now, we consider an explicit one-dimensional mapping with two or more control parameters to demonstrate the existence of extreme curves in its parameter space. For this purpose, we use the logistic-Gauss map, defined by the following function [31],

$$f(x, a_1, a_2, a_3) = \exp\{-a_1[x(1 - x^{a_2})]^2\} + a_3, \quad (2)$$

with $a_2 \in \mathbb{R}_+^*$. To investigate the fractal dimension of sets along the extreme curves e_k^{ij} , we define a two-dimensional cross section $\Gamma = \mathbb{R} \times \mathbb{R}_+^*$, by keeping $a_3 = 0.1$ constant. Hence, from Eq. (2), we first obtain the sequence of critical points:

$$x_1^* = 0, \quad x_2^* = (1 + a_2)^{-\frac{1}{a_2}}, \quad \text{and} \quad x_3^* = 1. \quad (3)$$

If a_2 is a rational number with an even numerator, two additional critical points emerge, $x_4 = -x_2$ and $x_5 = -x_3$. Next, by solving the equation $x_3^* = f(x_2^*, \mathbf{a})$, we obtain the extreme curve e_1^{23} :

$$a_1 = -\frac{\ln(0.9)(a_2 + 1)^{2+2/a_2}}{a_2^2}. \quad (4)$$

Similarly, we obtain the function of e_2^{32} :

$$a_1 = -\frac{\ln\left[(1 + a_2)^{-\frac{1}{a_2}} - 0.1\right]}{1.21(1 - 1.1a_2)^2}. \quad (5)$$

To associate the extreme curves with the periodicity cascades occurring in Γ , we estimate the Lyapunov exponent of the mapping defined in Eq. (2) for each parameter pair (a_1, a_2) using $\lambda = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \ln |f'(x_i)|$. In Fig. 1, we display Γ with a color code representing the amplitude of λ . Darker

shadings indicate periodic ($\lambda < 0$), while yellow-blue shadings indicate chaotic dynamics ($\lambda > 0$). The extreme curves e_1^{23} and e_2^{32} are traced in green in Fig. 1(a). In Fig. 1(b), we highlight a cascade of periodic windows relative to e_1^{23} . Note that this curve crosses over all periodic windows intersecting points corresponding to the *superstability locus*, where two white superstable curves are crossing [32,33]. Notably, if a stable periodic orbit contains extreme orbits in its extension, it automatically fits the criteria for a superstability locus. These extreme and superstable orbits¹ populate the superstability locus $S_1, S_2, \dots, S_\infty$ along the extreme curve e_1^{23} throughout this sequence of periodic windows. Moreover, the curve e_1^{23} also intersects the superstability locus of periodic windows, forming smaller sequences that accumulate at each periodic window. The infinitely many sequences of periodic windows, occurring at all scales of cross section Γ and traversed by the curve e_1^{23} , constitute the periodicity cascade. The same applies to e_2^{32} .

Exterior dimension. The self-similar organization of the periodicity cascades visualized in Fig. 1 suggests the presence of fractal sets embedded in the parameter cross section Γ . However, since the parameter sets leading to either chaotic or periodic behavior possess nonzero volume, their dimension is the same as the Euclidean dimension of the embedding space, i.e., an integer. To characterize the scaling of the cascades, different approaches have been proposed in the literature [3,8,10]. We focus on the framework of *exterior dimension* presented in Ref. [8], in which the scaling observed in Fig. 1 is attributed to the fractal geometry of the boundaries between the parameters sets leading to periodic or chaotic behavior. In our discussion, this fractality reflects the exterior dimension of fat fractal sets, formed by the parameters leading to chaotic behavior along specific curves. To introduce this concept more clearly, consider a region \mathcal{B} of the parameter set associated with chaotic behavior and extend this region by an arbitrarily small amount ε . Call this extended region as $\mathcal{B}(\varepsilon)$. The exterior dimension d_x is given by [8]

$$d_x = D - \lim_{\varepsilon \rightarrow 0} \frac{\ln V[\mathcal{B}(\varepsilon) - \mathcal{B}]}{\ln \varepsilon}, \quad (6)$$

where D is the unitary dimension of the curves and $V[\mathcal{B}(\varepsilon) - \mathcal{B}]$ is the remaining (*exterior*) volume excluding the volume of the original set \mathcal{B} . In accordance with Ref. [8], $V[\mathcal{B}(\varepsilon) - \mathcal{B}]$ can be estimated by computing the fraction of parameters $f(\varepsilon)$ that are uncertain over ε -size perturbations along specific curves of the cross section Γ . More specifically, we first choose a point $\varphi = (a_1, a_2)$ at random on the extreme curve of interest [Eqs. (4) or (5)] and perturb φ by considering an distance ε in both directions of the curve to obtain the points φ_- and φ_+ . Subsequently, we evaluate the Lyapunov exponent λ for each parameter value φ_- , φ , and φ_+ . If in these three parameter we identify chaotic ($\lambda > 0$) and periodic ($\lambda < 0$) attractors we record the central parameter φ as “uncertain” for the particular value of the perturbation ε . Otherwise, φ is said to be “certain” and is disregarded. We repeat this procedure

¹By superstability, we mean the parameter set in which stable periodic orbits exhibit a faster attraction to nearby trajectories, i.e., when x_i^* is part of a periodic orbit.

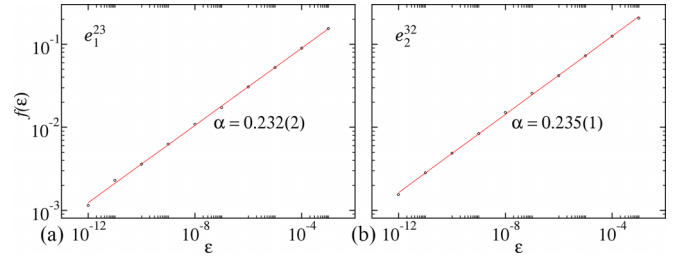


FIG. 2. Fraction of uncertain parameters $f(\varepsilon)$ as a function of the perturbation ε . A power-law regression $f(\varepsilon) = A\varepsilon^\alpha$ (red curve) provides the uncertainty exponent α along the extreme curve e_1^{23} (a) and e_2^{32} (b).

for a large number of φ values and calculate the fraction of uncertain parameters $f(\varepsilon)$ for each perturbation ε in the interval $10^{-12} \leq \varepsilon \leq 10^{-3}$. The fraction $f(\varepsilon)$ is known to depend on the perturbation ε as a power law, $f(\varepsilon) \sim \varepsilon^\alpha$, which the exponent α is simply $\alpha = \lim_{\varepsilon \rightarrow 0} \frac{\ln f(\varepsilon)}{\ln \varepsilon}$. Therefore, since $f(\varepsilon)$ approaches to $V[\mathcal{B}(\varepsilon) - \mathcal{B}]$ while $\varepsilon \rightarrow 0$, the exterior dimension d_x in Eq. (6) is determined as

$$d_x = D - \alpha, \quad (7)$$

where α is the uncertainty exponent, also known as the *exterior codimension*, that can be estimated from log-log plots of $f(\varepsilon)$ as a function of ε [8].

Extreme fractal dimension. Now, we demonstrate the existence of singular fractal sets along periodicity cascades in the parameter space of the map given by Eq. (2). To achieve this, we start by estimating the uncertainty exponent α of periodicity cascades along the extreme curves e_1^{23} and e_2^{32} . In Fig. 2, we present the fraction of uncertain parameters $f(\varepsilon)$ as a function of ε for parameters along the curves e_1^{23} [Fig. 2(a)] and e_2^{32} [Fig. 2(b)]. By fitting a power-law function to the data shown in these figures, we find $\alpha = 0.232(2)$ and $\alpha = 0.235(1)$ for the extreme curves e_1^{23} and e_2^{32} , respectively. These values of the uncertainty exponent are significantly different from the ones observed previously in the literature for one-dimensional maps in general ($\alpha \approx 0.41$) [8,10,11], or considering only the uncertainty of large attractors ($\beta \approx 0.51$) [3,11], and continuous-time flows ($\alpha \approx 0.41$) [29]. Interestingly, the values of α found here challenge the belief that these values are universal for one-dimensional maps with a given order of their maxima [10,11]. Furthermore, the lower values of α observed along the extreme curves indicate a higher sensitivity to small changes in the parameters on these curves.

Given the significant difference between the uncertainty exponent along the extreme curves and the values typically observed in the literature, we investigate the values of this exponent in the vicinity of the extreme curves, seeking transitions towards the typical values of α . In Fig. 3, we calculate α along parallel curves to the extreme curves. For this purpose, since the axes of the parameter spaces in Figs. 3(a) and 3(b) have different sizes, we apply the following spatial transformation to normalize these spaces,

$$T : \mathbb{R}^2 \longrightarrow [0, 1] \times [0, 1], \\ (x, y) \longmapsto \left(\frac{x - x_m}{\Delta x}, \frac{y - y_m}{\Delta y} \right),$$

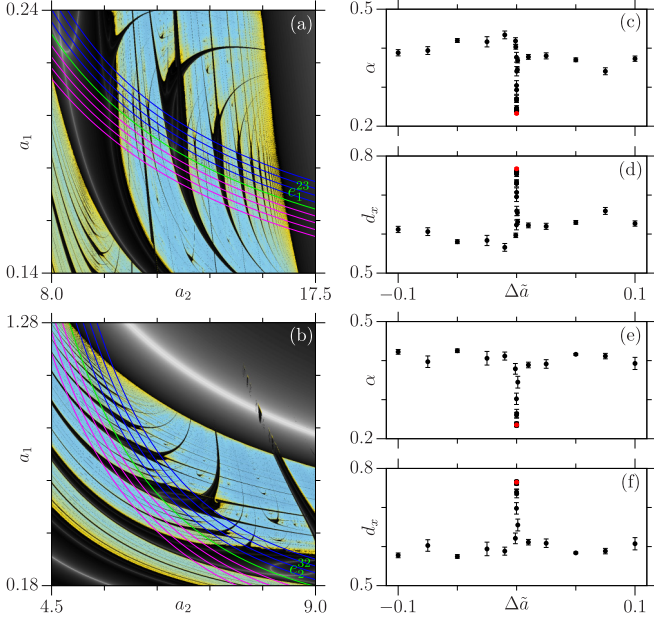


FIG. 3. (a) and (b) Parameter plane Γ illustrating the extreme curve e_1^{23} and e_2^{32} (green) and parallel curves distant positive (blue) and negative (pink) multiples of $\Delta\tilde{a} = 0.025$. (c) and (e) show the uncertainty exponent α and (d) and (f) present the exterior dimension d_x estimated along the parallel curves within $\Delta\tilde{a}$ from the curve e_1^{23} and e_2^{32} . Red markers show the singularities at $\Delta\tilde{a} = 0$ and the surrounded black markers are at $\Delta\tilde{a} = \pm 10^{-2}, \pm 10^{-3}, \pm 10^{-4}, \pm 10^{-5}, \pm 10^{-6},$ and $\pm 10^{-7}$.

where x_m and x_M are, respectively, the x minimum and maximum values in the abscissa axis with size $\Delta x = x_M - x_m$. The same applies to the y axis in the ordinate. By applying $T(a_1, a_2)$ in Eqs. (4) and (5), we represent the extreme curves e_1^{23} and e_2^{32} in the normalized space $[0, 1] \times [0, 1]$ and determine parallel curves distant $\Delta\tilde{a}$ from the extreme curve. Applying the inverse of the transformation T^{-1} to those parallel curves, we recover the original coordinate system, as shown in Figs. 3(a) and 3(b).

Following these definitions, we now consider parallel distances $\Delta\tilde{a} \in [-0.1, 0.1]$ centered on the extreme curves e_1^{23} and e_2^{32} . Within this interval, we obtain the uncertainty exponent along curves displaced from e_1^{23} [Fig. 3(c)] and e_2^{32} [Fig. 3(e)]. In addition, we employ Eq. (7) to calculate the exterior fractal dimension d_x of the sets dwelling along the curves analyzed within $\Delta\tilde{a}$ [see Figs. 3(d) and 3(f)]. In the interval surrounding both extreme curves e_1^{23} and e_2^{32} , we observe that the uncertainty exponent α , and consequently the exterior dimension d_x , undergo abrupt transitions assuming singular values exclusively along the extreme curves located at $\Delta\tilde{a} = 0$ (red markers). For the other curves within the interval, α and d_x assume the typical values previously found in the literature. The results presented in Fig. 3 demonstrate that periodicity cascades exhibit distinct features along the extreme curves, which give rise to the observed singularities. To elucidate these features, we first recall that the extreme curves intersect the periodic windows at all scales along the periodicity cascade in their superstability locus [as seen in the sequence $(S_i)_{i=1}^\infty$ in Fig. 1]. Since these superstability loci

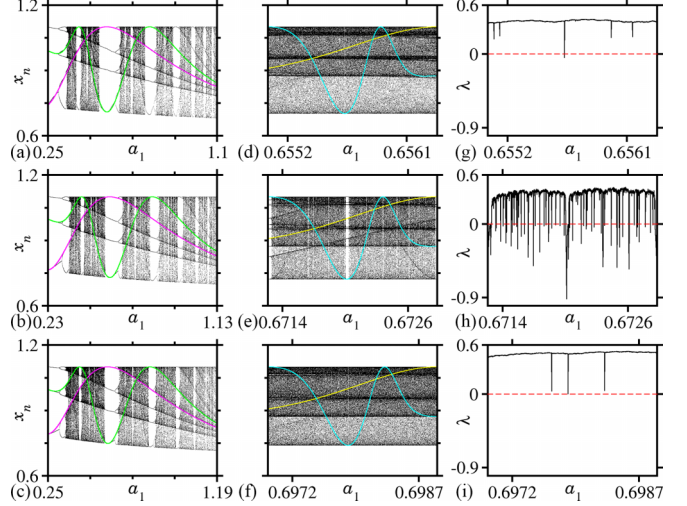


FIG. 4. (a)–(c) Bifurcation diagrams along curves at $\Delta\tilde{a} = -0.02, \Delta\tilde{a} = 0$ (extreme curve e_2^{32}), and $\Delta\tilde{a} = 0.02$, respectively. (d)–(f) Respective magnification of the bifurcation diagrams shown in the first column. (g)–(i) Lyapunov exponents corresponding to the bifurcation diagrams of the middle column. Colored curves correspond to the x value of the k th iteration of the critical point $x^* = 0, f^k(0)$, at each parameter. $k = 5$ (pink), 6 (green), 11 (yellow), and 13 (cyan).

serve as the structural foundation for the periodic windows, organizing their extension, it is guaranteed that the extreme curves intersect each window within the periodicity cascades. Conversely, any displaced curve, even though very close to the extreme one, may eventually bypass periodic windows at smaller scales within the parameter space. As a result, the increased density of periodic windows along the extreme curves signifies a more intricate relationship between parameters associated with periodic and chaotic behavior, leading to a decrease in the uncertainty exponent α . Furthermore, the heightened density of periodic windows also populates the one-dimensional curve more densely at a given scale, resulting in the fractal dimension of the boundaries being closer to unity. To better illustrate the increased density of periodic windows, in Fig. 4, we present the bifurcation diagrams of the state variable x in Eq. (2) along with their corresponding Lyapunov exponent.

In the bifurcation diagrams of Figs. 4(a)–4(c), the dynamics of x along the extreme curve e_2^{32} [Fig. 4(b)] appear similar to those of the displaced curves in Figs. 4(a) and 4(c). However, upon closer inspection at a finer scale, as demonstrated in the magnified bifurcation diagrams in Figs. 4(d)–4(f), the increased density of periodic windows along the extreme curve becomes evident. This higher density of periodic windows can be easily visualized by examining the Lyapunov exponents in Figs. 4(g)–4(i). Notably, the Lyapunov exponents along the extreme curve [Fig. 4(h)] assume negative values very often, in contrast to the displaced curves. The colored curves in all bifurcation diagrams of Fig. 4 establish a correspondence between the parameter intervals used for the extreme and displaced curves in the magnifications, ensuring comparability among the rows in this figure. These curves correspond to the k th iterate of the critical point $x^* = 0$, denoted as $f^k(0)$ at each

parameter. In the pink, green, yellow, and cyan curves, k takes the values of 5, 6, 11, and 13, respectively.

Summary. We consider a class of one-dimensional maps to explore the boundaries between parameters corresponding to self-similar structures of periodic behavior, periodicity cascades, and parameters that lead to chaos, forming fat fractal sets. In two-dimensional parameter spaces of these maps, we estimate the uncertainty exponent and exterior fractal dimension along parameter curves referred to as *extreme curves*. We observe that both measures assume values along these curves that differ significantly from those previously reported in the literature for arbitrary parameter curves, which were believed to be universal. We attribute this difference to a higher density of periodic windows along the extreme curves, which modifies the geometry of the fat fractal sets. Furthermore, in our attempt to identify the transition from the values observed along the extreme curves to the typical ones reported

in the literature, we discovered that these transitions occur abruptly near the extreme curves. Based on this observation, we propose that the uncertainty exponent and, consequently, the exterior fractal dimension exhibit singular values along the extreme curves, which we term the *extreme fractal dimension*.

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