

Optimizing leapover lengths of Lévy flights with resetting

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We consider a one-dimensional search process under stochastic resetting conditions. A target is located at $b \geq 0$ and a searcher, starting from the origin, performs a discrete-time random walk with independent jumps drawn from a heavy-tailed distribution. Before each jump, there is a given probability r of restarting the walk from the initial position. The efficiency of a “myopic search”—in which the search stops upon crossing the target for the first time—is usually characterized in terms of the first-passage time τ . On the other hand, great relevance is encapsulated by the leapover length $l = x_\tau - b$, which measures how far from the target the search ends. For symmetric heavy-tailed jump distributions, in the absence of resetting the average leapover is always infinite. Here we show instead that resetting induces a finite average leapover $\ell_b(r)$ if the mean jump length is finite. We compute exactly $\ell_b(r)$ and determine the condition under which resetting allows for nontrivial optimization, i.e., for the existence of r^* such that $\ell_b(r^*)$ is minimal and smaller than the average leapover of the single jump.

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In various situations found in nature on both macroscopic and microscopic scales, the analysis of the strategies that a searcher implements to reach a target proves to be of extreme interest [1]. Search strategies are relevant, for example, in the study of protein-DNA interactions [1], diffusion-controlled reactions [2], and the movement of animals or microorganisms searching for food [3,4]. The fundamental question common to all these different settings concerns the search efficiency and the existence of an optimal protocol to perform the task. The highly complex underlying dynamics are usually well described by random walk models, and thus the problem reduces to the study of optimization strategies for such stochastic processes.

In many contexts a central role is played by jump processes, which are defined as a sequence of independent jumps of random lengths [5–8]. For these processes a “myopic search” is typically implemented, where the search stops once the walker jumps over the target for the first time [9–12]. The final distance to the target, known in the literature as *leapover length* [13], is therefore an aspect that should not be underestimated in evaluating the efficiency of a myopic search [14]. This is especially relevant for Lévy flights, a well-known class of random walks with independent jumps drawn from a common probability density function (PDF) decaying for large η

as $\lambda(\eta) \propto \eta^{-1-\mu}$, with $0 < \mu < 2$. Indeed, it is known that for Lévy flights with symmetric jumps the PDF of the leapover length decays as $\wp(l) \propto l^{-1-\mu/2}$, leading to the seemingly paradoxical result that the average leapover diverges also in the range $1 < \mu < 2$, where the mean jump length is finite [13,15]. This feature is far from a subtlety, as Lévy flights are ubiquitous in nature and have found applications ranging from animal foraging [16,17] to human behavior [18]. Hence l is important to quantify, e.g., how far an animal gets from the resources it is looking for, if it crosses them in a *blind* phase of the search [1], or how far an infectious disease spreads after the first crossing of a border [18]. We also observe that leapover lengths are not only crucial in the context of search efficiency, but also play an important role in extreme value theory, where they can be interpreted as record increments [15], and in renewal theory, where they admit an interpretation as waiting times in the presence of aging [19].

In recent years, one of the most studied strategies for search optimization is *stochastic resetting*, which consists in restarting the search after random time intervals until the target is reached or crossed [20,21]. The main advantage of this strategy is the ability to control the search time τ through an external parameter (typically, the reset rate [20,21], but also different mechanisms have been studied [22,23]), which can be optimally chosen to minimize the average capture time of the target. This has been proven effective in a variety of areas, ranging from biology [24,25] to algorithm optimization [26], and has therefore generated increasing interest, both theoretical and experimental [9,27–35].

In this Letter, we show that stochastic resetting is an effective strategy to control the leapover length of Lévy flights. We obtain exactly the leapover distribution and show that for $\mu > 1$ resetting induces a finite average leapover, which we

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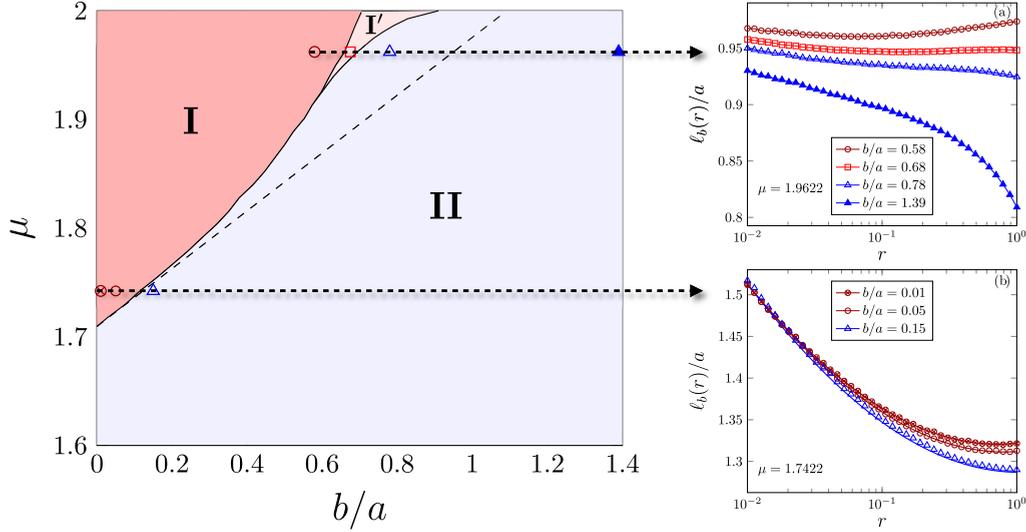


FIG. 1. Phase diagram for the case of Lévy stable jump processes, with $\hat{\lambda}(k) = e^{-|ak|^\mu}$. Region I is the phase where a nontrivial optimal resetting probability $0 < r^* < 1$ is guaranteed to exist: in this regime the optimal average leapover $\ell_b(r^*)$ is smaller than that obtained by a single jump. The boundary of this region is numerically computed from the exact condition given in Eq. (6). The dashed straight line corresponds to the analytical linear approximation of the boundary for small b , i.e., for targets close to the origin, given by Eq. (8a). In region I' the sufficient condition (6) is not satisfied but a nontrivial optimization probability is still observed, together with a local maximum for $\ell_b(r)$ [see panel (a) for details]. This region extends up to $b/a \approx 0.91$. Region II is the phase where the average leapover $\ell_b(r)$ assumes its minimum at the “trivial” value $r = 1$. The data displayed in panels (a) and (b), obtained by numerical simulations, show examples of transitions from one region to another as b increases, with μ kept fixed. In (b) we also show a comparison with the small- b approximation of $\ell_b(r)$ given by Eq. (9).

compute analytically (see Eq. (S2) in Supplemental Material [36]). We then determine a sufficient condition for the existence of a *nontrivial* optimization strategy, i.e., such that the minimal average leapover under resetting is strictly smaller than that obtained from a single jump; see Eq. (6). This condition, which strikingly involves properties of the reset-free random walk after 1 and 2 jumps only, makes it possible to identify regions in the (μ, b) plane where such a nontrivial optimization is possible; see Fig. 1 for a paradigmatic example.

To demonstrate these results, we consider a one-dimensional random walk x_n that for $n \geq 1$ evolves according to

$$x_n = \begin{cases} x_{n-1} + \eta_n & \text{with probability } 1 - r, \\ \eta_n & \text{with probability } r, \end{cases} \quad (1)$$

where $0 \leq r \leq 1$ and η_n are independent continuous random variables with common PDF $\lambda(\eta)$. The initial position is $x_0 = 0$. A target is placed at $b \geq 0$ and the evolution goes on until the *first-passage time* in (b, ∞) , denoted as τ . Consequently, $l = x_\tau - b$ is the leapover length. We call $f_r(x, n; b)$ the joint PDF of x_τ and τ . The leapover PDF is thus given by $\hat{\mathcal{G}}_r(l; b) = \sum_{n \geq 1} f_r(l + b, n; b)$. Let us also introduce $q_0(x, n; b)$, which is the PDF of the position x_n for walks that, in the absence of resetting, never cross b in n steps. We can then set up a first-renewal equation for $f_r(x, n; b)$ [9]:

$$f_r(x, n; b) = r \sum_{m=0}^{n-1} (1-r)^m Q_0(m; b) f_r(x, n-m; b) + (1-r)^n f_0(x, n; b), \quad (2)$$

where $Q_0(n; b) = \int_{-\infty}^b q_0(x, n; b) dx$ is the *survival probability* in n steps. By defining the double transform $\mathcal{G}(k, z) = \sum_{n \geq 0} z^n \int_{-\infty}^{\infty} e^{ikx} g(x, n) dx$, we obtain from Eq. (2) a relation between $\mathcal{F}_r(k, z; b)$ and $\mathcal{Q}_0(k, z; b)$, the transforms of $f_r(x, n; b)$ and $q_0(x, n; b)$ respectively. The Fourier transform of the leapover PDF is then obtained from $\hat{\mathcal{G}}_r(k; b) = e^{-ikb} \mathcal{F}_r(k, 1; b)$, and we arrive at

$$\hat{\mathcal{G}}_r(k; b) = e^{-ikb} \frac{1 - [1 - (1-r)\hat{\lambda}(k)]Q_0(k, 1-r; b)}{1 - rQ_0(0, 1-r; b)}, \quad (3)$$

where $\hat{\lambda}(k) = \int_{-\infty}^{\infty} e^{ik\eta} \lambda(\eta) d\eta$ is the characteristic function of the jumps. From this exact expression one can obtain, e.g., the large- l behavior of the leapover distribution and the average leapover length $\ell_b(r)$. We point out that the only unknown on the right-hand side (rhs) of Eq. (3) is $Q_0(k, z; b)$. By using a result of Spitzer [37], we compute exactly its Laplace transform with respect to b , which reads

$$\int_0^{\infty} e^{-wb} Q_0(k, z; b) db = Q_0(k, z) \frac{e^{\phi_+(k+iw, z)}}{w}, \quad (4)$$

where $Q_0(k, z)$ is a shorthand notation for $Q_0(k, z; b=0)$ and is known in the literature [38], whereas $\phi_+(\xi, z)$ is defined as

$$\phi_+(\xi, z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{ds}{\xi - s} \ln[1 - z\hat{\lambda}(s)]. \quad (5)$$

It can be shown that $\phi_+(\xi, z)$ is analytic for $\text{Im}(\xi) > 0$. We can thus compute the small- k expansion of the rhs in Eq. (4), and then transform it back to obtain the small- k expansion of $Q_0(k, z; b)$. Finally, we can derive the expansions for both the real and imaginary parts of $\hat{\mathcal{G}}_r(k; b)$.

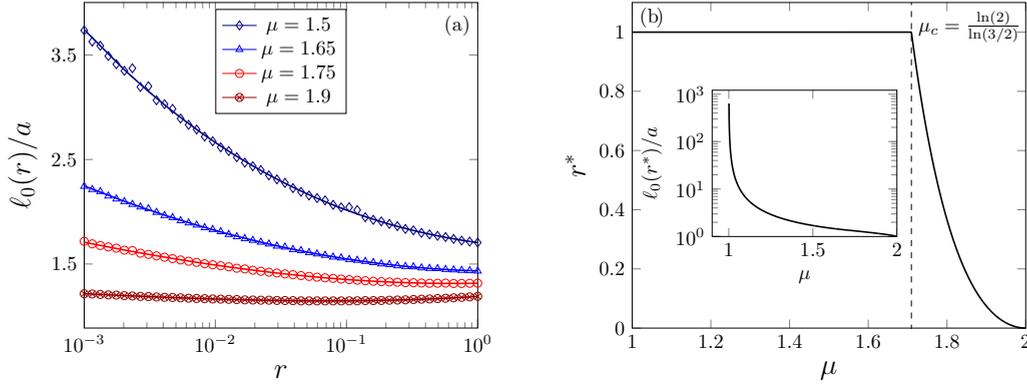


FIG. 2. (a) Average leapover distance and (b) optimal resetting probability for Lévy stable processes, case $b = 0$. In (a) the results of numerical simulations, displayed in function of the resetting probability, are compared with the theoretical curves (see Eq. (S55) in [36]), showing excellent agreement. In (b) we present r^* and (inset) the corresponding $\ell_0(r^*)$ versus μ . The dashed vertical line represents the critical value μ_c . The values of r^* are obtained by evaluating numerically the minimum of $\ell_0(r)$.

We consider the case of symmetric jump distributions that for large $|\eta|$ decay as $\hat{\lambda}(\eta) \propto |\eta|^{-1-\mu}$, with $0 < \mu < 2$. Consequently, $\hat{\lambda}(k) \sim 1 - |ak|^\mu$ for small k , where a is a scale parameter. For $r > 0$, we obtain $\text{Re}[\hat{\varphi}_r(k; b)] \sim 1 - A_r(b)|ak|^\mu$, where $A_r(b)$ is written in terms of the generating function $Q_0(z; b) = \sum_{n \geq 0} z^n Q_0(n; b)$ [36]. Hence $\hat{\varphi}_r(l; b) \sim A_r(b)\lambda(l)$ for large l . It follows that, as stated, for $\mu > 1$ the average leapover length $\ell_b(r)$ is finite and can be expressed as a well-defined function of r . Indeed, for small k and $\mu > 1$ we obtain $\text{Im}[\hat{\varphi}_r(k; b)] \sim k\ell_b(r)$, where $\ell_b(r)$ can be determined exactly [36]. For $r = 1$, this function is equal to the average leapover of a single jump. On the other hand, for $r \rightarrow 0$ we must recover the behavior of the reset-free walk, yielding a diverging $\ell_b(r)$ for $\mu < 2$. Hence, in the range $1 < \mu < 2$, since $\ell_b(r)$ diverges for $r \rightarrow 0$ and is finite for $r = 1$, a sufficient condition for the existence of a probability $0 < r^* < 1$ that minimizes $\ell_b(r)$ is that the derivative $\ell'_b(r)$ be positive as $r \rightarrow 1$. By using our exact result, we find that this condition can be recast as

$$\frac{\langle |x_2| \rangle}{\langle |x_1| \rangle} < \Psi_\mu(b), \quad (6)$$

where both sides of this inequality depend only on the first two steps of the reset-free walk, thus it is easy to evaluate numerically. Here $\langle |x_n| \rangle$ is the absolute first moment of the walk after n steps and $\Psi_\mu(b)$ has an exact expression (see Eq. (S4) in [36]) that involves $\lambda(\eta)$ and the survival probability in one and two steps. Interestingly, for $b = 0$ the rhs is a constant, $\Psi_\mu(0) = \frac{3}{2}$, independently of the jump distribution, as long as it is symmetric and continuous. We stress that, as both sides have an implicit dependence on μ , Eq. (6) allows one to determine the existence of a region in the (μ, b) plane where $\ell_b(r)$ has a global minimum at some $0 < r^* < 1$. In Fig. 1 we draw the phase diagram for the paradigmatic example of the family of Lévy stable processes, whose jumps have the characteristic function $\hat{\lambda}(k) = e^{-|ak|^\mu}$.

We now consider in detail the case $b = 0$. The case of a target at the origin is particularly significant because, thanks to the Sparre-Andersen universality [39], we are able to provide a compact expression for the average leapover length.

Indeed, the exact formula for $\ell_b(r)$ assumes a simple form when $b = 0$:

$$\ell_0(r) = \frac{\sqrt{r}}{\pi(1-\sqrt{r})} \int_0^\infty \frac{ds}{s^2} \ln \left(\frac{1 - (1-r)\hat{\lambda}(s)}{r} \right) \quad (7a)$$

$$= \frac{\sqrt{r}}{1-\sqrt{r}} \sum_{n=1}^{\infty} \frac{(1-r)^n}{2n} \langle |x_n| \rangle. \quad (7b)$$

We remark that both of these equations are valid for any symmetric and continuous jump distribution, not necessarily heavy-tailed, with finite first moment. The first gives $\ell_0(r)$ in terms of $\hat{\lambda}(k)$ only, while the second allows for an analytic computation once $\langle |x_n| \rangle$ is known for each n . For instance, in the case of Lévy stable processes, for $1 < \mu \leq 2$ we have $\langle |x_n| \rangle = 2a\Gamma(1 - 1/\mu)n^{1/\mu}/\pi$, where $\Gamma(z)$ is the Euler gamma function. One can then rewrite the series in Eq. (7b) in terms of the polylogarithm $\text{Li}_{1-1/\mu}(1-r)$ [40], and obtain an analytic expression for $\ell_0(r)$, which can be compared with simulations; see Fig. 2. Moreover, in this case Eq. (6) simplifies to $2^{1/\mu} < \frac{3}{2}$, which is solved for $\mu > \mu_c = \ln(2)/\ln(3/2) \approx 1.7095$. When $\mu > \mu_c$, one can show that $\ell_0(r)$ has indeed a unique minimum attained for $r^* < 1$. For μ below the critical value instead, $\ell_0(r)$ is a decreasing function of r , hence $r^* = 1$. Thus, we find explicitly that there exists a critical value μ_c defining a transition to a phase where, by choosing optimally the resetting probability, we obtain an average leapover that is smaller than the average length of a single jump. The whole situation is depicted in Fig. 2, where we plot the numerically computed r^* versus μ .

We now want to determine explicitly how μ_c changes by increasing b , at least for small b . This can be done by approximating the function $\Psi_\mu(b)$ for small b to get from Eq. (6) an explicit condition for the critical exponent $\mu_c(b)$. We linearize $\Psi_\mu(b) \approx \Psi_\mu(0) + \Psi'_\mu(0)b$ and then invert Eq. (6) to write $\mathcal{K}(\mu) = b$, where $\mathcal{K}(\mu) = [\langle |x_2| \rangle / \langle |x_1| \rangle - \Psi_\mu(0)] / \Psi'_\mu(0)$. By linearizing then $\mathcal{K}(\mu)$ around μ_c , we finally obtain $\mu_c(b) \approx b/\mathcal{K}'(\mu_c) + \mu_c$. Again, everything can be computed explicitly

for Lévy stable processes, and following the procedure we just described we find

$$\mu_c(b) \approx \frac{Cb}{a} + \frac{\ln(2)}{\ln(3/2)}, \quad (8a)$$

$$C = \frac{\ln(4)}{9\pi \ln^2(3/2)} \Gamma\left(\frac{\ln(3)}{\ln(2)}\right) \approx 0.265\,997\,3\dots \quad (8b)$$

The agreement between this linear approximation and the values obtained numerically from the exact condition of Eq. (6) is shown in Fig. 1. By following the same idea, we compute an explicit approximation of $\ell_b(r)$ for small b . To do this, we need the small- b expansion of $\mathcal{Q}_0(k, z; b)$, which can be obtained from the large- w expansion of its Laplace transform on the rhs of Eq. (4). For continuous and symmetric jump distributions, in the limit $w \rightarrow \infty$ the w -dependent term at the rhs tends to the Laplace transform of $\sqrt{1-z}\tilde{\mathcal{Q}}_0(z; b)$, hence we only need a small- b approximation of the generating function. This was calculated in [9] in the context of search time optimization of Lévy flights. It reads $\sqrt{1-z}\tilde{\mathcal{Q}}_0(z; b) \approx 1 + bI(z) + O(b^2)$, where $I(z)$ is an integral that depends on $\hat{\lambda}(k)$. It is then possible for us to write an explicit approximated expression for $\ell_b(r)$, which, by keeping only terms $O(b)$, reads

$$\ell_b(r) \approx \ell_0(r) + b\left(\frac{I(1-r)\ell_0(r)}{1-\sqrt{r}} - 1\right). \quad (9)$$

In the specific case of Lévy stable processes we can write explicitly $I(z)$ in terms of the polylogarithm $\text{Li}_{1+1/\mu}(z)$, hence we get an analytical curve that we can compare with simulations. The agreement of Eq. (9) with data is shown in Fig. 1. We find that the approximation reflects well the behavior of the numerical data for small b , although it tends to underestimate them, more so as b increases.

To summarize, we have shown that stochastic resetting is an effective mechanism to control the leapover lengths of one-dimensional Lévy flights performing a myopic search. We have demonstrated that, in contrast to the scenario without resetting, a finite average leapover length is attained and that, moreover, in some cases there exists an optimal resetting probability strictly smaller than one that minimizes the average leapover distance. Consequently, for probabilities close to this optimal value, the average leapover distance is smaller than that obtained by a single jump. A sufficient condition for this is given by Eq. (6). For Lévy stable processes, we have illustrated our findings in the phase diagram of Fig. 1. This diagram also shows an interesting phase in which nontrivial optimization is possible even if Eq. (6) is not verified. This work, in conjunction with other research in the literature, opens the way for the study of search strategies aimed at optimizing the coupling of spatial and temporal variables, such as first-passage times and leapover lengths. To expand the generality of our results, it would be beneficial in the future to investigate the phase diagram more thoroughly and for different types of jump distribution, as well as to conduct a rigorous study of the nontrivial phase violating Eq. (6), which in the present work is only investigated numerically.

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