


## Critical dimension for hydrodynamic turbulence

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Hydrodynamic turbulence exhibits nonequilibrium behavior with  $k^{-5/3}$  energy spectrum, and equilibrium behavior with  $k^{d-1}$  energy spectrum and zero viscosity, where  $d$  is the space dimension. Using recursive renormalization group in Craya-Herring basis, we show that the nonequilibrium solution is valid only for  $d < 6$ , whereas equilibrium solution with zero viscosity is the only solution for  $d > 6$ . Thus,  $d = 6$  is the critical dimension for hydrodynamic turbulence. In addition, we show that the energy flux changes sign from positive to negative near  $d = 2.15$ . We also compute the energy flux and Kolmogorov's constants for various  $d$ 's, and observe that our results are in good agreement with past numerical results.

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### I. INTRODUCTION

Field-theoretic tools help explain complex phenomena in high-energy physics, condensed-matter physics, statistical physics, and turbulence [1–4]. For example, Wilson and coworkers [5] constructed a theory for the second-order phase transition that goes beyond the mean field theory of Landau [6]. In Wilson's theory, the nonlinear term yields nontrivial scaling for  $d < 4$ , but it becomes irrelevant for  $d \geq 4$ . Therefore, the critical dimension for the second-order phase transition is 4. In this paper, we compute the critical dimension for hydrodynamic turbulence.

The frameworks of quantum field theory and statistical field theory have been extended to hydrodynamic turbulence. Prominent field-theoretic computations for hydrodynamic turbulence are direct interaction approximation (DIA) [7], renormalization group (RG) [8–10], generating functionals [10,11], Martin-Siggia-Rose (MSR) formalism [12], Recursive renormalization group [13–15], functional renormalization [16]. Other field theory works on hydrodynamic turbulence are Refs. [16–22]. These works are reviewed in Orszag [23] and Zhou [24]. Most of the prominent field theory works are for three dimensions (3D), where the RG analysis predicts that the energy spectrum  $E(k) \propto k^{-5/3}$ , and that the renormalization viscosity  $\nu(k)$  scales as  $k^{-4/3}$  with the renormalization constant around 0.40. Some calculations (e.g., Ref. [9]) employ particular forcing, whereas some others employ self-consistent procedure [13,24]. In comparison, RG works on 2D hydrodynamic turbulence is limited. In one such work, Olla [25] obtained two different spectral regimes:  $k^{-3}$  energy spectrum with a constant enstrophy flux at large wave numbers, and  $k^{-5/3}$  spectrum with a constant energy flux at small wave numbers. For the  $k^{-5/3}$  spectral regime, Olla [25] derived the renormalization constant to be 0.642 and the Kolmogorov constant to be 6.45. Nandy and Bhattacharjee [26] employed self-consistent mode-coupling scheme and obtained similar constants.

The energy transfers and fluxes of hydrodynamic turbulence are also computed using field theory. Kraichnan [7] employed DIA for these computations. Later, eddy-damped quasnormal Markovian approximation (EDQNM) and other schemes have been used for the flux calculations [23]. Verma [27,28] computed the energy fluxes using the mode-to-mode energy transfers. The equation for the energy flux yields Kolmogorov's constant [23].

Fournier and Frisch [29] employed EDQNM procedure to compute the stable energy spectra for various space dimensions, denoted by  $d$ . They showed that the energy flux changes sign from positive to negative near  $d = 2.05$  as  $d$  decreases from 3 to 2. Gotoh *et al.* [30] employed Lagrangian Renormalized Approximation and showed that the energy transfer in 4D is more efficient compared to that in 3D. Consequently, the Kolmogorov's constant for 4D,  $K_{K0} = 1.31$ , is smaller than that for 3D,  $K_{K0} = 1.72$ . Gotoh *et al.* [30] verified the field-theoretic predictions using numerical simulations. Berera *et al.* [31] observed similar results in their numerical simulations, for example,  $K_{K0} = 1.7$  and 1.3 for 3D and 4D respectively.

In statistical and quantum field theory, the parameters of theory (e.g., coupling constant and mass) depend critically on the space dimension [1–4]. For example, the fluctuations in  $\phi^4$  theory obey Gaussian property for  $d \geq 4$ . Hence,  $d = 4$  is called the upper critical dimension for the  $\phi^4$  theory. Theorists have been exploring whether such an upper critical dimension exists for hydrodynamic turbulence. For example, Adzhemyan *et al.* [32] showed that the Kolmogorov constant  $K_{K0} \propto d^{1/3}$ , which leads to the energy flux  $\epsilon_u \propto K_{K0}^{-3/2} \propto d^{-1/2} \rightarrow 0$  as  $d \rightarrow \infty$ . Similarly, Fournier *et al.* [33] showed that intermittency, which is a reflection of non-Gaussian nature of the fluctuations, vanishes as  $d \rightarrow \infty$ . These observations indicate that the velocity fluctuations possibly exhibit Gaussian behavior for large  $d$ . In this paper, we explore this issue for hydrodynamic turbulence using RG calculation in Craya-Herring basis that provides detailed picture of interactions in hydrodynamic turbulence.

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In this paper, we compute the renormalized viscosity using recursive renormalization, and Kolmogorov's constant using energy transfers for various dimensions. For these computations, we employ Craya-Herring basis [28,34–36] that simplifies the field-theoretic calculations of turbulent flows dramatically. In addition, this basis allows separate computations of the renormalized viscosities and energy transfers for each component, thus yielding finer details of turbulence without complex tensor algebra. In addition to the above simplification, we deviate from the conventional  $\int dpdq\delta(\mathbf{k} - \mathbf{p} - \mathbf{q})$  integrals to  $\int dpd\gamma$ , where  $\gamma$  is the angle between  $\mathbf{k}$  and  $\mathbf{p}$  in a triad  $(k, p, q)$ . This new scheme simplifies the asymptotic analysis, as well as the evaluation of the singular integrals of energy fluxes [37].

Using the above techniques, we compute the renormalized viscosity and Kolmogorov's constant for various  $d$ . We show that  $\nu(k) \sim k^{-4/3}$  solution exists only for  $d < 6$ , whereas  $\nu = 0$  is the solution of the RG equation beyond  $d = 6$ . Hence, the critical dimension for hydrodynamic turbulence is 6. In addition, we also compute the energy flux in the inertial range that yields Kolmogorov's constant. Our Kolmogorov constants for various dimensions are in good agreement with the past works [30,31].

The outline of the paper is as follows: In Sec. II, we introduce the relevant hydrodynamic equations in Craya-Herring basis. In Sec. III, we describe the renormalization group analysis for hydrodynamic turbulence using Craya-Herring basis. Section IV contains discussions on the energy transfers in a triad, as well as the energy fluxes for various  $d$ . Section V provides a brief discussion on the fractional energy transfers. Section VI reproduces the RG and energy flux computations for Kraichnan's  $k^{-3/2}$  energy spectrum. We conclude in Sec. VII.

## II. GOVERNING EQUATIONS AND FRAMEWORK

In Fourier space, the equations for the incompressible Navier-Stokes equations in  $d$  dimensions are [28,38]

$$(\partial_t + \nu k^2)\mathbf{u}(\mathbf{k}, t) = -i \int \frac{d\mathbf{p}}{(2\pi)^d} \{\mathbf{k} \cdot \mathbf{u}(\mathbf{q}, t)\}\mathbf{u}(\mathbf{p}, t) - ikp(\mathbf{k}, t) + \mathbf{F}_u(\mathbf{k}, t), \quad (1)$$

$$\mathbf{k} \cdot \mathbf{u}(\mathbf{k}, t) = 0, \quad (2)$$

where  $\mathbf{k} = \mathbf{p} + \mathbf{q}$ ;  $\mathbf{u}$ ,  $p$  are the velocity and pressure fields respectively;  $\nu$  is the kinematic viscosity; and  $\mathbf{F}_u$  is the external forcing, which is active at large scales, as in Kolmogorov's theory of turbulence. The transformation from real space to Fourier space and vice versa are as follows [1]:

$$\mathbf{u}(\mathbf{r}, t) = \int \frac{d\mathbf{k}}{(2\pi)^d} \mathbf{u}(\mathbf{k}, t) \exp(i\mathbf{k} \cdot \mathbf{r}), \quad (3)$$

$$\mathbf{u}(\mathbf{k}, t) = \int d\mathbf{r} [\mathbf{u}(\mathbf{r}, t) \exp(-i\mathbf{k} \cdot \mathbf{r})]; \quad (4)$$

and the pressure field is determined using the following equation:

$$p(\mathbf{k}, t) = -\frac{i}{k^2} \mathbf{k} \cdot \mathbf{F}_u(\mathbf{k}, t) - \frac{1}{k^2} \int \frac{d\mathbf{p}}{(2\pi)^d} \times \{\mathbf{k} \cdot \mathbf{u}(\mathbf{q}, t)\}\{\mathbf{k} \cdot \mathbf{u}(\mathbf{p}, t)\} \quad (5)$$

with  $\mathbf{k} = \mathbf{p} + \mathbf{q}$ .

The equation for the *modal energy*  $E(\mathbf{k}) = |\mathbf{u}(\mathbf{k})|^2/2$  is [27,39]

$$(\partial_t + 2\nu k^2)E(\mathbf{k}, t) = \int \frac{d\mathbf{p}}{(2\pi)^d} S^{uu}(\mathbf{k}|\mathbf{p}|\mathbf{q}) + \text{Re}[\mathbf{F}_u(\mathbf{k}, t) \cdot \mathbf{u}^*(\mathbf{k}, t)], \quad (6)$$

where

$$S^{uu}(\mathbf{k}|\mathbf{p}|\mathbf{q}) = \text{Im}[\{\mathbf{k} \cdot \mathbf{u}(\mathbf{q}, t)\}\{\mathbf{u}(\mathbf{p}, t) \cdot \mathbf{u}^*(\mathbf{k}, t)\}] \quad (7)$$

is the *mode-to-mode energy transfer rate* from the *giver* mode  $\mathbf{u}(\mathbf{p})$  to the *receiver* mode  $\mathbf{u}(\mathbf{k})$  with the mediation of mode  $\mathbf{u}(\mathbf{q})$ . Here  $\text{Im}$  denotes the imaginary part of the argument. The energy flux  $\Pi(R)$  is the net nonlinear energy transfer rate from all the modes residing inside the sphere of radius  $R$  to the modes outside the sphere. Hence, the ensemble average of  $\Pi(R)$  is [27,28,39]

$$\langle \Pi(R) \rangle = \int_R^\infty \frac{dk'}{(2\pi)^d} \int_0^R \frac{dp}{(2\pi)^d} \langle S^{uu}(\mathbf{k}'|\mathbf{p}|\mathbf{q}) \rangle. \quad (8)$$

In this paper, we will compute the renormalized viscosity, as well as  $\langle S^{uu}(\mathbf{k}'|\mathbf{p}|\mathbf{q}) \rangle$  and  $\langle \Pi(k_0) \rangle$ , using field theory in Craya-Herring basis. In this basis, the basis vectors in 3D are [34–36]:

$$\hat{e}_0(\mathbf{k}) = \hat{k}; \quad \hat{e}_1(\mathbf{k}) = \frac{\hat{k} \times \hat{n}}{|\hat{k} \times \hat{n}|}; \quad \hat{e}_2(\mathbf{k}) = \hat{e}_0(\mathbf{k}) \times \hat{e}_1(\mathbf{k}), \quad (9)$$

where the unit vector  $\hat{k}$  is along the wave number  $\mathbf{k}$ , and the unit vector  $\hat{n}$  is chosen along any direction. For space dimension  $d$  greater than 3, we choose additional  $d - 3$  orthogonal unit vectors that are perpendicular to  $\hat{e}_0(\mathbf{k})$ ,  $\hat{e}_1(\mathbf{k})$ , and  $\hat{e}_2(\mathbf{k})$ . For an incompressible flow,

$$\mathbf{u}(\mathbf{k}, t) = \sum_{j=1}^{d-1} u_j(\mathbf{k}, t) \hat{e}_j(\mathbf{k}). \quad (10)$$

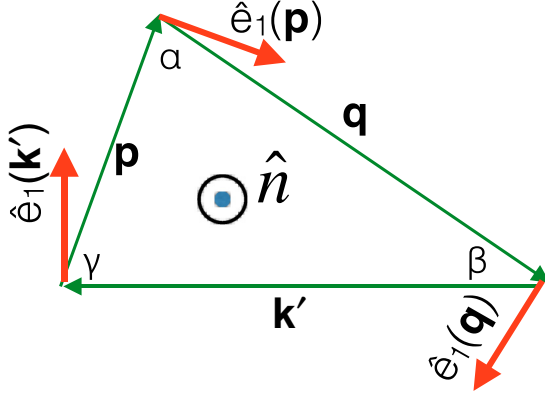
In this paper, we will derive the renormalized viscosity and energy flux by summing up contributions from all the interacting triads. Therefore, as a first step, we write down the evolution equations for  $u_j(\mathbf{k}, t)$  in a triad. For the same, we consider a wave-number triad  $(\mathbf{k}', \mathbf{p}, \mathbf{q})$  with  $\mathbf{k}' + \mathbf{p} + \mathbf{q} = 0$ , and choose  $\hat{n}$  as follows [28,40]:

$$\hat{n} = \frac{\mathbf{q} \times \mathbf{p}}{|\mathbf{q} \times \mathbf{p}|}. \quad (11)$$

Since  $\mathbf{k} = \mathbf{p} + \mathbf{q}$ , we deduce that  $\mathbf{k}' = -\mathbf{k}$ . The Craya-Herring basis vectors for the interacting wave numbers are illustrated in Fig. 1. Note that  $\alpha$ ,  $\beta$ , and  $\gamma$  are the angles in front of  $k$ ,  $p$ , and  $q$ , respectively. The net nonlinear interaction is a sum over all possible triads. Hence, the equations for the  $u_1$  components of a triad  $(\mathbf{k}', \mathbf{p}, \mathbf{q})$  are [28]:

$$(\partial_t + \nu k^2)u_1(\mathbf{k}', t) = ik' \int \frac{d\mathbf{p}}{(2\pi)^d} \sin(\beta - \gamma) u_1^*(\mathbf{p}, t) u_1^*(\mathbf{q}, t) + F_1(\mathbf{k}', t), \quad (12)$$

$$(\partial_t + \nu k^2)u_1(\mathbf{p}, t) = ip \int \frac{d\mathbf{q}}{(2\pi)^d} \sin(\gamma - \alpha) u_1^*(\mathbf{q}, t) u_1^*(\mathbf{k}', t) + F_1(\mathbf{p}, t), \quad (13)$$



$$\hat{e}_2(\mathbf{k}') = \hat{e}_2(\mathbf{p}) = \hat{e}_2(\mathbf{q}) = -\hat{n}$$

FIG. 1. Craya-Herring basis vectors for an interacting wave-number triad  $(\mathbf{k}', \mathbf{p}, \mathbf{q})$ . Reprinted with permission from Verma [28].

$$(\partial_t + \nu k^2)u_1(\mathbf{q}, t) = iq \int \frac{d\mathbf{k}'}{(2\pi)^d} \sin(\alpha - \beta)u_1^*(\mathbf{p}, t)u_1^*(\mathbf{k}', t) + F_1(\mathbf{q}, t), \quad (14)$$

with  $\mathbf{k}' + \mathbf{p} + \mathbf{q} = \mathbf{0}$ , and the angles  $\alpha, \beta, \gamma$  are computed for the respective triads. The equations for  $u_2(\mathbf{k}', t), u_3(\mathbf{k}', t), \dots, u_{d-1}(\mathbf{k}', t)$ , denoted by  $u_j(\mathbf{k}', t)$ , are similar:

$$(\partial_t + \nu k^2)u_j(\mathbf{k}', t) = ik' \int \frac{d\mathbf{p}}{(2\pi)^d} \{\sin \gamma u_1^*(\mathbf{p}, t)u_j^*(\mathbf{q}, t) - \sin \beta u_1^*(\mathbf{q}, t)u_j^*(\mathbf{p}, t)\} + F_j(\mathbf{k}', t), \quad (15)$$

$$(\partial_t + \nu k^2)u_j(\mathbf{p}, t) = ip \int \frac{d\mathbf{q}}{(2\pi)^d} \{\sin \alpha u_1^*(\mathbf{q}, t)u_j^*(\mathbf{k}', t) - \sin \gamma u_1^*(\mathbf{k}', t)u_j^*(\mathbf{q}, t)\} + F_j(\mathbf{p}, t), \quad (16)$$

$$(\partial_t + \nu k^2)u_j(\mathbf{q}, t) = iq \int \frac{d\mathbf{k}'}{(2\pi)^d} \{\sin \beta u_1^*(\mathbf{k}', t)u_j^*(\mathbf{p}, t) - \sin \alpha u_1^*(\mathbf{p}, t)u_j^*(\mathbf{k}', t)\} + F_j(\mathbf{q}, t), \quad (17)$$

with  $\mathbf{k}' + \mathbf{p} + \mathbf{q} = \mathbf{0}$ .

The energy flux is compactly captured by the following mode-to-mode energy transfers in the Craya-Herring basis [28,39]:

$$S^{uu}(\mathbf{k}'|\mathbf{p}|\mathbf{q}) = \sum_{j=1}^{d-1} S^{u_j u_j}(\mathbf{k}'|\mathbf{p}|\mathbf{q}), \quad (18)$$

with

$$S^{u_1 u_1}(\mathbf{k}'|\mathbf{p}|\mathbf{q}) = k' \sin \beta \cos \gamma \text{Im}\{u_1(\mathbf{q}, t)u_1(\mathbf{p}, t)u_1(\mathbf{k}', t)\}, \quad (19)$$

$$S^{u_j u_j}(\mathbf{k}'|\mathbf{p}|\mathbf{q}) = -k' \sin \beta \text{Im}\{u_1(\mathbf{q}, t)u_j(\mathbf{p}, t)u_j(\mathbf{k}', t)\} \quad (20)$$

for  $j \in [2..(d-1)]$ . An isotropic  $d$ -dimensional divergence-free flow field has  $d-1$  Craya-Herring components with

$$\langle |u_1(\mathbf{k}|^2) = \langle |u_2(\mathbf{k}|^2) = \dots = \langle |u_{d-1}(\mathbf{k}|^2) = C(\mathbf{k}). \quad (21)$$

In this paper, we denote  $\langle |u_j(\mathbf{k}|^2) = C_j(\mathbf{k})$ . The total kinetic energy is

$$\begin{aligned} \frac{\langle u^2 \rangle}{2} &= \int E(k)dk = \frac{1}{2} \int \frac{d\mathbf{k}}{(2\pi)^d} (d-1)C(\mathbf{k}) \\ &= \frac{1}{2} \frac{S_d}{(2\pi)^d} (d-1) \int dk k^{d-1} C(\mathbf{k}), \end{aligned} \quad (22)$$

where  $E(k)$  is the one-dimensional (1D) shell spectrum, and  $S_d = 2\pi^{d/2}/\Gamma(d/2)$  is the surface area of the  $d$ -dimensional sphere. The above equation yields the following relationship between the modal energy and 1D energy spectrum [7,27,41]:

$$E(k) = \frac{(d-1)}{2} C(\mathbf{k}) \frac{S_d k^{d-1}}{(2\pi)^d}. \quad (23)$$

After the above preliminary discussion on the relevant equations, we perform RG and energy transfer analysis for  $d$ -dimensional hydrodynamic turbulence.

### III. RENORMALIZATION GROUP ANALYSIS OF HYDRODYNAMIC TURBULENCE

In this section, we derive the renormalized viscosity using the Craya-Herring basis. We follow the recursive RG method proposed by McComb, Zhou, and coworkers [13,15,42]. Note that the coupling constant, the coefficient in front of the nonlinear term  $\mathbf{u} \cdot \nabla \mathbf{u}$ , is unchanged under renormalization due to the Galilean invariance [8,14]. Therefore, vertex renormalization is not required in hydrodynamic turbulence. In addition, in the recursive RG, the forcing or noise is introduced at large scales so as to produce a steady state with Kolmogorov spectrum. Hence *noise renormalization* too is avoided in this scheme [13,15,42], and the energy spectrum is taken as  $k^{-5/3}$ . Note that such a choice for  $E(k)$  is as arbitrary as the choice of noise that yields Kolmogorov's spectrum (as is done in noise renormalization [9]).

The evolution equations for  $u_1$  differs from the other components. Therefore, we expect that  $u_1$ 's renormalized viscosity, denoted by  $\nu_1(k)$ , differs from that of others, which is denoted by  $\nu_2(k)$ . Note that the renormalized viscosities of  $u_2, u_3, \dots, u_{d-1}$  are the same due to the symmetries of Eqs. (15)–(17).

#### A. Renormalization of $u_1$ component

In a recursive renormalization scheme, we divide the Fourier space into wave-number shells  $(k_m, k_{m+1})$ , where  $k_m = k_0 b^m$  with  $b > 1$ . We perform *coarse-graining* or averaging over a wave-numbers band, and compute its effects on the modes with lower wave number. Let us assume that we are at a stage with wave-number range of  $(k_0, k_{n+1})$ , among which the shell  $(k_n, k_{n+1})$  is coarse-grained. See Fig. 2 for an illustration.

We start with a dynamical equation for  $u_1^<(\mathbf{k}', t)$  Eq. (12). Note that  $\mathbf{q} = -\mathbf{k}' - \mathbf{p}$ . The convolution in the

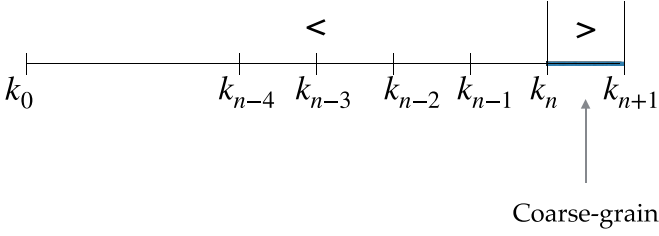


FIG. 2. In wave-number renormalization, the modes in the wave-number band  $(k_n, k_{n+1})$ , denoted by  $>$ , are coarse-grained. The coarse-graining leads to enhancement of effective viscosity for wave numbers  $k < k_n$ , denoted by  $<$ .

dynamical equation involves the following four sums:

$$\begin{aligned} & [\partial_t + v_1^{(n+1)}k^2]u_1^<(\mathbf{k}', t) \\ &= ik' \int \frac{d\mathbf{p}}{(2\pi)^d} \sin(\beta - \gamma) \\ & \quad \times [u_1^{*<}(\mathbf{p}, t)u_1^{*<}(\mathbf{q}, t) + u_1^{*<}(\mathbf{p}, t)u_1^{*>}(\mathbf{q}, t) \\ & \quad + u_1^{*>}(\mathbf{p}, t)u_1^{*<}(\mathbf{q}, t)] + u_1^{*>}(\mathbf{p}, t)u_1^{*>}(\mathbf{q}, t) \end{aligned} \quad (24)$$

because  $p$  and  $q$  may be either less than  $k_n$  or greater than  $k_n$ . As in large-eddy simulations (LES),  $v_1^{(n+1)}$  in Eq. (24) represents the renormalized viscosity for  $u_1$  in the wave-number range  $(k_0, k_{n+1})$  [38,43]. Now, we ensemble-average or coarse-grain the fluctuations at scales  $(k_n, k_{n+1})$ . After coarse-graining, the viscosity would be  $v_1^{(n)}$ , which acts on the wave numbers  $(k_0, k_n)$ .

For the coarse-graining process, we assume that  $u_1^>(\mathbf{k}, t)$  is time-stationary, homogeneous, isotropic, and Gaussian with zero mean, and that  $u_1^<(\mathbf{k}, t)$  are unaffected by coarse-graining [5,24,42]. That is,

$$\langle u_1^>(\mathbf{k}, t) \rangle = 0, \quad (25)$$

$$\langle u_1^<(\mathbf{k}, t) \rangle = u_1^<(\mathbf{k}, t). \quad (26)$$

Therefore, assuming weak correlation between  $<$  and  $>$  modes, we arrive at

$$\langle u_1^{*<}(\mathbf{p}, t)u_1^{*<}(\mathbf{q}, t) \rangle = u_1^{*<}(\mathbf{p}, t)u_1^{*<}(\mathbf{q}, t), \quad (27)$$

$$\langle u_1^{*<}(\mathbf{p}, t)u_1^{*>}(\mathbf{q}, t) \rangle = u_1^{*<}(\mathbf{p}, t)\langle u_1^{*>}(\mathbf{q}, t) \rangle = 0, \quad (28)$$

$$\langle u_1^{*>}(\mathbf{p}, t)u_1^{*<}(\mathbf{q}, t) \rangle = \langle u_1^{*>}(\mathbf{p}, t) \rangle u_1^{*<}(\mathbf{q}, t) = 0. \quad (29)$$

Substitution of the above relations in Eq. (24) yields

$$\begin{aligned} & [\partial_t + v_1^{(n+1)}k^2]u_1^<(\mathbf{k}', t) \\ &= ik' \int \frac{d\mathbf{p}}{(2\pi)^d} \sin(\beta - \gamma)u_1^{*<}(\mathbf{p}, t)u_1^{*<}(\mathbf{q}, t) \\ & \quad + ik' \int_{\Delta} \frac{d\mathbf{p}}{(2\pi)^d} \sin(\beta - \gamma)\langle u_1^{*>}(\mathbf{p}, t)u_1^{*>}(\mathbf{q}, t) \rangle, \end{aligned} \quad (30)$$

where  $\Delta$  represents the wave-number region  $(\mathbf{p}, \mathbf{q}) \in (k_n, k_{n+1})$ . The second term of Eq. (30) enhances or renormalizes the kinematic viscosity leading to the following equation:

$$\begin{aligned} & [\partial_t + v_1^{(n)}k^2]u_1^<(\mathbf{k}', t) = ik' \int \frac{d\mathbf{p}}{(2\pi)^d} \sin(\beta - \gamma) \\ & \quad \times [u_1^{*<}(\mathbf{p}, t)u_1^{*<}(\mathbf{q}, t)], \end{aligned} \quad (31)$$

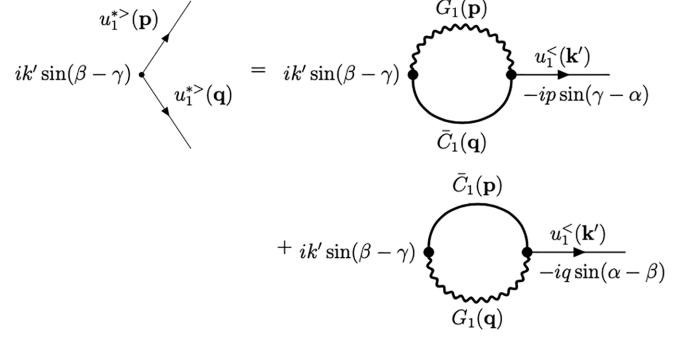


FIG. 3. Feynman diagrams associated with the renormalization of  $v_1$  for the  $u_1$  component.

where

$$v_1^{(n)}k^2 = v_1^{(n+1)}k^2 - \text{second integral of Eq. (30)}. \quad (32)$$

As we show below, Eq. (32) has two solutions. The first solution corresponds to the delta-correlated  $u_1$  for which the second integral of Eq. (30) is trivially zero [44–46]. For this case,

$$v_1^{(n)}k^2 = v_1^{(n+1)}k^2 = 0 \quad (33)$$

That is, the viscosity is not renormalized, and it remains 0 at all scales. This corresponds to the absolute equilibrium solution of Euler equation that has  $\nu = 0$  [44–46]. Hence, Euler equation and the corresponding field-theoretic equations satisfy time-reversal symmetry. The second solution, which is more complex and out of equilibrium, is computed as follows.

Under the quasi-Gaussian approximation, the second integral of Eq. (30) vanishes to the zeroth order. Hence, we expand the second term to the first order in perturbation that leads to the Feynman diagrams of Fig. 3. We compute the integral corresponding to the first loop diagram as follows. We expand  $u_1^{*>}(\mathbf{p}, t)$  using the Green's function [see Eq. (13)]:

$$\begin{aligned} u_1^{*>}(\mathbf{p}, t) &= \int_0^t dt' G_1(\mathbf{p}, t - t')(-ip) \int \frac{d\mathbf{h}}{(2\pi)^d} \sin(\gamma - \alpha) \\ & \quad \times u_1(\mathbf{h}, t')u_1(\mathbf{s}, t'), \end{aligned} \quad (34)$$

where  $\mathbf{p} + \mathbf{h} + \mathbf{s} = 0$ . We substitute the expression of Eq. (34) in the right-hand side of Eq. (30) and simplify the expression using the following relations [24,42]:

$$\langle u_1^*(\mathbf{q}, t)u_1(\mathbf{h}, t') \rangle = \bar{C}_1(\mathbf{q}, t - t')\delta(\mathbf{q} - \mathbf{h})(2\pi)^d, \quad (35)$$

$$G_1(\mathbf{k}, t - t') = \theta(t - t') \exp[-v_1(k)k^2(t - t')], \quad (36)$$

$$\bar{C}_1(\mathbf{k}, t - t') = C_1(\mathbf{k}) \exp[-v_1(k)k^2(t - t')]. \quad (37)$$

In the above equations,  $\bar{C}_1(\mathbf{k}, t - t')$  is the unequal time correlation, whereas  $C_1(\mathbf{k})$  is the equal-time correlation. Note that  $v_1(k)$  of Eqs. (36, 37) is the renormalized viscosity at wave number  $k$ . As in all field theories of turbulence, we assume that the timescales for  $G_1(\mathbf{k}, t - t')$  is same as that of  $\bar{C}_1(\mathbf{k}, t - t')$ . Equation (35) yields  $\mathbf{s} = -\mathbf{p} - \mathbf{h} = -\mathbf{p} - \mathbf{q} = \mathbf{k}'$ , using which we deduce that the

integral corresponding to the first loop diagram is

$$I_1 = \int_{\Delta} \frac{d\mathbf{p}}{(2\pi)^d} \int_0^t dt' G(\mathbf{p}, t-t')(k'p) \sin(\beta - \gamma) \times \sin(\gamma - \alpha) \bar{C}_1(\mathbf{q}, t-t') u_1^<(\mathbf{k}', t'). \quad (38)$$

Now, we employ Markovian approximation [23,38,41]. When  $v(k)k^2 \gg 1$ , the function  $\exp[-v(k)k^2(t-t')]$  rises sharply to unity near  $t' = t$ . Hence, the  $dt'$  integral gets maximal contribution near  $t' = t$ . Therefore,  $u_1(\mathbf{k}, t') \rightarrow u_1(\mathbf{k}, t)$ , and

$$I_1 = \int_{\Delta} \frac{d\mathbf{p}}{(2\pi)^d} \frac{kp \sin(\beta - \gamma) \sin(\gamma - \alpha) C_1(\mathbf{q})}{v_1(p)p^2 + v_1(q)q^2} u_1^<(\mathbf{k}', t). \quad (39)$$

Following similar steps, we compute the integral corresponding to the second loop diagram of Fig. 3 as

$$I_2 = \int_{\Delta} \frac{d\mathbf{p}}{(2\pi)^d} \frac{kq \sin(\beta - \gamma) \sin(\alpha - \beta) C_1(\mathbf{p})}{v_1(p)p^2 + v_1(q)q^2} u_1^<(\mathbf{k}', t). \quad (40)$$

Since  $I_1$  and  $I_2$  are proportional to  $u_1^<(\mathbf{k}', t)$ , these terms can be added to  $v_1^{(n+1)}k^2 u_1^<(\mathbf{k}', t)$  to yield the renormalized viscosity  $v_1^{(n)}$ . In particular, using Eqs. (31) and (32) we show that

$$\begin{aligned} v_1^{(n)}k^2 &= v_1^{(n+1)}k^2 - I'_1 - I'_2 \\ &= v_1^{(n+1)}k^2 - \int_{\Delta} \frac{d\mathbf{p}}{(2\pi)^d} \frac{k \sin(\beta - \gamma)}{v_1(p)p^2 + v_1(q)q^2} \\ &\quad \times [pC_1(\mathbf{q}) \sin(\gamma - \alpha) + qC_1(\mathbf{p}) \sin(\alpha - \beta)], \end{aligned} \quad (41)$$

where  $I'_1, I'_2$  are  $I_1, I_2$  without  $u_1^<(\mathbf{k}', t)$ .

To compute  $v_1^{(n)}$ , we choose  $k = k_n$  in Eq. (41). In addition, we make the following change of variables:

$$k = k_n; \quad \mathbf{p} = \mathbf{p}'k_n; \quad \mathbf{q} = \mathbf{q}'k_n \quad (42)$$

that yields a triad  $(1, p', q')$  with  $1 \leq p' \leq b$  and  $1 \leq q' \leq b$ . We choose  $b = 1.7$  for our calculation. Zhou *et al.* [47] showed that  $b \in (4/3, 1.8)$  yields a nearly constant value for the renormalized viscosity. McComb and Shanmugasundaram [13], and Zhou *et al.* [15] employed  $b$  in the same range. In our RG scheme, a modified version of Zhou *et al.* [47], we employ  $b = 1.7$  [which lies within  $(4/3, 1.8)$ ] so that the renormalized parameter and Kolmogorov's constant are close to the experimental values.

For the integral we employ  $p'$  and  $z = \cos \gamma$ , where  $\gamma$  is the angle between  $\mathbf{k}$  and  $\mathbf{p}$ , as the independent variables that yields

$$\int d\mathbf{p} = S_{d-1} \int_{\Delta} p'^{d-1} dp' \int_{\Delta'} dz (1-z^2)^{\frac{d-3}{2}}, \quad (43)$$

where  $\Delta, \Delta'$  are the domain of integrations:  $p' = [1, b]$  and  $z = [(p'^2 + 1 - b^2)/(2p'), p'/2]$ , in which the latter limits are obtained by setting  $q' = (1, b)$ . In this paper, we focus on  $k^{-5/3}$  spectral regime, for which  $C_1(\mathbf{k})$  is given by Eq. (23), and

$$E(k) = K_{\text{Ko}} \epsilon_u^{2/3} k^{-5/3}, \quad (44)$$

$$v_1^{(n)} = v_{1*} \sqrt{K_{\text{Ko}}} \epsilon_u^{1/3} k_n^{-4/3}, \quad (45)$$

where  $\epsilon_u$  is the energy flux,  $K_{\text{Ko}}$  is the Kolmogorov constant, and  $v_{1*}$  is the renormalization constant for  $u_1$ .

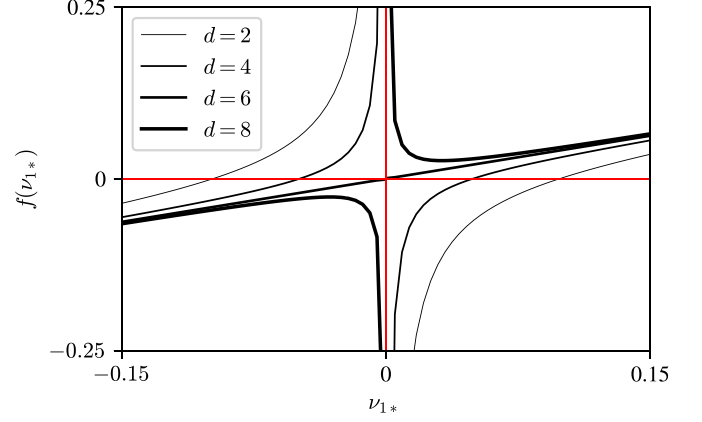


FIG. 4. Plot of  $f(v_{1*})$  vs.  $v_{1*}$  [Eq. (46)]. The solution  $v_{1*}$  is the root of  $f(v_{1*}) = 0$ . We have finite  $v_{1*}$  (both positive and negative) for  $d < 6$ , but it has no solution for  $d \geq 6$ . For  $d > 6$ , the allowed solution for the RG equation is  $v_{1*} = 0$ .

We substitute Eqs. (44) and (45) in Eq. (41), and simplify the expressions using trigonometric identities for the triad  $(1, p', q')$  (see Fig. 1). At  $k = k_n$ , these operations yield

$$\begin{aligned} v_{1*}(1 - b^{-4/3}) + \frac{2S_{d-1}}{(d-1)S_d} \frac{1}{v_{1*}} \int_1^b p'^{d-1} dp' \\ \int_{(p'^2+1-b^2)/(2p')}^{p'/2} dz (1-z^2)^{\frac{d-3}{2}} (F_1 + F_2) = f(v_{1*}) = 0, \end{aligned} \quad (46)$$

where

$$F_1(p', z) = \frac{(1-z^2)(p'-2z)(2p'z-1)p'q'^{-8/3-d}}{p'^{2/3} + q'^{2/3}}, \quad (47)$$

$$F_2(p', z) = \frac{(1-z^2)(1-p'^2)(2p'z-1)p'^{-2/3-d}q'^{-2}}{p'^{2/3} + q'^{2/3}}, \quad (48)$$

are functions of the independent variables  $p'$  and  $z$ . Equation (46) differs from those employed by McComb and Shanmugasundaram [13] and Zhou *et al.* [48] who computed the correction to  $v_1(k)$  [Eq. (41)] for all  $k$ 's that leads to a  $k$ -dependent  $v_{1*}$ . In our paper, we interpret  $v_1^{(n)}$  as the renormalized viscosity for wave numbers  $(k_0, k_n)$  that leads to a constant  $v_{1*}$ . Our scheme, which is motivated by LES [38,43], simplifies the computation of  $v_{1*}$  significantly.

The solution  $v_{1*}$  is the root of  $f(v_{1*}) = 0$  [see Eq. (46)], which is illustrated in Fig. 4 for  $d = 2, 4, 6, 8$ . For  $d < 6$ , we have positive and negative roots, out of which only the positive root is sensible because it leads to diminishing temporal correlation with the increase of  $t - t'$  [see Eq. (37)] and negative energy flux for  $d = 2$  in the  $k^{-5/3}$  regime. Hence, we work with positive  $v_{1*}$  for  $d < 6$ . In Table I we list  $v_{1*}$  for various  $d$ 's. Note that  $v_{1*}$  decreases gradually to zero as  $d \rightarrow 6$ .

For  $d \geq 6$ , Eq. (46) has no root. Therefore,  $v_{1*} = 0$ , the equilibrium solution of Euler equation, is the only solution for the RG equation. This is similar to Wilson's  $\phi^4$  theory [5], where the system transitions from nontrivial fixed point to Gaussian fixed point at  $d = 4$ . These observations indicate that  $d = 6$  is the *upper critical dimension* for hydrodynamic turbulence.

TABLE I. Renormalization constants,  $\nu_{1*}$  and  $\nu_{2*}$ , and Kolmogorov's constants,  $Ko$  and  $Ko'$ . For  $d = 6$ , the parameters correspond to the equilibrium solution.

$d$	$\nu_{1*}$	$\nu_{2*}$	$Ko$	$Ko'$
2	0.098	0.619	4.66	4.66
2.1	0.095	0.608	11.93	5.82
2.15	0.093	0.603	5.56	4.42
2.2	0.092	0.598	4.05	3.70
3	0.070	0.533	1.63	1.88
4	0.049	0.479	1.45	1.69
5	0.030	0.441	1.48	1.69
5.95	0.006	0.417	5.27	1.73
6*	0	0	-	-

To determine  $\nu_{1*}$ , we compute the integral of Eq. (46) numerically. For an accurate integration, we perform the  $dz$  integral using Gaussian quadrature and the  $dp'$  integral using a Romberg scheme. In addition, we employ *midpoint method* for computing the roots of Eq. (46). Refer to Appendix for details on the integration schemes used in this paper. We employ Python's `scipy.integrate.romberg` function whose tolerance limit is  $1.48 \times 10^{-8}$ . The values of  $\nu_{1*}$  for various  $d$ 's are listed in Table I and illustrated in Fig. 5.

For  $d = 2$ ,  $\nu_{1*}$  is the only renormalized parameter. However, higher dimensions have both  $\nu_{1*}$  and  $\nu_{2*}$ . Refer to Verma [37] for a detailed comparison of our  $\nu_{1*}$  with those reported earlier. In the next subsection, we will compute the renormalized viscosities for the  $u_2, \dots, u_{d-1}$  components.

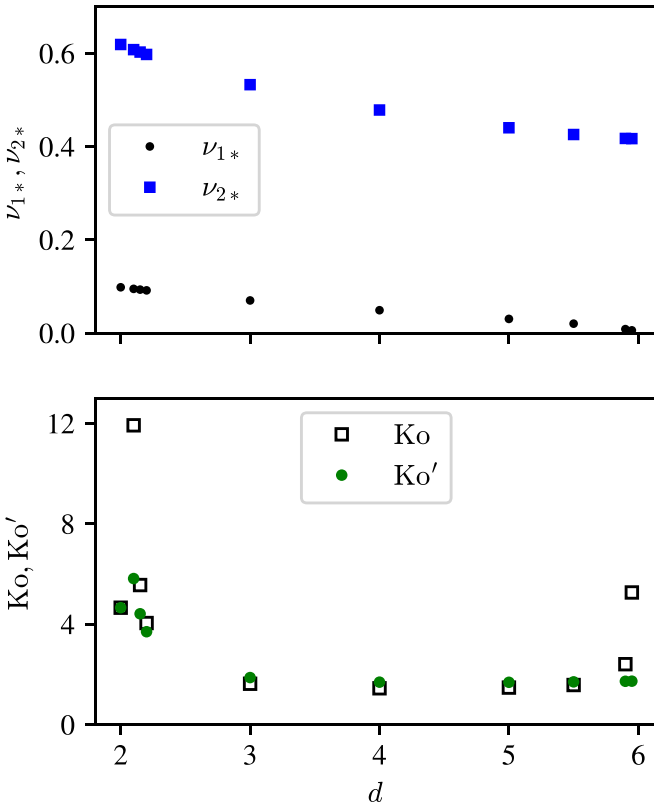


FIG. 5. Values of constants for various  $d$ 's: (a)  $\nu_{1*}$  and  $\nu_{2*}$  and (b)  $Ko$  and  $Ko'$ .

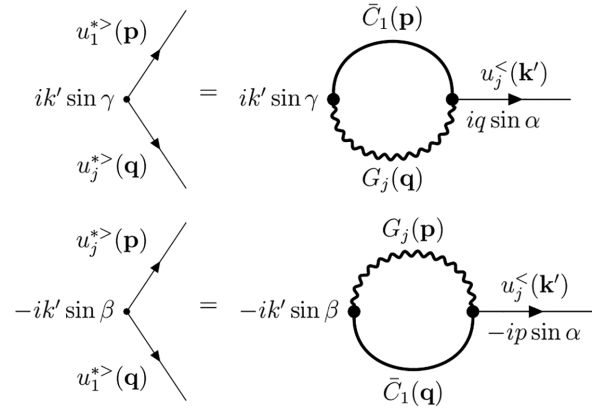


FIG. 6. Feynman diagrams associated with the renormalization of  $\nu_2$  for the  $u_j$  component.

### B. Renormalization of $u_j$ ( $j > 2$ ) components

For isotropic turbulence, the renormalized viscosities for the components  $u_2, \dots, u_{d-1}$  are the same. We denote this quantity as  $\nu_2^{(n)}$  and compute it following the same steps as in Sec. III A, but with Eq. (15). One of the intermediate steps in the derivation of  $\nu_2^{(n)}$  is

$$\begin{aligned}
 & (\partial_t + \nu_2^{(n+1)} k^2) u_j^{<}(\mathbf{k}', t) \\
 &= ik' \int \frac{d\mathbf{p}}{(2\pi)^d} \{ \sin \gamma u_1^{<}(\mathbf{p}, t) u_j^{<}(\mathbf{q}, t) \\
 &\quad - \sin \beta u_1^{<}(\mathbf{q}, t) u_j^{<}(\mathbf{p}, t) \} \\
 &\quad + ik' \int \frac{d\mathbf{p}}{(2\pi)^d} \{ \sin \gamma u_1^{>}(\mathbf{p}, t) u_j^{>}(\mathbf{q}, t) \\
 &\quad - \sin \beta u_1^{>}(\mathbf{q}, t) u_j^{>}(\mathbf{p}, t) \}, \quad (49)
 \end{aligned}$$

where  $j \geq 2$ . In the above equation, the terms of the form  $\int d\mathbf{p} (u_1^{>}(\mathbf{q}, t) u_j^{>}(\mathbf{p}, t))$  contribute to viscosity renormalization. In this subsection we show that  $\nu_2^{(n+1)} \neq \nu_1^{(n+1)}$ , which is expected because  $u_1$  and  $u_j$  with  $j \geq 2$  evolve differently [Eqs. (12) and (15)].

As in Sec. III A, we employ the isotropic correlation function of Eq. (23) and

$$\nu_2^{(n)} = \nu_{2*} \sqrt{K_{Ko}} \epsilon_u^{1/3} k_n^{-4/3}, \quad (50)$$

where  $\epsilon_u$  is the energy flux, and  $\nu_{2*}$  is the renormalization constant for  $\nu_j$  with  $j \geq 2$ . The second integral of Eq. (49) contributes to the viscosity renormalization, whose associated Feynman diagrams are shown in Fig. 6, and the corresponding integral is

$$\begin{aligned}
 I_3 &= -u_2^{<}(\mathbf{k}', t) \int_{\Delta} \frac{d\mathbf{p}}{(2\pi)^d} \\
 &\quad \times \left[ \frac{kq C_1(\mathbf{p}) \sin \gamma \sin \alpha}{\nu_1(p)p^2 + \nu_2(q)q^2} + \frac{kp C_1(\mathbf{q}) \sin \beta \sin \alpha}{\nu_2(p)p^2 + \nu_1(q)q^2} \right] \quad (51)
 \end{aligned}$$

that contributes to the viscosity renormalization as follows:

$$\begin{aligned}
 \nu_{2*} (1 - b^{-4/3}) &= \frac{2S_{d-1}}{(d-1)S_d} \int_1^b p^{d-1} dp' \\
 &\quad \times \int_{(p^2+1-b^2)/(2p')}^{p'/2} dz (1-z^2)^{\frac{d-3}{2}} F_3(p', z), \quad (52)
 \end{aligned}$$

with

$$F_3(p', z) = \frac{(1-z^2)p'^{-2/3-d}}{v_{1*}p'^{2/3} + v_{2*}q'^{2/3}} + \frac{(1-z^2)p'^2q'^{-8/3-d}}{v_{2*}p'^{2/3} + v_{1*}q'^{2/3}}. \quad (53)$$

We solve for  $v_{2*}$  by iterating Eq. (52) starting with a guess value of  $v_{2*}$ . The iterative process converges to  $v_{2*}$  listed in Table I and illustrated in Fig. 5. Since nonzero  $v_{1*}$  solution exists only for  $d < 6$ , Eq. (52) implies that  $v_{2*}$  too is valid for  $d < 6$ . For  $d \geq 6$ ,  $v_1 = v_2 = 0$  that corresponds to the equilibrium solution of Euler equation. Note  $v_{1*} \ll v_{2*}$ , as illustrated in Fig. 5.

For  $d \geq 3$ , hydrodynamic turbulence exhibits multitudes of triads, each of which have different  $\hat{n}$ . Hence, for a given  $\mathbf{k}$ , the Craya-Herring vectors of Fig. 1 transform to each other (depending on the triads). Note, however, that  $v_1(\mathbf{k}) \ll v_2(\mathbf{k})$ , hence we may estimate that  $v(k) \approx v_2(k)$ , which would be useful for LES. In spite of the above complications, independent evaluations of  $v_1(\mathbf{k})$  and  $v_2(\mathbf{k})$  yield valuable insights, chiefly that  $v_{1*} \rightarrow 0$  as  $d \rightarrow 6$ , leading to the upper critical dimension of hydrodynamic turbulence as 6. The earlier works, e.g., Ref. [29], could not reach this result because they did not resolve the renormalized viscosities for the different components of the velocity field.

Using Eqs. (45) and (50), we derive that for both  $v_1^{(n)}$  and  $v_2^{(n)}$ ,

$$\frac{v_{1,2}^{(n)}}{v_{1,2}^{(n)}} = \left(\frac{k_{n+1}}{k_n}\right)^{-4/3} = b^{-4/3}. \quad (54)$$

In quantum field theory, we express the running coupling constant in terms of  $b = \exp(l)$  [1]. Using  $b^{-4/3} \approx 1 - 4l/3$  (for small  $l$ ), we derive the beta function for  $v$  using

$$\frac{dv}{dl} \approx -\frac{4}{3}v, \quad (55)$$

or

$$\beta(v) = \frac{d \ln v}{d \ln k} \approx -\frac{4}{3}. \quad (56)$$

Note that the beta function for the coupling constant is

$$\beta(\lambda) = \frac{d \ln \lambda}{d \ln k} = 0 \quad (57)$$

due to Galilean invariance. These relations would be useful in relating field theory of turbulence and quantum field theory [1].

In the next section, we compute the energy flux using field theory.

#### IV. ENERGY TRANSFERS AND FLUXES IN $d$ DIMENSIONS

In this section, we compute the energy transfer rates and energy flux in the inertial range of hydrodynamic turbulence.

##### A. Energy transfers and flux for $u_1$ component

In this subsection, we will compute the mode-to-mode energy transfers among the  $u_1$  components within a triad. We start with Eq. (19) and present the ensemble-averaged

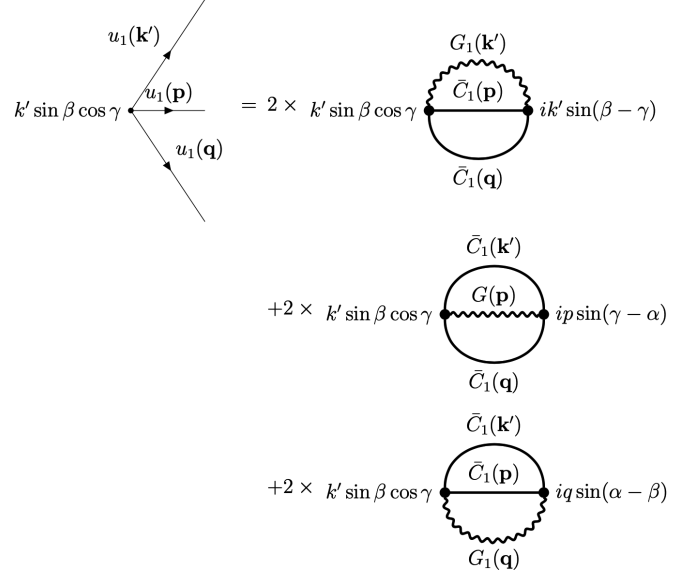


FIG. 7. Feynman diagrams associated with the energy transfers between the  $u_1$  components. Equation (61) illustrates two ways to get the second-order correlation functions, which leads to the factor 2 in all the diagrams.

mode-to-mode energy transfer from  $u_1(\mathbf{p}, t)$  to  $u_1(\mathbf{k}', t)$  with the mediation of  $u_1(\mathbf{q}, t)$ , which is

$$\langle S^{u_1 u_1}(\mathbf{k}'|\mathbf{p}|\mathbf{q}) \rangle = k' \sin \beta \cos \gamma \times \Im \{ \langle u_1(\mathbf{q}, t) u_1(\mathbf{p}, t) u_1(\mathbf{k}', t) \rangle \} \quad (58)$$

with  $\mathbf{k}' + \mathbf{p} + \mathbf{q} = 0$ . Following earlier literature [7,23], we assume that the variables  $u_1(\mathbf{p}, t)$ ,  $u_1(\mathbf{k}', t)$ , and  $u_1(\mathbf{q}, t)$  are quasnormal. Under this assumption, the triple correlation of Eq. (58) vanishes to the zeroth order. However, the first-order expansion of the triple correlation of Eq. (58) leads to a fourth-order correlation, that is expanded as a sum of products of two second-order correlations. The corresponding Feynman diagrams are given in Fig. 7.

Let us evaluate the integral corresponding to the first Feynman diagram of Fig. 7. Here,  $u_1(\mathbf{k}', t)$  is expanded using the Green's function as [see Eq. (12)]

$$u_1(\mathbf{k}', t) = i \int_0^t dt' G_1(\mathbf{k}', t-t') k' \int \frac{d\mathbf{h}}{(2\pi)^d} \sin(\beta - \gamma) \times [u_1^*(\mathbf{h}, t') u_1^*(\mathbf{s}, t')] \quad (59)$$

with  $\mathbf{k}' + \mathbf{h} + \mathbf{s} = 0$ . Substitution of the above in Eq. (58) leads to a fourth-order correlation, which is expanded as a sum of products of two second-order correlations:

$$\begin{aligned} & \langle u_1(\mathbf{q}, t) u_1(\mathbf{p}, t) u_1(\mathbf{h}, t') u_1(\mathbf{s}, t') \rangle \\ &= \langle u_1(\mathbf{q}, t) u_1(\mathbf{p}, t) \rangle \langle u_1(\mathbf{h}, t') u_1(\mathbf{s}, t') \rangle \\ &+ \langle u_1(\mathbf{q}, t) u_1(\mathbf{h}, t') \rangle \langle u_1(\mathbf{p}, t) u_1(\mathbf{s}, t') \rangle \\ &+ \langle u_1(\mathbf{q}, t) u_1(\mathbf{s}, t') \rangle \langle u_1(\mathbf{p}, t) u_1(\mathbf{h}, t') \rangle. \end{aligned} \quad (61)$$

Note that  $\langle u_1(\mathbf{q}, t) u_1(\mathbf{p}, t) \rangle = \langle u_1(\mathbf{h}, t') u_1(\mathbf{s}, t') \rangle = 0$  because  $\mathbf{p} + \mathbf{q} = \mathbf{k} \neq 0$  and  $\mathbf{r} + \mathbf{s} = \mathbf{k} \neq 0$ . Using the above

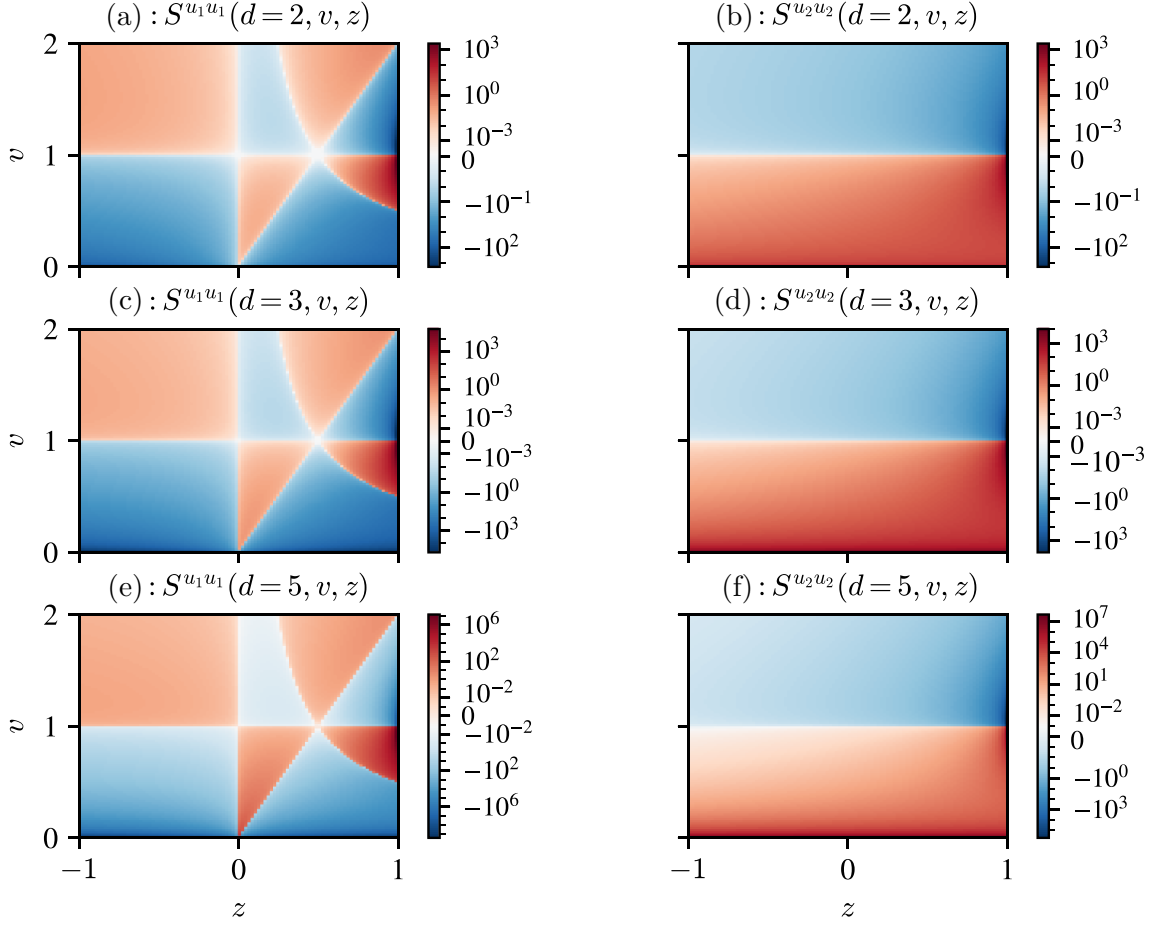


FIG. 8. The density plots of  $\langle S^{u_1 u_1}(v, z) \rangle$  and  $\langle S^{u_2 u_2}(v, z) \rangle$  for  $d = 2$  (top row),  $d = 3$  (middle row), and  $d = 5$  (bottom row).

correlations, we deduce that

$$\begin{aligned} & \langle u_1(\mathbf{q}, t) u_1(\mathbf{p}, t) u_1(\mathbf{k}', t) \rangle_a \\ &= \int_0^t dt' G_1(\mathbf{k}', t - t') i k' \\ & \quad \times \sin(\beta - \gamma) 2 \bar{C}_1(\mathbf{p}, t - t') \bar{C}_1(\mathbf{q}, t - t'). \end{aligned} \quad (62)$$

Using the properties of temporal relations of Eqs. (36) and (37), we deduce that

$$\langle u_1(\mathbf{q}, t) u_1(\mathbf{p}, t) u_1(\mathbf{k}', t) \rangle = \frac{i 2 k' \sin(\beta - \gamma) C_1(\mathbf{p}) C_1(\mathbf{q})}{v_1(k) k^2 + v_1(p) p^2 + v_1(q) q^2}. \quad (63)$$

This term plus other two terms of Fig. 7 yields

$$\langle S^{u_1 u_1}(\mathbf{k}' | \mathbf{p} | \mathbf{q}) \rangle = \frac{\text{numr}_1}{v_1(k) k^2 + v_1(p) p^2 + v_1(q) q^2}, \quad (64)$$

where

$$\begin{aligned} \text{numr}_1 &= 2 k' \sin \beta \cos \gamma [k' \sin(\beta - \gamma) C_1(\mathbf{p}) C_1(\mathbf{q}) \\ & \quad + p \sin(\gamma - \alpha) C_1(\mathbf{k}') C_1(\mathbf{q}) \\ & \quad + q \sin(\alpha - \beta) C_1(\mathbf{k}') C_1(\mathbf{p})]. \end{aligned} \quad (65)$$

The physics in the inertial range is scale invariant, hence we employ the following transformations [7,41]:

$$k = \frac{R}{u}; \quad p = \frac{Rv}{u}; \quad q = \frac{Rw}{u}; \quad (66)$$

that leads to

$$\langle S^{u_1 u_1}(\mathbf{k}' | \mathbf{p} | \mathbf{q}) \rangle = (2\pi)^{2d} \epsilon_u k^{-2d} K_{\text{Ko}}^{3/2} \frac{4}{S_d^2 (d-1)^2} \langle S^{u_1 u_1}(v, z) \rangle, \quad (67)$$

where

$$\langle S^{u_1 u_1}(v, z) \rangle = \frac{\text{numr}_2}{v_{1*} (1 + v^{2/3} + w^{2/3})} \quad (68)$$

with

$$\begin{aligned} \text{numr}_2 &= 2(2vz - 1)(1 - z^2) z v w^{-2} (vw)^{-2/3-d} \\ & \quad + 2(v - 2z)(1 - z^2) z v^2 w^{-8/3-d} \\ & \quad + 2(1 - v^2)(1 - z^2) z w^{-2} v^{-1/3-d} \end{aligned} \quad (69)$$

and  $w^2 = 1 + v^2 - 2vz$ . Note that  $\langle S^{u_1 u_1}(\mathbf{k}' | \mathbf{p} | \mathbf{q}) \rangle$  has a dimension of  $k^{-2d}$ , whereas  $\langle S^{u_1 u_1}(v, z) \rangle$  is dimensionless. In Figs. 8(a), 8(c), and 8(e) we illustrate the density plots of  $\langle S^{u_1 u_1}(v, z) \rangle$  for  $d = 2, 3, 5$  respectively.

As is evident in Figs. 8(a), 8(c), and 8(e), the function  $\langle S^{u_1 u_1}(v, z) \rangle$  exhibits the following interesting properties:

(1)  $\langle S^{u_1 u_1}(v, z) \rangle \rightarrow \infty$  as  $z \rightarrow 1$  and  $v \rightarrow 1$ , hence  $\langle S^{u_1 u_1}(v, z) \rangle$  is singular near this region. Note that  $\langle S^{u_1 u_1}(v, z) \rangle$  in the figure is bounded due to the finite resolution. For  $z \approx 1$ ,  $\langle S^{u_1 u_1}(v, z) \rangle$  is positive when  $v < 1$  and negative otherwise. This feature illustrates forward energy transfers in hydrodynamic turbulence [37,49,50].



(2)  $\langle S^{u_1 u_1}(v, z) \rangle$  takes large negative values for all  $z$ 's when  $v \rightarrow 0$ . These are nonlocal reverse energy transfers from  $k'$  to  $p$  when  $p \ll k'$ . These transfers are responsible for the inverse energy cascade in 2D hydrodynamic turbulence when  $E(k) \sim k^{-5/3}$ .

(3) The singularity of  $\langle S^{u_1 u_1}(v, z) \rangle$  become more severe with the increase of  $d$ . For example, asymptotically  $\langle S^{u_1 u_1}(v, z) \rangle \rightarrow (1-v)^{-8/3-d}$  as  $v \rightarrow 1$  and  $z \rightarrow 1$  [37,50].

The above properties are in agreement with the earlier results [37,50].

After a brief discussion on  $\langle S^{u_1 u_1}(v, z) \rangle$  we compute the energy flux in 2D arising from the cumulative energy transfers:

$$\langle \Pi_{u_1}(R) \rangle = \int_R^\infty \frac{d\mathbf{k}'}{(2\pi)^2} \int_0^R \frac{d\mathbf{p}}{(2\pi)^2} \langle S^{u_1 u_1}(\mathbf{k}'|\mathbf{p}|\mathbf{q}) \rangle. \quad (70)$$

Substitution of Eq. (67) transforms the energy flux equation to [7,37]

$$\frac{\langle \Pi_{u_1}(R) \rangle}{\epsilon_u} = A \int_0^1 dv [\ln(1/v)] v^{d-1} \int_{-1}^1 dz (1-z^2)^{\frac{d-3}{2}} \times \langle S^{u_1 u_1}(v, z) \rangle, \quad (71)$$

where

$$A = K_{\text{Ko}}^{3/2} \frac{4}{(d-1)^2} \frac{S_{d-1}}{S_d}. \quad (72)$$

We compute the double integral of Eq. (71) numerically. We employ Gauss-Jacobi quadrature for the  $dz$  integral, and Romberg iterative scheme for the  $dv$  integral. Refer to Section A for a brief discussion on the integration procedure.

Two-dimensional hydrodynamics has only  $u_1$  component. Hence,  $\Pi_{u_1}(R)$  is the energy flux for 2D turbulence. In the  $k^{-5/3}$  regime of 2D turbulence, we observe that  $\Pi_{u_1}(R) < 0$  indicating an inverse cascade of energy, consistent with the predictions of Kraichnan [51]. Using  $\Pi_{u_1}(R) = -\epsilon_u$  and  $v_{1*} = 0.098$ , we deduce that  $K_{\text{Ko}} = 1.19$ . This  $K_{\text{Ko}}$  is lower than that reported in experiments and numerical situations, which is approximately 6. This inconsistency is possibly due to the inability of the recursive RG schemes to capture the nonlocal interactions [37]. Note that 2D hydrodynamic turbulence involves local forward energy transfer and nonlocal inverse energy transfer, which is difficult to incorporate in RG procedure. Verma [37] employed a temporary fix for this problem by increasing the lower cutoff of the flux integral to 0.22. We find that  $\int_{0.22}^1 dv \dots$  yields  $K_{\text{Ko}} = 4.46$ , which is close to the earlier numerical and experimental results. Hopefully, in future we will understand the reason for the cutoff better.

### B. Energy transfers and fluxes for the $u_j$ components with $j \geq 2$

As shown in Eq. (20), the mode-to-mode energy transfer from  $u_j(\mathbf{p})$  to  $u_j(\mathbf{k}') [j \geq 2]$  with the mediation of  $u_1(\mathbf{q})$  is

$$\langle S^{u_j u_j}(\mathbf{k}'|\mathbf{p}|\mathbf{q}) \rangle = -k' \sin \beta \Im \{ \langle u_1(\mathbf{q}, t) u_j(\mathbf{p}, t) u_j(\mathbf{k}', t) \rangle \}. \quad (73)$$

Note that  $\langle S^{u_j u_j}(\mathbf{k}'|\mathbf{p}|\mathbf{q}) \rangle$  are the same for all  $j$ 's from  $j = 2$  to  $d-1$  due to isotropy. To compute this quantity we employ the scheme described in Sec. IV A. For simplicity, we restrict ourselves to flows for which  $\langle u_1 u_j \rangle = 0$  when  $j > 1$ .

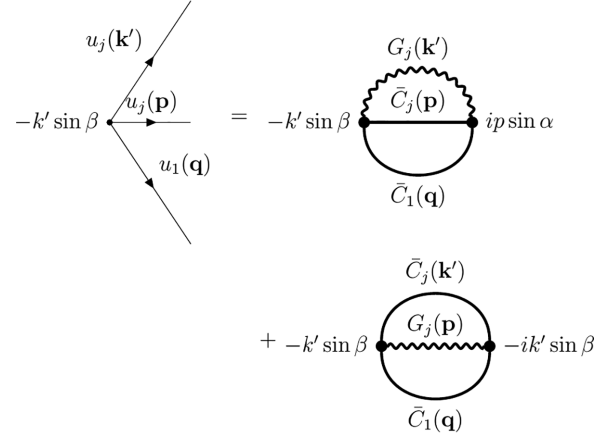


FIG. 9. Feynman diagrams associated with the energy transfers between the  $u_j$  component ( $j \geq 2$ ).

Consequently, the expansion of  $u_j$  components in terms of the Green's function yields nonzero values, whereas the terms arising from the expansion of  $u_1$  component vanishes identically. The Feynman diagrams associated with  $\langle S^{u_j u_j}(\mathbf{k}'|\mathbf{p}|\mathbf{q}) \rangle$  (for  $j \geq 2$ ) are illustrated in Fig. 9.

Following the same steps as in Sec. IV A, the field-theoretic estimate for  $\langle S^{u_j u_j}(\mathbf{k}'|\mathbf{p}|\mathbf{q}) \rangle$  is

$$\langle S^{u_j u_j}(\mathbf{k}'|\mathbf{p}|\mathbf{q}) \rangle = (k \sin \beta)^2 \frac{C_1(\mathbf{q})[C_j(\mathbf{p}) - C_j(\mathbf{k}')] }{v_2(k)k^2 + v_2(p)p^2 + v_1(q)q^2}. \quad (74)$$

We assume that turbulence is isotropic, hence  $C_j(\mathbf{k}') = C(\mathbf{k}')$ . In addition, we transform  $\langle S^{u_j u_j}(\mathbf{k}'|\mathbf{p}|\mathbf{q}) \rangle$  to  $\langle S^{u_j u_j}(v, z) \rangle$  as follows:

$$\langle S^{u_j u_j}(\mathbf{k}'|\mathbf{p}|\mathbf{q}) \rangle = (2\pi)^{2d} \epsilon_u k^{-2d} K_{\text{Ko}}^{3/2} \frac{4}{S_d^2(d-1)^2} \times \langle S^{u_j u_j}(v, z) \rangle \quad (75)$$

with

$$\langle S^{u_j u_j}(v, z) \rangle = \frac{v^2 w^{-8/3-d} (v^{-2/3-d} - 1)(1 - z^2)}{v_{2*}(1 + v^{2/3}) + v_{1*} w^{2/3}}. \quad (76)$$

In Figs. 8(b), 8(d), and 8(f), we illustrate  $\langle S^{u_2 u_2}(v, z) \rangle$  for  $d = 2, 3, 5$ . As shown in the figure,  $\langle S^{u_2 u_2}(v, z) \rangle$  exhibits the following interesting properties:

(1)  $\langle S^{u_j u_j}(v, z) \rangle$  is singular when  $z \rightarrow 1$  and  $v \rightarrow 1$ .  $\langle S^{u_j u_j}(v, z) \rangle$  is positive when  $v < 1$  and negative otherwise, implying forward energy transfers when  $p \rightarrow k'$  [37,49,50].

(2)  $\langle S^{u_j u_j}(v, z) \rangle \gg 1$  for all  $z$ 's when  $v \rightarrow 0$ . These transfers represent forward nonlocal energy transfers from  $p$  to  $k'$ .

(3) The severity of the singularity of  $\langle S^{u_j u_j}(v, z) \rangle$  increases with  $d$ , with  $\langle S^{u_2 u_2}(v, z) \rangle \rightarrow (1-v)^{-8/3-d}$  as  $v \rightarrow 1$  and  $z \rightarrow 1$  [37,50].

Now we compute the energy flux that receives contributions from the  $u_j$  components as given below:

$$\langle \Pi_{u_j}(R) \rangle = \int_R^\infty \frac{d\mathbf{k}'}{(2\pi)^3} \int_0^R \frac{d\mathbf{p}}{(2\pi)^3} \langle S^{u_j u_j}(\mathbf{k}'|\mathbf{p}|\mathbf{q}) \rangle, \quad (77)$$

hence

$$\frac{\langle \Pi_{u_j}(R) \rangle}{\epsilon_u} = A \int_0^1 dv [\ln(1/v)] v^{d-1} \int_{-1}^1 dz (1-z^2)^{\frac{d-3}{2}} \times \langle S^{u_j u_j}(v, z) \rangle, \quad (78)$$

where  $A$  is given by Eq. (72). For Eq. (78), we perform the  $dz$  integral using Gauss-Jacobi quadrature, whereas the  $dv$  integral using Romberg iterative scheme. Note that the total energy flux in the inertial range is

$$\Pi(R) = \langle \Pi_{u_1}(R) \rangle + (d-2) \langle \Pi_{u_2}(R) \rangle, \quad (79)$$

which equals the dissipation range  $\epsilon_u$ .

Now a brief discussion on the 3D energy flux, which is

$$\langle \Pi(R) \rangle = \langle \Pi_{u_1}(R) \rangle + \langle \Pi_{u_2}(R) \rangle = \epsilon_u \quad (80)$$

Note that  $\nu_{1*}$  and  $\nu_{2*}$  appear in the denominators of Eqs. (68) and (76) respectively. Since  $\nu_{1*} \ll \nu_{2*}$ , the negative energy flux  $\Pi_{u_1}$  dominates positive  $\Pi_{u_2}$  leading to  $\Pi(R) < 0$ . This is a problem. Fortunately, this issue is easily resolved by employing  $\int_{0,22}^1 dv$  for  $\Pi_{u_1}(R)$  of Eq. (70) (as in Sec. IV A); this procedure yields  $K_{K_0} = 1.64$ , which is in good agreement with earlier field-theoretic computations, as well as numerical and experimental results. As discussed in Sec. III B,  $\hat{e}_1(\mathbf{k})$  and  $\hat{e}_2(\mathbf{k})$  are transformed to each other under the change of triads ( $\hat{n}$ ). Therefore, we may also use  $\nu_{1*} \leftarrow \nu_{2*}$  that yields  $K'_{K_0} = 1.89$ .

For  $d$  dimensions, the solution of Eq. (79) yields  $K_{K_0}$  and  $K'_{K_0}$  for various  $d$ 's. These results are listed in Table I and illustrated in Fig. 5. The constant  $K_{K_0}$  increases from  $d = 2$  to  $d = 2.1$ , then decreases up to  $d = 4$ , and finally increases again up to  $d = 6$ . Note that these constants are not defined for  $d \geq 6$ , where the equilibrium solution [ $E(k) \sim k^{d-1}$  with zero flux] is valid. Thus,  $d = 6$  is the critical dimension for hydrodynamic turbulence. Also, for a given  $E(k)$ ,  $\epsilon_u \propto K_{K_0}^{-3/2}$ . Hence,  $\epsilon_u$  is inversely proportional to  $K_{K_0}^{3/2}$ .

We compare our predictions with those in the past literature. Using Lagrangian renormalized approximation, Gotoh *et al.* [30] showed that the Kolmogorov's constant for 3D and 4D are 1.72 and 1.31, respectively. Berera *et al.* [31] reported the corresponding constants to be 1.7 and 1.3, respectively. The corresponding numbers in our calculations, 1.63 and 1.45, are in general agreement with the earlier results.

Exploration of turbulence in fractal dimension remains a challenge. Lanotte *et al.* [52] simulated hydrodynamic turbulence in fractal dimension between 2.5 to 3. They studied variations of energy spectrum and probability distribution function of vorticity as function of fractional dimensions. It will be interesting to employ similar ideas to dimensions close to 2 and for much higher dimensions, but we expect these numerical experiments to be very expensive. It is possible that the fractional dimensions is related to the quasi-2D anisotropic turbulence that shows a transition from positive energy flux to negative energy flux with the decrease of vertical dimension [53].

## V. FRACTIONAL ENERGY TRANSFERS

To disentangle the energy transfers between various wave-number regimes, as well as to quantify the dependence of the

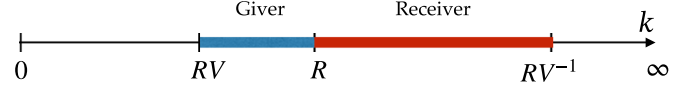


FIG. 10. The fractional energy transfer  $\langle \Pi_V(R) \rangle / \epsilon_u$  is the energy transfer from wave-number shell  $(RV, R)$  to  $(R, R/V)$ , where  $V < 1$ .

energy flux on  $d$ , we define *fractional energy flux* as follows:

$$\begin{aligned} \frac{\langle \Pi_V(R) \rangle}{\epsilon_u} &= A \int_V^1 dv [\ln(1/v)] v^{d-1} \int_{-1}^1 dz (1-z^2)^{\frac{d-3}{2}} \\ &\quad \times [\langle S^{u_1 u_1}(v, z) \rangle + (d-1) \langle S^{u_2 u_2}(v, z) \rangle] \\ &= \int_V^1 dv T(v), \end{aligned} \quad (81)$$

where  $0 < V < 1$ . Based on the relations of Eq. (66), we deduce that  $\langle \Pi_V(R) \rangle$  represents the net energy transfer from the giver modes in the band  $(RV, R)$  to the receiver modes in the band  $(R, R/V)$ . See Fig. 10 for an illustration.

We compute  $\int_V^1 dv T(v)$  for  $V = (0, 1)$  and  $d = 2, 2.1, 2.15, 2.5, 3, 4, 5.9$ , and plot them in Figs. 11(a) and 11(b). The plots in the figure reveal that  $\Pi(R)/\epsilon_u = \int_0^1 dv T(v)$  is negative for  $d < 2.15$ , positive for  $d > 2.15$ , and 0 for  $d \approx 2.15$ . Hence, the energy flux  $\langle \Pi(R) \rangle$  changes sign at  $d \approx 2.15$ . The spiking in  $K_{K_0}$  near  $d = 2.15$  is due to the vanishing of the energy flux (see Fig. 5). Our results are in a reasonable agreement with those of Fournier and Frisch [29] who reported the transition dimension for the energy flux to be approximately 2.06.

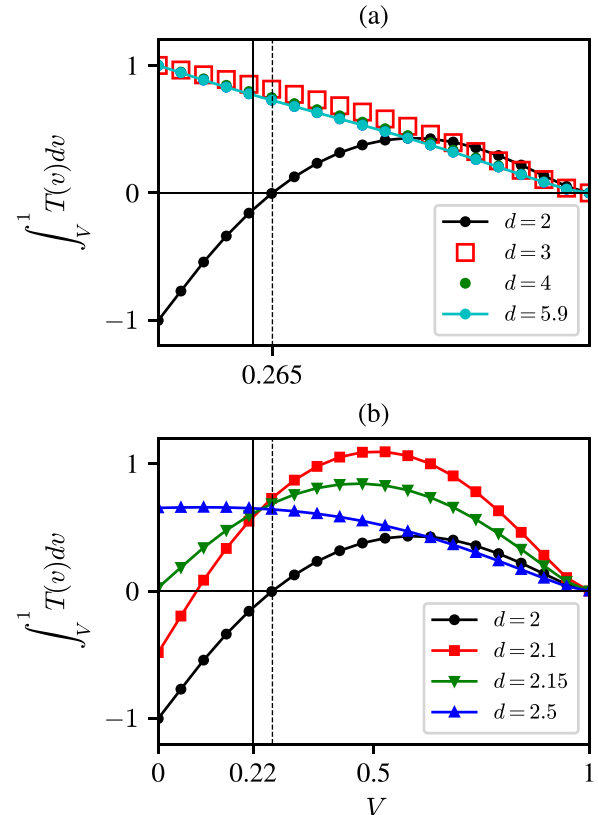


FIG. 11. Fractional energy transfer  $\langle \Pi_V(R) \rangle / \epsilon_u$  for various dimensions.

In 2D,  $\int_V^1 dvT(v)$  changes sign from negative to positive as  $V$  crosses 0.265 from left to right [see Fig. 11(a)]. This feature arises due to the positive local transfers, but significant negative (inverse) nonlocal energy transfers [37,50]. The inverse energy cascade in 2D hydrodynamics is due to the above nonlocal reverse energy transfers. For more details on local and nonlocal energy transfers, refer to Verma [37].

In the next section, we compute the renormalization and Kolmogorov's constants for Kraichnan's  $k^{-3/2}$  spectrum.

## VI. RENORMALIZATION AND ENERGY FLUX COMPUTATIONS FOR KRAICHNAN'S $k^{-3/2}$ SPECTRUM

Kraichnan [54] argued that the sweeping effect may lead to  $k^{-3/2}$  energy spectrum for hydrodynamic turbulence. However, experiments, numerical simulations, and analytical works rule out this spectrum, and strongly support Kolmogorov's  $k^{-5/3}$  spectrum. Still, for mathematical curiosity we explore whether  $k^{-3/2}$  spectrum satisfies the RG equation.

In the  $k^{-3/2}$  framework,

$$\bar{E}(k) = K_{\text{Kr}}(\epsilon_u U_0)^{1/2} k^{-3/2}, \quad (82)$$

$$\bar{\nu}_1(k_n) = \nu_{\text{Kr}1*} K_{\text{Kr}}^{1/2} (\epsilon_u U_0)^{1/4} k_n^{-5/4}, \quad (83)$$

$$\bar{\nu}_2(k_n) = \nu_{\text{Kr}2*} K_{\text{Kr}}^{1/2} (\epsilon_u U_0)^{1/4} k_n^{-5/4}, \quad (84)$$

where  $U_0$  is the large-scale fluid velocity;  $\nu_{\text{Kr}1*}$  and  $\nu_{\text{Kr}2*}$  are the renormalization constants for  $u_1$  and  $u_2$  components; and  $K_{\text{Kr}}$  is Kraichnan's constant (corresponding to Kolmogorov's constant). For the  $k^{-3/2}$  energy spectrum, the Feynman diagrams and all the equations of Secs. III and IV, except those for  $F_1(p', z)$ ,  $F_2(p', z)$ ,  $F_3(p', z)$ ,  $\langle S^{u_i u_j}(v, z) \rangle$ ,  $\langle S^{u_i u_j}(v, z) \rangle$ ,  $\text{num}_2$ ,  $E(k)$ ,  $\nu_1(k_n)$ , and  $\nu_2(k_n)$ , are unchanged. The above equations are modified to the following form (with bar):

$$\bar{F}_1(p', z) = \frac{(1 - z^2)(p' - 2z)(2p'z - 1)p'q'^{-5/2-d}}{p'^{3/4} + q'^{3/4}}, \quad (85)$$

$$\bar{F}_2(p', z) = \frac{(1 - z^2)(1 - p^2)(2p'z - 1)p'^{-1/2-d}q'^{-2}}{p'^{3/4} + q'^{3/4}}, \quad (86)$$

$$\bar{F}_3(p', z) = \frac{(1 - z^2)p'^{-1/2-d}}{\nu_{\text{Kr}1*}p'^{3/4} + \nu_{\text{Kr}2*}q'^{3/4}} + \frac{(1 - z^2)p'^2q'^{-5/2-d}}{\nu_{\text{Kr}2*}p'^{3/4} + \nu_{\text{Kr}1*}q'^{3/4}}, \quad (87)$$

$$\langle \bar{S}^{u_i u_j}(v, z) \rangle = \frac{v^2 w^{-5/2-d}(v^{-1/2-d} - 1)(1 - z^2)}{\nu_{\text{Kr}2*}(1 + v^{3/4}) + \nu_{\text{Kr}1*}w^{3/4}}, \quad (88)$$

$$\langle \bar{S}^{u_i u_i}(v, z) \rangle = \frac{\overline{\text{num}}_2}{\nu_{\text{Kr}1*}(1 + v^{3/4} + w^{3/4})} \quad (89)$$

$$\begin{aligned} \overline{\text{num}}_2 &= 2(2vz - 1)(1 - z^2)zv w^{-2}(vw)^{-1/2-d} \\ &\quad + 2(v - 2z)(1 - z^2)zv^2 w^{-5/2-d} \\ &\quad + 2(1 - v^2)(1 - z^2)zv w^{-2}v^{-1/2-d} \end{aligned} \quad (90)$$

for  $j \geq 2$ . Using the revised equations we compute the new renormalization and Kraichnan's constants for various space dimensions. We observe that  $\nu_{\text{Kr}1*}$  has nonzero solution for  $d < 6$ , and it has no solution for  $d > 6$ . However,  $\nu_* = 0$

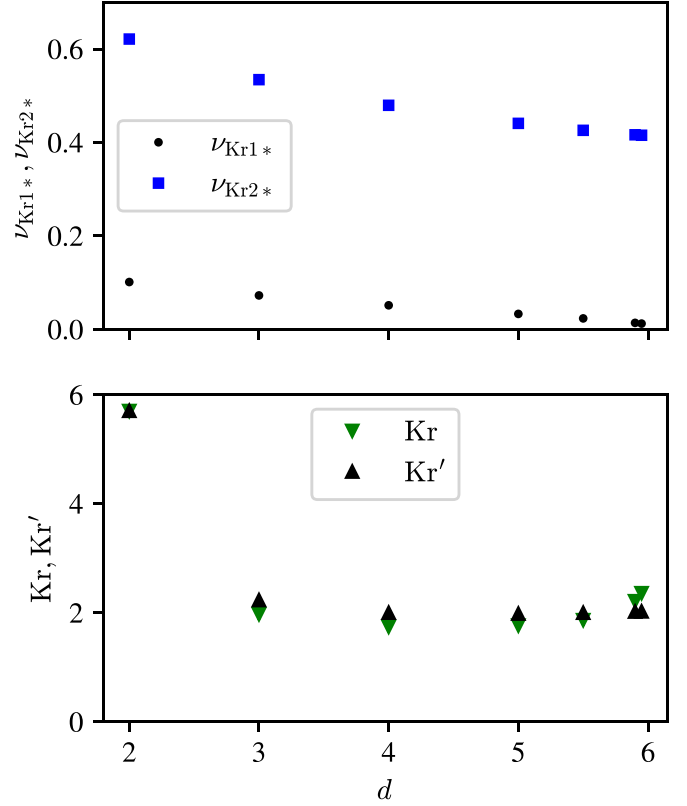


FIG. 12. For Kraichnan's  $k^{-3/2}$  spectrum, the values of constants for various  $d$ 's: (a)  $\nu_{\text{Kr}1*}$  and  $\nu_{\text{Kr}2*}$ ; (b)  $Kr$  and  $Kr'$ .

is a valid solution for  $d > 6$ . Hence,  $d = 6$  is the critical dimension for the Kraichnan's spectrum as well. In Fig. 12 we present the renormalized parameters and Kraichnan's constant for various dimensions. Note that the constants  $Kr$  and  $Kr'$  correspond respectively to cases when  $\nu_{\text{Kr}1*} \neq \nu_{\text{Kr}2*}$  and  $\nu_{\text{Kr}1*} = \nu_{\text{Kr}2*}$ .

Thus, surprisingly, Kraichnan's  $k^{-3/2}$  spectrum and the corresponding viscosity formulas are solutions to the RG equations for  $d < 6$ . We conclude in the next section.

## VII. CONCLUSIONS

In this paper, we employ perturbative field theory to the incompressible Navier-Stokes equation and compute the renormalized viscosities and Kolmogorov's constant for various space dimensions. We employ Craya-Herring basis that simplifies the calculations considerably. We summarize our findings as follows.

(1) For space dimension less than 6, Kolmogorov's spectrum  $E(k) \sim k^{-5/3}$  is a solution of the RG equation with the renormalized viscosity scaling as  $\nu_{(1,2)*} \sqrt{\text{Ko}} \epsilon_u^{1/3} k^{-4/3}$ , where  $\nu_{(1,2)*}$  are the prefactors for the components of the Craya-Herring basis. These constants are computed using the recurrence relation for the renormalized viscosity. Our computed constants are in general agreement with earlier results. These solutions are out of equilibrium when the energy flux is nonzero.

(2) The renormalization constants  $\nu_{(1,2)*}$  are functions of space dimension. Interestingly  $\nu_{1*}$  gradually decreases to zero

at  $d = 6$ . Our detailed calculations show that the solutions for  $d < 6$  are out of equilibrium, but they merge with equilibrium solution at  $d = 6$ . Thus,  $d = 6$  is the critical upper dimension for hydrodynamic turbulence.

(3) For  $d \geq 6$ , the viscosity remains *unnormalized* ( $\nu = 0$ ), and the equilibrium solution of Euler equation,  $E(k) \sim k^{d-1}$ , satisfies the RG equation. Note that the energy flux vanishes under this condition. Adzhemyan *et al.* [32] showed that the Kolmogorov constant  $K_{\text{Ko}} \propto d^{1/3}$  which leads the vanishing energy flux,  $\epsilon_u \propto K_{\text{Ko}}^{-3/2} \propto d^{-1/2} \rightarrow 0$ , as  $d \rightarrow \infty$ . In similar lines, Fournier *et al.* [33] showed that intermittency vanishes as  $d \rightarrow \infty$ . These observations too indicate Gaussianity of the velocity field at large  $d$ , consistent with our results. Our renormalization calculation indicates that the upper critical dimension for hydrodynamic turbulence is 6. Thus, we can argue that the nonequilibrium solution with nonzero energy flux transitions to the equilibrium solution with  $\epsilon_u = 0$ . Note that the equilibrium solution with  $\nu = 0$  respects time-reversal symmetry. However, the nonequilibrium solution with finite  $\nu$  and nonzero energy flux breaks the time-reversal symmetry [55].

(4) Using field theory, we compute the mode-to-mode energy transfers, energy fluxes, and Kolmogorov's constant for  $d < 6$ . The energy flux is negative for  $d < 2.15$ , whereas it is positive for  $d > 2.15$ . The transition dimension  $d = 2.15$  is in reasonable agreement with the predictions of Fournier and Frisch [29], according to which the energy flux changes sign near  $d \approx 2.05$ .

(5) Our results, in particular Kolmogorov's constant, are in agreement with previous works for 4D turbulence simulations [30,31]. Note that simulation of turbulent flows for  $d \geq 4$  is very expensive due to large grid size. Simulation of turbulence in fractional dimension is of interest [52], but these simulations too require considerable computing resources.

(6) The present work does not include intermittency correction, which is more complex to compute. Researchers have employed multiloop field-theoretic calculations (e.g., Ref. [56]) and Lagrangian field-theory calculations (e.g., Ref. [57]) to quantify intermittency in turbulence. It will be interesting to use Craya-Herring basis for intermittency computations.

(7) Interestingly, Kraichnan's  $k^{-3/2}$  energy spectrum, which is inspired by the sweeping effect, too satisfies the recursive RG equation for hydrodynamic turbulence. Note, however, that  $k^{-3/2}$  energy spectrum is ruled out based on

numerical and experimental findings, as well as from analytical works such as Kolmogorov's K41 theory [58].

In summary, field-theoretic tools provide valuable insights into hydrodynamic turbulence.

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## APPENDIX: EVALUATION OF THE RENORMALIZATION AND ENERGY FLUX INTEGRALS

For the RG procedure, the integrals of Eqs. (46) and (52) are finite because they are performed in the band  $1 \leq p' \leq b$  and  $1 \leq q' \leq b$ . Here, we employ the Gaussian quadrature for the  $dq'$  integral and a Romberg scheme for the  $dp'$  integral. This procedure yields finite and accurate results. However, the integrals for the energy flux are singular [1] and they need special attention.

The energy-flux integral is of the following form:

$$I = \int_0^1 dv [\ln(1/v)] v^{d-1} \int_{-1}^1 dz (1-z^2)^{(d-1)/2} f(v, z), \quad (\text{A1})$$

where  $f(v, z)$  involves singularities. For accurate evaluation of  $dz$  integration, we employ the Gauss-Jacobi quadrature:

$$\int_{-1}^1 dz f(v, z) (1-z)^{(d-1)/2} (1+z)^{(d-1)/2} \approx \sum_k f(v, z_k) w_k, \quad (\text{A2})$$

where  $z_k$  is the  $k$ th root of Jacobi polynomials, and  $w_k$  is the corresponding *weight*. Note that  $f(v, z_k)$  is evaluated at  $z = z_k$ . The Gauss-Jacobi quadrature yields finite answer for these singular integrals. We employ a Romberg iterative scheme for the subsequent  $dv$  integration.

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