Methodological notes on gauge invariance in the treatment of waves and oscillations in plasmas via the Einstein-Vlasov-Maxwell system: Fundamental equations

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The theory of gauge transformations in linearized gravitation is investigated. After a brief discussion of the fundamentals of the kinetic theory in curved spacetime, the Einstein-Vlasov-Maxwell (EVM) system of equations in terms of gauge-invariant quantities is established without neglecting the equations of motion associated with the dynamics of the nonradiative components of the metric tensor. The established theory is applied to a noncollisional electron-positron plasma, leading to a dispersion relation for gravitational waves in this model system. The problem of Landau damping is addressed and some attention is given to the issue of the energy exchanges between the plasma and the gravitational wave. In a future paper, a more complete set of approximate dispersion relations for waves and oscillations in plasmas will be presented, including the dynamics of nonradiative components of the metric tensor, with special attention to the problems of the Landau damping and of the energy exchanges between matter, the electromagnetic field and the gravitational field.

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I. INTRODUCTION

The questions about the propagation and even the existence of gravitational waves go back to the very foundations of the general theory of relativity [1,2]. These waves have been well investigated (both experimentally and theoretically [3–16]) over the past decades, culminating in their detection some years ago by the LIGO system [17–20]. From a theoretical point of view, the discussion of gravitational waves is very facilitated and simplified in the linear regime [7–10]. In this limiting case, it is relatively simple to show that gravitation behaves most like electromagnetism, in the sense that Einstein equations exhibits gauge freedom, in the same way as Maxwell equations, and that the gravitational waves possesses, as electromagnetic waves do, two independent states of polarization. On the other hand, it is well established in the realm of the classic field theory that only gauge-invariant quantities could have a physical significance [21,22], and one of the objectives of this paper is to exploit this subject and properly apply it to the study of waves and oscillations in fully relativistic plasmas in the context of the kinetic theory [23].

The history of the theoretical study of the propagation of gravitational waves in plasmas began long before the experimental confirmation of their existence. In the 1970s, the problem of gravitational wave propagation in a medium was addressed by considering a hot noncollisional system of particles through kinetic theory [24]. In this study, a dispersion relation was derived by the resolution of the Einstein-Boltzmann (in its noncollisional version, that is, Vlasov equation) system and it was concluded, on the one hand, that the impact of the plasma on the propagation of gravitational waves should be small and, on the other hand, that there is a possibility of wave-particle resonances in this physical system. In later works, the controversy over the possibility of Landau damping of gravitational waves was discussed at several levels [25], and the effect of gravitational radiation from distant sources on electromagnetic waves in dispersive media was investigated [26]. A complete system of macroscopic equations for the electromagnetic and gravitational fields in magnetized plasmas was presented in 1982 and the propagation of gravitational waves in this medium was investigated using kinetic theory in the following year [27,28]. In the late 1990s, a system of equations governing the nonlinear dynamics of interacting neutrinos and gravitons in plasmas was established [29] and the excitation of electromagnetic and Langmuir waves by gravitational waves was considered [30].

At the turn of the century, the effects of intense gravitational waves (i.e., in the nonlinear regime) on cold plasmas have been investigated [31]. Then, in a series of works, the interactions between gravitational waves, electromagnetic waves and a magnetized plasma were explored, and the issue of gravitational Landau damping was raised again [32–36]. In the same period, some studies were carried out to examine the excitation of magnetosonic waves and the transfer of energy from gravitational waves to the plasma, especially in neutron star merger events [37–40]. Furthermore, cosmological issues involving the excitation of plasma waves by gravitational waves have been considered in 2002, employing relativistic hydrodynamic equations [41]. In 2004, also with a hydrodynamic approach, the nonlinear coupling between Alfvén waves and gravitational waves in strongly magnetized plasmas was scrutinized [42]. Spherically symmetric solutions of the Einstein-Vlasov-Maxwell system have been discussed in 2004–2005 [43,44], and the transfer of gravitational energy to plasma particles in a system of astrophysical interest and the interaction between gravitational waves in strongly magnetized plasmas were topics revisited in 2006 [45,46]. In 2010, in a sequence of works, coupling constants for the nonlinear interactions between gravitational, electrostatic and electromagnetic waves in a relativistic nonmagnetized and noncollisional plasma were obtained, and a system of coupled equations for gravitational and electromagnetic waves in a relativistic plasma have been analyzed allowing the identification of several resonances between gravitational waves and plasma [47,48].

The effects of dispersion and scattering through the interstellar medium on the detection of low-frequency gravitational waves have been analyzed in 2013 [49]. In 2015, vorticity generation in a plasma around Schwarzschild and Kerr black holes was investigated employing a magnetofluid and Arnowitt-Deser-Misner (ADM) formalisms [50]. In this work, a prescribed metric was assumed and some results of astrophysical interest were obtained, such as a proposed mechanisms for collimation of plasma jets and for vortex formation in the protoplanetary disks of a supermassive star, presumably related to planet formation [51]. In 2017, the damping of gravitational waves in collisional material media was investigated through the Boltzmann equation, evidencing two distinct mechanisms for the process [52]. In the same year, the coupling between the electromagnetic and gravitational fields was once again the subject of study, possibly motivated by (at the time) recent observations of the LIGO-Virgo collaboration [53,54]. In 2018, the magnetofluid formalism was revisited in a work where plasmas was taken as a system of multiple perfect charged fluids, and the formalism proposed provided suitable tools for characterizing plasmas in a given curved spacetime, including equilibrium states [55]. No directly related to gravitational radiation (but still a very interesting subject), the energy extraction from a spinning Kerr black hole via magnetic reconection was theoretically investigated, showing that the process is possible for black holes of high spin surrounded by a strongly magnetized plasma [56]. In recent years, some research topics in the area have been the excitation of magnetohydrodynamic waves by gravitational waves in strongly magnetized plasmas [57], the dissipation of gravitational waves [58] and the propagation of gravitational waves in magnetized dielectric media [59]. Very recently, some works have focused on the important issue of the polarization of gravitational waves [60,61], and a mechanism for the reflection of electromagnetic waves by a gravitational wave background in plasma media was proposed [62].

Some excellent theses developed since the early 2000s and which certainly also deserve mention are the works of M. Servin [63], J. B. Moortgat [64], and O. Janson [65]. These texts all deal essentially with the same topic: the propagation and interactions of electromagnetic and gravitational waves in plasmas in the context of general relativity. As we have seen, in the present work we intend to answer questions about the behavior of this same class of physical system and, therefore, these three works are fundamental to us.

Our aim in this work is to bring together, in a clear, concise, direct and (as far as possible) self-contained way, the fundamentals of the classical theory of fields—especially the issue of the gauge freedom—and the most profound theory of the plasma state (in a nonquantum level) which, in our understanding, is the general relativistic kinetic theory. This approach allows the safe and unambiguous determination of the radiative components of the gravitational field, without neglecting the dynamics associated with the other (nonradiative) components. Furthermore, an approximate dispersion relation for gravitational waves in an electron-positron plasma, obtained from the EVM system of equations, will be treated and discussed in order to illustrate the established methodology and, to some extent, to elucidate the behavior of gravitational waves in plasmas. In short, our goal is to establish a complete picture of the dynamics and interactions of gravitation, electromagnetism, and matter in plasma state using only gauge-invariant quantities whenever possible, and apply the formalism to find out dispersion relations for any kind of gravitational oscillation, including the calculation of their damping and energy exchange rates.

We need to reinforce that the present observational and experimental status of plasma physics does not require a covariant formulation (since gravitational effects are generally small), except when the aim is to describe the dynamics of components of the metric tensor that do not have a Newtonian counterpart, as is the case of the present paper. In addition, our formalism can have application, e.g., to cosmological plasmas involving small-scale dynamos in Riemannian spaces. This is the case of Lobachevsky or spherical geometries, which are possible geometries for the spatial part of the Friedmann cosmological models [66].

The paper is organized in the following way. In Sec. II, Einstein field equations (that is, the equations of gravitation) are established in the linear (weak-field) regime, and the gauge freedom of the theory is discussed in some detail. The equations of motion are then written in terms of gauge invariants analogous to the electric and magnetic fields of the Maxwell theory, and the general theory of gravitational waves in terms of these invariants is discussed. Some attention is payed to classical field theoretical aspects of gravitation and to the energy carried and exchanged by the gravitational field. In Sec. III, we digress into the general relativistic kinetic theory of noncollisional plasmas and write down the Vlasov equation which, together with Einstein and Maxwell equations, constitute a complete system of equations for the description of plasma state phenomena. In that section, the equations are written in terms of the aforementioned gauge invariants whenever possible. Finally, in Sec. IV, an dispersion relations for gravitational waves in an electron-positron plasma is obtained and discussed.

When writing this paper we keep in mind a wide public, specially physicist that, like us, are interested in both branches of physics (plasma physics and general relativity). The step-by-step construction shown in the text reflect very much the path taken by the authors in preparing these methodological notes and are presented precisely because, presumably, other researchers in plasma physics interested in general relativity (especially in gravitational waves) but without much experience in the area can benefit from this. Therefore, it is important to emphasize that a reader specialized in general relativity can safely skip some parts of the text, especially Sec. II of Sec. III A.

We follow the convention in which the signature of the metric tensor is $(+\,-\,-\,-)$ and the Ricci tensor $R_{\mu\nu}$ is obtained from contracting the first and the last indices of the Riemann-Christoffel tensor, $R^{\sigma}{}_{\mu\nu\rho}$. Latin indices can assume values from 1 to 3 and are used do label spatial coordinates. Greek indices can assume values from 0 to 3 and are used to label space-time coordinates. The temporal coordinate is denoted by $x^0 = ct$, where c is the velocity of light and t is the time. Einstein summation convention is assumed. It should be clear from the context when a superscript symbolizes an index or an exponent.

II. GAUGE TRANSFORMATIONS IN LINEARIZED GRAVITATION

A. On the weak-field limit of the Einstein equations

We proceed to a brief discussion of the equations of gravitation in the weak-field limit [7–9]. This treatment provides a linearized version of Einstein equations, facilitating—or even allowing—a general and detailed study of gravitational waves, the main object of study of this work. In particular, the Fourier decomposition of the field is legitimate only in the linearized theory. In addition to the nonlinearity of Einstein equations, the theoretical study of gravitational waves is hampered by a subtlety whose origin lies in the very general covariance of the theory: As the choice of coordinate system is completely arbitrary, it can be difficult to distinguish which wave solutions represent real physical effects and which are mere artifices resulting from a particular choice of coordinates. This last difficulty will be properly addressed in Sec. II B.

Usually, the full Einstein equations (neglecting the cosmological term) are written in the form

$$G_{\mu\nu} = -\frac{8\pi G}{c^4}T_{\mu\nu}, \tag{1}$$

with the Einstein tensor $G_{\mu\nu}$ given by

$$G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R\,, \tag{2}$$

where $R_{\mu\nu}$, R, $g_{\mu\nu}$, and $T_{\mu\nu}$ are, respectively, the Ricci curvature tensor, the Ricci curvature scalar, the metric tensor (whose components are the gravitational potentials), and the stress-energy tensor of the physical system, which constitute the source of the gravitational field. As usual, c is the velocity of light in vacuum and G is the Newtonian gravitational constant. As is well known, in general relativity gravitation is conceived properly as a manifestation of the curvature of spacetime caused by a distribution of mass and energy, both contained in $T_{\mu\nu}$. Curvature itself is encapsulated primarily in the metric tensor $g_{\mu\nu}$ and second in the Ricci mathematical objects $R_{\mu\nu}$ and R (among others). Under the exclusive influence of a gravitational field, a point particle of mass m moves in spacetime along geodesic trajectories, which are the curved space analogous of straight lines of the flat (or empty) space. The equations of motion of the point particle are in this case

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = 0, \tag{3}$$

where the derivatives of the coordinates x^μ are taken with respect to the proper time τ, and $\Gamma^\mu_{\nu\rho}$ are the Christoffel symbols of the second kind, given in terms of derivatives of the metric by

$$\Gamma^\mu_{\nu\rho} = \frac{1}{2}g^{\mu\sigma}\left(\frac{\partial g_{\nu\sigma}}{\partial x^\rho} + \frac{\partial g_{\rho\sigma}}{\partial x^\nu} - \frac{\partial g_{\nu\rho}}{\partial x^\sigma}\right). \tag{4}$$

If, besides the gravitation, the particle is subjected to an electromagnetic field $F_{\mu\nu}$, then the equations of motion must be modified to incorporate the electromagnetic force, taking the form

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = \frac{q}{m}F^\mu{}_\nu\frac{dx^\nu}{d\tau}\,, \tag{5}$$

where q is the electric charge of the point particle. The equations governing the dynamics of the electromagnetic tensor $F_{\mu\nu}$ (that is, Maxwell equations) in presence of a gravitation field and the influence of electricity and magnetism on gravitation will be briefly discussed in Sec. III A. It should be mentioned that all the equations so far—including Maxwell's—can be derived from a variational principle. For details in the derivation, one could consult Refs. [7–9] and [67].

Our next task is to establish the weak-field limit of the Einstein equations. We assume a *quasi*-Minkowskian metric, that is, a metric in the form

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\,, \tag{6}$$

where $\eta_{\mu\nu}$ is the Minkowski (that is, flat space-time) metric tensor, represented by the matrix

$$[\eta_{\mu\nu}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{7}$$

and

$$|h_{\mu\nu}| \ll 1\,. \tag{8}$$

The quantities $h_{\mu\nu}$ will henceforth be called *perturbations of the metric*. It can be shown [7] that in this linear limit the components of the Einstein tensor are given by

$$G_{\mu\nu} = \frac{1}{2}\Bigg(\Box h_{\mu\nu} - \eta_{\mu\nu}\Box h + \frac{\partial^2 h}{\partial x^\mu \partial x^\nu} + \eta_{\mu\nu}\eta^{\rho\alpha}\eta^{\sigma\beta}\frac{\partial^2 h_{\alpha\beta}}{\partial x^\rho \partial x^\sigma} - \eta^{\rho\alpha}\frac{\partial^2 h_{\alpha\mu}}{\partial x^\nu \partial x^\rho} - \eta^{\rho\alpha}\frac{\partial^2 h_{\alpha\nu}}{\partial x^\mu \partial x^\rho}\Bigg), \tag{9}$$

where $h = \eta^{\mu\nu}h_{\mu\nu} = h^\mu{}_\mu$ is the trace of the tensor $h_{\mu\nu}$ and $\Box$ is the D'Alembert wave operator. Substituting Eq. (9) in the left-hand side of Eq. (1) gives us the desired weak-field equations.

It is well known that in general we are, to a certain extent, free to choose conveniently the properties of the coordinate system to be used in a specific physical problem. Particularly in dealing with gravitational waves in vacuum, usually the preferred one is the *transverse-traceless* (or TT) system of coordinates [8], in which the metric perturbations obeys both the transversality condition

$$\frac{\partial h^\mu{}_\nu}{\partial x^\mu} = 0 \tag{10}$$

and the traceless condition

$$h = 0\,. \tag{11}$$

In fact, the TT coordinates are a subcategory of harmonic coordinates, for which

$$\frac{\partial h^\mu{}_\nu}{\partial x^\mu} - \frac{1}{2}\frac{\partial h}{\partial x^\nu} = 0\,. \tag{12}$$

With conditions (10) and (11) enforced in Eq. (9) applied to empty space, Eq. (1) assumes the very simple and suggestive form

$$\Box h_{\mu\nu} = 0\,. \tag{13}$$

Field Eq. (13) gives us the impression that generally all 10 independent components of the metric tensor can exhibit a wavelike behavior, which by no means corresponds to a physical reality. The solution of Eq. (13) for a plane gravitational wave propagating in a specific direction furnishes the well-known *plus* and *cross* independent tensor polarization states for these waves, correctly showing that gravitational waves in vacuum just have two degrees of freedom associated solely to the space-space transverse components of the metric, the other components being zero [8]. Furthermore, we stress that the aforementioned TT system of coordinates can be chosen only in vacuum, so the dynamics of the metric in a material medium—particularly in a plasma—cannot be taken into account in this oversimplified formulation. Fortunately, all the difficulties pointed out above can be removed in a gauge-free version of the linear theory of general relativity, as we shall see below.

B. Gauge freedom, gauge invariance, and Einstein field equations

In electromagnetism we write field equations for invariant gauge quantities and express the electromagnetic force in terms of these same quantities: The invariant quantities are the vector fields **E** and **B**, which satisfy Maxwell equations, and the force law is derived from the Lorentz force [21]. It is desirable, as far as possible, that the same can be done in the linear theory of gravity. Furthermore, it is worth mentioning that a theory following the same spirit for nonrelativistic quantum plasmas has already been obtained [68]. In this theory, the gauge-invariant Wigner function is taken as the basis of a fluidlike system describing the plasma. So our main objectives in this subsection are as follows: (i) to express Einstein field equations in the linear regime in terms of gauge-invariant quantities and (ii) to show that the only radiative gravitational modes, in any gauge and in presence of sources (inclusive), correspond precisely to the pure spatial part of $h_{\mu\nu}$ satisfying conditions of null trace and transversality.

With this, it is removed from the theory any kind of difficulty and obscurity involving the nondistinction between objective gravitational waves and mere artifices resulting from a given choice of coordinate system (which, in general relativity, is the same as a choice of gauge). We begin the discussion by establishing expressions for the components of the metric tensor which will allow us to find gauge-invariant quantities in the linear theory as directly as possible. Except for conventions, for some notation and for brevity in our approach, our rationale in this subsection follows closely that of Refs. [69] and [70]. Furthermore, a more general alternative approach—although more abstract—is found in Ref. [71].

The scalar part of the perturbation tensor—that is, the time-time component of $h_{\mu\nu}$, invariant under spatial rotations—can be identified with the Newtonian gravitational potential ϕ according to

$$h_{00} = \frac{2\phi}{c^2}\,. \tag{14}$$

This identification goes back to the origins of general relativity, where the equations for the gravitational field were constructed based on the equivalence principle and forced to reproduce the results of the Newtonian theory in the limit of weak fields and bodies moving at low speeds [7].

The vector part of $h_{\mu\nu}$—that is, the part that transforms as a three-vector under spatial rotations—is identified with the set of time-space type components of $h_{\mu\nu}$:

$$[h_{01} \quad h_{02} \quad h_{02}] = -\frac{1}{c}[A_x \quad A_y \quad A_z] = -\frac{1}{c}\mathbf{A}\,. \tag{15}$$

We proceed a little further now by employing the Helmholtz decomposition for the vector **A**. The Helmholtz theorem states that the vector **A**, subject to the appropriate boundary condition at spatial infinity

$$\lim_{\mathbf{x}\to\infty} \mathbf{A} = 0 \tag{16}$$

can be expressed as

$$\mathbf{A} = \nabla\psi + \mathbf{h}\,, \tag{17}$$

where ψ is a specific scalar field and $\mathbf{h}$ is a vector field satisfying

$$\nabla\cdot\mathbf{h} = 0\,. \tag{18}$$

Equations (17) and (18) allows expressing Eq. (15) in terms of components as

$$h_{0i} = -\frac{1}{c}\left(\frac{\partial\psi}{\partial x^i} + \mathsf{h}_i\right), \tag{19}$$

with the condition

$$\frac{\partial \mathsf{h}^i}{\partial x^i} = 0\,. \tag{20}$$

The quantities h_i are the components of the three-vector $\mathbf{h}$, and so here the operations of raising and lowering indices must be performed by contraction with the Krönecker delta, as in $\mathsf{h}^i = \delta^{ij}\mathsf{h}_j$ (for three-vectors, the lowering and raising indices operations must be performed with δ_{ij} and δ^{ij}, respectively).

A strategy similar to the one outlined above can be employed in structuring the tensor part of $h_{\mu\nu}$—that is, the part that transforms according to a three-tensor in spatial rotations. It can be shown that this tensor part of the perturbation admits a Helmholtz decomposition of the form

$$h_{ij} = h_{ij}^{TT} + \frac{1}{3}\delta_{ij}H + \frac{1}{2}\left(\frac{\partial\mathscr{E}_i}{\partial x^j} + \frac{\partial\mathscr{E}_j}{\partial x^i}\right) + \left(\frac{\partial^2}{\partial x^i\partial x^j} - \frac{1}{3}\delta_{ij}\nabla^2\right)\lambda\,, \tag{21}$$

where h_{ij}^{TT} is the *transverse traceless* part of h_{ij}, satisfying the conditions

$$\frac{\partial h^{i\,TT}_{\;j}}{\partial x^i} = 0, \tag{22}$$

$$\delta^{ij}h_{ij}^{TT} = 0\,. \tag{23}$$

Here λ and $\mathscr{E}_i$ are, respectively, the scalar and vector potential for h_{ij}, the last one satisfying

$$\frac{\partial \mathscr{E}^i}{\partial x^i} = 0\,. \tag{24}$$

By contracting Eq. (21) with δ^{ij} and employing Eqs. (23) and (24), we see that $H = \delta^{ij} h_{ij}$, that is, the (three-dimensional) trace of h_{ij}.

A second-order symmetric four-tensor has 10 independent components. But calculating the total number of independent functions given by Eqs. (14), (19), and (21) does not give this value. In fact, in the proposed decomposition we have four scalar functions (ϕ, ψ, H, and λ), three vector components h_i, three vector components $\mathscr{E}_i$ and the six components of the symmetric three-tensor h_{ij}^{TT}, giving 16 functions. On the other hand, there are six constraints—one given by Eq. (20), three given by Eq. (22), one given by Eq. (23), plus one given by equation—which gives us 16 – 6 = 10 independent functions, as expected. Note that Eqs. (22) and (23) are closely related to the traceless transverse coordinate condition usually employed in the discussion of the propagation and properties of gravitational waves in free space. However, there is no gauge fixing here: All we have done is to establish an appropriate decomposition for the perturbation tensor $h_{\mu\nu}$, with the correct number of independent components (10), without any kind of restriction on the physical system in question or on the adopted coordinate system (except, of course, that we are working in the linear regime of the field equations). It is also worth mentioning that it can be proved that the decomposition presented here is unique when the quantities ψ, λ, and $\mathscr{E}_i$ are subjected to certain fairly simple and reasonable physical universal boundary conditions. However, we will not deal with the details of this matter here [69].

Let us now look for invariant gauge quantities in the linear theory, in a sense the gravitational analogs of the **E** and **B** fields of electromagnetism. As is well known, in the linear regime the perturbations of the metric transforms, in a coordinate system change (or gauge transformation), as [7]

$$h'_{\mu\nu} = h_{\mu\nu} - \frac{\partial \chi_\nu}{\partial x^\mu} - \frac{\partial \chi_\mu}{\partial x^\nu}\,. \tag{25}$$

Under spatial rotations, the transformation four-vector χ_μ can also be conveniently decomposed into a scalar and a vector part according to

$$\chi_\mu = (\chi_0, \chi_i) = \left(\frac{\eta}{c}, \frac{\partial a}{\partial x^i} + b_i\right), \tag{26}$$

with the condition

$$\frac{\partial b^i}{\partial x^i} = 0\,. \tag{27}$$

Note that in Eqs. (26) and (27) above we employ Helmholtz's theorem again, now for the three-vector χ_i. Like χ_μ, the transformation functions η, a, and b^i [subject to constraint (27)] are completely arbitrary *except for keeping the field weak.*

For the scalar component of the perturbation tensor, the transformation (25) gives

$$\phi' = \phi - \frac{\partial \eta}{\partial t}\,, \tag{28}$$

as can be verified by substituting Eq. (14) in the transformation (25) and employing Eq. (26). Substituting the components given in Eq. (19) into Eq. (25), we find after some algebra

$$\frac{\partial \psi'}{\partial x^i} + \mathsf{h}'_i = \frac{\partial}{\partial x^i}\left(\psi + \eta + \frac{\partial a}{\partial t}\right) + \mathsf{h}_i + \frac{\partial b_i}{\partial t}\,. \tag{29}$$

Operating with $\delta^{ij}\frac{\partial}{\partial x^j}$ on Eq. (29) and taking into account the constraints (20) and (27), we find

$$\psi' = \psi + \eta + \frac{\partial a}{\partial t}\,, \tag{30}$$

which, when inserted back into Eq. (29), gives

$$\mathsf{h}'_i = \mathsf{h}_i + \frac{\partial b_i}{\partial t}\,. \tag{31}$$

Finally, performing the transformation (25) of the pure spatial components of $h_{\mu\nu}$, given by Eq. (21), we now find

$$\begin{aligned} h'^{TT}_{ij} &+ \frac{1}{3}\delta_{ij}H' + \frac{1}{2}\left(\frac{\partial \mathscr{E}'_i}{\partial x^j} + \frac{\partial \mathscr{E}'_j}{\partial x^i}\right) \\ &+ \left(\frac{\partial^2}{\partial x^i \partial x^j} - \frac{1}{3}\delta_{ij}\nabla^2\right)\lambda' \\ &= h^{TT}_{ij} + \frac{1}{3}\delta_{ij}H + \frac{1}{2}\left(\frac{\partial \mathscr{E}_i}{\partial x^j} + \frac{\partial \mathscr{E}_j}{\partial x^i}\right) \\ &+ \left(\frac{\partial^2}{\partial x^i \partial x^j} - \frac{1}{3}\delta_{ij}\nabla^2\right)\lambda \\ &- 2\frac{\partial^2 a}{\partial x^i \partial x^j} - \frac{\partial b_i}{\partial x^j} - \frac{\partial b_j}{\partial x^i}\,. \end{aligned} \tag{32}$$

Contracting Eq. (32) with δ^{ij} and taking into account the constraints (22), (23), (24), and (27), we find

$$H' = H - 2\nabla^2 a\,. \tag{33}$$

Returning with the result (33) into Eq. (32) and rearranging, we obtain

$$\begin{aligned} h'^{TT}_{ij} &+ \frac{1}{2}\left(\frac{\partial \mathscr{E}'_i}{\partial x^j} + \frac{\partial \mathscr{E}'_j}{\partial x^i}\right) + \left(\frac{\partial^2}{\partial x^i \partial x^j} - \frac{1}{3}\delta_{ij}\nabla^2\right)\lambda' \\ &= h^{TT}_{ij} + \frac{1}{2}\left[\frac{\partial(\mathscr{E}_i - 2b_i)}{\partial x^j} + \frac{\partial(\mathscr{E}_j - 2b_j)}{\partial x^i}\right] \\ &+ \left(\frac{\partial^2}{\partial x^i \partial x^j} - \frac{1}{3}\delta_{ij}\nabla^2\right)(\lambda - 2a). \end{aligned} \tag{34}$$

By comparing the terms to the left- and right-hand sides of Eq. (34), we conclude that

$$h'^{TT}_{ij} = h^{TT}_{ij}\,, \tag{35}$$

$$\mathscr{E}'_i = \mathscr{E}_i - 2b_i\,, \tag{36}$$

$$\lambda' = \lambda - 2a\,, \tag{37}$$

which completes the picture of the behavior of the functions ϕ, ψ, H, λ, h_i, $\mathscr{E}_i$, and h_{ij}^{TT} under the gauge transformations of the linear gravitational theory. Note that, so far, the only invariant quantities are the tensor components h_{ij}^{TT}, just those related to gravitational radiation in the traditional approach using the transverse traceless gauge. From the set of transformations (28), (30), (31), (33), (35), (36), and (37) we can find two more scalar invariants and a vector invariant. The first scalar invariant, Φ, is obtained by adding Eq. (28) with the first time derivative of Eq. (30) and with one half of the second time derivative of Eq. (37). That gives us

$$\Phi = \phi' + \frac{\partial \psi'}{\partial t} + \frac{1}{2}\frac{\partial^2 \lambda'}{\partial t^2} = \phi + \frac{\partial \psi}{\partial t} + \frac{1}{2}\frac{\partial^2 \lambda}{\partial t^2}. \tag{38}$$

The second scalar invariant, Θ, is obtained by taking the Laplacian of Eq. (33) and subtracting Eq. (37), giving

$$\Theta = \tfrac{1}{3}(H' - \nabla^2\lambda') = \tfrac{1}{3}(H - \nabla^2\lambda)\,. \tag{39}$$

The factor $\frac{1}{3}$ in Eq. (39) has been included for convenience. Finally, a vector invariant Ξ_i is obtained by adding Eq. (31) with one half of the first time derivative of Eq. (36), which results in

$$\Xi_i = \mathsf{h}_i' + \frac{1}{2}\frac{\partial \mathscr{E}_i'}{\partial t} = \mathsf{h}_i + \frac{1}{2}\frac{\partial \mathscr{E}_i}{\partial t}\,. \tag{40}$$

Note that, due to divergenceless conditions (20) and (24), the vector Ξ_i satisfies

$$\frac{\partial \Xi^i}{\partial x^i} = 0\,. \tag{41}$$

Our next goal is to express the metric tensor in terms of the 11 invariant quantities Φ, Θ, Ξ_i, and h_{ij}^{TT} and to establish the equations of motion for these quantities. In fact, considering the five constraints given by Eqs. (22), (23), and (41), of these quantities we have just $11 - 5 = 6$ independent functions, meaning that of the 10 components of the tensor $h_{\mu\nu}$, in general only six are physically significant.

By isolating the functions ϕ, H, and h_i respectively in Eqs. (38), (39), and (40), and adequately substituting the results in Eqs. (14), (19), and (21), we can show that the scalar, vector, and tensor components of the perturbation tensor $h_{\mu\nu}$ can be expressed in terms of gauge invariants in the form:

$$h_{00} = \frac{2}{c^2}\left(\Phi - \frac{\partial \psi}{\partial t} - \frac{1}{2}\frac{\partial^2\lambda}{\partial t^2}\right), \tag{42}$$

$$h_{0i} = -\frac{1}{c}\left(\Xi_i + \frac{\partial \psi}{\partial x^i} - \frac{1}{2}\frac{\partial \mathscr{E}_i}{\partial t}\right), \tag{43}$$

$$h_{ij} = h_{ij}^{TT} + \delta_{ij}\Theta + \frac{1}{2}\left(\frac{\partial \mathscr{E}_i}{\partial x^j} + \frac{\partial \mathscr{E}_j}{\partial x^i}\right) + \frac{\partial^2\lambda}{\partial x^i \partial x^j}. \tag{44}$$

To clarify, the metric perturbation $h_{\mu\nu}$ depends on the 11 invariants Φ, Θ, Ξ_i, and h_{ij}^{TT} subjected to the five conditions (22), (23), and (41)—giving six physical degrees of freedom, as we saw—and on the five gauge-dependent quantities ψ, λ, and $\mathscr{E}_i$ subjected to the condition (24), totaling $5 - 1 = 4$ gauge degrees of freedom. The total number of degrees of freedom is $6 + 4 = 10$, as expected for a symmetric second-order four-tensor.

By Eqs. (42)–(44) we see that it is not possible to express the metric tensor only in terms of invariant gauge quantities, and this conclusion reflects the equivalence principle: At any point in spacetime it is always possible to find a locally inertial frame of reference, which can be mathematically achieved with appropriate choices of the functions ψ, λ, and $\mathscr{E}_i$.

It can be shown from Eq. (9) that, despite the gauge dependence of the metric, the components of the Einstein tensor $G_{\mu\nu}$ *are* expressed only in terms of the gauge-invariant quantities as follows:

$$G_{00} = -\nabla^2\Theta, \tag{45}$$

$$G_{0i} = -\frac{1}{c}\left(\frac{\partial^2\Theta}{\partial t\,\partial x^i} - \frac{1}{2}\nabla^2\Xi_i\right), \tag{46}$$

$$\begin{aligned} G_{ij} = {} & \frac{1}{2}\Box h_{ij}^{TT} - \delta_{ij}\frac{1}{c^2}\frac{\partial^2\Theta}{\partial t^2} \\ & + \frac{1}{2c^2}\left(\frac{\partial}{\partial x^j}\frac{\partial \Xi_i}{\partial t} + \frac{\partial}{\partial x^i}\frac{\partial \Xi_j}{\partial t}\right) \\ & + \frac{1}{2}\left(\frac{\partial^2}{\partial x^i \partial x^j} - \delta_{ij}\nabla^2\right)\left(\frac{2\Phi}{c^2} - \Theta\right). \end{aligned} \tag{47}$$

By checking Eq. (47) we see that the wave operator $\Box$ appears just in the first term at the right-hand side, acting on the transverse traceless part of the metric perturbation. As we will see soon, this implies that these components are the only obeying a wave equation and so are radiative, corresponding to the wave modes of the gravitational field.

The Einstein field Eq. (1) can now be split in a series of simpler equations, one for each of the gauge-invariant quantities Φ, Θ, Ξ_i, and h_{ij}^{TT}. The equation for Θ is readily obtained simply by taking the time-time component of Eq. (1), that is, by writing

$$G_{00} = -\frac{8\pi G}{c^4}T_{00}\,. \tag{48}$$

From Eq. (45) substituted in Eq. (48) above, we obtain

$$\nabla^2\Theta = \frac{8\pi G}{c^4}T_{00}\,. \tag{49}$$

Next, to establish an equation for the gauge-invariant vector Ξ_i, we take the time-space type components of Eq. (1), getting

$$G_{0i} = -\frac{8\pi G}{c^4}T_{0i}\,. \tag{50}$$

For the left-hand side of Eq. (50) we substitute Eq. (46), and for the right-hand side we employ a Helmholtz decomposition of the form

$$T_{0i} = c\left(\frac{\partial S}{\partial x^i} + S_i\right), \tag{51}$$

with the condition

$$\frac{\partial S^i}{\partial x^i} = 0\,. \tag{52}$$

So, the Eq. (50) can now be written as

$$\frac{\partial^2\Theta}{\partial t\,\partial x^i} - \frac{1}{2}\nabla^2\Xi_i = \frac{8\pi G}{c^2}\left(\frac{\partial S}{\partial x^i} + S_i\right). \tag{53}$$

It can be shown that, by taking the three-divergence of Eq. (53), we can recover the result (49). On the other hand, we can readily put Eq. (53) in the explicit vector form

$$\nabla\frac{\partial\Theta}{\partial t} - \frac{1}{2}\nabla^2\boldsymbol{\Xi} = \frac{8\pi G}{c^2}(\nabla S + \mathbf{S})\,. \tag{54}$$

Then, by taking the curl of Eq. (54) and, remembering that the curl of any gradient is zero, we are led to

$$\nabla \times \left(\frac{1}{2}\nabla^2 \boldsymbol{\Xi} + \frac{8\pi G}{c^2}\mathbf{S}\right) = 0\,. \tag{55}$$

Equation (55) implies that

$$\frac{1}{2}\nabla^2 \boldsymbol{\Xi} + \frac{8\pi G}{c^2}\mathbf{S} = \nabla f(\mathbf{x}, t)\,, \tag{56}$$

where $f(\mathbf{x}, t)$ is some scalar function of coordinates and time. Since the three-vectors $\boldsymbol{\Xi}$ and $\mathbf{S}$ are both divergenceless and are assumed to vanish as $\mathbf{x} \to \infty$, Eq. (56) implies that

$$\nabla^2 \boldsymbol{\Xi} = -\frac{16\pi G}{c^2}\mathbf{S}\,, \tag{57}$$

or, in component form,

$$\nabla^2 \Xi_i = -\frac{16\pi G}{c^2}S_i\,. \tag{58}$$

Equation (58) is the differential equation for the gauge-invariant vector Ξ_i. It is worthwhile to mention that the equations of motion for Θ and for Ξ_i [Eqs. (49) and (58)] are both inhomogeneous Poisson-like equations (not wavelike) and so have only (nonradiative) solutions decaying with r^{-2}, r being a radial coordinate. Thus, neither Θ nor Ξ_i can be associated with gravitational waves. Furthermore, notice that the source for Θ is T_{00} and for Ξ_i is just the *rotational* part of T_{0i}, given by S_i.

To obtain equations of motion for the Φ and h_{ij}^{TT}, we must take into account the space-space components of Eq. (1). Substituting Eq. (47) in the right-hand side of Eq. (1) we obtain

$$\frac{1}{2}\Box h_{ij}^{TT} - \delta_{ij}\frac{1}{c^2}\frac{\partial^2\Theta}{\partial t^2} + \frac{1}{2c^2}\left(\frac{\partial}{\partial x^j}\frac{\partial \Xi_i}{\partial t} + \frac{\partial}{\partial x^i}\frac{\partial \Xi_j}{\partial t}\right) + \frac{1}{2}\left(\frac{\partial^2}{\partial x^i \partial x^j} - \delta_{ij}\nabla^2\right)\left(\frac{2\Phi}{c^2} - \Theta\right) = -\frac{8\pi G}{c^4}T_{ij}. \tag{59}$$

To find the differential equation to Φ, we contract Eq. (59) with δ^{ij} and use the conditions given by Eqs. (23) and (41). We obtain, after some algebraic manipulations,

$$\frac{1}{c^2}\frac{\partial^2\Theta}{\partial t^2} + \frac{2}{3c^2}\nabla^2\Phi - \frac{1}{3}\nabla^2\Theta = \frac{8\pi G}{c^4}P\,, \tag{60}$$

where P is given by

$$P = \tfrac{1}{3}\delta^{ij}T_{ij}. \tag{61}$$

Substituting Eq. (49) for $\nabla^2\Theta$ in the third term at left-hand side in Eq. (60) and taking the Laplacian of the resulting equation, we get

$$\nabla^2\frac{\partial^2\Theta}{\partial t^2} + \frac{2}{3}\nabla^2\nabla^2\Phi - \frac{8\pi G}{3c^2}\nabla^2 T_{00} = \frac{8\pi G}{c^2}\nabla^2 P\,. \tag{62}$$

The first term at left-hand side in Eq. (62) may be tamed by taking the divergence of Eq. (54) and employing the divergenceless conditions (41) and (52). The result is

$$\nabla^2\frac{\partial\Theta}{\partial t} = \frac{8\pi G}{c^2}\nabla^2 S\,. \tag{63}$$

Now, taking the time derivative of Eq. (63), it gives

$$\nabla^2\frac{\partial^2\Theta}{\partial t^2} = \nabla^2\left(\frac{8\pi G}{c^2}\frac{\partial S}{\partial t}\right). \tag{64}$$

Returning with the result (64) in Eq. (62) and rearranging we obtain

$$\nabla^2\left(\frac{4\pi G}{c^2}\frac{\partial S}{\partial t} + \frac{1}{3}\nabla^2\Phi - \frac{4\pi G}{3c^2}T_{00} - \frac{4\pi G}{c^2}P\right) = 0\,. \tag{65}$$

Again, since the functions S, Φ, T_{00}, and P are all assumed to vanish as $\mathbf{x} \to \infty$, Eq. (65) implies that

$$\frac{4\pi G}{c^2}\frac{\partial S}{\partial t} + \frac{1}{3}\nabla^2\Phi - \frac{4\pi G}{3c^2}T_{00} - \frac{4\pi G}{c^2}P = 0\,, \tag{66}$$

that is,

$$\nabla^2\Phi = \frac{4\pi G}{c^2}\left[T_{00} + 3\left(P - \frac{\partial S}{\partial t}\right)\right]. \tag{67}$$

which is the differential equation for Φ – again, a Poisson equation.

Finally, the differential equation for h_{ij}^{TT} will be found, again by properly manipulating the Eq. (59). However, at this point of the development, it is useful first to exploit the conservation law

$$\frac{\partial T^{\mu}{}_{\nu}}{\partial x^{\mu}} = 0, \tag{68}$$

which is valid in this form to the first order. For $\nu = 0$ we obtain from Eq. (68) an equation for energy conservation in the form

$$\frac{1}{c^2}\frac{\partial T_{00}}{\partial t} = \nabla^2\mathbf{S}, \tag{69}$$

where use was made of Eqs. (51) and (52). To exploit the case with $\nu = i$, it is convenient to express the components of the stress tensor T_{ij} in the Helmholtz decomposed form

$$T_{ij} = \sigma_{ij} + \delta_{ij}P + \left(\frac{\partial\sigma_i}{\partial x^j} + \frac{\partial\sigma_j}{\partial x^i}\right) + \left(\frac{\partial^2}{\partial x^i\partial x^j} - \frac{1}{3}\delta_{ij}\nabla^2\right)\sigma, \tag{70}$$

with

$$\frac{\partial\sigma^i{}_j}{\partial x^i} = 0, \tag{71}$$

$$\delta^{ij}\sigma_{ij} = 0, \tag{72}$$

$$\frac{\partial\sigma^i}{\partial x^i} = 0\,. \tag{73}$$

By virtue of Eqs. (71) and (72), the σ_{ij} part of T_{ij} is called the *transverse traceless* part of the stress tensor. Now, setting $\nu = i$ in Eq. (68) and taking into account Eqs. (51) and (70), we obtain a conservation equation that can be readily written in the explicit vector form,

$$\nabla\frac{\partial S}{\partial t} + \frac{\partial\mathbf{S}}{\partial t} = \nabla P + \nabla^2\boldsymbol{\Sigma} + \frac{2}{3}\nabla^2\nabla\sigma, \tag{74}$$

with $\mathbf{\Sigma} = [\sigma_1 \quad \sigma_2 \quad \sigma_3]$. Taking the divergence of Eq. (74) we obtain

$$\nabla^2\left[\frac{3}{2}\left(\frac{\partial S}{\partial t} - P\right) - \nabla^2\sigma\right] = 0\,, \tag{75}$$

where use was made of conditions (52) and (71). On the other hand, taking the curl of Eq. (74) and using the fact that the curl of a gradient is always zero, we get

$$\nabla \times \left[\frac{\partial \mathbf{S}}{\partial t} - \nabla^2\mathbf{\Sigma}\right] = 0. \tag{76}$$

Imposing the universal boundary conditions $S = P = \sigma = S_i = \sigma_i = 0$ as $\mathbf{x} \to \infty$, Eqs. (75) and (76) gives, respectively,

$$\frac{3}{2}\left(\frac{\partial S}{\partial t} - P\right) = \nabla^2\sigma \tag{77}$$

and

$$\frac{\partial S_i}{\partial t} = \nabla^2\sigma_i. \tag{78}$$

Yet, from Eq. (77) we can write

$$\frac{\partial S}{\partial t} = \frac{2}{3}\nabla^2\sigma + P, \tag{79}$$

which will be used latter. Now, by taking the Laplacian of Eq. (59) we find

$$\begin{aligned}&\frac{1}{2}\nabla^2\Box h_{ij}^{TT} - \delta_{ij}\frac{1}{c^2}\nabla^2\frac{\partial^2\Theta}{\partial t^2}\\&+\frac{1}{2c^2}\left(\frac{\partial}{\partial x^j}\frac{\partial\nabla^2\Xi_i}{\partial t} + \frac{\partial}{\partial x^i}\frac{\partial\nabla^2\Xi_j}{\partial t}\right)\\&+\frac{1}{2}\left(\frac{\partial^2}{\partial x^i\partial x^j} - \delta_{ij}\nabla^2\right)\nabla^2\left(\frac{2\Phi}{c^2} - \Theta\right)\\&+\frac{8\pi G}{c^4}\nabla^2 T_{ij} = 0.\end{aligned} \tag{80}$$

Substituting Eqs. (49), (58), (64), and (67), respectively, for $\nabla^2\Theta$, $\nabla^2\Xi_i$, $\nabla^2\frac{\partial^2\Theta}{\partial t^2}$, and $\nabla^2\Phi$ in Eq. (80) we find, after rearranging,

$$\begin{aligned}&-\frac{c^4}{16\pi G}\nabla^2\Box h_{ij}^{TT} + \delta_{ij}\nabla^2\frac{\partial S}{\partial t} + \left(\frac{\partial}{\partial x^j}\frac{\partial S_i}{\partial t} + \frac{\partial}{\partial x^i}\frac{\partial S_j}{\partial t}\right)\\&+\left(\frac{\partial^2}{\partial x^i\partial x^j} - \delta_{ij}\nabla^2\right)\frac{3}{2}\left(\frac{\partial S}{\partial t} - P\right) - \nabla^2 Tij = 0.\end{aligned} \tag{81}$$

At this point, substituting Eqs. (77), (78), and (79), respectively, for $\frac{3}{2}(\frac{\partial S}{\partial t} - P)$, $\frac{\partial S_i}{\partial t}$, and $\frac{\partial S}{\partial t}$ in Eq. (81) and grouping similar terms, we are led to

$$\begin{aligned}&-\frac{c^4}{16\pi G}\nabla^2\Box h_{ij}^{TT} + \delta_{ij}\nabla^2 P + \left(\frac{\partial^2}{\partial x^i\partial x^j} - \frac{1}{3}\delta_{ij}\nabla^2\right)\nabla^2\sigma\\&+\left(\frac{\partial\nabla^2\sigma_i}{\partial x^j} + \frac{\partial\nabla^2\sigma_j}{\partial x^i}\right) - \nabla^2 T_{ij} = 0\,.\end{aligned} \tag{82}$$

Finally, by the Helmholtz decomposition for T_{ij}, Eq. (70), we recognize that the second, third, fourth, and fifth terms at left-hand side in Eq. (82) when added gives exactly $-\sigma_{ij}$, that is,

$$\frac{c^4}{16\pi G}\nabla^2\Box h_{ij}^{TT} + \nabla^2\sigma_{ij} = 0 \tag{83}$$

or

$$\nabla^2\left(\Box h_{ij}^{TT} + \frac{16\pi G}{c^4}\sigma_{ij}\right) = 0\,. \tag{84}$$

Assuming, as usual, that both h_{ij}^{TT} and σ_{ij} goes to zero as $\mathbf{x} \to \infty$, Eq. (84) implies that

$$\Box h_{ij}^{TT} = -\frac{16\pi G}{c^4}\sigma_{ij}\,, \tag{85}$$

completing our set of equations for the gauge invariants Φ, Θ, Ξ_i, and h_{ij}^{TT}. Summarizing, in the weak-field approximation of general relativity, the Einstein equations are split up in a set of linear inhomogeneous differential equations given by

$$\nabla^2\Theta = \frac{8\pi G}{c^4}T_{00}, \tag{86}$$

$$\nabla^2\Phi = \frac{4\pi G}{c^2}\left[T_{00} + 3\left(P - \frac{\partial S}{\partial t}\right)\right], \tag{87}$$

$$\nabla^2\Xi_i = -\frac{16\pi G}{c^2}S_i, \tag{88}$$

$$\Box h_{ij}^{TT} = -\frac{16\pi G}{c^4}\sigma_{ij}\,, \tag{89}$$

where T_{00}, P, S, S_i, and σ_{ij} are related to the stress-energy tensor of the physical system under consideration and represents the sources for the gravitational field. Solving Eqs. (86)–(89) completely determine the six physical degrees of freedom of the metric. The metric perturbation itself, as discussed previously, are given by Eqs. (42)–(44) and are clearly gauge dependent, as it has to be. The Christoffel symbols appearing in the equations of motion (3) and (5) are also gauge dependent, reflecting the equivalence principle. That is, the objects $\Gamma^{\mu}_{\nu\rho}$ can be locally made to vanish by an appropriate choice of coordinates. Chosen a gauge (or a system of coordinates) with $\psi = \lambda = \mathscr{E}_i = 0$ Eqs. (42)–(44) assumes the very simple form

$$h_{00} = \frac{2\Phi}{c^2}, \tag{90}$$

$$h_{0i} = -\frac{1}{c}\Xi_i, \tag{91}$$

$$h_{ij} = h_{ij}^{TT} + \delta_{ij}\Theta, \tag{92}$$

and this will be our preferred gauge in explicit calculations involving the metric tensor. The Einstein field Eqs. (86)–(89), of course, are not affected by this or any other choice of gauge, as they are written only in terms of gauge-invariant quantities. Just to mention some interesting features of field Eqs. (86)–(89) and the metric (90)–(92), note that in free space, all field equations are homogeneous and so the Eqs. (86) and (87) are the same, leading to $\Theta = \frac{2\Phi}{c^2}$. This identification is closely related to the Schwarzschild solution expressed in isotropic coordinates and correct to the first order, in which $h_{00} = \frac{2\Phi}{c^2}$, $h_{ij} = \delta_{ij}\frac{2\Phi}{c^2}$, and $h_{ij}^{TT} = h_{0i} = 0$, where $\Phi = -\frac{GM}{r}$ is the Newtonian gravitational potential of a point source of mass M. Furthermore, observe that Eq. (89) is the only wave equation of the set of field Eqs. (86)–(89). This means that only the h_{ij}^{TT} components of the metric perturbation behaves as waves and can be correctly identified as the radiative degrees of freedom of the gravitational field, as mentioned before. For the sake of completeness, in the next subsection this aspect will be briefly revised.

C. Gravitational waves in free space

It was pointed out in the Sec. II B that gravitational radiation is associated to the h_{ij}^{TT} part of the metric perturbation, which satisfies the traceless condition (22) and the divergenceless condition (23). In analogy with the wave solutions for electric and magnetic fields in electromagnetism, the general solution of the wave Eq. (89) is

$$h_{ij}^{TT}(\mathbf{x},t) = -\frac{4G}{c^4}\int \frac{\sigma_{ij}(\mathbf{x}',t')}{|\mathbf{x}-\mathbf{x}'|}d^3x', \tag{93}$$

where we use the definition of the retarded time, $t' = t - \frac{|\mathbf{x}-\mathbf{x}'|}{c}$ and the integration is performed over the source. Furthermore, in Eq. (93) it is understood that σ_{ij} are taken to the leading order. To the solutions given in Eq. (93), which describes the generation of the field by a gravitational source manifested in $\sigma_{ij}(\mathbf{x}',t')$, we can always add the solution of the homogeneous equation associated to Eq. (89)

$$\Box h_{ij}^{TT} = 0\,, \tag{94}$$

satisfying to the conditions (22) and (23). Equation (94) admits as solutions plane waves of the form

$$h_{ij}^{TT} = \epsilon_{ij}e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}, \tag{95}$$

where ϵ_{ij} are the components of the polarization three-tensor, which is a symmetric tensor. By imposing the condition (22) to the plane-wave solution (95) we find

$$\epsilon_{ij}k^i = \epsilon_{1j}k^1 + \epsilon_{2j}k^2 + \epsilon_{3j}k^3 = 0. \tag{96}$$

Furthermore, the traceless condition (23) implies that

$$\epsilon_{11} + \epsilon_{22} + \epsilon_{33} = 0\,. \tag{97}$$

As in the discussion of plane electromagnetic waves, an understanding of the features of the solution given in Eqs. (95)–(97) is facilitated when considering the propagation along a specific coordinate axis, say z axis, in the direction of increasing z. In this case, $k^1 = k^2 = 0$ and Eq. (96) furnishes $\epsilon_{3j}k^3 = 0$, that is,

$$\epsilon_{31} = \epsilon_{32} = \epsilon_{33} = 0\,, \tag{98}$$

meaning that every component ϵ_{ij} related to the direction of propagation (the z direction) has to be zero. This shows that, indeed, gravitational waves are transverse. With $\epsilon_{33} = 0$, from Eq. (97) we get

$$\epsilon_{22} = -\epsilon_{11}\,. \tag{99}$$

Finally, collecting all the results (95), (98), and (99) together in matrix form we obtain

$$\begin{aligned}\left[h_{ij}^{TT}\right] &= \begin{bmatrix}\epsilon_{11} & \epsilon_{12} & 0\\ \epsilon_{12} & -\epsilon_{11} & 0\\ 0 & 0 & 0\end{bmatrix}e^{i(kz-\omega t)}\\ &= \begin{bmatrix}h_+ & 0 & 0\\ 0 & -h_+ & 0\\ 0 & 0 & 0\end{bmatrix} + \begin{bmatrix}0 & h_\times & 0\\ h_\times & 0 & 0\\ 0 & 0 & 0\end{bmatrix}.\end{aligned} \tag{100}$$

The two polarization states of gravitational waves are represented by the two matrices at right-hand side in Eq. (100): the "plus" polarization are associated to the function $h_+(z,t) = \epsilon_{11}e^{i(kz-\omega t)}$ and the "cross" polarization to $h_\times(z,t) = \epsilon_{12}e^{i(kz-\omega t)}$. So, there are two different gravitational wave modes (transverse and mutually independent) contained in h_{ij}, oscillating in the xy plane. In analogy with the electromagnetic case, the values of the amplitudes ϵ_{11} and ϵ_{12} completely specify the field.

Concluding this subsection, it is worthwhile to mention that the result given in Eq. (100) are independent of any gauge, being just firmly based in the Helmholtz decomposition in the way that the tensor components h_{ij}^{TT} must be taken as real physical objects. Furthermore, as the traceless and divergenceless (or transversality) conditions (22) and (23) imposed to h_{ij}^{TT} are not related to a gauge choice for waves propagating in free space (as in the traditional approach using the Lorentz gauge), these conditions surely apply to gravitational waves propagating in material media, especially in a plasma.

D. The energy carried by the gravitational field

One of the most profound aspects of the Lagrangian formulation of field theory is the way conservation laws arises. From an appropriate Lagrangian density $\mathscr{L}$, the equations of motion for all fields of a physical system can be readily derived by imposing that the action S obeys the *variational principle*

$$\delta S = \delta\int d^4x\sqrt{-g}\mathscr{L} = 0, \tag{101}$$

where g is the determinant of the metric tensor. According to Noether's theorem [22], if the action S of a system is invariant under certain continuous transformation of the space-time coordinates and physical fields, then there will be a conserved quantity associated with this transformation (the transformation itself is called a *symmetry*). Consider then a system composed by the gravitational field plus other fields (e.g., the electromagnetic and matter fields). When the symmetry of the action of this system under space-time translations is properly analyzed and established, then we are led to a momentum-energy conservation law. One of the mentioned conservation laws that can be derived from this procedure is

$$\frac{\partial}{\partial x^\mu}\left[\sqrt{-g}\left(t^\mu{}_\nu + T^\mu{}_\nu\right)\right] = 0\,, \tag{102}$$

where $T^\mu{}_\nu$ are the mixed components of the stress-energy tensor of the system (including electromagnetic and matter terms, but not gravity) and $t^\mu{}_\nu$ is a gravitational stress-energy *pseudotensor* given by

$$t^\mu{}_\nu = \frac{c^4}{16\pi G}\frac{1}{\sqrt{-g}}\left[\left(\Gamma^\mu_{\alpha\beta} - \delta^\mu_\beta\Gamma^\gamma_{\alpha\gamma}\right)\frac{\partial(\sqrt{-g}g^{\alpha\beta})}{\partial x^\nu} - \delta^\mu_\nu\bar{R}\right], \tag{103}$$

with

$$\bar{R} = g^{\mu\nu}\left(\Gamma^\sigma_{\mu\nu}\Gamma^\rho_{\rho\sigma} - \Gamma^\sigma_{\mu\rho}\Gamma^\rho_{\nu\sigma}\right). \tag{104}$$

The form given for $t^\mu{}_\nu$ in Eq. (103) and the related conservation law, Eq. (102), were obtained following Dirac's approach [67], although several other explicit forms can be chosen (e.g., the Landau-Lifshitz pseudotensor and related conservation law). For reasons of scope and space, we will not concern ourselves here with the derivation of the Eqs. (102) and (103), neither make an effort to clarify why the gravitational stress-energy tensor is, in fact, a pseudotensor and why it is nonunique. For in-depth discussions of the subject, the reader can consult Refs. [67], [72], and [73]. The point that really must be clear hereafter is as follows: because of the invariance of the action S under space-time translations, the conservation of the *total* energy and momentum (gravitation plus other fields) is guaranteed by Eq. (102), with gravitational contributions to the total stress-energy tensor given by Eq. (103). Observe that the leading term in $t^{\mu}{}_{\nu}$ is of second order, which justifies the conservation law given in Eq. (68), correct to the first order. It is important to emphasize that the entire discussion that we will undergo later on about the energy exchanges between matter and the gravitational field could also be carried out using other choices of the pseudotensor, leading to physically equivalent results. We chose the expressions given by Eqs. (102) and (103) because they follow naturally (or, at least, more naturally) from the variational principle, which, in turn, provides a unified way of investigating physical systems (although we recognize that, in part, these choices were made for reasons of personal taste considering the background of the authors in their studies of general relativity).

Taking the covariant divergence of Einstein equation [Eq. (1)] and using that $\nabla_{\mu}T^{\mu}{}_{\nu}=0$, it can be shown from Eq. (102) that

$$\frac{\partial(\sqrt{-g}t^{\mu}{}_{\nu})}{\partial x^{\mu}}=\frac{c^4}{16\pi G}\sqrt{-g}\frac{\partial g_{\rho\sigma}}{\partial x^{\nu}}G^{\rho\sigma}\,. \tag{105}$$

Especially, with $\nu=0$, Eq. (105) becomes the (nearest of a) gravitational analogous of the Poynting theorem of electromagnetism. Indeed, identifying

$$\sqrt{-g}t^{0}{}_{0}=\mathscr{U}_g \tag{106}$$

and

$$c\sqrt{-g}t^{i}{}_{0}=\mathscr{J}_g^{i}, \tag{107}$$

respectively, as the gravitational energy density and the gravitational energy flux, Eq. (105) is rewritten as

$$\frac{\partial\mathscr{U}_g}{\partial t}+\frac{\partial\mathscr{J}_g^{i}}{\partial x^i}=\frac{c^4}{16\pi G}\sqrt{-g}\frac{\partial g_{\mu\nu}}{\partial t}G^{\mu\nu}. \tag{108}$$

Furthermore, considering the weak-field approximation, Eq. (108) acquires the form

$$\frac{\partial\mathscr{U}_g}{\partial t}+\frac{\partial\mathscr{J}_g^{i}}{\partial x^i}=\frac{c^4}{16\pi G}\frac{\partial h_{\mu\nu}}{\partial t}G^{\mu\nu}, \tag{109}$$

with $G^{\mu\nu}$ being the contravariant components of the Einstein tensor, given by Eq. (9). For gravitational waves in free space (or for any gravitational field in free space), where $G^{\mu\nu}=0$ from the Einstein field Eqs. (1), Eqs. (108) and (109) guarantees gravitational energy conservation. Just to clarify the picture, we mention that, in the weak-field approximation, Eqs. (106) and (107) for energy density and energy flux of a plane gravitational wave propagating in the z direction in vacuum gives

$$<\mathscr{U}_g>=\frac{c^2\omega^2}{32\pi G}[(\epsilon_{11})^2+(\epsilon_{12})^2] \tag{110}$$

and

$$<\mathscr{J}_g>=\frac{c^3\omega^2}{32\pi G}[(\epsilon_{11})^2+(\epsilon_{12})^2], \tag{111}$$

where the symbol $<>$ represents the mean value over a period and $\omega=c\,k$ is the angular frequency of the wave. It is nice to observe that the results given in Eqs. (110) and (111) greatly resembles that for electromagnetic waves. On the other hand, for a medium other than empty space, the right-hand side of Eqs. (108) and (109) are associated to the coupling between gravity and the other constituents of the physical system under consideration, giving the rate of exchange of gravitational energy. So, Eqs. (108) and (109) constitutes suitable tools to study these exchanges.

Our last task in this section is to furnish some useful formulas to calculate the Christoffel symbols in terms of the gauge-invariant quantities obtained in part B of this section (they are needed, among other things, to calculate the densities $\mathscr{U}_g$ and $\mathscr{J}_g^{i}$). For this purpose, we employ our preferred gauge with $\psi=\lambda=\mathscr{E}_i=0$ and, with this, from Eq. (4) we obtain the complete list of Christoffel symbols of the second kind, as follows:

$$\Gamma^0_{00}=\frac{1}{c^3}\frac{\partial\Phi}{\partial t}, \tag{112}$$

$$\Gamma^0_{0i}=\Gamma^0_{i0}=\frac{1}{c^2}\frac{\partial\Phi}{\partial x^i}, \tag{113}$$

$$\Gamma^0_{ij}=\Gamma^0_{ji}=-\frac{1}{2c}\left(\frac{\partial\Xi_i}{\partial x^j}+\frac{\partial\Xi_j}{\partial x^i}+\frac{\partial h_{ij}}{\partial t}\right), \tag{114}$$

$$\Gamma^i_{00}=\frac{1}{c^2}\delta^{ij}\left(\frac{\partial\Xi_j}{\partial t}+\frac{\partial\Phi}{\partial x^j}\right), \tag{115}$$

$$\Gamma^i_{j0}=\Gamma^i_{0j}=\frac{1}{2c}\delta^{ik}\left(\frac{\partial\Xi_k}{\partial x^j}-\frac{\partial\Xi_j}{\partial x^k}-\frac{\partial h_{jk}}{\partial t}\right), \tag{116}$$

$$\Gamma^i_{jk}=\Gamma^i_{kj}=\frac{1}{2}\delta^{il}\left(\frac{\partial h_{jk}}{\partial x^l}-\frac{\partial h_{jl}}{\partial x^k}-\frac{\partial h_{kl}}{\partial x^j}\right), \tag{117}$$

with h_{ij} given by Eq. (92). For the contracted Christoffel symbol appearing in Eq. (103) we obtain, considering the linear regime,

$$\Gamma^{\gamma}_{\alpha\gamma}=\frac{1}{2}\frac{\partial\ln(-g)}{\partial x^{\alpha}}\approx\frac{1}{2}\frac{\partial\ln(1+h)}{\partial x^{\alpha}}\approx\frac{1}{2}\frac{\partial h}{\partial x^{\alpha}}\,. \tag{118}$$

On the other hand, the four-dimensional trace h of the metric perturbation is given by

$$h=\eta^{\mu\nu}h_{\mu\nu}=\frac{2\Phi}{c^2}-3\Theta\,, \tag{119}$$

where use was made of Eqs. (23), (90), and (92). Substituting Eq. (119) in Eq. (118) we find

$$\Gamma^{\gamma}_{\alpha\gamma}=\frac{\partial}{\partial x^{\alpha}}\left(\frac{\Phi}{c^2}-\frac{3\Theta}{2}\right), \tag{120}$$

which is correct to the first order. It is not very useful to insert all the formulas (112)–(117) and (120) in Eqs. (106) and (107). The result would be cumbersome and difficult to manage. It is much more valuable to use the obtained formulas whenever they are necessary, depending on what dynamical mode of the gravitational field we are dealing. For example, to obtain Eqs. (110) and (111) for plane gravitational waves propagating in the z direction the only nonzero Christoffel symbols are those depending on h_{ij}^{TT}, and $\Gamma^{\gamma}_{\alpha\gamma} = 0$, which facilitates calculations.

III. NONCOLLISIONAL PLASMAS: THE EINSTEIN-VLASOV-MAXWELL SYSTEM

A plasma can be defined as a large quasineutral collection of charged particles whose dynamics is governed by collective interactions instead of simple pair interactions. Plasma kinetic theory, in turn, can be viewed as an attempt to establish, in a statistical way, the physics involved in the mutual effects between the electromagnetic field and matter in the plasma state. It is natural, in general relativity, to try this statistical approach to study the mutual effects between the gravitational field, the electromagnetic field and the plasma. So, in this section we will briefly summarize the basic concepts of kinetic theory in curved spacetime and write down the fundamental equation governing the dynamics of the one-body distribution function for a noncollisional plasma—the Vlasov equation.

A. The Einstein-Maxwell system

For the sake of completeness and clarity of the text, before we undergo into kinetic theory, it is instructive to review briefly the Einstein and Maxwell equations in absence of matter—the Einstein-Maxwell system—in the linear regime. For a more detailed treatment, one can consult Refs. [7–9,67]. In a material medium, these equations do not constitute a closed system and so cannot provide a complete description of any physical system, as we will see.

The right-hand side of the Einstein Eq. (1) depends on the distribution of mass and energy of the physical system under consideration. In the same way, the right-hand side of the equations governing the dynamics of the electromagnetic tensor $F_{\mu\nu}$, given by

$$\frac{1}{\sqrt{-g}}\frac{\partial(\sqrt{-g}F^{\mu\nu})}{\partial x^{\mu}} = \mu_0 J^{\nu}, \tag{121}$$

depends on the electric four-current J^{ν} defined by

$$J^{\nu} = (\rho c, \mathbf{J}), \tag{122}$$

where ρ is the electric charge density and $\mathbf{J}$ is the electric current density three-vector. As usual, μ_0 in Eq. (121) symbolizes the magnetic permeability of free space, whose value is $4\pi \times 10^{-7}$ H/m. With $\nu = 0$ Eq. (121) corresponds to Gauss law of electricity, and with $\nu = i$ to the Ampère-Maxwell law. The equations corresponding to the nonexistence of magnetic monopoles (or the Gauss law of magnetism) and to the Faraday-Lenz induction law are given by

$$\frac{\partial F_{\nu\rho}}{\partial x^{\mu}} + \frac{\partial F_{\rho\mu}}{\partial x^{\nu}} + \frac{\partial F_{\mu\nu}}{\partial x^{\rho}} = 0\,. \tag{123}$$

Equations (121) and (123) are the Maxwell equations in curved spacetime and, except for the factors $\sqrt{-g}$ in Eq. (121) are the same as in the flat spacetime. Maxwell equations could too be written in terms of the electromagnetic four-potential A_{μ}, but the forms adopted here involves only the gauge-free quantities $F_{\mu\nu}$ given in matrix representation by

$$\mathbb{F} = [F_{\mu\nu}] = \begin{bmatrix} 0 & \frac{E_x}{c} & \frac{E_y}{c} & \frac{E_z}{c} \\ -\frac{E_x}{c} & 0 & -B_z & B_y \\ -\frac{E_y}{c} & B_z & 0 & -B_x \\ -\frac{E_z}{c} & -B_y & B_x & 0 \end{bmatrix}, \tag{124}$$

that is, the components of the conventional electric and magnetic fields. So, as our aim is a gauge-free treatment, the Eqs. (121) and (123) are most adequate for our purposes. Furthermore, in the linear regime Eq. (121) can be written in the form

$$\frac{\partial F^{\mu\nu}}{\partial x^{\mu}} = \mu_0 J^{\nu} - \frac{1}{2}\frac{\partial h}{\partial x^{\mu}}F^{\mu\nu}, \tag{125}$$

in which the second term at the right-hand side represents an explicit first-order influence of the gravitational field over the electromagnetic field. Equation (125) alone provides a nice insight: The direct coupling of electromagnetism to gravity involves only the scalar components of the gravitational field [see Eq. (119)] and, therefore, there is no direct influence of gravitational wave modes on the dynamics of electromagnetic fields.

To investigate the reciprocal effects, that is, the influence of electromagnetism in the dynamics of the gravitational field, we proceed by conveniently writing the electromagnetic stress-energy tensor in the matrix form

$$\mathbb{T}_{\text{em}} = \frac{1}{\mu_0}\left[\mathbb{F}\,\mathbb{g}^{-1}\mathbb{F} - \frac{1}{4}\mathbb{g}\,\text{Tr}(\tilde{\mathbb{F}}\mathbb{F})\right], \tag{126}$$

where $\mathbb{T}_{\text{em}}$, $\mathbb{g}$, $\mathbb{F}$ stands for the matrix representations of the electromagnetic stress-energy tensor, the metric tensor and the electromagnetic tensor in terms of covariant components, and $\tilde{\mathbb{F}}$ is given by

$$\tilde{\mathbb{F}} = [F^{\mu\nu}] = \mathbb{g}^{-1}\mathbb{F}\,\mathbb{g}^{-1}, \tag{127}$$

where $\mathbb{g}^{-1}$ is the inverse of $\mathbb{g}$. In view of Eq. (6), to the first order we have

$$\mathbb{g} = \eta + \mathbb{h} \tag{128}$$

and

$$\mathbb{g}^{-1} = \eta - \mathbb{h}\,, \tag{129}$$

where η is the matrix representation of the Minkowski metric. So, to the first order we have

$$\tilde{\mathbb{F}} = \begin{bmatrix} 0 & -\frac{E_x}{c} & -\frac{E_y}{c} & -\frac{E_z}{c} \\ \frac{E_x}{c} & 0 & -B_z & B_y \\ \frac{E_y}{c} & B_z & 0 & -B_x \\ \frac{E_z}{c} & -B_y & B_x & 0 \end{bmatrix} \tag{130}$$

and

$$\frac{1}{4\mu_0}\text{Tr}(\tilde{\mathbb{F}}\mathbb{F}) = \frac{\epsilon_0 E^2}{2} - \frac{B^2}{2\mu_0}\,. \tag{131}$$

For convenience, in Eq. (131) we introduce the electric permittivity of free space, $\epsilon_0 = \frac{1}{\mu_0 c^2}$. With Eqs. (128), (129), and (131) inserted in Eq. (126) we obtain, to the first order,

$$\mathbb{T}_{\text{em}} = \mathbb{T}^{(\text{flat})}_{\text{em}} + \mathbb{T}^{(\text{curved})}_{\text{em}}, \tag{132}$$

where $\mathbb{T}_{\text{em}}^{(\text{flat})}$ is the flat space-time contribution to $\mathbb{T}_{\text{em}}$, given by

$$\mathbb{T}_{\text{em}}^{(\text{flat})} = \begin{bmatrix} \mathscr{U}_{\text{em}}^{(\text{flat})} & -\frac{s_x}{c} & -\frac{s_y}{c} & -\frac{s_z}{c} \\ -\frac{s_x}{c} & -\tau_{11} & -\tau_{12} & -\tau_{13} \\ -\frac{s_y}{c} & -\tau_{21} & -\tau_{22} & -\tau_{23} \\ -\frac{s_z}{c} & -\tau_{31} & -\tau_{32} & -\tau_{33} \end{bmatrix}, \tag{133}$$

where $\mathscr{U}_{\text{em}}^{(\text{flat})} = \frac{\epsilon_0 E^2}{2} + \frac{B^2}{2\mu_0}$ is the zero-order electromagnetic energy density. In Eq. (133) we employ the definition of the Maxwell stress tensor τ_{ij},

$$-\tau_{ij} = T_{em,ij}^{(\text{flat})} = -\epsilon_0 E_i E_j - \frac{B_i B_j}{\mu_0} + \mathscr{U}_{\text{em}}^{(\text{flat})}\delta_{ij}, \tag{134}$$

and indicate the components of the Poynting vector $\mathbf{s} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}$ [the lowercase $\mathbf{s}$ must not to be confused with the capital $\mathbf{S}$ defined in Eq. (51)]. In turn, the first-order correction $\mathbb{T}_{\text{em}}^{(\text{curved})}$ including the gravitational-electromagnetic coupling is given by

$$\mathbb{T}_{\text{em}}^{(\text{curved})} = -\frac{1}{\mu_0}\left[\mathbb{F}\mathbb{h}\mathbb{F} + \frac{1}{4}\mathbb{h}\,\text{Tr}(\tilde{\mathbb{F}}\mathbb{F})\right]. \tag{135}$$

The general form of $\mathbb{T}_{\text{em}}^{(\text{curved})}$ is hard to handle, and a better and cleaner job can be made evaluating the first-order terms in specific cases and then identify the quantities T_{00} S, S_i, P, and σ_{ij}, necessary to write Einstein Eqs. (86)–(89). A general procedure to extract these quantities from the stress-energy tensor in reciprocal space will be discussed in Sec. III D.

As we can see, the Einstein-Maxwell system is a far rich system of equations. On the other hand, it suffers for not taking into account the mutual effects between the gravitational and electromagnetic fields and matter in a material medium, and so cannot provide a complete description of the field-matter interactions. Therefore, we will now move on to the kinetic theory as an attempt to describe these interaction and formulate a complete theory of plasmas in curved spacetime. Later, we will apply this theory to the problem of wave propagation in this complex medium.

B. The Vlasov equation in curved spacetime

Plasmas are systems of many electrically charged particles (mostly partially or fully ionized gases). So plasmas are, at the same time, affected by the gravitational and electromagnetic fields and sources of these field. It is then necessary to add to the Einstein-Maxwell system one more equation to take into account the effects of the fields on the plasma, an objective that can be achieved through kinetic theory [23].

In analogy to the phase space of nonrelativistic mechanics, the phase space of general relativity is also the space of all coordinates and momenta of a physical system, as shown in Fig. 1. There are, however, some subtleties to consider in the relativistic case. They are as follows:

(i) In theory of relativity time is, naturally, a coordinate, and the construction of the phase space must also incorporate this fact naturally.

(ii) The components of the three-momentum of a particle and its total (rest plus kinetic) energy are, respectively, the spatial and temporal components of the momentum four-vector, whose magnitude is constant; that is, there is a constraint between the components of the four-momentum, which reduces the dimensionality of the phase space.

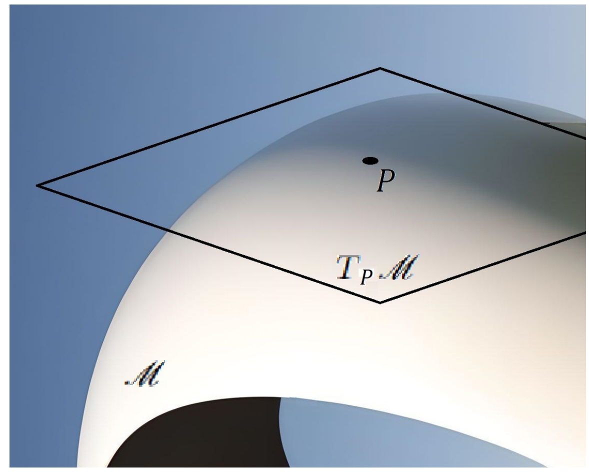

FIG. 1. The phase space of general relativistic kinetic theory is made up of spacetime (a four-dimensional manifold) and the tangent bundle (the set of all tangent spaces).

(iii) Some care is needed in defining the relevant volume elements, in order to preserve the covariance of the theory.

With the precautions mentioned above, we can define the relativistic phase space of an N-particle system as the space of the $4N$ space-time coordinates (say, x, y, z, and $x^0 = ct$ of each particle) and the $3N$ spatial components of the four-moments of the system (say, p_x, p_y, and p_z of each particle) of the system (the reason why we are apparently neglecting the temporal components of the four-momenta is the constraint mentioned in item (ii) and will be discussed shortly). It is therefore a $7N$-dimensional space. We will see that, despite the difference in dimensionality between relativistic and nonrelativistic phase spaces (the first is $7N$-dimensional and the second is $6N$-dimensional), the one-body distribution function here also depends on seven variables: In relativistic theory, x^μ and p^i take the place of $\mathbf{x}$, t and $\mathbf{p}$. From a geometric point of view, the $7N$-dimensional phase space can be visualized as a kind of union between spacetime (a four-dimensional manifold $\mathscr{M}$ in which the dynamics of all N particles develop) and the set of tangent spaces at every point of $\mathscr{M}$ (in which the four-momentum of the N particles resides). Usually, the tangent space of $\mathscr{M}$ at point P is designated by $T_P\mathscr{M}$ and the set of all tangent spaces of $\mathscr{M}$ is called the tangent bundle. However, as a result of the constraint

$$p^\mu p_\mu = p^0 p_0 + g_{ij} p^i p^j = m^2 c^2, \tag{136}$$

we can express one of the four-momentum components in terms of the others, and the most natural choice (i.e., the one that makes everything more like the usual nonrelativistic theory) is to express the temporal component as a function of the spatial components. Thus, the "slice" of the tangent bundle that interests us in the construction of the phase space is the one in which Eq. (136) is satisfied for each particle of the system. Observe that we are symbolizing the contravariant components of the four-momentum and the three-momentum equally by p^i, as they are identical. On the other hand, writing p_i we are indicating only the covariant components of the three-momentum.

Having defined the relevant phase space, we can move on to the determination of the Vlasov equation in curved spacetime, that is, the differential equation governing the dynamics of the one-body distribution function for a collisionless plasma in general relativity. For brevity, as the details of this subject can be found and are cleanly discussed elsewhere [23], we will present the mentioned equation by simple analogy to its nonrelativistic counterpart, much in the same way Maxwell equations can be adapted to general relativity. Thus, by imposing the covariance requirement, we write down relativistic equations that fall into the nonrelativistic equations in appropriate limits. The nonrelativistic Vlasov equation can be written in the form

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{dx^i}{dt}\frac{\partial f}{\partial x^i} + \frac{dp^i}{dt}\frac{\partial f}{\partial p^i} = 0\,, \tag{137}$$

where $f = f(\mathbf{x}, \mathbf{p}, t)$ is the one-body distribution function. In general relativity, the dependencies of f must be replaced by x^μ and p^i. So, the relativistic Vlasov equation can be readily written in the form

$$\frac{df}{d\tau} = \frac{dx^\mu}{d\tau}\frac{\partial f}{\partial x^\mu} + \frac{dp^i}{d\tau}\frac{\partial f}{\partial p^i} = 0\,, \tag{138}$$

where $f = f(x^\mu, p^i) \equiv f(x, p)$. On the other hand, from Eq. (5) and using that $p^\mu = m\frac{dx^\mu}{d\tau}$, we obtain

$$\frac{dp^i}{d\tau} = -\Gamma^i_{\nu\rho}\, p^\nu \frac{dx^\rho}{d\tau} + qF^i{}_\nu \frac{dx^\nu}{d\tau}\,. \tag{139}$$

Finally, substituting Eq. (139) in Eq. (138) we find the Vlasov equation in the form

$$p^\mu \frac{\partial f}{\partial x^\mu} - \Gamma^i_{\nu\rho}\, p^\nu p^\rho \frac{\partial f}{\partial p^i} + qF^i{}_\nu p^\nu \frac{\partial f}{\partial p^i} = 0\,. \tag{140}$$

The weak-field limit of the Vlasov equation will be discussed in Sec. III E. However, it is interesting to keep in mind right away that the Christoffel symbols (which personify the gravitational force field), in our preferred gauge can all be expressed solely in terms of the gravitational gauge invariants, as pointed out in Sec. II. It is also nice to check that the nonrelativistic Vlasov Eq. (137) is readily recovered from its general relativistic counterpart, Eq. (140), by taking the appropriate limits.

The Vlasov equation determines the influence of gravity and electromagnetism over the distribution function—that is, over the matter behavior—via the Christoffel symbols $\Gamma^\mu_{\nu\rho}$ and the electromagnetic tensor $F^\mu{}_\nu$, being the third cornerstone in the theory discussed. In the next subsection we will see how to properly formulate expressions for the matter stress-energy tensor and charge four-current in terms of the distribution function f. For now, it is worth to mention that for a system with many particle species, there is one distribution function for each of these species, each described by an appropriate Vlasov equation.

C. Charge four-current and matter stress-energy tensor in kinetic theory

In nonrelativistic kinetic theory we can obtain several interesting physical quantities by performing integrations of the one-body distribution function in momentum space. In particular, the number density, the charge density, the charge current density and the matter stress tensor are given, respectively, by

$$n(\mathbf{x}, t) = \int f(\mathbf{x}, \mathbf{p}, t)\, d^3p, \tag{141}$$

$$\rho(\mathbf{x}, t) = q \int f(\mathbf{x}, \mathbf{p}, t)\, d^3p, \tag{142}$$

$$J^i(\mathbf{x}, t) = \frac{q}{m} \int p^i f(\mathbf{x}, \mathbf{p}, t)\, d^3p, \tag{143}$$

$$T^{ij}(\mathbf{x}, t) = \frac{1}{m} \int p^i p^j f(\mathbf{x}, \mathbf{p}, t)\, d^3p. \tag{144}$$

For the general relativistic counterparts of Eqs. (141)–(144) above, it can be shown [23] that we must replace the momentum space volume element d^3p by the invariant $\frac{\sqrt{-g}}{p_0}d^3p$, taking care with the other factors in order to maintain the dimensional consistency of the formalism. Furthermore, the charge density and current combine to form the four-vector J^μ, and the stress tensor is replaced by the most general object $T^{\mu\nu}$. It gives us

$$n(x) = mc \int f(x, p)\, \frac{\sqrt{-g}}{p_0} d^3p, \tag{145}$$

$$J^\mu(x) = qc \int p^\mu f(x, p)\, \frac{\sqrt{-g}}{p_0} d^3p, \tag{146}$$

$$T^{\mu\nu}(x) = c \int p^\mu p^\nu f(x, p)\, \frac{\sqrt{-g}}{p_0} d^3p. \tag{147}$$

It should be noted that, in view of Eq. (136), in all the equations above we have

$$p_0 = \sqrt{g_{00}m^2c^2 + (g_{0i}g_{0j} - g_{00}g_{ij})p^ip^j}, \tag{148}$$

with the contravariant component of the temporal part of the four-momentum given by

$$p^0 = \frac{p_0 - g_{0i}p^i}{g_{00}}\,. \tag{149}$$

Our next task is to establish Eqs. (146) and (147) in the weak field limit. For this, we must first obtain the linearized versions of $\frac{\sqrt{-g}}{p_0}$, $p^0\frac{\sqrt{-g}}{p_0}$ and $p^0p^0\frac{\sqrt{-g}}{p_0}$. From Eqs. (148) and (149), we find

$$\frac{\sqrt{-g}}{p_0} = \frac{1 + \alpha(x, p)}{\bar{p}_0}, \tag{150}$$

$$p^0\frac{\sqrt{-g}}{p_0} = 1 + \beta(x, p), \tag{151}$$

$$p^0p^0\frac{\sqrt{-g}}{p_0} = \bar{p}_0(1 + \gamma(x, p)). \tag{152}$$

where $\bar{p}_0 = \sqrt{m^2c^2 + \delta_{ij}p^ip^j}$ is the flat space-time covariant time component of the four-momentum and α, β, and γ are the following nondimensional first-order quantities

$$\alpha(x, p) = \frac{h}{2} - \frac{h_{00}(\bar{p}_0)^2 - h_{ij}p^ip^j}{2(\bar{p}_0)^2}, \tag{153}$$

$$\beta(x, p) = \frac{h}{2} - \frac{h_{00}\bar{p}_0 + h_{0i}p^i}{\bar{p}_0}, \tag{154}$$

$$\gamma(x, p) = -\alpha(x, p) + 2\beta(x, p). \tag{155}$$

Here, too, the general expressions of the above equations in terms of the gauge invariants are very complicated and difficult to handle, making it more practical to obtain explicit expressions only in specific calculations. For example, for gravitational waves, we have $h_{00} = h_{0i} = h = 0$ and $h_{ij} = h_{ij}^{TT}$. Thus, in this case,

$$\alpha(x,p) = -\gamma(x,p) = \frac{h_{ij}^{TT} p^i p^j}{2(\bar{p}_0)^2}, \tag{156}$$

$$\beta(x,p) = 0. \tag{157}$$

As another example, consider the Schwarzschild-like metric in isotropic coordinates for points far away from the event horizon, given by $h_{00} = \frac{2\Phi}{c^2}$, $h_{ij} = \delta_{ij}\frac{2\Phi}{c^2}$, $h = -\frac{4\Phi}{c^2}$, and $h_{ij}^{TT} = h_{0i} = 0$. In this case we have

$$\alpha(x,p) = \frac{\Phi}{c^2}\left[\frac{\delta_{ij}p^ip^j}{(\bar{p}_0)^2} - 3\right], \tag{158}$$

$$\beta(x,p) = -\frac{4\Phi}{c^2}, \tag{159}$$

$$\gamma(x,p) = -\frac{\Phi}{c^2}\left[\frac{\delta_{ij}p^ip^j}{(\bar{p}_0)^2} + 5\right]. \tag{160}$$

With formulas (150)–(152), in weak-field limit the components of the charge four-current and of the stress-energy tensor are expressed as

$$J^i(\mathbf{x},t) = q\,c\int p^i\frac{[1+\alpha(x,p)]}{\bar{p}_0}f(x,p)d^3p, \tag{161}$$

$$\rho(\mathbf{x},t) = \frac{J^0(\mathbf{x},t)}{c} = q\int [1+\beta(x,p)]f(x,p)d^3p, \tag{162}$$

$$T^{ij}(\mathbf{x},t) = c\int p^ip^j\frac{[1+\alpha(x,p)]}{\bar{p}_0}f(x,p)d^3p, \tag{163}$$

$$T^{0i}(\mathbf{x},t) = c\int p^i[1+\beta(x,p)]f(x,p)d^3p, \tag{164}$$

$$T^{00}(\mathbf{x},t) = c\int \bar{p}_0[1+\gamma(x,p)]f(x,p)d^3p. \tag{165}$$

Equations (161) and (162) represents the source terms for Maxwell equations, whereas Eqs. (163)–(165) play that role in Einstein equations. In the next subsection we will see how to extract from the components T^{0i} and T^{ij} the objects S, S_i, P, and σ_{ij} (necessary to write Einstein equations), a job that proves to be simpler in the reciprocal space.

D. Fourier transformed source terms for gravity

In order to extract S, S_i, P, and σ_{ij} from T^{0i} and T^{ij}, it is convenient to take the Fourier transformed versions of Eqs. (51) and (52) for T^{0i} and (70)–(73) for T^{ij}. Furthermore, taking Fourier transforms is the usual starting point for the study of waves in a material media, providing a simple way to obtain dispersion relations, as will be done in Sec. IV. We employ the convention in which the Fourier transform of a function $f(\mathbf{x})$ and the related Fourier integral are respectively given by

$$\hat{f}(\mathbf{k}) = \int f(\mathbf{x})e^{-i\mathbf{k}\cdot\mathbf{x}}d^3x \tag{166}$$

and

$$f(\mathbf{x}) = \frac{1}{(2\pi)^3}\int \hat{f}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}d^3k. \tag{167}$$

Hereafter, a hat over any letter will symbolize a spatial Fourier transform, as above. By Fourier transforming Eqs. (51), (52) and (70)–(73) we obtain

$$\hat{T}_{0i} = c(ik_i\hat{S} + \hat{S}_i), \tag{168}$$

$$k^i\hat{S}_i = 0, \tag{169}$$

$$\hat{T}_{ij} = \hat{\sigma}_{ij} + \delta_{ij}\hat{P} + i(k_j\hat{\sigma}_i + k_i\hat{\sigma}_j) + \left(k_ik_j - \tfrac{1}{3}\delta_{ij}k^2\right)\hat{\sigma}, \tag{170}$$

$$k^i\hat{\sigma}_{ij} = 0, \tag{171}$$

$$\delta^{ij}\hat{\sigma}_{ij} = 0, \tag{172}$$

$$k^i\hat{\sigma}_i = 0, \tag{173}$$

where $k^2 = k^ik_i = \delta_{ij}k^ik^j$ (remember that $\mathbf{k}$ is a three-vector, not a four-vector). Contracting Eq. (168) with k^i and taking into account Eq. (169) we find

$$\hat{S} = \frac{k^i}{ick^2}\hat{T}_{0i}. \tag{174}$$

Returning with the result (174) in (168) and rearranging, we get

$$\hat{S}_i = \frac{1}{c}\left(\delta_i^j - \frac{k_ik^j}{k^2}\right)\hat{T}_{0j}. \tag{175}$$

By imposing conditions (171)–(173) we can also solve Eq. (170) for $\hat{P}$, $\hat{\sigma}$, $\hat{\sigma}_i$, and $\hat{\sigma}_{ij}$. The results are

$$\hat{P} = \frac{1}{3}\delta^{ij}\hat{T}_{ij}, \tag{176}$$

$$\hat{\sigma} = -\frac{3}{2}\left(\frac{\delta^{ij}}{3k^2} - \frac{k^ik^j}{k^4}\right)\hat{T}_{ij}, \tag{177}$$

$$\hat{\sigma}_i = \frac{1}{ik^2}\left(k^k\delta_i^j - \frac{k_ik^jk^k}{k^2}\right)\hat{T}_{jk}, \tag{178}$$

$$\begin{aligned}\hat{\sigma}_{ij} = {}& \left(\delta_i^k\delta_j^l - \frac{1}{2}\delta_{ij}\delta^{kl}\right)\hat{T}_{kl};\\ &+ \frac{1}{2k^2}\left(k_ik_j\delta^{kl} + \delta_{ij}k^kk^l - 2k_jk^k\delta_i^l - 2k_ik^k\delta_j^l\right)\hat{T}_{kl}\\ &+ \frac{1}{2k^4}k_ik_jk^kk^l\hat{T}_{kl}.\end{aligned} \tag{179}$$

Although σ and σ_i do not appears in the Eqs. for the gravitational field, obtaining Eqs. (177) and (178) for these quantities was important for writing Eq. (179) for σ_{ij}. We now have at hand all the expressions we need to write the field Eqs. (86)–(89).

E. Three-dimensional forms of Maxwell and Vlasov equations in the weak field regime: The EVM system

From Eqs. (123), (124), (125), and (130), to the first order in the gravitational perturbation, Maxwell equations can be written in the familiar three-dimensional form

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} - \frac{1}{2}\nabla h \cdot \mathbf{E}, \tag{180}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} + \frac{1}{2}\left(\frac{1}{c^2}\frac{\partial h}{\partial t}\mathbf{E} - \nabla h \times \mathbf{B}\right), \tag{181}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{182}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{183}$$

where $\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$, as in flat spacetime. Equations (180)–(183) corresponds to the well-known equations of the electromagnetic theory, corrected by a few terms proportional to space-time derivatives of h and to the electric and magnetic fields themselves. In this same three-dimensional formalism, Vlasov Eq. (140) can be written in the following friendly form:

$$\frac{\partial f}{\partial t} + \frac{c\mathbf{p}}{p^0} \cdot \nabla f + (\mathbf{F}_{\rm em} + \mathbf{F}_g) \cdot \nabla_{\mathbf{p}} f = 0, \tag{184}$$

where

$$\mathbf{F}_{\rm em} = q\left(\mathbf{E} + \frac{c\mathbf{p}}{p^0} \times \mathbf{B}\right) \tag{185}$$

is the electromagnetic force and

$$\mathbf{F}_g = \frac{p^0}{c}\left[-\nabla\Phi - \frac{\partial \boldsymbol{\Xi}}{\partial t} + \frac{c\mathbf{p}}{p^0} \times (\nabla \times \boldsymbol{\Xi})\right] + \mathbf{F}_T \tag{186}$$

is the gravitational force, both correct to the first order. $\mathbf{F}_T$ comes from the tensor part of the metric and has components given by

$$F_{T,i} = \frac{\partial h_{ij}}{\partial t} p^j + \frac{c}{p^0}[jk, i] p^j p^k, \tag{187}$$

with

$$[jk, i] = \frac{1}{2}\left(\frac{\partial h_{ij}}{\partial x^k} + \frac{\partial h_{ik}}{\partial x^j} - \frac{\partial h_{jk}}{\partial x^i}\right). \tag{188}$$

For consistency, in the whole set of Eqs. (184)–(187), p^0 must be taken correct to the zero order. From Eqs. (148) and (149) we obtain for p^0 in this limit

$$p^0 = \bar{p}_0 = \sqrt{m^2c^2 + \delta_{ij} p^i p^j}\ . \tag{189}$$

In Eq. (186), the terms collected in brackets closely resembles the electromagnetic force—given by Eq. (185)—written in terms of electromagnetic potentials (note that just the first one appears in the Newtonian theory of gravitation). However, it is important to stress that this similarity is somewhat superficial, since the potentials Φ and $\boldsymbol{\Xi}$ always satisfy the inhomogeneous Poisson Eqs. (87) and (88) (not first-order coupled equations), and there is not a gravitational analogous to the Faraday and Maxwell induction terms in the gravitational field Eqs. (86)–(89). Furthermore, the gravitational potentials we are dealing are gauge invariant in the linear theory of gravity, whereas electromagnetic potentials are not invariant with respect to the electromagnetic gauge transformations. It is also worth to mention that, while the electromagnetic force is invariant to the gauge transformations of electromagnetism, the gravitational force is dependent on the chosen gravitational gauge, as required by the principle of equivalence.

Einstein Eqs. (86)–(89), Maxwell Eqs. (180)–(183) and Vlasov Eq. (184) constitutes a complete system to describe the behavior of a collisionless plasma in a general relativistic framework—the EVM system. In the next section, the dispersion relation for gravitational waves in an homogeneous plasma will be derived from these system of equations.

IV. GRAVITATIONAL AND ELECTROSTATIC WAVES IN AN ELECTRON-POSITRON PLASMA

So far we have discussed Einstein, Maxwell, and Vlasov equations just in the gravitational weak-field limit, that is, no approximations were made for the electromagnetic field and for the distribution function. Furthermore, we have yet to introduce in the formalism the fact that plasmas are quasineutral systems of (at least) two charged particle species. Hereafter, the whole set of field Eqs. (86)–(89) and (180)–(183) and correspondent source terms, plus the Vlasov Eq. (184), elaborated in the gravitational weak-field limit, will be taken as exact equations. The dispersion function for any kind of oscillation or wave can then be found employing perturbation theory.

A. The Einstein-Vlasov-Poisson system for an homogeneous neutral electron-positron plasma

Relativistic electron-positron pair plasmas are principal constituents of high-energy astrophysical environments, such as neutron stars and black holes surroundings [74]. Thus, a theoretical description of this type of plasma is fundamental in understanding and interpreting several processes and phenomena that occur in these environments, such as pair creation and annihilation, relativistic jets from active galactic nuclei, and γ-ray bursts [74,75]. In these systems, collective plasma processes are responsible for determining the magnetic field dynamics, energy partition, and radiation emission [74]. Furthemore, this type of pair plasma appeared in the very early universe [76].

Although there are several theoretical works on relativistic electron-positron pair plasmas dynamics, to our knowledge there is still a gap regarding an understading the particularities of gravitational oscillatory modes in this medium—especially gravitational waves—employing a kinetic approach. For example, in Ref. [77] a quantum hidrodynamical model for a multicomponent electron-positron-ion plasma was proposed for studying their gravitational instability. In this work, however, gravitation was taken in the Newtonian sense, that is, via the Poisson equation for a scalar potential. Employing a two-fluid model, electromagnetic wave instability in unmagnetized electron-positron pair plasma was discussed in Ref. [78], and in Ref. [79] again a two-fluid approach was established for describing nonlinear waves in an inhomogeneous collisionless magnetized relativistic electron-positron plasma in a prescribed gravitational field. The nonlinear interaction between magnetic field-aligned electromagnetic waves and electrostatic oscillation in a electron-positron-ion plasma was considered in Ref. [80], and in Ref. [81] relativistic collisionless shock waves, associated with GBRs, propagating in inhomogeneous electron-positron plasmas was studied. In these last two works, however, nothing related to gravity was considered.

In view of the above, as an application illustrating the generality, security and simplicity of the gauge-invariant formalism, we now apply it to an electron-positron homogeneous plasma where gravitational and electrostatic wave perturbations takes place. First, in order to write down the appropriate equations, we must to establish the relevant physical variables of the problem. For gravitational waves, the scalar and vector components of the metric perturbation are $\Phi = \Theta = \Xi_i = 0$. The transversal traceless tensor components are $h_{ij}^{TT} = h_{ij}$ by virtue of Eq. (92), and so, hereafter we will omit the superscript TT in h_{ij} for simplicity. As pointed out in Sec. II C, the transversality and traceless conditions for gravitational waves are not restricted to vacuum propagation, being solely based in the Helmholtz decomposition of the metric tensor. So, here, too, considering a plane gravitational wave propagating along the z axis, in the direction of increasing z, we have

$$\epsilon_{31} = \epsilon_{32} = \epsilon_{33} = 0 \tag{190}$$

and

$$\epsilon_{22} = -\epsilon_{11}\,, \tag{191}$$

which leads to

$$[h_{ij} = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & 0 \\ \epsilon_{12} & -\epsilon_{11} & 0 \\ 0 & 0 & 0 \end{bmatrix} e^{i(kz-\omega t)}, \tag{192}$$

with $k^1 = k^2 = 0$ and $k^3 = k$. The relevant Einstein equations for the system are Eq. (89), which here reads as

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial z^2}\right)h_{11} = -\frac{16\pi G}{c^4}\sigma_{11} \tag{193}$$

and

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial z^2}\right)h_{12} = -\frac{16\pi G}{c^4}\sigma_{12}. \tag{194}$$

The transverse traceless tensor components σ_{ij} are obtained in a straightforwardly manner by employing the general procedure outlined in Sec. III D. From Eq. (179), with $k^1 = k^2 = 0$ and $k^3 = k$, we find that the only nonzero σ_{ij} objects are

$$\sigma_{11} = \tfrac{1}{2}(T_{11} - T_{22}), \tag{195}$$

$$\sigma_{22} = -\sigma_{11}, \tag{196}$$

$$\sigma_{12} = T_{12}\,. \tag{197}$$

In turn, the matter stress tensor components T_{ij} are given by Eq. (163), with the α function given by Eq. (156). In order to construct these object, we must sum up the positron and the electron contributions. So, symbolizing the positron ant the electron distribution functions respectively by f and g, we find

$$T_{11} = n_0 c \int \frac{(p_1)^2}{\bar{p}_0}(1+\alpha)(f+g)d^3p\,, \tag{198}$$

$$T_{22} = n_0 c \int \frac{(p_2)^2}{\bar{p}_0}(1+\alpha)(f+g)d^3p \tag{199}$$

and

$$T_{12} = n_0 c \int \frac{p_1 p_2}{\bar{p}_0}(1+\alpha)(f+g)d^3p, \tag{200}$$

with

$$\alpha = \frac{[(p^1)^2 - (p^2)^2]h_{11} + 2\,p^1p^2h_{12}}{2(\bar{p}_0)^2}. \tag{201}$$

In Eqs. (198)–(200) (and in every equation hereafter), as usual in plasma physics, we conveniently renormalize the distribution functions in the way that the momentum space integration of its nonrelativistic version in an homogeneous equilibrium medium results in the unity [see Eq. (141)], which, in turn, entails the multiplication of every momentum space integrals of the distribution functions f and g by n_0, the equilibrium particle density for positrons and electrons (the same density for both species in a neutral homogeneous plasma). We must not worry about the electromagnetic stress tensor because, as we will see shortly, it is a second-order object and thus negligible in the first-order approach we begin to discuss below.

Now, as usual in plasma theory, we employ perturbation theory assuming the following first-order approximations for the distribution functions and for the metric perturbations:

$$f(\mathbf{x},\mathbf{p},t) = f^{(0)}(\mathbf{p}) + f^{(1)}(\mathbf{x},\mathbf{p},t), \tag{202}$$

$$g(\mathbf{x},\mathbf{p},t) = g^{(0)}(\mathbf{p}) + g^{(1)}(\mathbf{x},\mathbf{p},t), \tag{203}$$

$$h_{ij}(\mathbf{x},t) = h_{ij}^{(0)} + h_{ij}^{(1)}(\mathbf{x},t). \tag{204}$$

A superscript (0) indicates zero order (or unperturbed equilibrium quantities) and a superscript (1) indicates first-order corrections, assumed small. As indicated in Eqs. (202) and (203), the unperturbed distribution functions are assumed to be independent of positions (as the system is homogeneous) and time independent. In view of Eqs. (199)–(202), taking now the unperturbed metric satisfying $h_{ij}^{(0)} = 0$ and assuming that the equilibrium distributions functions for electrons and positrons are equal, that is, $g^{(0)} = f^{(0)}$, in view of Eqs. (195)–(204), Einstein Eqs. (193) and (194) correct to the first order are written as

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial z^2}\right)h_{11} = -\frac{8\pi G n_0}{c^3}\int \frac{(p_1)^2 - (p_2)^2}{\bar{p}_0}(\mathscr{F} + 2\alpha f^{(0)})d^3p \tag{205}$$

and

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial z^2}\right)h_{12} = -\frac{8\pi G n_0}{c^3}\int \frac{2p_1p_2}{\bar{p}_0}(\mathscr{F} + 2\alpha f^{(0)})d^3p\,, \tag{206}$$

where we define a first auxiliary distribution function

$$\mathscr{F} = f^{(1)} + g^{(1)} \tag{207}$$

and omit the superscript (1) in h_{ij} and α to simplify the notation. Equations (205) and (206), complemented by the formulas (201) and (207), completes our construction of gravitational field equations for the system we are dealing.

We pass now to the elaboration of the relevant Maxwell equations for the electron-positron plasma. As we are interested in an electrostatic problem, we must assume $\mathbf{B} = 0$. For the electric field, we assume the first-order perturbation expansion

$$\mathbf{E} = \mathbf{E}^{(1)}(\mathbf{x}, t) \tag{208}$$

with $\mathbf{E}^{(0)} = 0$. So, from Eqs. (180)–(183), with $h = 0$ we are left with

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \tag{209}$$

and

$$\nabla \times \mathbf{E} = 0, \tag{210}$$

where we again omit the superscript (1) in $\mathbf{E}$ and ρ for simplicity. The equations for $\mathbf{B}$ are not relevant here. Observe that, coherent with the assumption $\mathbf{E}^{(0)} = 0$, the zero-order charge density vanishing is guaranteed by the condition of equal equilibrium densities for positrons and electrons and by the assumption $g^{(0)} = f^{(0)}$. With $h = \beta = 0$ [see Eq. (157)] the first-order charge density is given by

$$\rho = \rho_{\text{positrons}} + \rho_{\text{electrons}} = en_0 \int \mathscr{G} d^3p\,, \tag{211}$$

where was defined a second auxiliary distribution function,

$$\mathscr{G} = f^{(1)} - g^{(1)}. \tag{212}$$

As is well known, Eq. (210) ensures that the electric field can be expressed as the gradient of a scalar potential Ψ as

$$\mathbf{E} = -\nabla\Psi\,. \tag{213}$$

Inserting Eqs. (211) and (213) in Eq. (209), we are led to the Poisson equation

$$\nabla^2\Psi = -\frac{n_0 e}{\epsilon_0}\int \mathscr{G} d^3p \tag{214}$$

for the first-order potential Ψ. As mentioned before, as the electric field $\mathbf{E}$ and the metric perturbations h_{ij} are first-order quantities, Eqs. (132)–(135) shows that, indeed, the electromagnetic stress tensor is a second-order object, thus being discarded in the construction of the Einstein Eqs. (205) and (206).

Concluding the construction of the Einstein-Vlasov-Poisson system, we pass now to the relevant Vlasov equations. Following the usual steps of perturbation theory, to the first order the Vlasov equations for positrons and electrons are respectively given by

$$\frac{\partial f^{(1)}}{\partial t} + \frac{c\mathbf{p}}{\bar{p}_0}\cdot\nabla f^{(1)} + (\mathbf{F}_g - e\nabla\Psi)\cdot\nabla_{\mathbf{p}} f^{(0)} = 0 \tag{215}$$

and

$$\frac{\partial g^{(1)}}{\partial t} + \frac{c\mathbf{p}}{\bar{p}_0}\cdot\nabla g^{(1)} + (\mathbf{F}_g + e\nabla\Psi)\cdot\nabla_{\mathbf{p}} f^{(0)} = 0\,, \tag{216}$$

where use was made of Eq. (213). We stress that, as the difference of electrons and positrons are just the sign of its charges, the gravitational force is the same for the two types of particles. In view of Eqs. (205) and (206) and (214), it is convenient to rewrite the Vlasov Eqs. (215) and (216) for the auxiliary distributions $\mathscr{F}$ and $\mathscr{G}$ defined by Eqs. (207) and (212). Thus, adding and subtracting Eq. (216) from Eq. (215) we get

$$\frac{\partial \mathscr{F}}{\partial t} + \frac{c\mathbf{p}}{\bar{p}_0}\cdot\nabla\mathscr{F} + 2\,\mathbf{F}_g\cdot\nabla_{\mathbf{p}} f^{(0)} = 0 \tag{217}$$

and

$$\frac{\partial \mathscr{G}}{\partial t} + \frac{c\mathbf{p}}{\bar{p}_0}\cdot\nabla\mathscr{G} - 2e\,\nabla\Psi\cdot\nabla_{\mathbf{p}} f^{(0)} = 0, \tag{218}$$

completing the system of equations.

Note that, to the first order, electricity and gravity are completely decoupled in the electron-positron plasma. First, there is one pair of equations [namely, Eqs. (214) and (218)] to describe electrostatic oscillations in the plasma. Second, there are three equations [Eqs. (205) and (206) and (217)] doing the same job for gravitational waves. In the former case the potential function whose oscillations are considered is Ψ and the relevant distribution is the auxiliary function $\mathscr{G}$. In the last one, the potential functions are h_{11} and h_{12} and the relevant distribution is the auxiliary function $\mathscr{F}$. The problem of electrostatic oscillations thus reduces to the special relativistic case, which is discussed in details elsewhere [82] and so will not be carried forward here. In Sec. III C, we will just deal with the problem of propagation of gravitational waves in the electron-positron plasma. However, before we undergo in this subject, to complete the picture, it is important to write down expressions for the components of the gravitational force.

B. The force exerted by gravitational waves

From Eq. (186) it is clear that, for the gravitational oscillations we are dealing, the only nonzero terms of the gravitational force are that related to the tensor part of the metric, given by Eq. (187) (observe that this components of the gravitational force are just those that have not a Newtonian counterpart and electromagnetic analogs). To find the desired force components, we must first compute the Christoffel symbols of the first kind, given by Eq. (188). From the twenty seven objects $[jk, i]$, the only nonzero are

$$[1\,3, 1] = [3\,1, 1] = \frac{1}{2}\frac{\partial h_{11}}{\partial z}, \tag{219}$$

$$[2\,3, 1] = [3\,2, 1] = \frac{1}{2}\frac{\partial h_{12}}{\partial z}, \tag{220}$$

$$[1\,3, 2] = [3\,1, 2] = \frac{1}{2}\frac{\partial h_{12}}{\partial z}, \tag{221}$$

$$[2\,3, 2] = [3\,2, 2] = -\frac{1}{2}\frac{\partial h_{11}}{\partial z}, \tag{222}$$

$$[1\,2, 3] = [2\,1, 3] = -\frac{1}{2}\frac{\partial h_{12}}{\partial z}, \tag{223}$$

$$[1\,1,3] = -\frac{1}{2}\frac{\partial h_{11}}{\partial z}, \tag{224}$$

$$[2\,2,3] = \frac{1}{2}\frac{\partial h_{11}}{\partial z}. \tag{225}$$

With Eqs. (219)–(225) inserted in Eq. (187) we find for the gravitational force components the expressions

$$F_{g,1} = p^1\Lambda h_{11} + p^2\Lambda h_{12}, \tag{226}$$

$$F_{g,2} = -p^2\Lambda h_{11} + p^1\Lambda h_{12}, \tag{227}$$

$$F_{g,3} = -\frac{c}{2\bar{p}_0}[(p^1)^2 - (p^2)^2]\frac{\partial h_{11}}{\partial z} - \frac{cp^1p^2}{\bar{p}_0}\frac{\partial h_{12}}{\partial z}, \tag{228}$$

where Λ is the differential operator

$$\Lambda = \frac{cp^3}{\bar{p}_0}\frac{\partial}{\partial z} + \frac{\partial}{\partial t}. \tag{229}$$

We are now in position to pursue the dispersion relation of gravitational waves in electron-positron plasma.

C. Dispersion relation for gravitational waves

To find the dispersion relation for gravitational waves in the studied medium, we could proceed as in the nonrelativistic theory by taking the Fourier-Laplace transform (that is, the Fourier transform in space and Laplace transform in time) of the Vlasov and field equations, properly treating the problem as an initial value one. Alternatively, a simpler although equivalent procedure is to Fourier transform the equations, allowing a complex angular frequency with a (presumably) small imaginary part. So, employing the usual prescriptions

$$\frac{\partial}{\partial t} \to -i\omega \tag{230}$$

and

$$\frac{\partial}{\partial z} \to ik \tag{231}$$

applying the second method to Einstein Eqs. (205) and (206) and to Vlasov Eq. (217), we are led to the following set of transformed equations:

$$(\omega^2 - c^2k^2)\tilde{h}_{11} = \frac{8\pi G n_0}{c}\int \frac{(p_1)^2 - (p_2)^2}{\bar{p}_0}(\tilde{\mathscr{F}} + 2\tilde{\alpha} f^{(0)})d^3p \tag{232}$$

and

$$(\omega^2 - c^2k^2)\tilde{h}_{12} = \frac{8\pi G n_0}{c}\int \frac{2p_1p_2}{\bar{p}_0}(\tilde{\mathscr{F}} + 2\tilde{\alpha} f^{(0)})d^3p, \tag{233}$$

$$\tilde{\Lambda}\tilde{\mathscr{F}} = -2\,\tilde{\mathbf{F}}_g\cdot\nabla_{\mathbf{p}} f^{(0)}. \tag{234}$$

Furthermore, by Fourier transforming the α function [Eq. (201)] and the gravitational force components [Eqs. (226)–(228)] we find

$$\tilde{\alpha} = \frac{[(p^1)^2 - (p^2)^2]\tilde{h}_{11} + 2\,p^1p^2\tilde{h}_{12}}{2(\bar{p}_0)^2}, \tag{235}$$

$$\tilde{F}_{g,1} = p^1\tilde{\Lambda}\tilde{h}_{11} + p^2\tilde{\Lambda}\tilde{h}_{12}, \tag{236}$$

$$\tilde{F}_{g,2} = -p^2\tilde{\Lambda}\tilde{h}_{11} + p^1\tilde{\Lambda}\tilde{h}_{12}, \tag{237}$$

$$\tilde{F}_{g,3} = -i\bar{p}_0 c\,k\tilde{\alpha}. \tag{238}$$

In the set of equations above it was defined

$$\tilde{\Lambda} = i\left(\frac{cp^3k}{\bar{p}_0} - \omega\right) \tag{239}$$

and a tilde over any letter symbolizes a space-time Fourier transform, defined as

$$\tilde{f}(\mathbf{k},\omega) = \int f(\mathbf{x},t)e^{-i(\mathbf{k}\cdot\mathbf{x}-\omega t)}dt\,d^3x \tag{240}$$

for a function f. The correspondent Fourier integral (or inverse Fourier transform) is, therefore, given by

$$f(\mathbf{x},t) = \frac{1}{(2\pi)^4}\int \hat{f}(\mathbf{k},\omega)e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}d\omega\,d^3k\,. \tag{241}$$

Just to mention, in view of Eq. (241) and the form (192) assumed for the gravitational waves, we identify the components of the gravitational polarization tensor as

$$\epsilon_{ij} = \frac{1}{(2\pi)^4}\tilde{h}_{ij}. \tag{242}$$

Our efforts now relies in writing the functions $\tilde{\alpha}$, $\tilde{\mathscr{F}}$, and $\tilde{F}_{g,i}$ in terms of $\tilde{h}_{11}$ and $\tilde{h}_{12}$ and then to substitute the results in the field Eqs. (232) and (233). From Eqs. (234)–(239), we obtain

$$\begin{aligned}\tilde{\mathscr{F}} + 2\tilde{\alpha}f^{(0)} =& 2\left[\frac{(p^1)^2 - (p^2)^2}{2(\bar{p}_0)^2} - \left(p^1\frac{\partial}{\partial p^1} - p^2\frac{\partial}{\partial p^2}\right)\right]f^{(0)}\tilde{h}_{11} \\ &+ 2\left[\frac{p^1p^2}{(\bar{p}_0)^2} - \left(p^2\frac{\partial}{\partial p^1} + p^1\frac{\partial}{\partial p^2}\right)\right]f^{(0)}\tilde{h}_{12} \\ &+ 2\frac{ck}{\left(\frac{cp^3}{\bar{p}_0}k - \omega\right)}\frac{(p^1)^2 - (p^2)^2}{2\bar{p}_0}\frac{\partial f^{(0)}}{\partial p^3}\tilde{h}_{11} \\ &+ 2\frac{ck}{\left(\frac{cp^3}{\bar{p}_0}k - \omega\right)}\frac{p^1p^2}{\bar{p}_0}\frac{\partial f^{(0)}}{\partial p^3}\tilde{h}_{12}\,.\end{aligned} \tag{243}$$

At first sight, it could seems that, when substituted in Eqs. (232) and (233), Eq. (243) would leave to a linear homogeneous system for $\tilde{h}_{11}$ and $\tilde{h}_{12}$. However, assuming an even zero-order distribution function satisfying $\partial f^{(0)}/\partial p^i \sim p^i$ (the case of Maxwell and Synge-Jüttner distributions, for example), many of the several resulting momentum space integrals vanishes by virtue of parity, and the only nonvanishing resulting integrals are given by

$$\mathscr{A} = \int \frac{[(p^1)^2 - (p^2)^2]^2}{2(\bar{p}_0)^3}f^{(0)}\,d^3p, \tag{244}$$

$$\mathscr{A}' = \int \frac{(p^1)^2 - (p^2)^2}{\bar{p}_0}\left(p^1\frac{\partial}{\partial p^1} - p^2\frac{\partial}{\partial p^2}\right)f^{(0)}\,d^3p, \tag{245}$$

$$\mathscr{B} = \int \frac{2(p^1)^2(p^2)^2}{(\bar{p}_0)^3}f^{(0)}\,d^3p, \tag{246}$$

$$\mathscr{B}' = \int \frac{2p^1p^2}{\bar{p}_0}\left(p^2\frac{\partial}{\partial p^1} + p^1\frac{\partial}{\partial p^2}\right)f^{(0)}\,d^3p, \tag{247}$$

$$\mathscr{R} = c\int \frac{1}{u-\omega/k}\frac{[(p^1)^2-(p^2)^2]^2}{2(\bar{p}_0)^2}\frac{\partial f^{(0)}}{\partial p^3}\,d^3p, \tag{248}$$

$$\mathscr{S} = c\int \frac{1}{u-\omega/k}\frac{2(p^1)^2(p^2)^2}{(\bar{p}_0)^2}\frac{\partial f^{(0)}}{\partial p^3}\,d^3p, \tag{249}$$

with $u = cp^3/\bar{p}_0$, the z component of the three-velocity. In terms of the above integrals, Eqs. (232) and (233) gives the two *independent* dispersion relations for $\tilde{h}_{11}$ and $\tilde{h}_{12}$, respectively:

$$\omega^2 - c^2k^2 = \frac{16\pi G n_0}{c}(\mathscr{A} - \mathscr{A}' + \mathscr{R}) \tag{250}$$

and

$$\omega^2 - c^2k^2 = \frac{16\pi G n_0}{c}(\mathscr{B} - \mathscr{B}' + \mathscr{S}). \tag{251}$$

Indeed, there is no system of equations to be solved here. Shortly, we will deal with the integrals (244)–(249) and show that the dispersion relations (250) and (251) are exactly the same, as one could expect. For now, it is important to observe that the integrals $\mathscr{A}$, $\mathscr{A}'$, $\mathscr{B}$, and $\mathscr{B}'$ results in just real functions of the physical parameters of the plasma (as the temperature and the electron mass), while $\mathscr{R}$ and $\mathscr{S}$ contains the denominator $u - \omega/k$, causing the function do be integrated to become singular for $u = \omega/k$, the phase velocity of the wave (as in the nonrelativistic theory). This singularity, as is well known, is related to the Landau damping. In the following subsection we will solve the integrals found in the limit of low temperatures and, with this, we will investigate the possibility of the Landau damping in the electron-positron plasma.

D. Evaluation of the integrals: On the Landau damping

We now proceed to approximately evaluate the integrals (244)–(248) and to investigate the dispersion relations (250) and (251). First, to show that the two dispersion relations are indeed the same, we adopt a spherical coordinate system for $\mathbf{p}$, with the wave vector $\mathbf{k}$ oriented along the z axis. Furthermore, to perform more concrete calculations, when appropriate, we will assume the Synge-Jüttner zero-order distribution function

$$f^{(0)}_{\rm SJ}(\mathbf{p}) = \frac{1}{4\pi\, m^3c^3}\frac{\mu}{K_2(\mu)}e^{-\mu\gamma}, \tag{252}$$

where $\mu = mc^2/k_BT$ is the temperature parameter and $K_2(\mu)$ is the modified Bessel function of second kind, of order 2. Observe that $\gamma(p) = \sqrt{1 + p^2/m^2c^2}$ is the usual Lorentz factor [not to be confused with the gamma defined in Eq. (155)], with $\mathbf{p} = \gamma m\mathbf{v}$, and $\mathbf{v}$ is the three-velocity. With the mentioned choice of coordinates we have

$$p^1 = p_1 = p_x = p\sin\theta\cos\phi, \tag{253}$$

$$p^2 = p_2 = p_y = p\sin\theta\sin\phi, \tag{254}$$

$$p^3 = p_3 = p_z = p\cos\theta, \tag{255}$$

where θ and ϕ are the polar and azimuthal angles, respectively. Thus, from Eqs. (244)–(248) with $\bar{p}_0 = \gamma mc$ we get, after some algebraic manipulations and integration in the azimuthal angle,

$$\mathscr{A} = \mathscr{B} = \frac{\pi}{2\,m^3c^3}\int_0^\infty dp\,\frac{p^6 f^{(0)}}{\gamma^3}\int_0^\pi d\theta\,\sin^5\theta, \tag{256}$$

$$\mathscr{A}' = \mathscr{B}' = -\frac{\pi\,\mu}{m^3c^3}\int_0^\infty dp\,\frac{p^6 f^{(0)}}{\gamma^2}\int_0^\pi d\theta\,\sin^5\theta, \tag{257}$$

$$\mathscr{R} = \mathscr{S} = \frac{\pi\,c}{2\,m^2c^2}\int_0^\infty dp\,\frac{p^6}{\gamma^2}\int_0^\pi d\theta\,\frac{\sin^5\theta}{u-\omega/k}\frac{\partial f^{(0)}}{\partial p_z}, \tag{258}$$

with $u = p_z/\gamma m$. The result (257) was obtained employing the relativistic distribution (252) but remains valid in the nonrelativistic limit with $\gamma \to 1$, for which we can use the Maxwell distribution. With the results (256)–(258), and taking into account Eqs. (250) and (251), it is clear that, in fact, the two polarization states of gravitational waves obeys the same dispersion relation, as expected.

To analyze the behavior of the dispersion relation (250), it is instructive do perform the integrals (256)–(258) assuming low particle speeds, that is, a low-temperature plasma so that $k_BT \ll mc^2$ (and $\mu \gg 1$). In this nonrelativistic limit it can be shown that the Synge-Jüttner function becomes the Maxwell distribution,

$$f^{(0)}_M(\mathbf{p}) = \frac{1}{(2\pi\, m\,k_BT)^{3/2}}e^{-p^2/2m\,k_BT}. \tag{259}$$

With the distribution (259) and $\gamma = 1$, the integrals (256) and (257) can be solved quickly by elementary methods and employing the tabulated integral

$$\int_0^\infty x^{2n}e^{-x^2/a}dx = \frac{(2n-1)!!\sqrt{\pi}}{2^{(n+1)}}a^{(2n+1)/2} \tag{260}$$

for n even. The integration given by Eq. (258) is more tractable in its original rectangular forms [Eqs. (248) and (249)] as the Maxwell distribution can be factorized in the form

$$f^{(0)}_M(\mathbf{p}) = f^{(0)}_x(p_x)f^{(0)}_y(p_y)f^{(0)}_z(p_z), \tag{261}$$

with

$$f^{(0)}_i(p_i) = \frac{1}{(2\pi\, m\,k_BT)^{1/2}}e^{-p_i^2/2m\,k_BT}. \tag{262}$$

The results are

$$\mathscr{A} = \frac{2\,(k_BT)^2}{mc^3}, \tag{263}$$

$$\mathscr{A}' = -\frac{4\mu\,(k_BT)^2}{mc^3}, \tag{264}$$

$$\mathscr{R} = \frac{2\,(k_BT)^2}{mc}\int_{-\infty}^\infty \frac{dF(u)/du}{u-\omega/k}du. \tag{265}$$

In Eq. (265) was used the distribution function $F(u)$ for the velocity u, defined by

$$F(u)\,du = f^{(0)}(p_z)\,dp_z. \tag{266}$$

With this, as is well known, we have

$$F(u) = mf^{(0)}(mu) = \left(\frac{m}{2\pi\,k_BT}\right)^{1/2}e^{-mu^2/2\,k_BT}. \tag{267}$$

We are now in place to return to the distribution function and discuss the issue of the noncollisional damping of gravitational waves for a low-temperature electron-positron plasma. Substituting the Eqs. (263)–(265) in Eq. (250), we find

$$\omega^2 - c^2k^2 = 2\,\omega_g^2\left(\frac{2}{\mu} + \frac{1}{\mu^2}\right) + \frac{2\,\omega_g^2 c^2}{\mu^2} \times \left(\mathscr{P}\int_{-\infty}^{\infty} \frac{dF/du}{u-\omega/k}du + i\pi\frac{dF}{du}\bigg|_{u=\omega/k}\right), \tag{268}$$

where we used the usual prescription

$$\int_{-\infty}^{\infty} \frac{F(z)}{z-z_0}dz = \mathscr{P}\int_{-\infty}^{\infty}\frac{F(z)}{z-z_0}dz + i\pi F(z_0)\,, \tag{269}$$

in which $\mathscr{P}$ stands for Cauchy principal value. Furthermore, we define the gravitational plasma frequency ω_g as

$$\omega_g^2 = 16\pi G m n_0\,. \tag{270}$$

Compare the definition above with that given in Refs. [63] and [83], and observe that our definition of ω_g would almost be obtained in the form (270) by taking the electron plasma frequency given by $\omega_p^2 = e^2 n_0/m\,\epsilon_0$ and making some formal substitutions based on the comparison between the Newton universal gravitation law and Coulomb law, namely $\epsilon_0 \to 1/4\pi G$ and $e^2 \to m^2$.

Now, assuming $(\omega/k)^2 \gg k_B T/m$, it is known that [84,85]

$$\mathscr{P}\int_{-\infty}^{\infty}\frac{dF/du}{u-\omega/k}du = \frac{k^2}{\omega^2} + \frac{3}{2}\frac{k^4}{\omega^4}\frac{c^2}{\mu} + \dots\,. \tag{271}$$

Inserting the series expansion (271) in Eq. (268) and retaining only terms proportional to $1/\mu$ and $1/\mu^2$, we are led to

$$\omega^2 - c^2k^2 = \frac{4\,\omega_g^2}{\mu} + \frac{2\,\omega_g^2}{\mu^2}\frac{(\omega^2 + c^2k^2)}{\omega^2} + \frac{2i\pi\,\omega_g^2 c^2}{\mu^2}\frac{dF}{du}\bigg|_{u=\omega/k}\,. \tag{272}$$

Following the usual procedure, we now write $\omega = \omega_R + i\omega_I$, with ω_R and ω_I respectively the real and imaginary parts of ω, and assume $\omega_I \ll \omega_R$. With ω substituted in Eq. (272) we arrive at

$$\omega_R^2 - c^2k^2 = \frac{4\,\omega_g^2}{\mu} \tag{273}$$

and

$$\omega_I = \frac{\pi\,\omega_g^2 c^2}{\mu^2 \omega_R}\frac{dF}{du}\bigg|_{u=\omega_R/k} \tag{274}$$

by discarding terms proportional to $1/\mu^2$ and to any power of ω_I in the expression for ω_R and retaining only the leading terms to determine ω_I. To find the damping parameter ω_I, we must first find the expression for the phase velocity $v_\phi = \omega_R/k$ of the gravitational waves from Eq. (272), and then calculate the derivative in Eq. (274) applied at v_ϕ. It happens, however, that Eq. (273) leads to a phase velocity greater than c (corresponding to a refractive index less than unity):

$$v_\phi = c\left(1 + \frac{4\,\omega_g^2}{\mu c^2 k^2}\right)^{1/2} > c\,. \tag{275}$$

Thus, as the relativistic particle dynamics does not allows $u > c$ and no physical distribution function can really extend to this superluminal regime (although Maxwell's does, since it is a nonrelativistic distribution), we conclude that $\omega_I = 0$, and there is no Landau damping for gravitational waves. Physically, it just means that, as electrons and positrons are not allowed to travel in the direction of propagation of the wave with the same speed as it, the resonant wave-particle coupling cannot occur, and no energy exchange between particles and waves can take place (the same as for electromagnetic waves).

Let us discuss a little deeper this issue. Despite the fact that for the system we are studying there is no Landau damping, let us pretend for a moment that $\omega_I \neq 0$. From Eq. (109) (see Sec. II D) we can readily establishes the following expression for the gravitational instantaneous energy exchange rate (in watts per cubic meter), in the weak-field approximation:

$$\Gamma_g = \frac{c^4}{16\pi G}\frac{\partial h_{\mu\nu}}{\partial t}G^{\mu\nu}\,. \tag{276}$$

It is the density of work realized by (or on) the gravitational waves, that is, the rate of gravitational energy loss (or gain) per unit volume. Now, employing Eq. (47) for the spatial components of the Einstein tensor, we find for gravitational waves

$$\Gamma_g = \frac{c^4}{32\pi G}\frac{\partial h_{ij}}{\partial t}\Box h^{ij}\,. \tag{277}$$

As usual when dealing with wave phenomena, we are just interested in the mean value of Γ_g over a period. To achieve this value, we can proceed in two ways. The first, is to take the real part of the complex plane wave (192), that could be

$$h_{ij} = \epsilon_{ij}e^{\omega_I t}\cos(kz - \omega_R t) \tag{278}$$

if we assume real ϵ_{ij}, substitute their relevant derivatives in Eq. (277), and take the mean values of the trigonometric functions that appear in the calculations. The second approach is a bit more economic and clean, for it does not requires any explicit calculation of mean values. In this method, we persist in writing h_{ij} in the complex form (192), and apply the prescription [21]

$$\left\langle \frac{\partial h_{ij,\mathrm{Real}}}{\partial t}\Box h^{ij}_{\mathrm{Real}}\right\rangle = \frac{1}{2}\mathrm{Re}\left(\frac{\partial h^*_{ij}}{\partial t}\Box h^{ij}\right), \tag{279}$$

where the symbol $<>$ represents the mean value over a period. With formula (279) inserted in Eq. (277) we find

$$\langle\Gamma_g\rangle = \frac{c^4}{64\pi G}\mathrm{Re}\left(\frac{\partial h^*_{ij}}{\partial t}\Box h^{ij}\right). \tag{280}$$

Whatever the method used, the result obtained is

$$\langle\Gamma_g\rangle = \frac{\omega_I c^2}{32\pi G}\left(\omega_R^2 - \frac{2\omega_g^2}{\mu}\right)\epsilon_{ij}\epsilon^{ij}e^{2\omega_I t}, \tag{281}$$

where only leading terms were retained and the dispersion relation (273) was used. Note that, as the smallest possible value of ω_R^2 is $4\,\omega_g^2/\mu$, the bracket in Eq. (281) is always positive, and the sign of $\langle\Gamma_g\rangle$ is linked to the sign of ω_I. If $\omega_I < 0$, then the gravitational wave is damped and $\langle\Gamma_g\rangle < 0$, indicating a decrease of the gravitational energy density with time. If, on the other hand, $\omega_I > 0$, then the gravitational wave would presents an instability and $\langle\Gamma_g \gg 0$, indicating an increase of the gravitational energy density with time. Of course, as the correct value for the Landau parameter is $\omega_I = 0$, then $\langle\Gamma_g\rangle = 0$, agreeing with the previous affirmation about the nonexistence of energy exchange between the gravitational wave and the particles of the plasma. Just to mention, observe that the opposite limit, with $(\omega/k)^2 \ll k_B T/m$, cannot be described properly employing the Maxwell distribution, for it would requires particle speeds typically greater than the phase velocity of the wave which, in turn, is greater than c.

The physical conclusions found so far are interesting and reasonable but apply only to a low-temperature plasma in the limit $(\omega/k)^2 \gg k_B T/m$. A general, fully relativistic treatment valid for a wide range of temperatures and frequencies, requires to solve the integrals (256)-(258) using the Synge-Jüttner distribution (252). These calculations, although difficult, can potentially reveal a richer structure for the dispersion relation, and will be presented and discussed in a future opportunity.

V. CONCLUSION

The linear regime of Einstein field equations and the problem of gauge invariance of the underlying theory were revised and meticulously analysed employing the Helmholtz decomposition scheme for vectors and second-order tensors. In this regime, the field equations are split up in a set of differential equations for two scalar, one vector, and one tensor gauge-invariant gravitational potentials, the first three obeying Poisson-type equations and the last satisfying a nonhomogeneous wave equation, being associated to gravitational radiation. Although the metric is dependent on a choice of gauge (as required by the principle of equivalence), the Einstein tensor is clearly a gauge-invariant object, being physical significant. The problem of gravitational waves propagating in free space were revised under the gaze of the gauge-invariant theory, showing that this methodology it very simple and physically illuminating, much better than be worried about choosing this or that gauge and embarrassed for differentiating real gravitational wave effects from spurious coordinate choice ones. From the theory we can quickly obtain the relevant field equation for a given system or problem. We set a general equation describing gravitational energy exchanges and some useful formulas to write down the source terms for gravitation employing Fourier transforms. Furthermore, we set some general expressions for the Christoffel symbols and for volume elements in terms of gauge-invariant potential, necessary to correctly develop the relevant Einstein, Vlasov, and Maxwell equations and evaluate momentum space integrals.

After briefly revising the Einstein-Vlasov-Maxwell system in the linear regime, we apply the theory for describing electrostatic and gravitational waves in an electron-positron plasma, showing that, in this case, to the first order, there is a complete decoupling between electric and gravitational oscillation. For low temperatures, we find an dispersion relation for the gravitational waves assuming $(\omega/k)^2 \gg k_B T/m$, showing that these waves are not damped, and so have their energy conserved. The momentum space integrals involving the Synge-Jüttner distribution were not solved exactly, a job that is left to a future work and promises to reveal a structurally richer dispersion relation.

As mentioned in the Introduction, our aim was to consistently bring together plasma kinetic theory and the linear theory of gravitation in terms of gauge-invariant potentials, establishing a general, simple and secure methodology to deal in equal footing with oscillations of any (radiative or not) components of the gravitational field. In future works, the methodology presented here will be applied to several problems with isotropic and nonisotropic plasmas, magnetized or not, involving the whole set of gravitational gauge-invariant potentials. We hope that the present paper will help to clarify, to anyone interested in the subject, some issues involving the gauge invariance in general relativity, motivating some plasma physicist to take a tour and make some adventures in general relativity and general relativistic physicist to make some trials in plasma physics. We hope too that our systematic and (more or less) complete presentation could facilitates every physicist interested in general relativistic plasma and kinetic theory to pursue their goals, and motivates futures works, in the way the theory discussed here be extended to the quantum level.

Finally, it is worth mentioning that there are several alternative formulations of the theory of general relativity, such as the ADM formalism. In this version, the covariant general relativistic plasma equations can be cast into more familiar special relativistic forms. As a promising avenue, it can be investigated more deeply, on how the theory fits in this (or other) alternative formulation and how to employ it in order to explore gravitational field and plasma dynamics in terms of gauge-invariant quantities. Unfortunately, we have not yet been able to make efforts in this direction and, while we can apologize for this, we also extend an invitation to other researchers to get involved with this subject.

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