Modulational instability and collapse of internal gravity waves in the atmosphere

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Nonlinear two-dimensional (IGWs) in the atmospheres of the Earth and the Sun are studied. The resulting two-dimensional nonlinear equation has the form of a generalized nonlinear Schrödinger equation with nonlocal nonlinearity, that is, when the nonlinear response depends on the wave intensity at some spatial domain. The modulation instability of IGWs is predicted, and specific cases for the Earth's atmosphere are considered. In a number of particular cases, the instability thresholds and instability growth rates are analytically found. Despite the nonlocal nonlinearity, we demonstrate the possibility of critical collapse of IGWs due to the scale homogeneity of the nonlinear term in spatial variables.

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I. INTRODUCTION

The spectrum of acoustic-gravity waves in the atmosphere of planets and the Sun consists of acoustic and internal gravity waves [1,2], as well as evanescent wave modes [3–5]. Internal gravity waves, which are the lower branch of acoustic-gravity atmospheric waves, have been intensively studied in the physics of the Earth's and the Sun's atmosphere for more than 60 years [1,2,6–8]. Space missions have provided an additional incentive to study these waves also in the atmospheres of other planets, for example, Mars and Venus [9,10]. Interest in internal gravity waves (IGWs) is largely due to the important contribution that these waves make to the dynamics and energetics of the atmospheres of planets and the Sun, ensuring effective redistribution of disturbance energy on a global scale. In the Earth's atmosphere, these waves can be generated by various sources of natural and anthropogenic origin. In particular, IGWs are associated with the sources localized in the upper atmosphere and on the Sun, for example, precipitation of charged particles at high latitudes, ionospheric currents, solar terminator, etc. [11–13]. In addition, IGWs in the upper atmosphere and ionosphere caused by tropospheric or ground-based sources are currently being intensively studied [14,15]. The linear theory of IGWs has been developed in great detail (see, e.g., Ref. [16], reviews [17,18], and references therein). In particular, the Coriolis force caused by rotation, the presence of the magnetic field of the Earth and the Sun [19], the effect of random temperature inhomogeneity [20], etc. were taken into account. As IGWs propagate upward in the atmosphere, their amplitudes rapidly increase with altitude due to an exponential decrease in the background density [1,6,21]. In this connection, when considering such waves, it becomes necessary to take into account nonlinear effects. Nonlinear effects during the propagation of IGWs have been studied in a number of works (see also Ref. [22] and references therein). In particular, based on fluid equations with a term taking into account force of gravity and an adiabatic equation of state, nonlinear equations for IGWs in the atmosphere were derived in Refs. [23–25]. The three-wave interaction of IGWs and nonlinear responses were considered in Refs. [26–28]. Interaction of atmospheric gravity solitary waves with ion acoustic solitary waves was studied in Ref. [29]. Nonlinear structures in the form of convective cells of IGW waves in the Earth's atmosphere [30], tripole vortices and vortex chains [31,32], dust-acoustic gravity vortices [33], and so-called dust devils (rotating columns of rising dust) [34,35] were considered. Nonlinear IGWs waves in a weakly ionized atmosphere in the form of dipole vortices (cyclone-anticyclone pairs) were found in Refs. [36,37]. In a recent paper [38], nonlinear equations were obtained to describe the dynamics of IGWs using the reductive perturbation method. In the one-dimensional case, the corresponding solutions are presented in the form of breather solitons, rogue waves, and dark solitons. Some aspects of nonlinear IGW, including intensive numerical modeling, have also been studied in Refs. [39–42].

In this paper, we study two-dimensional (2D) nonlinear IGWs based on equations obtained with the aid of the reductive perturbation method proposed in Ref. [38]. Unlike other works on nonlinear atmospheric IGWs, we use this method to derive a 2D nonlinear equation with a nonlocal nonlinearity when the nonlinear response depends on the wave packet intensity at some extensive spatial domain. This equation may be treated as a generalized nonlinear Schrödinger equation, but the linear part is essentially anisotropic and, moreover,

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the corresponding linear operator can be either elliptic or hyperbolic. This equation, in contrast to the original fluid equations describing a stratified atmosphere, is quite adequately amenable to analytical analysis and, despite its model nature, predicts the modulation instability of IGWs and the possibility of wave collapse.

The 2D nonlinear Schrödinger (NLS) equation with nonlocal nonlinearity was considered in a number of works. An important property of spatially nonlocal nonlinear response is that it prevents a catastrophic collapse of multidimensional wave packets which usually occurs in local self-focusing media with a cubic nonlinearity. In particular, a rigorous proof of absence of collapse in the model of the nonlocal NLS equation with sufficiently general symmetric real-valued response kernel was presented in Refs. [43,44]. It was shown that nonlocal nonlinearity arrests the collapse and results in the existence of stable coherent structures, not only the fundamental soliton (which can collapse in the NLS with local nonlinearity), but also dipole solitons, the so-called azimuthal solitons (azimuthons), and vortex solitons [45–49]. It is important that in these models the term with nonlocal nonlinearity is not scale homogeneous in spatial variables. This is precisely the reason for the absence of collapse. Despite the nonlocal nonlinearity in our model, we predict the possibility of IGW collapse, similar to the collapse of Langmuir waves in a plasma [50,51], and self-focusing of nonlinear beams in optics [52]. Collapse (since the model is two-dimensional, collapse is critical) is possible due to the fact that the nonlocal nonlinearity under consideration is scale homogeneous in spatial variables. We also study the modulation instability of IGWs, which is a precursor of collapse that occurs at the nonlinear stage of instability.

The paper is organized as follows. In Sec. II, the model two-dimensional nonlinear equations for IGWs are presented. The modulation instability is studied in Sec. III. In Sec. IV, we demonstrate the possibility of collapse of IGWs. The conclusion is made in Sec. V.

II. MODEL EQUATIONS

2D Stenflo equations [23,25] to govern the dynamics of nonlinear atmospheric IGWs have the form

$$\frac{\partial}{\partial t}\left(\Delta\psi-\frac{1}{4H^2}\psi\right)+\{\psi,\Delta\psi\}+\frac{\partial\chi}{\partial x}=0, \qquad (1)$$

$$\frac{\partial\chi}{\partial t}+\{\psi,\chi\}-\omega_g^2\frac{\partial\psi}{\partial x}=0, \qquad (2)$$

where $\Delta=\partial^2/\partial x^2+\partial^2/\partial z^2$ is the two-dimensional Laplacian, and the Poisson bracket (Jacobian) $\{f,g\}$ is defined by

$$\{f,g\}=\frac{\partial f}{\partial x}\frac{\partial g}{\partial z}-\frac{\partial f}{\partial z}\frac{\partial g}{\partial x}. \qquad (3)$$

Here, $\psi(x,z)$ is the velocity stream function, $\chi(x,z)$ is the normalized density perturbation, H is the equivalent atmospheric height, $\omega_g=(g/H)^{1/2}$ is the Brunt-Väisälä, and g is the free fall acceleration. The coordinates x and z correspond to the horizontal and vertical coordinates, respectively, the z axis is directed upward against the gravitational acceleration $\mathbf{g}=-g\hat{\mathbf{z}}$, $\hat{\mathbf{z}}$ is the unit vector along the z direction, and the x axis lies in a plane perpendicular to the z axis. In the linear approximation, taking $\psi\sim\exp(i\mathbf{K}\cdot\mathbf{x}-i\omega t)$ and $\chi\sim\exp(i\mathbf{K}\cdot\mathbf{x}-i\omega t)$, where $\mathbf{x}=(x,z)$, ω and $\mathbf{K}=(K_x,K_z)$ are the frequency and wave number respectively, Eqs. (1) and (2) yield the dispersion relation of the IGWs,

$$\omega^2=\frac{\omega_g^2K_x^2}{K^2+1/(4H^2)}, \qquad (4)$$

where $K^2=K_x^2+K_z^2$. In Eqs. (1) and (2), characteristic frequencies $\omega\gg\Omega_0$ are considered, where Ω_0 is the angular rotation velocity of the planet, and then the Coriolis force can be neglected. The Ampère force is also neglected, that is, the influence of the magnetic field in the ionized atmosphere, which is justified at sufficiently high altitudes [19]. Further, we consider an isothermal atmosphere, that is, Brunt-Väisälä frequency is assumed to be independent of the vertical coordinate z. For the Earth's atmosphere, in particular, this corresponds to altitudes $\gtrsim$200 km. Due to the dissipation of short-wave harmonics, the lower limit for wavelengths for IGWs is $\sim$10 km at altitudes $\sim$200-300 km, while typical wavelength values are hundreds of kilometers.

The reductive perturbation method (the multiscale expansion method) [53] for Eqs. (1) and (2) to study the behavior of nonlinear IGWs was elaborated in a recent paper [38]. This method is often used in the theory of nonlinear waves and leads to evolution equations, which in many cases turn out to be more suitable for analysis than the original problem. Following this technique, the space and time variables were expanded in Ref. [38] as $\mathbf{x}=\mathbf{x}+\varepsilon\mathbf{X}+\cdots$ and $t=t+\varepsilon T+\varepsilon^2\tau+\cdots$, respectively, where $\mathbf{X}=(X,Z)$, and ε is a small dimensionless parameter that scales weak dispersion and nonlinearity. As was shown, to obtain a nontrivial evolution, it was enough to restrict ourselves to expanding the time variable up to the second order and the space variable up to the first order in ε and, in that case,

$$\frac{\partial}{\partial\mathbf{x}}\to\frac{\partial}{\partial\mathbf{x}}+\varepsilon\frac{\partial}{\partial\mathbf{X}},\quad \frac{\partial}{\partial t}\to\frac{\partial}{\partial t}+\varepsilon\frac{\partial}{\partial T}+\varepsilon^2\frac{\partial}{\partial\tau}. \qquad (5)$$

In turn, the fields ψ and χ were expanded in powers in ε as

$$\psi=\varepsilon\psi^{(1)}+\varepsilon^2\psi^{(2)}+\varepsilon^3\psi^{(3)}+\cdots, \qquad (6)$$

$$\chi=\varepsilon\chi^{(1)}+\varepsilon^2\chi^{(2)}+\varepsilon^3\chi^{(3)}+\cdots, \qquad (7)$$

where $\psi^{(1)}=\tilde{\psi}^{(1)}+\bar{\psi}$, $\chi^{(1)}=\tilde{\chi}^{(1)}+\bar{\chi}$,

$$\tilde{\psi}^{(1)}=\Psi(\mathbf{X},T,\tau)e^{i\mathbf{K}\cdot\mathbf{x}-i\omega t}+\text{c.c.}, \qquad (8)$$

$$\tilde{\chi}^{(1)}=\Phi(\mathbf{X},T,\tau)e^{i\mathbf{K}\cdot\mathbf{x}-i\omega t}+\text{c.c.}. \qquad (9)$$

Secondary mean flows $\bar{\psi}$ and $\bar{\chi}$ depend only on slow variables $\mathbf{X}$, T, and τ. As a result, using the standard procedure for eliminating secular terms, the following system of nonlinear equations for the envelope Ψ and the secondary mean flow $\bar{\psi}$ was obtained in Ref. [38]:

$$\hat{L}\Psi+K_z\left(K_x\frac{\omega_g^2}{v_{gx}}-\omega K^2-\frac{K_x^2\omega_g^2}{\omega}\right)\Psi\frac{\partial\bar{\psi}}{\partial X}$$
$$+K_x\left(\omega K^2+\frac{K_x^2\omega_g^2}{\omega}\right)\Psi\frac{\partial\bar{\psi}}{\partial Z}=0, \qquad (10)$$

where

$$\hat{L} = 2\omega\left(K^2 + \frac{1}{4H^2}\right)\left(i\frac{\partial}{\partial\tau} + \frac{1}{2}\frac{\partial^2\omega}{\partial K_x^2}\frac{\partial^2}{\partial X^2} + \frac{1}{2}\frac{\partial^2\omega}{\partial K_z^2}\frac{\partial^2}{\partial Z^2} + \frac{\partial^2\omega}{\partial K_x\partial K_z}\frac{\partial^2}{\partial X\partial Z}\right), \tag{11}$$

$$\omega_g^2\frac{\partial^2\bar{\psi}}{\partial X^2} - \frac{1}{4H^2}\left(v_{gx}^2\frac{\partial^2\bar{\psi}}{\partial X^2} + v_{gz}^2\frac{\partial^2\bar{\psi}}{\partial Z^2}\right) = \left(\omega K^2 + \frac{K_x^2\omega_g^2}{\omega}\right)\left(K_z\frac{\partial|\Psi|^2}{\partial X} - K_x\frac{\partial|\Psi|^2}{\partial Z}\right), \tag{12}$$

and for the group velocities $v_{gx} = \partial\omega/\partial K_x$ and $v_{gz} = \partial\omega/\partial K_z$ we have

$$v_{gx} = \frac{\omega_g(K_z^2 + 1/4H^2)}{(K^2 + 1/4H^2)^{3/2}}, \quad v_{gz} = -\frac{\omega_g K_x K_z}{(K^2 + 1/4H^2)^{3/2}}. \tag{13}$$

In Ref. [38], only the one-dimensional case was studied, when the system (10) and (12) was reduced to either the focusing or defocusing one-dimensional NLS equation. In the presented paper, we consider the two-dimensional system.

We introduce dimensionless variables τ', x, z, Ψ' and $\bar{\psi}'$ by

$$\tau' = \omega_g\tau, \quad x = \frac{X}{H}, \quad z = \frac{Z}{H}, \quad \Psi' = \frac{\Psi}{\omega_g H^2}, \quad \bar{\psi}' = \frac{\bar{\psi}}{\omega_g H^2}, \tag{14}$$

and further the primes are omitted. Inserting Eq. (14) into Eqs. (10) and (12) yields

$$i\frac{\partial\Psi}{\partial\tau} + A\frac{\partial^2\Psi}{\partial x^2} + B\frac{\partial^2\Psi}{\partial z^2} + 2C\frac{\partial^2\Psi}{\partial x\partial z} + D\Psi\frac{\partial\bar{\psi}}{\partial x} + E\Psi\frac{\partial\bar{\psi}}{\partial z} = 0, \tag{15}$$

$$F\frac{\partial^2\bar{\psi}}{\partial x^2} - G\frac{\partial^2\bar{\psi}}{\partial z^2} = M\left(q_z\frac{\partial|\Psi|^2}{\partial x} - q_x\frac{\partial|\Psi|^2}{\partial z}\right), \tag{16}$$

where $q_x = K_x H$, $q_z = K_z H$, and the coefficients A, B, C, D, E, F, G, and M are determined in the Appendix. Note that the coefficients E, F, G, and M are positive, A is negative, while B, C, and D are of indefinite sign and their sign depends on the specific values of q_x and q_z. The properties of solutions to the system of nonlinear equations (15) and (16) largely depend on the linear part of Eq. (15). The corresponding linear partial differential equation with constant coefficients A, B, and C has elliptic type if

$$C^2 - AB < 0, \tag{17}$$

and hyperbolic type if

$$C^2 - AB > 0. \tag{18}$$

In case

$$C^2 - AB = 0, \tag{19}$$

the equation is of parabolic type. Note that the classification of equation types depends only on the coefficients of the second derivatives and the sign of the first term in Eq. (15) does not affect it [54]. As is known, the elliptic type of equation corresponds to the boundary value problem, the hyperbolic type to the wave equation, and the parabolic type to the diffusion problem. In this paper, we restrict ourselves to the case (17), that is, an elliptic operator in the linear part of the equation (15). The parabolic case (19) is not considered due to restrictions on the relationship between q_x and q_z, which greatly limits its practical significance. The hyperbolic case (18) is expected to be considered in the future. Using explicit expressions for A, B, and C, conditions (17)–(19) can be rewritten in equivalent forms $Q < 0$, $Q > 0$ and $Q = 0$, respectively, where the function $Q(q_x, q_z)$ is determined by

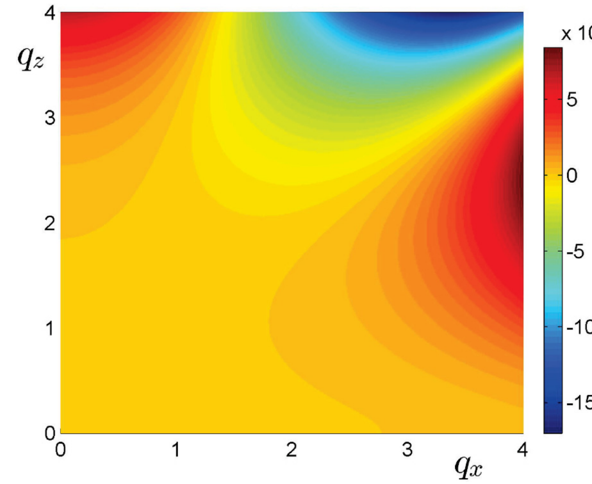

FIG. 1. The contour plot of the function $Q(q_x, q_z)$ in Eq. (20). Negative Q corresponds to the elliptic operator in the linear part of Eq. (15).

$$Q(q_x, q_z) = q_z^2(8q_x^2 - 4q_z^2 - 1)^2 - 3q_x^2(1 + 4q_z^2) \times (8q_z^2 - 4q_x^2 - 1). \tag{20}$$

The contour plot of the function $Q(q_x, q_z)$ on the plane (q_x, q_z) is shown in Fig. 1. By introducing the angle α between the vector $\mathbf{q} = (q_x, q_z)$ and the vector $q_x\hat{\mathbf{x}}$, where $\hat{\mathbf{x}}$ is a unit vector in the horizontal direction, one can also define another function $\tilde{Q}(q_x, \alpha)$ with the same sign as $Q(q_x, q_z)$ by

$$\tilde{Q}(q_x, \alpha) = 16q_x^4\tan^2\alpha(\tan^4\alpha - 10\tan^2\alpha + 7) + 4q_x^2(2\tan^4\alpha - 7\tan^2\alpha + 3) + 3. \tag{21}$$

The curve $\tilde{Q}(q_x, \alpha) = 0$ separating the plane (q_x, α) into the elliptic and hyperbolic regions of the linear operator in Eq. (15) is shown in Fig. 2. The ellipticity region of the linear operator in Eq. (15) is visible from Figs. 1 and 2. Note that since $A < 0$, it follows from (17) that $B < 0$.

Using the convolution identity,

$$(fg)_{\mathbf{p},\omega} = \int f_{\mathbf{p}_1,\omega_1}g_{\mathbf{p}_2,\omega_2}\delta(\mathbf{p} - \mathbf{p}_1 - \mathbf{p}_2)\delta(\omega - \omega_1 - \omega_2) \times d\mathbf{p}_1 d\mathbf{p}_2 d\omega_1 d\omega_2, \tag{22}$$

connecting the Fourier transforms of the product of arbitrary functions $f(\mathbf{r}, t)$ and $g(\mathbf{r}, t)$ expressed in physical space with the corresponding Fourier transforms of these

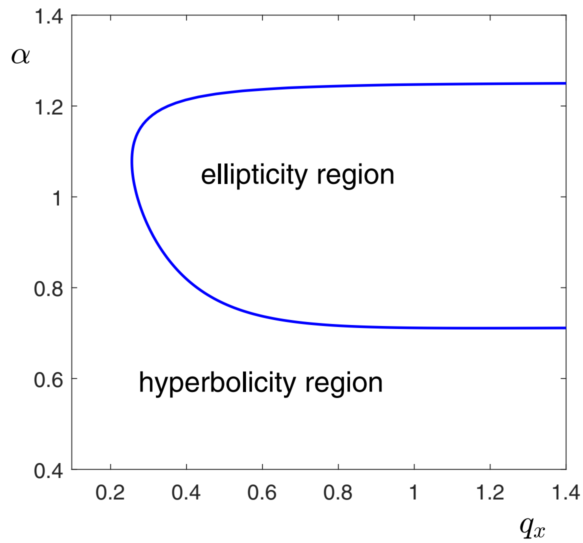

FIG. 2. The regions of ellipticity and hyperbolicity in Eq. (15) on the plane (q_x, α). These areas are separated by the curve determined by the equation $\tilde{Q}(q_x, \alpha) = 0$.

functions,

$$f_{\mathbf{p},\omega} = \int f(\mathbf{r}, t)e^{-i\mathbf{p}\cdot\mathbf{r}+i\omega t}\, d\mathbf{p}d\omega, \tag{23}$$

$$g_{\mathbf{p},\omega} = \int g(\mathbf{r}, t)e^{-i\mathbf{p}\cdot\mathbf{r}+i\omega t}\, d\mathbf{p}d\omega, \tag{24}$$

where $\delta(x)$ is the Dirac delta function, we can rewrite Eqs. (15) and (16) in Fourier space as

$$(\omega - \omega_{\mathbf{p}})\Psi_p = -i\int (Dp_{2x} + Ep_{2z})\Psi_{p_1}\bar{\psi}_{p_2}\delta \times (p - p_1 - p_2)dp_1dp_2, \tag{25}$$

where

$$\omega_{\mathbf{p}} = Ap_x^2 + Bp_z^2 + 2Cp_xp_z \tag{26}$$

and

$$\bar{\psi}_p = \frac{iM(q_zp_x - q_xp_z)}{Gp_z^2 - Fp_x^2}\int \Psi_{p_1}\Psi^*_{p_2}\delta(p - p_1 - p_2)dp_1dp_2, \tag{27}$$

respectively. Here and below, we use the shorthand notation $p \equiv (\mathbf{p}, \omega)$, so

$$\delta(p - p_1 - p_2) \equiv \delta(\mathbf{p} - \mathbf{p}_1 - \mathbf{p}_2)\delta(\omega - \omega_1 - \omega_2), \tag{28}$$

and $dp_1dp_2 \equiv d\mathbf{p}_1d\mathbf{p}_2d\omega_1d\omega_2$. Note that from Eq. (17), it follows that $\omega_{\mathbf{p}}$ is a negative definite quadratic form (despite the fact that the coefficient C is indefinite in sign), that is, $\omega_{\mathbf{p}} < 0$. Substituting Eq. (27) into Eq. (25), we have one equation for the Fourier transform Ψ_p,

$$(\omega - \omega_{\mathbf{p}})\Psi_p = \int V(p, p_1, p_2, p_3)\Psi_{p_1}\Psi_{p_2}\Psi^*_{p_3}dp_1dp_2dp_3, \tag{29}$$

where the interaction matrix element $V(p, p_1, p_2, p_3)$ is determined by

$$V(p, p_1, p_2, p_3) = \frac{M}{2}\left[\frac{(Dp_{1x} + Ep_{1z})(q_zp_{1x} - q_xp_{1z})}{Gp_{1z}^2 - Fp_{1x}^2} + \frac{(Dp_{2x} + Ep_{2z})(q_zp_{2x} - q_xp_{2z})}{Gp_{2z}^2 - Fp_{2x}^2}\right] \times \delta(p - p_1 - p_2 - p_3), \tag{30}$$

and symmetrization in p_1 and p_1 is taken into account. The system of nonlinear equations (15) and (16), to the best of our knowledge, has apparently never been considered in problems in nonlinear physics before. In appearance (and physical meaning), this system resembles the Zakharov equations (and their generalizations) describing the interaction of high-frequency and low-frequency waves in plasma [48,50,55,56] and the equations for the interaction of short-wave and long-wave disturbances on the surface of shallow water [57–59]. The system (15) and (16) is, however, much more difficult to analyze than the analogues mentioned above.

Equation (29) has an exact solution in the form of a monochromatic plane wave,

$$\Psi_p = \Psi_0V(p, p_0, p_0, -p_0), \tag{31}$$

where

$$V(p, p_0, p_0, -p_0) = S_{\mathbf{k}_0}|\Psi_0|^2\delta(\mathbf{p} - \mathbf{k}_0)\delta(\omega - \omega_0), \tag{32}$$

and we have introduced the notation

$$S_{\mathbf{k}_0} = \frac{M(Dk_{0x} + Ek_{0z})(q_zk_{0x} - q_xk_{0z})}{Gk_{0z}^2 - Fk_{0x}^2}. \tag{33}$$

In physical space, this corresponds to the solution

$$\Psi(\mathbf{r}, t) = \Psi_0 \exp(i\mathbf{k}_0 \cdot \mathbf{r} - i\omega_0t), \tag{34}$$

with a frequency depending on the amplitude Ψ_0,

$$\omega_0 = \omega_{\mathbf{k}_0} - S_{\mathbf{k}_0}|\Psi_0|^2, \tag{35}$$

where

$$\omega_{\mathbf{k}_0} = Ak_{0x}^2 + Bk_{0z}^2 + 2Ck_{0x}k_{0z}. \tag{36}$$

In the next section, we consider the stability of such a plane wave.

III. NONLINEAR DISPERSION RELATION AND MODULATIONAL INSTABILITY

The perturbed plane wave solution in physical space has the form

$$\Psi = (\Psi_0 + \delta\Psi)\exp(i\mathbf{k}_0 \cdot \mathbf{r} - i\omega_0t), \tag{37}$$

where

$$\delta\Psi = \Psi^+e^{i\mathbf{k}\cdot\mathbf{r}-i\Omega t} + \Psi^-e^{-i\mathbf{k}\cdot\mathbf{r}+i\Omega t} \tag{38}$$

is a linear modulation with the frequency Ω and the wave vector $\mathbf{k}$. In Fourier space, Eqs. (37) and (38) correspond to

$$\Psi_p = (\Psi_0 + \delta\Psi_p)\delta(p - p_0), \tag{39}$$

and

$$\delta\Psi_p = \Psi^+\delta(\mathbf{p}-\mathbf{k})\delta(\omega-\Omega) + \Psi^-\delta(\mathbf{p}+\mathbf{k})\delta(\omega+\Omega), \quad (40)$$

respectively. Linearizing Eq. (29) in $\delta\Psi_p$, we get the nonlinear dispersion relation:

$$1 - |\Psi_0|^2\left[\frac{S_{\mathbf{k}_0+\mathbf{k}}}{\omega_{\mathbf{k}_0+\mathbf{k}} - \omega_{\mathbf{k}_0} - \Omega} + \frac{S_{\mathbf{k}_0-\mathbf{k}}}{\omega_{\mathbf{k}_0-\mathbf{k}} - \omega_{\mathbf{k}_0} + \Omega}\right] = 0. \quad (41)$$

Equation (41) is a quadratic equation in Ω,

$$\Omega^2 + \Omega\left[\omega_{\mathbf{k}_0-\mathbf{k}} - \omega_{\mathbf{k}_0+\mathbf{k}} + \left(S_{\mathbf{k}_0+\mathbf{k}} - S_{\mathbf{k}_0-\mathbf{k}}\right)|\Psi_0|^2\right] + \left[S_{\mathbf{k}_0+\mathbf{k}}\left(\omega_{\mathbf{k}_0-\mathbf{k}} - \omega_{\mathbf{k}_0}\right) + S_{\mathbf{k}_0-\mathbf{k}}\left(\omega_{\mathbf{k}_0+\mathbf{k}} - \omega_{\mathbf{k}_0}\right)\right]|\Psi_0|^2 - \left(\omega_{\mathbf{k}_0+\mathbf{k}} - \omega_{\mathbf{k}_0}\right)\left(\omega_{\mathbf{k}_0-\mathbf{k}} - \omega_{\mathbf{k}_0}\right) = 0, \quad (42)$$

and it can be easily solved. The negativity of the discriminant of the equation corresponds to instability with the growth rate $\gamma = |\mathrm{Im}\,\Omega|$. It can also be seen that the instability has a threshold character with respect to the amplitude Ψ_0. As noted above, the coefficients C and D are indefinite in sign and their sign depends on the specific values of q_x and q_z. In the general case, the dependence of the instability growth rate on the wave vector of a plane wave $\mathbf{k}_0$, the wave vector of perturbations $\mathbf{k}$, and the values of q_x and q_z is quite complex. Equation (42) is greatly simplified in a number of important limiting cases. In the limit of long-wave modulations $\mathbf{k} \ll \mathbf{k}_0$, using

$$\omega_{\mathbf{k}_0\pm\mathbf{k}} \sim \omega_{\mathbf{k}_0} \pm \frac{\partial\omega_{\mathbf{k}_0}}{\partial\mathbf{k}_0}\cdot\mathbf{k} + \frac{1}{2}\frac{\partial^2\omega_{\mathbf{k}_0}}{\partial\mathbf{k}_0^2}k^2, \quad (43)$$

from Eq. (42) we obtain

$$(\Omega - \mathbf{v}_g\cdot\mathbf{k})^2 = \tfrac{1}{4}\left(\omega''_{\mathbf{k}_0}\right)^2 k^4 - \omega''_{\mathbf{k}_0}k^2 S_{\mathbf{k}_0}|\Psi_0|^2, \quad (44)$$

where $\mathbf{v}_g = \partial\omega_{\mathbf{k}_0}/\partial\mathbf{k}_0$ and $\omega''_{\mathbf{k}_0} = \partial^2\omega_{\mathbf{k}_0}/\partial\mathbf{k}_0^2$. Since $\omega_{\mathbf{k}_0} < 0$, then if $S_{\mathbf{k}_0} < 0$ and the amplitude threshold is exceeded,

$$4\left|S_{\mathbf{k}_0}\right||\Psi_0|^2 > \left|\omega''_{\mathbf{k}_0}\right|k^2, \quad (45)$$

Equation (44) corresponds to convective instability when growing disturbances are carried away with the group velocity $\mathbf{v}_g$, and the instability growth rate is given by

$$\gamma = k\sqrt{4\left|S_{\mathbf{k}_0}\right|\left|\omega''_{\mathbf{k}_0}\right||\Psi_0|^2 - \left(\omega''_{\mathbf{k}_0}\right)^2 k^2}. \quad (46)$$

Note that in this case the instability with respect to the wave numbers of perturbations k_x and k_z is isotropic.

More interesting is the opposite case of short-wave modulations $\mathbf{k} \gg \mathbf{k}_0$. This instability is an instability of a uniform field (in the limit $\mathbf{k}_0 \to 0$) leading to the splitting of this field into clumps, which ultimately results in the emergence of coherent structures at the nonlinear stage, which generally speaking can be both nonstationary (collapsing cavitons) [50] and stationary (stable solitons). Then, taking into account that $\omega_{\mathbf{k}}$ and $S_{\mathbf{k}}$ are even functions, Eq. (42) becomes

$$\Omega^2 = \omega_{\mathbf{k}}(\omega_{\mathbf{k}} - 2S_{\mathbf{k}}|\Psi_0|^2). \quad (47)$$

Since $\omega_{\mathbf{k}} < 0$, then Eq. (47) predicts a purely growing instability (modulational instability) if $S_{\mathbf{k}} < 0$ and if the amplitude threshold is exceeded:

$$2|S_{\mathbf{k}}||\Psi_0|^2 > |\omega_{\mathbf{k}}|. \quad (48)$$

The instability growth rate γ is given by

$$\gamma = \sqrt{2|S_{\mathbf{k}}||\omega_{\mathbf{k}}||\Psi_0|^2 - \omega_{\mathbf{k}}^2}. \quad (49)$$

From Eqs. (33) and (36) in which $\mathbf{k}_0$ is replaced by $\mathbf{k}$, it is evident that instability has a substantially anisotropic character and, depending on the relationship between k_x and k_z, the mode of modulation instability can change (longitudinal or transverse instability of the envelope wave packets). Moreover, it follows from Eq. (33) that the parametric coupling is nonlocal. By introducing the angle β between the vector $\mathbf{k} = (k_x, k_z)$ and the vector $k_x\hat{\mathbf{x}}$, where $\hat{\mathbf{x}}$ is a unit vector in the x direction, one can rewrite $S_{\mathbf{k}}$ in Eq. (33) as

$$S_{\mathbf{k}} = \frac{M(D + E\tan\beta)(q_z - q_x\tan\beta)}{F(\varepsilon\tan^2\beta - 1)}, \quad (50)$$

where $\varepsilon = G/F$ and for typical values of carrier wave numbers q_x and q_z we have $\varepsilon \ll 1$. Equation (50) is not valid for $\varepsilon\tan^2\beta \sim 1$ (in this case, the conditions for deriving Eq. (16) are violated). First, we consider the angles β satisfying condition $\varepsilon\tan^2\beta \ll 1$. Note that this case corresponds to both longitudinal $k_x \gg k_z$ and transverse $k_z \gg k_x$ modulation instability, except for very small longitudinal wave numbers k_x corresponding to angles $\beta \gg \arctan(\sqrt{1/\varepsilon})$. The stability region with respect to the wave numbers of perturbations and the amplitude of the plane wave depends on the sign of the function $\mathcal{Q}(k_x, \beta)$ defined by

$$\mathcal{Q}(k_x, \beta) = 2|\Psi_0|^2 M(D + E\tan\beta)(q_x\tan\beta - q_z)/F + k_x^2(A + B\tan^2\beta + 2C\tan\beta). \quad (51)$$

The curves $\mathcal{Q}(k_x, \beta) = 0$, dividing the plane (k_x, β) into stable and unstable regions for different values of the plane-wave amplitude Ψ_0 and for specific values of the carrier wave numbers $q_x = 1$ and $q_z = 1.5$ are shown in Fig. 3. The picture does not qualitatively depend on the specific values of the carrier wave numbers q_x and q_z (in the considered region of ellipticity) and the amplitude Ψ_0. It is evident from Fig. 3 that the instability is anisotropic, and a mode of change of modulation instability from longitudinal to transverse is possible. In the case $\varepsilon\tan^2\beta \gg 1$, the modulation instability is only transverse. It can be seen that $S_{\mathbf{k}}$ does not depend on $\mathbf{k}$ for perturbations with $k_z \ll k_x$ (pure longitudinal instability) or $k_z \gg k_x$ (pure transverse instability). In these cases, the expression for $S_{\mathbf{k}}$ is reduced to S_1 and S_2, defined as

$$S_1 = -\frac{MDq_z}{F}, \quad S_2 = -\frac{MEq_x}{G}. \quad (52)$$

In the ellipticity region, we have $D > 0$, and all other coefficients in Eq. (52) are always positive, so the necessary conditions for instability $S_1 < 0$ and $S_2 < 0$ are satisfied. Then the longitudinal and transverse instability growth rates are obtained from Eq. (49), and have the form, respectively,

$$\gamma = \sqrt{2S_1Ak_x^2|\Psi_0|^2 - A^2k_x^4}, \quad \text{if } k_z \ll k_x \quad (53)$$

and

$$\gamma = \sqrt{2S_2Bk_z^2|\Psi_0|^2 - B^2k_z^4}, \quad \text{if } k_z \gg k_x. \quad (54)$$

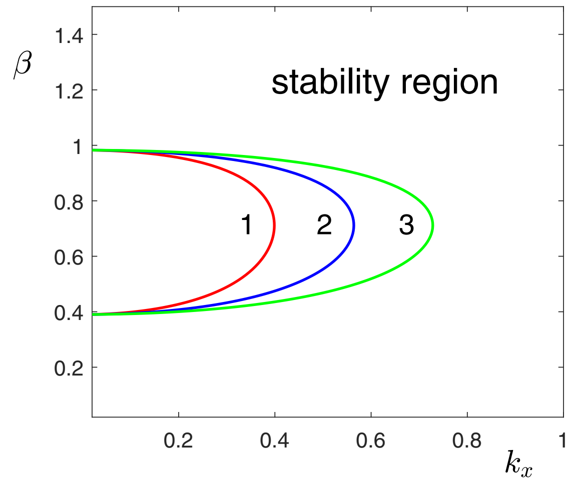

FIG. 3. Stability and instability regions on the plane (k_x, β) for different amplitude values Ψ_0. The outer regions to the right of the curves correspond to the stability regions. The numbers near the curves correspond to different amplitudes: 1: $\Psi_0 = 0.003$, 2: $\Psi_0 = 0.006$, and 3: $\Psi_0 = 0.01$.

The optimal horizontal and vertical wave numbers of perturbations corresponding to the maximum instability growth rates in (53) and (54) are

$$k_{x,\text{opt}} = |\Psi_0|\sqrt{\frac{S_1}{|A|}} \quad \text{and} \quad k_{z,\text{opt}} = |\Psi_0|\sqrt{\frac{S_2}{|B|}}, \tag{55}$$

respectively. It is at such scales that instability most contributes to the emergence of coherent nonlinear entities (stationary or nonstationary). In fact, just on such scales a two-dimensional soliton (which apparently turns out to be unstable) or a collapsing caviton can arise. The dependence of the instability growth rate γ on the vertical wave number of perturbations k_z in the case $k_z \gg k_x$ for different amplitude values Ψ_0 and for specific values $q_x = 1$ and $q_z = 1.5$ is shown in Fig. 4. For example, in the Earth's atmosphere,

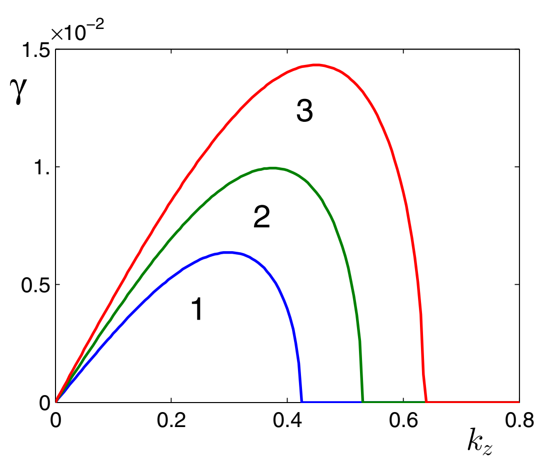

FIG. 4. Dependence of the instability growth rate γ on the vertical wave number k_z in the case $k_z \gg k_x$ for different amplitude values Ψ_0. The numbers under the curves correspond to different amplitudes: 1: $\Psi_0 = 0.004$, 2: $\Psi_0 = 0.005$, and 3: $\Psi_0 = 0.006$.

the equivalent atmospheric height at the altitudes $\gtrsim$200 km (i.e., for an isothermal atmosphere) is $H \sim 40$ km, that is, the values of q_x and q_z correspond to horizontal and vertical wavelengths ∼40 km and ∼30 km, respectively. Note that the wave numbers of perturbation (envelope) are much less than the characteristic wave numbers of the IGW (carrier). For the value $\Psi_0 = 0.004$, which corresponds to the perturbation velocity ∼10 m/s, the optimal values $k_{z,\text{opt}} = 0.3$ correspond to the characteristic vertical size of the perturbation region ∼130 km.

IV. COLLAPSE OF INTERNAL GRAVITY WAVES

The system of Eqs. (15) and (16) can be written in the form of the generalized nonlinear Schrödinger equation with the nonlocal nonlinearity,

$$i\frac{\partial\Psi}{\partial\tau} + A\frac{\partial^2\Psi}{\partial x^2} + B\frac{\partial^2\Psi}{\partial z^2} + 2C\frac{\partial^2\Psi}{\partial x\partial z} + \Psi\int R(\mathbf{r}-\mathbf{r}')M\left(q_z\frac{\partial}{\partial x} - q_x\frac{\partial}{\partial z}\right)|\Psi(\mathbf{r}')|^2 d^2\mathbf{r}' = 0, \tag{56}$$

where the kernel $R(\mathbf{r})$ is the Green's function of the equation

$$\left(F\frac{\partial^2}{\partial x^2} - G\frac{\partial^2}{\partial z^2}\right)R(\mathbf{r}-\mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}'), \tag{57}$$

that in Fourier space corresponds to

$$R_{\mathbf{k}} = \frac{1}{Gk_z^2 - Fk_x^2}. \tag{58}$$

Equation (56) conserves the 2D norm

$$\mathcal{N} = \int |\Psi|^2 d^2\mathbf{r}, \tag{59}$$

and Hamiltonian

$$\mathcal{H} = \int \left\{A\left|\frac{\partial\Psi}{\partial x}\right|^2 + B\left|\frac{\partial\Psi}{\partial z}\right|^2 + \frac{C}{2}\left(\frac{\partial\Psi}{\partial x}\frac{\partial\Psi^*}{\partial z} + \frac{\partial\Psi}{\partial z}\frac{\partial\Psi^*}{\partial x}\right) - \frac{|\Psi|^2}{2}\int R(\mathbf{r}-\mathbf{r}')M\left(q_z\frac{\partial}{\partial x} - q_x\frac{\partial}{\partial z}\right)|\Psi(\mathbf{r}')|^2 d^2\mathbf{r}'\right\}d^2\mathbf{r}, \tag{60}$$

and can be written in the Hamiltonian form

$$i\frac{\partial\Psi}{\partial t} = \frac{\delta\mathcal{H}}{\delta\Psi^*}. \tag{61}$$

The nonlinear term in Eq. (56) is somewhat reminiscent of the nonlocal nonlinearity in previously studied models with a kernel depending on the difference in spatial coordinates. The expressions for the kernels $R(\mathbf{r})$ in these models were dictated by the corresponding physical problems and were quite different from each other. For example, in Ref. [46] the kernel has the form of a quadratic exponential, in Ref. [47] a Hankel function of the first kind of zero order, and in Ref. [49] it contains the complementary error function. In all these cases, the nonlocal nonlinearity was scale inhomogeneous in spatial variables, and this is what resulted in the absence of collapse and the existence of stable coherent structures in the form of not only the 2D fundamental soliton (the ground state), but also in the form of a dipole soliton, rotating multisolitons and vortex solitons.

The key point for further analysis is the scale homogeneity of Eq. (56) in spatial variables, which is easily seen from Eq. (58), and, as a consequence, the Hamiltonian $\mathcal{H}$. The stationary solution of Eq. (56) in the form of $\Psi(\mathbf{r},t) = \Phi(\mathbf{r})\exp(i\lambda^2 t)$ corresponds to a stationary point of the Hamiltonian $\mathcal{H}$ for a fixed 2D norm $\mathcal{N}$ and resolves the variational problem $\delta\mathcal{S}[\Phi] = 0$ for the functional

$$\mathcal{S}[\Phi] = \mathcal{H} + \lambda^2\mathcal{N}. \tag{62}$$

Solving this variational problem is equivalent to finding a solution of the stationary equation:

$$-\lambda^2\Phi + A\frac{\partial^2\Phi}{\partial x^2} + B\frac{\partial^2\Phi}{\partial z^2} + 2C\frac{\partial^2\Phi}{\partial x\partial z} + \Phi\int R(\mathbf{r}-\mathbf{r}')M\left(q_z\frac{\partial}{\partial x} - q_x\frac{\partial}{\partial z}\right)|\Phi(\mathbf{r}')|^2 d^2\mathbf{r}' = 0. \tag{63}$$

Multiplying Eq. (63) by Φ^*, and then integrating over the whole space (taking into account zero boundary conditions at infinity), we obtain

$$-\lambda^2\mathcal{N} - I_1 + I_2 = 0, \tag{64}$$

where

$$I_1 = A\left|\frac{\partial\Phi}{\partial x}\right|^2 + B\left|\frac{\partial\Phi}{\partial z}\right|^2 + \frac{C}{2}\left(\frac{\partial\Phi}{\partial x}\frac{\partial\Phi^*}{\partial z} + \frac{\partial\Phi}{\partial z}\frac{\partial\Phi^*}{\partial x}\right) \tag{65}$$

and

$$I_2 = |\Phi|^2\int R(\mathbf{r}-\mathbf{r}')M\left(q_z\frac{\partial}{\partial x} - q_x\frac{\partial}{\partial z}\right)|\Phi(\mathbf{r}')|^2 d^2\mathbf{r}' d^2\mathbf{r}. \tag{66}$$

On the other hand, one can write

$$\mathcal{H} = I_1 - \frac{I_2}{2}. \tag{67}$$

With the scale homogeneity in mind, we consider an $\mathcal{N}$-preserving scaling transformation $\Phi^{(\alpha)} = \Phi(\alpha\mathbf{r})$ and obtain for the corresponding values:

$$\mathcal{N}^{(\alpha)} = \alpha^2\mathcal{N}, \quad I_1^{(\alpha)} = I_1, \quad I_2^{(\alpha)} = \alpha^2 I_2. \tag{68}$$

We use the approach developed by Hobart and Derrick [60,61] (see also further development in Ref. [62]). For the functional $V[\phi]$ of the form

$$V[\phi] = \sum_{\nu=-n_1}^{n_2} V^{(\nu)}(\alpha), \tag{69}$$

where $V^{(\nu)}(\alpha)$ is a homogeneous function of the scale parameter α of degree ν, and with a stationary point $\phi = u(\mathbf{r})$, that is, $\delta V[u] = 0$, with a scale transformation $\phi_\alpha = u(\alpha\mathbf{r})$, the following equality

$$\frac{\delta V[u]}{\delta\alpha} = \sum_{\nu=-n_1}^{n_2}\frac{\partial V^{(\nu)}}{\partial\alpha}\bigg|_{\alpha=1} = \sum_{\nu=-n_1}^{n_2}\nu V^{(\nu)}\bigg|_{\alpha=1} = 0 \tag{70}$$

is true (the so-called virial theorem). Since the functional $\mathcal{S}$ is scale homogeneous in spatial variables, from the Hobart-Derrick virial theorem, we can immediately write

$$\frac{\partial}{\partial\alpha}(\mathcal{H}^{(\alpha)} + \lambda^2\mathcal{N}^{(\alpha)})\bigg|_{\alpha=1} = 0, \tag{71}$$

and then from Eqs. (68) and (71) we have an additional restriction for the stationary states:

$$2\lambda^2\mathcal{N} + I_2 = 0. \tag{72}$$

Combining Eqs. (64), (67), and (72), one can obtain that $\mathcal{H} = 0$. Thus, the Hamiltonian at any stationary solution is equal to zero. This fact in the model under consideration is not accidental. This is typical for the 2D models with a cubic local nonlinearity and scale homogeneity in both spatial variables. Then, one can conclude that an arbitrary initial localized field distribution with $\mathcal{H} \neq 0$ never reaches a stationary state in the course of evolution, that is, either spreads out or collapses. Collapse in the model of the two-dimensional NLS equation is usually called critical, since (unlike the three-dimensional case) it occurs when the 2D norm of the wave field exceeds a certain critical value (in this case, the Hamiltonian is negative) [63]. The same is true for our model. By analogy with the two-dimensional NLS equation, one can expect for the critical value $\mathcal{N}_c = \int\Phi_0^2 d^2\mathbf{r}$, where $\Phi_0(\mathbf{r})$ is a nodeless solution (ground state) of Eq. (63). Thus, a sufficiently intense disturbance results in the collapse of internal gravity waves.

V. CONCLUSION

We have studied the dynamics of 2D nonlinear IGWs. The analysis was carried out on the basis of a system of 2D nonlinear equations for the velocity stream function and secondary mean flow, obtained with the aid of the reductive perturbation method in Ref. [38]. We have obtained one equation for the envelope in the form of 2D generalized nonlinear Schrödinger equation with nonlocal nonlinearity when the nonlinear response depends on the wave intensity at some spatial domain. Only the elliptic type of this equation has been considered. The instability of a monochromatic plane wave, which is an exact solution of the corresponding equation, has been studied, and a nonlinear dispersion equation has been found. In the limit of long-wave modulations, when the wave vector of modulations can be neglected compared to the wave vector of the plane wave, the instability is of a convective type. In the opposite case of short-wave modulations, we have a purely growing instability (modulation instability). In both cases, the corresponding instability thresholds and instability growth rates have been found. Numerical estimates for a characteristic region of localization of unstable perturbations, consistent with the results of possible predictions of experimental observations on nonlinear IGWs, are given for the real Earth's atmosphere. It is usually believed that modulation instability at the nonlinear stage results in the formation of a soliton or collapsing caviton. We have shown that, due to scaling homogeneity in spatial variables, the Hamiltonian of the resulting nonlinear equation with nonlocal nonlinearity is equal to zero, which leads to the critical collapse of atmospheric IGWs. In reality, no singularity occurs and the collapse arrests due to the dissipation of short-wave harmonics corresponding to the lower limit of wavelengths for IGWs ($\lesssim$10 km for the Earth's atmosphere at altitudes ~200 km).

APPENDIX

In this Appendix, we write the explicit expressions for the coefficients A, B, C, D, E, F, G, and M in Eqs. (15) and (16):

$$A = -\frac{12q_x\left(1+4q_z^2\right)}{(1+4q^2)^{5/2}}, \quad B = -\frac{4q_x\left(8q_z^2-4q_x^2-1\right)}{(1+4q^2)^{5/2}}, \quad \text{(A1)}$$

$$C = \frac{4q_z\left(8q_x^2-4q_z^2-1\right)}{(1+4q^2)^{5/2}}, \quad D = \frac{2q_z\left(4q_x^4-4q_z^4-q_z^2\right)}{\left(1+4q_z^2\right)(1+4q^2)}, \quad \text{(A2)}$$

$$E = \frac{q_x(1+8q^2)}{2(1+4q^2)}, \quad F = \left[1-\frac{\left(1+4q_z^2\right)^2}{(1+4q^2)^3}\right], \quad \text{(A3)}$$

$$G = \frac{16q_x^2q_z^2}{(1+4q^2)^3}, \quad M = \frac{q_x(1+8q^2)}{2(1+4q^2)^{1/2}}. \quad \text{(A4)}$$

[1] C. O. Hines, Internal atmospheric gravity waves at ionospheric heights, Can. J. Phys. **38**, 1441 (1960).

[2] I. Tolstoy, Long-period gravity waves in the atmosphere, J. Geophys. Res. **72**, 4605 (1967).

[3] R. L. Walterscheid and J. H. Hecht, A reexamination of evanescent acoustic-gravity waves: Special properties and aeronomical significance, J. Geophys. Res. **108**, 4340 (2005)

[4] O. K. Cheremnykh, A. K. Fedorenko, E. I. Kryuchkov, and Y. A. Selivanov, Evanescent acoustic-gravity modes in the isothermal atmosphere: Systematization, applications to the Earth and solar atmospheres, Ann. Geophys. **37**, 405 (2019).

[5] O. Cheremnykh, A. Fedorenko, Y. Selivanov, and S. Cheremnykh, Continuous spectrum of evanescent acoustic-gravity waves in an isothermal atmosphere, Mon. Not. R. Astron. Soc. **503**, 5545 (2021).

[6] K. C. Yeh and C. H. Liu, Acoustic-gravity waves in the upper atmosphere, Rev. Geophys. Space Phys. **12**, 193 (1974).

[7] T. Beer, *Atmospheric Waves* (John Wiley, New York, 1974).

[8] E. E. Gossard and W. H. Hooke, *Waves in the Atmosphere: Atmospheric Infrasound and Gravity Waves: Their Generation and Propagation* (Elsevier Scientific Publishing Company, New York, 1975).

[9] G. Schubert and R. L. Walterscheid, Propagation of small-scale acoustic-gravity waves in the Venus atmosphere, J. Atmos. Sci. **41**, 1202 (1984).

[10] J. M. Forbes and Y. Moudden, Solar terminator wave in a Mars general circulation model, Geophys. Res. Lett. **36**, 17201 (2009).

[11] P. Prikryl, D. B. Muldren, S. J. Sofko, and J. M. Ruohoniemi, Solar wind Alfven waves: A source of pulsed ionospheric convection and atmospheric gravity waves, Ann. Geophys. **23**, 401 (2005).

[12] D. C. Fritts and S. L. Vadas, Gravity wave penetration into the thermosphere: Sensitivity to solar cycle variations and mean winds, Ann. Geophys. **26**, 3841 (2008).

[13] A. V. Bespalova, A. K. Fedorenko, O. K. Cheremnykh, and I. T. Zhuk, Satellite observations of wave disturbances caused by moving solar terminator, J. Atmos. Sol. Terr. Phys. **140**, 79 (2016).

[14] Yu. G. Rapoport, O. E. Gotynyan, V. M. Ivchenko, L. V. Kozak, and M. Parrot, Effect of acoustic-gravity wave of the lithospheric origin on the ionospheric F region before earthquakes, Phys. Chem. Earth **29**, 607 (2004).

[15] E. Yiğit, A. D. Aylward, and A. S. Medvedev, Parameterization of the effects of vertically propagating gravity waves for thermosphere general circulation models: Sensitivity study, J. Geophys. Res. **113**, D19106 (2008).

[16] B. R. Sutherland, *Internal Gravity Waves* (Cambridge University Press, Cambridge, 2015).

[17] S. H. Francis, Global propagation of atmospheric gravity waves: A review, J. Atmos. Sol. Terr. Phys. **37**, 1011 (1975).

[18] D. C. Fritts and M. J. Alexander, Gravity wave dynamics and effects in the middle atmosphere, Rev. Geophys. **41**, 1003 (2003).

[19] T. D. Kaladze, O. A. Pokhotelov, H. A. Shah, M. I. Khan, and L. Stenflo, Acoustic-gravity waves in the Earth's ionosphere, J. Atmos. Sol. Terr. Phys. **70**, 1607 (2008).

[20] V. M. Lashkin and O. K. Cheremnykh, Acoustic-gravity waves in quasi-isothermal atmospheres with a random vertical temperature profile, Wave Motion **119**, 103140 (2023).

[21] A. Roy, S. Roy, and A. P. Misra, Dynamical properties of acoustic-gravity waves in the atmosphere, J. Atmos. Sol. Terr. Phys. **186**, 78 (2019).

[22] Yu. Z. Miropol'sky, *Dynamics of Internal Gravity Waves in the Ocean* (Kluwer, Dordrecht, 2001).

[23] L. Stenflo, Acoustic solitary waves, Phys. Fluids **30**, 3297 (1987).

[24] L. Stenflo, Acoustic gravity vortices, Phys. Scr. **41**, 641 (1990).

[25] L. Stenflo and P. K. Shukla, Nonlinear acoustic-gravity waves, J. Plasma Phys. **75**, 841 (2009).

[26] B. Dong and K. C. Yeh, Resonant and nonresonant wave-wave interactions in an isothermal atmosphere, J. Geophys. Res. **93**, 3729 (1988).

[27] D. C. Fritts, S. Sun, and D.-Y. Wang, Wave-wave interactions in a compressible atmosphere 1. A general formulation including rotation and wind shear, J. Geophys. Res. **97**, 9975 (1992).

[28] C.-S. Huang and J. Li, Weak nonlinear theory of the ionospheric response to atmospheric gravity waves in the *F*-region, J. Atmos. Terr. Phys. **53**, 903 (1991).

[29] C.-S. Huang and J. Li, Interaction of atmospheric gravity solitary waves with ion acoustic solitary waves in the ionospheric *F*-region, J. Atmos. Terr. Phys. **54**, 951 (1992).

[30] O. Onishchenko, O. Pokhotelov, and V. Fedun, Convective cells of internal gravity waves in the earth's atmosphere with finite temperature gradient, Ann. Geophys. **31**, 459 (2013).

[31] D. Jovanović, L. Stenflo, and P. K. Shukla, Acoustic gravity tripolar vortices, Phys. Lett. A **279**, 70 (2001).

[32] D. Jovanović, L. Stenflo, and P. K. Shukla, Acoustic-gravity nonlinear structures, Nonlinear Proc. Geophys. **9**, 333 (2002).

[33] P. K. Shukla and A. A. Shaikh, Dust-acoustic gravity vortices in a nonuniform dusty atmosphere, Phys. Scr. T **75**, 247 (1998).

[34] O. G. Onishchenko, W. Horton, O. A. Pokhotelov, and V. Fedun, "Explosively growing" vortices of unstably stratified atmosphere, J. Geophys. Res. Atmos. **121**, 11264 (2016).

[35] O. Onishchenko, V. Fedun, I. Ballai, A. Kryshtal, and G. Verth, Generation of localised vertical streams in unstable stratified atmosphere, Fluids **6**, 454 (2021).

[36] A. P. Misra, A. Roy, D. Chatterjee, and T. D. Kaladze, Internal gravity waves in the Earth's ionosphere, IEEE Trans. Plasma Sci. **50**, 2603 (2022).

[37] T. D. Kaladze, A. P. Misra, A. Roy, and D. Chatterjee, Nonlinear evolution of internal gravity waves in the Earth's ionosphere: Analytical and numerical approach, Adv. Space Res. **69**, 3374 (2022).

[38] V. M. Lashkin and O. K. Cheremnykh, Nonlinear internal gravity waves in the atmosphere: Rogue waves, breathers and dark solitons, Commun. Nonlinear Sci. Numer. Simulat. **130**, 107757 (2024).

[39] K. M. Huang, S. D. Zhang, F. Yi, C. M. Huang, Q. Gan, Y. Gong, and Y. H. Zhang, Nonlinear interaction of gravity waves in a nonisothermal and dissipative atmosphere, Ann. Gephys. **32**, 263 (2014).

[40] D. C. Fritts, B. Laughman, T. S. Lund, and J. B. Snively, Self-acceleration and instability of gravity wave packets: 1. Effects of temporal localization, J. Geophys. Res. Atmos. **120**, 8783 (2015).

[41] J. B. Snively, Nonlinear gravity wave forcing as a source of acoustic waves in the mesosphere, thermosphere, and ionosphere, Geophys. Res. Lett. **44**, 12020 (2017).

[42] T. Mixa, D. Fritts, T. Lund, B. Laughman, L. Wang, and L. Kantha, Numerical simulations of high-frequency gravity wave propagation through fine structures in the mesosphere, J. Geophys. Res. Atmos. **124**, 9372 (2019).

[43] S. K. Turitsyn, Spatial dispersion of nonlinearity and stability of multidimensional solitons, Theor. Math. Phys. **64**, 797 (1985).

[44] W. Królikowski, O. Bang, N. I. Nikolov, D. Neshev, J. Wyller, J. J. Rasmussen, and D. Edmundson, Modulational instability, solitons and beam propagation in spatially nonlocal nonlinear media, J. Opt. B: Quantum Semiclassical Opt. **6**, S288 (2004).

[45] C. Rotschild, M. Segev, Z. Xu, Y. V. Kartashov, L. Torner, and O. Cohen, Two-dimensional multipole solitons in nonlocal nonlinear media, Opt. Lett. **31**, 3312 (2005).

[46] A. I. Yakimenko, V. M. Lashkin, and O. O. Prikhodko, Dynamics of two-dimensional coherent structures in nonlocal nonlinear media, Phys. Rev. E **73**, 066605 (2006).

[47] V. M. Lashkin, A. I. Yakimenko, and O. O. Prikhodko, Two-dimensional nonlocal multisolitons, Phys. Lett. A **366**, 422 (2007).

[48] V. M. Lashkin, Two-dimensional ring-like vortex and multisoliton nonlinear structures at the upper-hybrid resonance, Phys. Plasmas **14**, 102311 (2007).

[49] V. M. Lashkin, Two-dimensional nonlocal vortices, multipole solitons, and rotating multisolitons in dipolar Bose-Einstein condensates, Phys. Rev. A **75**, 043607 (2007).

[50] V. E. Zakharov, The collapse of Langmuir waves, Sov. Phys. JETP **35**, 908 (1972).

[51] V. E. Zakharov and E. A. Kuznetsov, Solitons and collapses: Two evolution scenarios of nonlinear wave systems, Phys. Usp. **55**, 535 (2012).

[52] Y. S. Kivshar and G. P. Agrawal, *Optical Solitons: From Fibers to Photonic Crystals* (Academic Press, San Diego, 2003).

[53] R. K. Dodd, J. C. Eilbeck, J. D. Gibbon, and H. C. Morris, *Solitons and Nonlinear Wave Equations* (Academic Press, London, 1982).

[54] F. John, *Partial Differential Equations* (Springer-Verlag, New York, 1982).

[55] E. A. Kuznetsov, A. M. Rubenchik, and V. E. Zakharov, Soliton stability in plasmas and hydrodynamics, Phys. Rep. **142**, 103 (1986).

[56] V. M. Lashkin, Stable three-dimensional Langmuir vortex soliton, Phys. Plasmas **27**, 042106 (2020).

[57] D. J. Benney, A general theory for interactions between short and long waves, Stud. Appl. Math. **56**, 81 (1977).

[58] A. C. Newell, Long waves-short waves; a solvable model, SIAM J. Appl. Math. **35**, 650 (1978).

[59] T. Kanna, M. Vijayajayanthi, and M. Lakshmanan, Mixed solitons in a (2+1)-dimensional multicomponent long-wave-short-wave system, Phys. Rev. E **90**, 042901 (2014).

[60] R. H. Hobart, On the instability of a class of unitary field models, Proc. Phys. Soc. (London) **82**, 201 (1963).

[61] G. H. Derrick, Comments on nonlinear wave equations as models for elementary particles, J. Math. Phys. **5**, 1252 (1964).

[62] V. G. Makhankov, Yu. P. Rybakov, and V. I. Sanyuk, *The Skyrme Model: Fundamentals, Methods, Applications* (Springer-Verlag, New York, 1993).

[63] C. Sulem and P.-L. Sulem, *The Nonlinear Schrödinger Equation: Self-Focusing and Wave Collapse* (Springer-Verlag, New York, 1999).