Impact of environmental stochastic fluctuations on the evolutionary stability of imitation dynamics

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To show the impact of environmental noise on imitation dynamics, the stochastic stability and stochastic evolutionary stability of a discrete-time imitation dynamics with random payoffs are studied in this paper. Based on the stochastic local stability of fixation states and constant interior equilibria in a two-phenotype model, we extend the concept of stochastic evolutionary stability to the stochastic imitation dynamics, which is defined as a strategy such that, if all the members of the population adopt it, then the probability for any mutant strategy to invade the population successfully under the influence of natural selection is arbitrarily low. Our main results show clearly that the stochastic evolutionary stability of the system depends only on the properties of the mean matrix of the random payoff matrix and is independent of the randomness of the random payoff matrix. Moreover, as two examples, we show also that under the framework of stochastic imitation dynamics, the noise intensity affects the evolution of cooperative behavior in a stochastic prisoner's dilemma game and the system's nonlinear dynamic behavior.

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I. INTRODUCTION

It is well known that imitation dynamics is an important theoretical branch of evolutionary game dynamics, which mainly involves the process that the time evolution of strategies (or behaviors) in population is more likely to be based on the mutual imitation among individuals than on inheritance [1,2]. Recently, we investigated the impact of the sensitivity of individuals to the expected payoff differences between paired individuals on the stability of a discrete-time imitation dynamics and on the evolutionary stability of the imitation dynamics [3]. However, it is still unclear how the environmental stochastic fluctuations (or the environmental uncertainty) will influence the evolutionary stability of the imitation dynamics.

Similar to the classic evolutionary game theory, it is also assumed that the environmental conditions do not change over time in the standard imitation dynamics. However, this assumption cannot be thought to be always true since environmental conditions in the real world are changing and uncertain. As pointed out by May [4], for the population dynamics in ecology, because of the uncertainty of real environments, the birth rates, carrying capacities, competition coefficients, and other parameters which characterize natural biological systems all, to a greater or lesser degree, exhibit random fluctuations (see also [5]). This also implies that the stochastic fluctuations in the surrounding environment of a population may also cause changes in the occurrence of interactions between individuals and, more importantly, changes in the payoffs received by the interacting individuals. Therefore, there is no *a priori* reason to assume that the payoffs in an imitation dynamics are constants if the environment is actually stochastic.

In fact, in order to explore the impact of environmental stochastic fluctuations on the evolutionary game dynamics, Zheng *et al.* [6,7] (see also Feng *et al.* [8]) developed the concept of stochastic evolutionary stability based on the stochastic stability of the stochastic recurrence equation and stochastic replicator dynamics. Similar to the standard definition of evolutionarily stable strategy (ESS) developed by Maynard Smith [12] (see also [2,13]), a stochastically evolutionarily stable (SES) strategy is defined as a strategy such that, if all the members of the population adopt it, then the probability for at least any slightly perturbed strategy to invade the population under the influence of natural selection is arbitrarily low (see also [6,8–11]). In this study, based on the framework of discrete-time imitation dynamics (in which the Fermi function is taken to measure the probability of reciprocal imitation between paired individuals [3,14]), we will extend the concept of stochastic evolutionary stability to the imitation dynamics with random payoffs (i.e., the stochastic imitation dynamics). Our main goal is not only to reveal how environmental stochastic fluctuations affect the stability of imitation dynamics, especially on the evolution of cooperation and the nonlinear dynamic behavior of the system, but also to elucidate the conditions and properties of stochastic evolutionary stability in imitation dynamics.

II. BASIC MODEL AND STOCHASTIC STABILITY ANALYSIS

Consider a discrete-time imitation dynamics with two pure strategies S_1 and S_2, respectively. To show the influence of the environmental noise on the imitation dynamics, we here

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assume that for the pairwise interactions, the payoff matrix at time step $t \geqslant 1$ is a random matrix, which is given by

$$\mathbf{\Pi}(t) = \begin{pmatrix} c_{11}(t) & c_{12}(t) \\ c_{21}(t) & c_{22}(t) \end{pmatrix} = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix}, \quad (1)$$

where $c_{ij}(t)$ denotes the payoff to S_i against S_j at time step $t \geqslant 1$ and it is a random variable for $i, j = 1, 2$. We also assume that the probability distributions of these payoffs do not depend on the time step $t \geqslant 1$ [6]. The means, variances, and covariances of all these random payoffs are given by $\langle c_{ij}(t)\rangle = \bar{c}_{ij}$, $\langle [c_{ij}(t) - \bar{c}_{ij}]^2\rangle = \sigma_{ij}^2$, and $\langle [c_{ij}(t) - \bar{c}_{ij}][c_{kl}(t) - \bar{c}_{kl}]\rangle = \sigma_{ij,kl}$, respectively, for $i, j, k, l = 1, 2$ with $(i, j) \neq (k, l)$. Let u_t denote the frequency of strategy S_1 in the population at time step $t \geqslant 1$. Then, under the assumption of random pairwise interactions [2], the expected payoffs of S_1 and S_2 at time step $t \geqslant 1$, denoted by $\pi_{1,t}$ and $\pi_{2,t}$, respectively, are given by $\pi_{1,t} = u_t a(t) + (1 - u_t)b(t)$ and $\pi_{2,t} = u_t c(t) + (1 - u_t)d(t)$, and the mean payoff of the population at time step $t \geqslant 1$ is $\bar{\pi}_t = u_t\pi_{1,t} + (1 - u_t)\pi_{2,t}$.

Let $N_{1,t}$ denote the number of individuals using strategy S_1 at time stem $t \geqslant 1$, and similarly, $N_{2,t}$ the number of individuals using strategy S_2. Based on the input-output model, the proportional imitation rule, and the assumption of random pairwise interactions [2], the numbers of individuals using S_1 and individuals using S_2 at time step $t + 1$ can be given by

$$\begin{aligned} N_{1,t+1} &= N_{1,t} - N_{1,t}(1 - u_t)g_{21,t} + N_{2,t}u_t g_{12,t} \\ &= N_{1,t}[1 + (1 - u_t)(g_{12,t} - g_{21,t})], \\ N_{2,t+1} &= N_{2,t} - N_{2,t}u_t g_{12,t} + N_{1,t}(1 - u_t)g_{21,t} \\ &= N_{2,t}[1 + u_t(g_{21,t} - g_{12,t})], \end{aligned} \quad (2)$$

respectively, where $u_t = N_{1,t}/(N_{1,t} + N_{2,t})$ and $g_{ji,t}$ denotes the probability that an individual using S_i switches his strategy to S_j when he interacts with an individual using S_j at time step $t \geqslant 1$ for $i, j = 1, 2$. Here, $g_{ji,t}$ is defined as a Fermi function [3,14], that is, $g_{ji,t} = \frac{1}{1+e^{-\beta(\pi_{j,t}-\pi_{i,t})}}$ for $i, j = 1, 2$, where the parameter β is a positive constant that measures the individual sensitivity to the payoff difference between paired individuals [3]. Obviously, the term $1 + (1 - u_t)(g_{12,t} - g_{21,t})$ in Eq. (2) represents the change rate of the number of individuals using S_1 and, similarly, the term $1 + u_t(g_{21,t} - g_{12,t})$ the change rate of the numbers of individuals using S_2. Based on Eq. (2), the frequency of strategy S_1 at time step $t + 1$ can be expressed as

$$\begin{aligned} u_{t+1} &= u_t[1 + (1 - u_t)(g_{12,t} - g_{21,t})] \\ &= u_t - u_t(1 - u_t)\frac{1 - e^{\beta(\pi_{1,t}-\pi_{2,t})}}{1 + e^{\beta(\pi_{1,t}-\pi_{2,t})}}. \end{aligned} \quad (3)$$

This is a stochastic recurrence equation. It provides a basic model for understanding the dynamic stability and evolutionary stability of imitation dynamics in stochastic environments.

For the asymptotic (or long-run) behavior of Eq. (3), let $\hat{u}$ represent a constant (nonrandom) equilibrium, that is, an equilibrium that does not depend on the randomness of the payoff matrix $\mathbf{\Pi}(t)$. For example, the fixation states $\hat{u} = 0$ and $\hat{u} = 1$ are both the constant equilibria of the system. Following Karlin and Liberman [15,16] (see also [6]), a constant equilibrium $\hat{u}$ is considered to be stochastically locally stable (SLS) if for any $\epsilon > 0$ there exists δ_0 such that $\mathbb{P}(u_t \to \hat{u}) \geqslant 1 - \epsilon$ as soon as $|u_0 - \hat{u}| < \delta_0$. On the other hand, a constant equilibrium $\hat{u}$ is stochastically locally unstable (SLU) if $\mathbb{P}(u_t \to \hat{u}) = 0$ as soon as $|u_0 - \hat{u}| > 0$.

A. Stochastic local stability of fixation states

Consider first the stochastic local stability of the fixation state $\hat{u} = 0$. Note that

$$\begin{aligned} \frac{du_{t+1}}{du_t} &= 1 - (1 - 2u_t)\frac{1 - e^{\beta(\pi_{1,t}-\pi_{2,t})}}{1 + e^{\beta(\pi_{1,t}-\pi_{2,t})}} \\ &\quad + u_t(1 - u_t)\frac{2e^{\beta(\pi_{1,t}-\pi_{2,t})}\beta[a(t)-b(t)-c(t)+d(t)]}{(1 + e^{\beta(\pi_{1,t}-\pi_{2,t})})^2}. \end{aligned} \quad (4)$$

Thus, for $\mathbb{P}[b(t) = d(t)] < 1$, when the system state is near $\hat{u} = 0$, Eq. (3) can be approximated as

$$\begin{aligned} u_{t+1} &\approx u_t\left(1 - \frac{1 - e^{\beta[b(t)-d(t)]}}{1 + e^{\beta[b(t)-d(t)]}}\right) \\ &= \frac{2u_t}{1 + e^{-\beta[b(t)-d(t)]}}. \end{aligned} \quad (5)$$

Note also that

$$\begin{aligned} & u_n = u_0\prod_{t=0}^{n-1}\frac{2}{1 + e^{-\beta[b(t)-d(t)]}} \\ \Rightarrow\quad & \frac{1}{n}\ln\frac{u_n}{u_0} = \frac{1}{n}\sum_{t=0}^{n-1}\ln\frac{2}{1 + e^{-\beta[b(t)-d(t)]}} \\ \Rightarrow\quad & \lim_{n\to\infty}\frac{1}{n}\ln\frac{u_n}{u_0} = \lim_{n\to\infty}\frac{1}{n}\sum_{t=0}^{n-1}\ln\frac{2}{1 + e^{-\beta[b(t)-d(t)]}} \\ & = \left\langle\ln\frac{2}{1 + e^{-\beta[b(t)-d(t)]}}\right\rangle, \end{aligned} \quad (6)$$

where $\langle\cdot\rangle$ denotes the mathematical expectation. Here we need to point out that the last step in the above equation is determined by the strong law of large numbers, and the probability distributions of the random variables $b(t)$ and $d(t)$ do not depend on the time step $t \geqslant 1$. Thus, for $\mathbb{P}[b(t) = d(t)] < 1$, the fixation state $\hat{u} = 0$ is SLS if

$$\left\langle\ln\frac{2}{1 + e^{-\beta[b(t)-d(t)]}}\right\rangle < 0 \quad (7)$$

(the proof is shown in the Appendix; see also Zheng *et al.* [6]).

Furthermore, for the degenerate case with $\mathbb{P}[b(t) = d(t)] = 1$ [i.e., $b(t) = d(t)$ at any time step $t \geqslant 1$], when the system state is near $\hat{u} = 0$, Eq. (2) can be approximated as

$$\begin{aligned} u_{t+1} &= u_t - u_t(1 - u_t)\frac{1 - e^{\beta[a(t)-c(t)]u_t}}{1 + e^{\beta[a(t)-c(t)]u_t}} \\ &\approx \frac{2u_t}{1 + e^{-\beta[a(t)-c(t)]u_t}}. \end{aligned} \quad (8)$$

Note also that when u_t is small, we have $e^{-\beta[a(t)-c(t)]u_t} \approx 1 - \beta[a(t) - c(t)]u_t$. So, the above equation can also be reapproximated as

$$u_{t+1} \approx \frac{2u_t}{2 - \beta[a(t) - c(t)]u_t}. \quad (9)$$

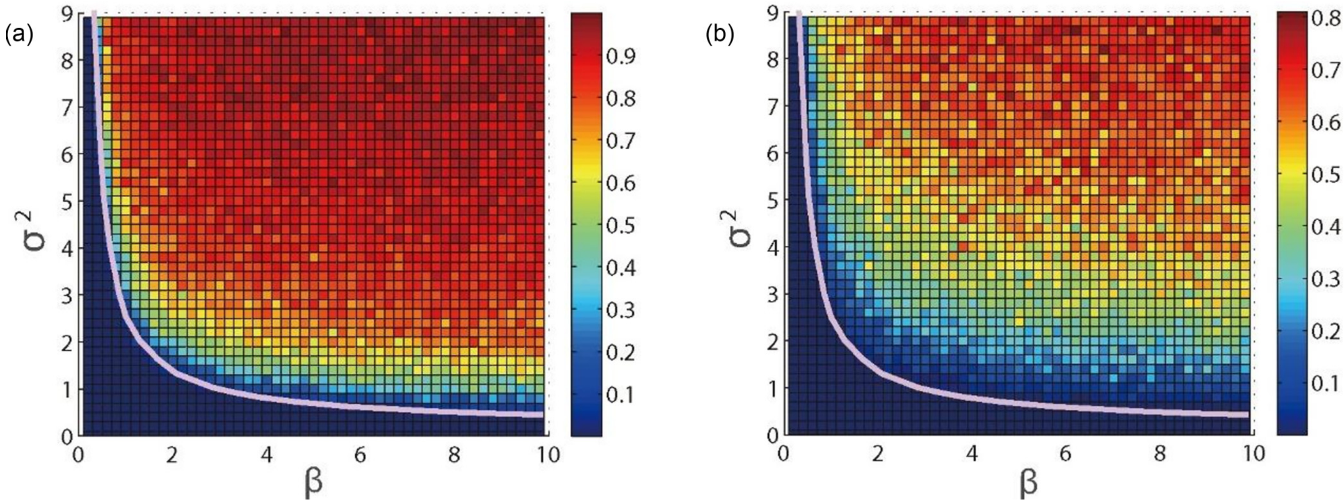

FIG. 1. Stochastic simulations for the stochastic local stability of cooperation in the imitation dynamics based on SPD game. We here take $b = 2$ and $c = 1$, and that ϑ_{ij} for $i, j = 1, 2$ are independent and identically distributed Gaussian noises with variance σ^2. Moreover, the initial frequencies of cooperation are taken as 0.99 [in panel (a)] and 0.9 [in panel (b)], respectively. The simulation results are shown on the β-σ^2 plane, in which the color of each point represents the probability that the cooperation will be fixed, which is calculated from a thousand independent stochastic simulations. The light purple curve corresponds to the theoretical critical condition $\langle \ln \frac{2}{1+e^{\beta[-c+\eta(t)]}} \rangle = 0$ for the stochastic local stability of the fixation state of cooperation. Obviously, these results show clearly a mutual compensation effect between the noise intensity and the size of parameter β for the evolution of cooperation.

Let $v_t = 1/u_t$, that is, $v_t \to \infty$ if $u_t \to 0$. Then, the above equation can be equivalently expressed as

$$v_{t+1} = v_t - \frac{\beta[a(t) - c(t)]}{2}. \tag{10}$$

This implies that

$$\begin{aligned} v_n - v_0 &= -\frac{\beta}{2} \sum_{t=0}^{n-1} [a(t) - c(t)] \\ \Rightarrow \quad \frac{1}{n}(v_n - v_0) &= \frac{\beta}{2} \frac{1}{n} \sum_{t=0}^{n-1} [c(t) - a(t)] \\ \Rightarrow \quad \lim_{n\to\infty} \frac{1}{n}(v_n - v_0) &= \frac{\beta}{2} \lim_{n\to\infty} \frac{1}{n} \sum_{t=0}^{n-1} [c(t) - a(t)] \\ &= \frac{\beta}{2} \langle c(t) - a(t) \rangle. \end{aligned} \tag{11}$$

Therefore, for the degenerate case with $\mathbb{P}[b(t) = d(t)] = 1$, the fixation state $\hat{u} = 0$ is SLS if $\bar{c} > \bar{a}$ (the proof of this result is similar to that in the Appendix, and see also [6]).

By symmetry, we can see that the fixation state $\hat{u} = 1$ is SLS if $\langle \ln \frac{2}{1+e^{\beta[a(t)-c(t)]}} \rangle < 0$, and for the degenerate case with $\mathbb{P}[a(t) = c(t)] = 1$, $\hat{u} = 1$ is SLS if $\bar{b} > \bar{d}$.

Example 1. In this example, we consider a stochastic imitation dynamics based on a stochastic prisoner's dilemma (SPD) game, which was proposed by Bereby-Meyer and Roth [17] (see also [18]). The payoff matrix $\mathbf{\Pi}(t)$ is taken as

$$\mathbf{\Pi}(t) = \begin{pmatrix} b - c & -c \\ b & 0 \end{pmatrix} + \begin{pmatrix} \vartheta_{11}(t) & \vartheta_{12}(t) \\ \vartheta_{21}(t) & \vartheta_{22}(t) \end{pmatrix},$$

where the first matrix on the right side of the above corresponds to a standard prisoner's dilemma (PD) game with $b > c$ [2,12,13], and the second matrix is a random payoff matrix, which represents the additive effect of environmental noises on the pairwise interactions [18]. For convenience, we here assume the random variables $\vartheta_{ij}(t)$ $(i, j = 1, 2)$ to be independent and identically distributed Gaussian noises if $\mathbb{P}[\vartheta_{ij}(t) = \vartheta_{kl}(t)] < 1$ for $ij \neq kl$, that is, $\langle \vartheta_{ij}(t) \rangle = 0$ and $\langle \vartheta_{ij}(t)\vartheta_{ij}(t') \rangle = 2D\delta(t - t')$ for $i, j = 1, 2$, and $\langle \vartheta_{ij}(t)\vartheta_{kl}(s) \rangle = 0$ for $ij \neq kl$ and $s \neq t$. Let u_t denote the frequency of cooperation (C) and $1 - u_t$ the frequency of defection (D) at time step $t \geqslant 1$. Then, it is easy to see that the fixation of D $(\hat{u} = 0)$ is SLS if $\langle \ln \frac{2}{1+e^{-\beta[-c+\xi(t)]}} \rangle < 0$, where $\xi(t) = \vartheta_{12}(t) - \vartheta_{22}(t)$ with $\langle \xi(t) \rangle = 0$ and $\langle \xi(t)\xi(t') \rangle = 4D\delta(t - t')$, and the fixation of C $(\hat{u} = 1)$ is SLS if $\langle \ln \frac{2}{1+e^{\beta[-c+\eta(t)]}} \rangle < 0$, where $\eta(t) = \vartheta_{11}(t) - \vartheta_{21}(t)$ with $\langle \eta(t) \rangle = 0$ and $\langle \eta(t)\eta(t') \rangle = 4D\delta(t - t')$. The stochastic simulation results show clearly that for a given value of β, a higher noise intensity is more conducive to promoting the evolution of cooperation, and similarly, for a given noise intensity, a higher β is also more conducive to promoting the evolution of cooperation [see Figs. 1(a) and 1(b)]. This implies that for the evolution of cooperation, there may be a mutual compensation effect between the noise intensity and the size of parameter β.

B. Stochastic local stability of a constant interior equilibrium

Let $\hat{u} \in (0, 1)$ be a constant interior equilibrium of the system, which is a solution of $\pi_{1,t} - \pi_{2,t} = 0$ independent of the randomness of the payoff matrix $\mathbf{\Pi}(t)$ in the interval $0 < u_t < 1$ [6]. For example, if $a(t) = d(t)$ and $b(t) = c(t)$ at any time step $t \geqslant 1$, then $\hat{u} = 1/2$ must be a constant interior equilibrium of Eq. (3). This implies that for Eq. (3), the existence of constant interior equilibrium strongly depends on the nature of the random payoff matrix $\mathbf{\Pi}(t)$; that is, we cannot guarantee the constant interior equilibrium always exists in general.

Note that if a constant interior equilibrium $\hat{u}$ exists, then we have

$$\left.\frac{\mathrm{d}u_{t+1}}{\mathrm{d}u_t}\right|_{u_t=\hat{u}} = 1 + \hat{u}(1-\hat{u})\frac{\beta[a(t)-b(t)-c(t)+d(t)]}{2}. \tag{12}$$

When u_t is near $\hat{u}$, we have the approximation

$$\begin{aligned} u_{t+1} &= \hat{u} + \left(1 + \hat{u}(1-\hat{u})\frac{\beta[a(t)-b(t)-c(t)+d(t)]}{2}\right)(u_t - \hat{u}) \\ \Rightarrow \quad u_{t+1} - \hat{u} &= \left(1 + \hat{u}(1-\hat{u})\frac{\beta[a(t)-b(t)-c(t)+d(t)]}{2}\right)(u_t - \hat{u}) \\ \Rightarrow \quad (u_{t+1} - \hat{u})^2 &= \left(1 + \hat{u}(1-\hat{u})\frac{\beta[a(t)-b(t)-c(t)+d(t)]}{2}\right)^2 (u_t - \hat{u})^2. \end{aligned} \tag{13}$$

Let $z_t = (u_t - \hat{u})^2$ for all possible $t \geqslant 0$. Then, we have

$$\begin{aligned} z_{t+1} &= z_t\left(1 + \hat{u}(1-\hat{u})\frac{\beta[a(t)-b(t)-c(t)+d(t)]}{2}\right)^2 \\ \Rightarrow \quad z_n &= z_0\prod_{t=0}^{n-1}\left(1 + \hat{u}(1-\hat{u})\frac{\beta[a(t)-b(t)-c(t)+d(t)]}{2}\right)^2 \\ \Rightarrow \quad \lim_{n\to\infty}\frac{1}{n}\ln\left(\frac{z_n}{z_0}\right) &= \lim_{n\to\infty}\frac{1}{n}\sum_{t=0}^{n-1}\ln\left(1 + \hat{u}(1-\hat{u})\frac{\beta[a(t)-b(t)-c(t)+d(t)]}{2}\right)^2 \\ &= \left\langle \ln\left(1 + \hat{u}(1-\hat{u})\frac{\beta[a(t)-b(t)-c(t)+d(t)]}{2}\right)^2 \right\rangle. \end{aligned} \tag{14}$$

This implies that a constant interior equilibrium $\hat{u}$ is SLS if

$$\left\langle \ln\left(1 + \hat{u}(1-\hat{u})\frac{\beta[a(t)-b(t)-c(t)+d(t)]}{2}\right)^2 \right\rangle < 0 \tag{15}$$

(the proof of this result is also similar to that in the Appendix, and see also [6]).

Example 2. From the inequality in Eq. (15), if all $a(t)$, $b(t)$, $c(t)$, and $d(t)$ are constants, and $a(t) = d(t) = a$ and $b(t) = c(t) = b$ with $b > a$, then Eq. (3) degenerates to a deterministic recurrence equation, and the constant interior equilibrium $\hat{u} = 1/2$ is locally asymptotically stable if $\beta < \beta_{cr} = 8/(b-a)$, where β_{cr} denotes the bifurcation value of β such that for $\beta > \beta_{cr}$, with the increase on β, the system will exhibit the periodic bifurcations and chaos, and also the symmetry breaking in this process [3,19–21]. For $a = 0$ and $b = 4$, when $\beta = 6$, the system will exhibit a periodic two-cycle with symmetric breaking [see Figs. 2(a) and 2(e)] (see also [3]). This naturally raises a more challenging question: how will the environmental noise affect the nonlinear dynamic behavior of the system? As a special case, we here assume that $a(t) = d(t) = 0$ and $b(t) = c(t)$ at any time step $t \geqslant 1$, where $\langle b(t)\rangle = \bar{b} = 4$ and the variance of $b(t)$ is given by $\langle [b(t) - \bar{b}]^2\rangle = \sigma_b^2$. From the inequality in Eq. (15), the constant interior equilibrium $\hat{u} = 1/2$ is SLS if $\langle \ln[4 - \beta b(t)]^2\rangle < \ln 16$. If σ_b^2 is small but $\sigma_b^2 \neq 0$, then the condition that $\hat{u} = 1/2$ is SLS can be approximately expressed as $\ln(1-\beta)^2 < \frac{\sigma_b^2}{2(1-\beta)^2}$; that is, $\hat{u} = 1/2$ must be SLS if $\beta < 2$. So, for small σ_b^2, the distribution characteristics of u_t for given different values of β should match the bifurcation diagram corresponding to $\sigma_b^2 = 0$. For example, when $\sigma_b^2 = 0.01$, the distribution of u_t is a bimodal distribution for $\beta = 6$, which corresponds to the periodic two-cycle with symmetric breaking in the bifurcation diagram [see Figs. 2(b) and 2(f)]. However, for $\beta = 6$, as σ_b^2 increases, the distribution of u_t undergoes a transition from a bimodal distribution to a 4-peak distribution (i.e., the symmetry breaking disappears due to the increase in σ_b^2) [see Figs. 2(c) and 2(g)], then from a 4-peak distribution to a bimodal distribution [see Figs. 2(d) and 2(h)]. This strongly suggests that for the stochastic imitation dynamics with constant interior equilibrium, the increase in noise intensity may change the nonlinear dynamic behavior of the system. This result should also be consistent with findings in a previous study [21].

III. STOCHASTIC EVOLUTIONARY STABILITY

For the stochastic imitation dynamics, we define a stochastically evolutionarily stable (SES) strategy as a strategy such that, if all the members of the population adopt it, then the probability for any mutant strategy to invade the population successfully under the influence of natural selection is arbitrarily low (see also [6,8]). This also implies that if a strategy is a SES strategy, then the fixation state of this SES strategy must be SLS for any one possible mutant strategy.

To show the condition for the stochastic evolutionary stability in the stochastic imitation dynamics, we consider a population consisting of only two mixed strategies $\mathbf{u} = (u, 1-u)$ and $\hat{\mathbf{u}} = (\hat{u}, 1-\hat{u})$. The payoff matrix at time step

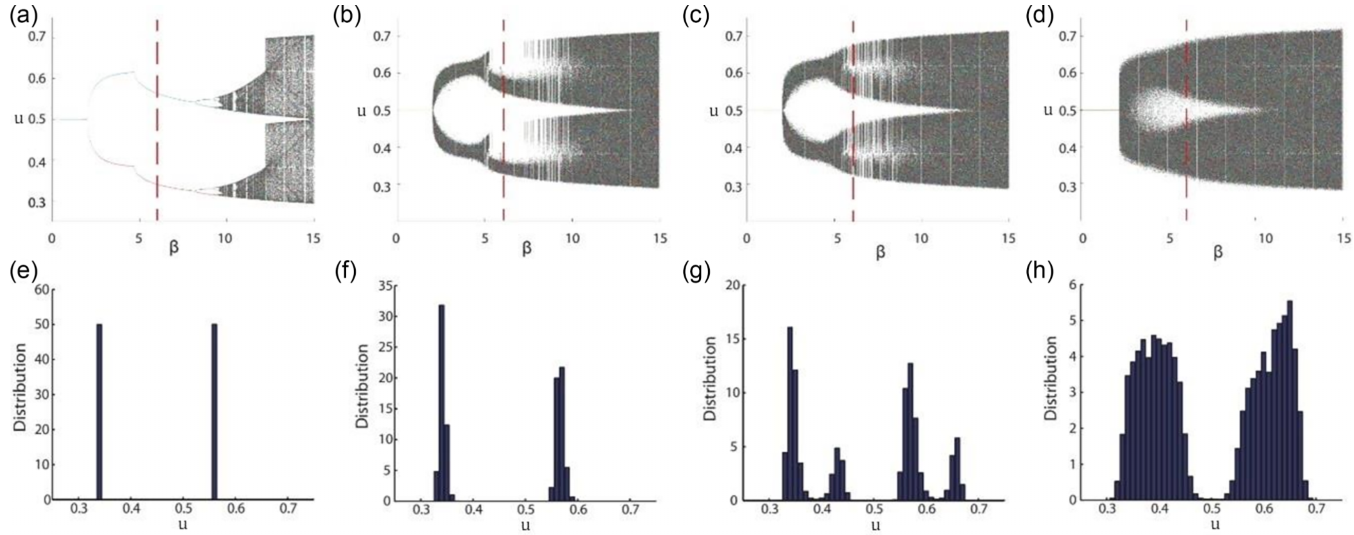

FIG. 2. Stochastic simulations for the impact of the noise intensity σ_b^2 on the symmetric breaking. The bifurcation diagrams of the system corresponding to $\sigma_b^2 = 0$, $\sigma_b^2 = 0.01$, $\sigma_b^2 = 0.1$, and $\sigma_b^2 = 1$, as well as the corresponding distribution of u_t at $\beta = 6$, are presented in panels (a) and (e), (b) and (f), (c) and (g), and (d) and (h), respectively. All these results show how the nonlinear dynamical properties of the imitation dynamics are affected by the environmental noise. For given σ_b and initial value $u_0 = 0.6$, the stochastic simulations are performed corresponding to the different values of β, respectively. For the simulation corresponding to a given value of β, the initial 3×10^3 steps are discarded as transients, and the iterative results from the next 10^4 steps are recorded as the data points plotted in the bifurcation diagrams.

$t \geqslant 1$ for these two mixed strategies is given by

$$\begin{pmatrix} \mathbf{u}\cdot\mathbf{\Pi}(t)\mathbf{u} & \mathbf{u}\cdot\mathbf{\Pi}(t)\hat{\mathbf{u}} \\ \hat{\mathbf{u}}\cdot\mathbf{\Pi}(t)\mathbf{u} & \hat{\mathbf{u}}\cdot\mathbf{\Pi}(t)\hat{\mathbf{u}} \end{pmatrix}, \tag{16}$$

where $\mathbf{u}\cdot\mathbf{\Pi}(t)\mathbf{u}$ [or $\mathbf{u}\cdot\mathbf{\Pi}(t)\hat{\mathbf{u}}$] is the payoff to strategy $\mathbf{u}$ against strategy $\mathbf{u}$ (or $\tilde{\mathbf{u}}$) at time step $t \geqslant 1$, and $\hat{\mathbf{u}}\cdot\mathbf{\Pi}(t)\mathbf{u}$ [or $\hat{\mathbf{u}}\cdot\mathbf{\Pi}(t)\hat{\mathbf{u}}$] the payoff to strategy $\hat{\mathbf{u}}$ against strategy $\mathbf{u}$ (or $\hat{\mathbf{u}}$). Let y_t be the frequency of strategy $\mathbf{u}$ at time step $t \geqslant 0$, and $1 - y_t$ the frequency of strategy $\hat{\mathbf{u}}$. Then, the expected payoffs of $\mathbf{u}$ and $\hat{\mathbf{u}}$ at time step $t \geqslant 1$ can be given by $\pi_{\mathbf{u},t} = y_t\mathbf{u}\cdot\mathbf{\Pi}(t)\mathbf{u} + (1-y_t)\mathbf{u}\cdot\mathbf{\Pi}(t)\hat{\mathbf{u}}$ and $\pi_{\hat{\mathbf{u}},t} = y_t\hat{\mathbf{u}}\cdot\mathbf{\Pi}(t)\mathbf{u} + (1-y_t)\hat{\mathbf{u}}\cdot\mathbf{\Pi}(t)\hat{\mathbf{u}}$, respectively. Thus, the frequency of $\mathbf{u}$ at time step $t+1$, y_{t+1} can be described as

$$y_{t+1} = y_t - y_t(1-y_t)\frac{1 - e^{\beta(\pi_{\mathbf{u},t} - \pi_{\hat{\mathbf{u}},t})}}{1 + e^{\beta(\pi_{\mathbf{u},t} - \pi_{\hat{\mathbf{u}},t})}}. \tag{17}$$

Based on the definition of stochastic evolutionary stability, if strategy $\hat{\mathbf{x}}$ is said to be a SES strategy, then the fixation state of $\hat{\mathbf{u}}$, denoted by $\hat{y} = 0$, is SLS for all possible $\mathbf{u} \neq \hat{\mathbf{u}}$. For the stochastic local stability of $\hat{y} = 0$, we need to consider two possible cases:

(i) If strategy $\hat{\mathbf{u}}$ satisfies $\mathbb{P}\{[\mathbf{\Pi}(t)\hat{\mathbf{u}}]_1 = [\mathbf{\Pi}(t)\hat{\mathbf{u}}]_2\} = 1$, that is, $[\mathbf{\Pi}(t)\hat{\mathbf{u}}]_1 = [\mathbf{\Pi}(t)\hat{\mathbf{u}}]_2$ at any time step $t \geqslant 1$, where $[\mathbf{\Pi}(t)\hat{\mathbf{u}}]_1 = \hat{u}a(t) + (1-\hat{u})b(t)$ and $[\mathbf{\Pi}(t)\hat{\mathbf{u}}]_2 = \hat{u}c(t) + (1-\hat{u})d(t)$, then Eq. (17) can be rewritten as

$$y_{t+1} = y_t - y_t(1-y_t)\frac{1 - e^{\beta[\mathbf{u}\cdot\mathbf{\Pi}(t)\mathbf{u} - \hat{\mathbf{u}}\cdot\mathbf{\Pi}(t)\mathbf{u}]y_t}}{1 + e^{\beta[\mathbf{u}\cdot\mathbf{\Pi}(t)\mathbf{u} - \hat{\mathbf{u}}\cdot\mathbf{\Pi}(t)\mathbf{u}]y_t}}. \tag{18}$$

When y_t is near 0, the above equation can be approximated as

$$y_{t+1} \approx \frac{2y_t}{1 + e^{-\beta[\mathbf{u}\cdot\mathbf{\Pi}(t)\mathbf{u} - \hat{\mathbf{u}}\cdot\mathbf{\Pi}(t)\mathbf{u}]y_t}}. \tag{19}$$

Therefore, similar to the analysis in Eqs. (8)–(11), $\hat{y} = 0$ is SLS if $\langle \hat{\mathbf{u}}\cdot\mathbf{\Pi}(t)\mathbf{u}\rangle > \langle \mathbf{u}\cdot\mathbf{\Pi}(t)\mathbf{u}\rangle$ (the proof is similar to that in the Appendix). This implies that a strategy $\hat{\mathbf{u}}$ satisfying $\mathbb{P}\{[\mathbf{\Pi}(t)\hat{\mathbf{u}}]_1 = [\mathbf{\Pi}(t)\hat{\mathbf{u}}]_2\} = 1$ is said to be a SES strategy if $\hat{\mathbf{u}}\cdot\bar{\mathbf{\Pi}}\mathbf{u} > \mathbf{u}\cdot\bar{\mathbf{\Pi}}\mathbf{u}$ for all possible $\mathbf{u} \neq \hat{\mathbf{u}}$, where $\bar{\mathbf{\Pi}} = \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix}$ is called the mean payoff matrix of $\mathbf{\Pi}(t)$. This condition is exactly the so-called stability condition for ESS in classical evolutionary game theory [2]. For example, if $a(t) = d(t)$ and $b(t) = c(t)$ at any time step $t \geqslant 1$, then it is easy to see that strategy $\hat{\mathbf{u}} = (\frac{1}{2}, \frac{1}{2})$ is a SES strategy if $\bar{b} > \bar{a}$.

(ii) If strategy $\hat{\mathbf{u}}$ satisfies $\mathbb{P}\{[\mathbf{\Pi}(t)\hat{\mathbf{u}}]_1 = [\mathbf{\Pi}(t)\hat{\mathbf{u}}]_2\} < 1$, then, similarly to the analysis in Eqs. (5)–(7), $\hat{y} = 0$ is SLS if

$$\left\langle \ln \frac{2}{1 + e^{-\beta[\mathbf{u}\cdot\mathbf{\Pi}(t)\hat{\mathbf{u}} - \hat{\mathbf{u}}\cdot\mathbf{\Pi}(t)\hat{\mathbf{u}}]}} \right\rangle < 0 \tag{20}$$

(the proof is similar to that in the Appendix). For convenience, we take

$$H(\mathbf{u}, \hat{\mathbf{u}}) = \left\langle \ln \frac{2}{1 + e^{-\beta[\mathbf{u}\cdot\mathbf{\Pi}(t)\hat{\mathbf{u}} - \hat{\mathbf{u}}\cdot\mathbf{\Pi}(t)\hat{\mathbf{u}}]}} \right\rangle. \tag{21}$$

Note that

$$\begin{aligned} \frac{\partial H(\mathbf{u}, \hat{\mathbf{u}})}{\partial u} &= \beta\left\langle \frac{\{[\mathbf{\Pi}(t)\hat{\mathbf{u}}]_1 - [\mathbf{\Pi}(t)\hat{\mathbf{u}}]_2\}e^{-\beta[\mathbf{u}\cdot\mathbf{\Pi}(t)\hat{\mathbf{u}} - \hat{\mathbf{u}}\cdot\mathbf{\Pi}(t)\hat{\mathbf{u}}]}}{1 + e^{-\beta[\mathbf{u}\cdot\mathbf{\Pi}(t)\hat{\mathbf{u}} - \hat{\mathbf{u}}\cdot\mathbf{\Pi}(t)\hat{\mathbf{u}}]}} \right\rangle, \\ \frac{\partial^2 H(\mathbf{u}, \hat{\mathbf{u}})}{\partial u^2} &= -\beta^2\left\langle \frac{\{[\mathbf{\Pi}(t)\hat{\mathbf{u}}]_1 - [\mathbf{\Pi}(t)\hat{\mathbf{u}}]_2\}^2 e^{-\beta[\mathbf{u}\cdot\mathbf{\Pi}(t)\hat{\mathbf{u}} - \hat{\mathbf{u}}\cdot\mathbf{\Pi}(t)\hat{\mathbf{u}}]}}{\left(1 + e^{-\beta[\mathbf{u}\cdot\mathbf{\Pi}(t)\hat{\mathbf{u}} - \hat{\mathbf{u}}\cdot\mathbf{\Pi}(t)\hat{\mathbf{u}}]}\right)^2} \right\rangle < 0. \end{aligned} \tag{22}$$

Thus, we can see that if $\hat{\mathbf{u}}$ is a SES strategy, then it must be the solution of the equation

$$\left.\frac{\partial H(\mathbf{u},\hat{\mathbf{u}})}{\partial u}\right|_{\mathbf{u}=\hat{\mathbf{u}}} = 0$$
$$\Rightarrow \quad \langle[\boldsymbol{\Pi}(t)\hat{\mathbf{u}}]_1 - [\boldsymbol{\Pi}(t)\hat{\mathbf{u}}]_2\rangle = 0$$
$$\Rightarrow \quad (\bar{\boldsymbol{\Pi}}\hat{\mathbf{u}})_1 - (\bar{\boldsymbol{\Pi}}\hat{\mathbf{u}})_2 = 0, \tag{23}$$

which is $\hat{u} = (\bar{b}-\bar{d})/(\bar{b}-\bar{d}+\bar{c}-\bar{a})$. This means that the pure strategy S_1 is SES for all possible $\mathbf{u} \neq S_1$ if $\hat{u} \geqslant 1$; the pure strategy S_2 is SES for all possible $\mathbf{u} \neq S_2$ if $\hat{u} \leqslant 0$; and if $\hat{u}$ is in the interval $0 < \hat{u} < 1$, that is, $\bar{a} > \bar{c}$ and $\bar{d} > \bar{b}$, or $\bar{a} < \bar{c}$ and $\bar{d} < \bar{b}$, then mixed strategy $\hat{\mathbf{u}} = (\hat{u}, 1-\hat{u})$ is SES for all possible $\mathbf{u} \neq \hat{\mathbf{u}}$.

All of the above results show clearly that for the stochastic imitation dynamics described by Eq. (2), the stochastic evolutionary stability depends only on the mean payoff matrix $\bar{\boldsymbol{\Pi}}$ of the random payoff matrix $\boldsymbol{\Pi}(t)$.

IV. DISCUSSION

In this paper, the stochastic stability and stochastic evolutionary stability of a discrete-time imitation dynamics with random payoffs are studied. As is well known, randomness (or uncertainty) in the environment is one of the main characteristics of nature, and this environmental noise will generally affect the results of interactions between species and between individuals [6], for example, the impact of environmental stochastic fluctuations on the interspecific competition, or more theoretically, the stochastic stability of the Lotka-Volterra equation under environmental stochastic fluctuations [4]. Furthermore, as shown in the introduction, in order to explore the impact of environmental noise on evolutionary game dynamics, some previous studies not only investigated the stochastic stability of the evolutionary game dynamics with random payoffs but also developed the concept of stochastic evolutionary stability [6,8]. So, as a natural extension of the study on the evolutionary game dynamics with random payoffs, it is also a very interesting and challenging question as to how the imitation dynamics is affected by the environmental fluctuations.

For the stochastic stability of the imitation dynamics with random payoffs, we mainly focus on the stochastic local stability of the system's constant equilibrium, that is, the stochastic local stability of the boundary (or fixation state) and constant interior equilibrium. Our main results show that the boundary $\hat{u} = 0$ is SLS if $\langle \ln \frac{2}{1+e^{-\beta[b(t)-d(t)]}}\rangle < 0$, and similarly, the boundary $\hat{u} = 1$ is SLS if $\langle \ln \frac{2}{1+e^{\beta[a(t)-c(t)]}}\rangle < 0$; if a constant interior equilibrium $\hat{u} \in (0, 1)$ exists, then it is SLS if $\langle \ln[1 + \hat{u}(1-\hat{u})\frac{\beta[a(t)-b(t)-c(t)+d(t)]}{2}]^2\rangle < 0$. On the other hand, we have also to point out that for the degenerate case with $\mathbb{P}[b(t) = d(t)] = 1$ (or $\mathbb{P}[a(t) = c(t)] = 1$), $\hat{u} = 0$ (or $\hat{u} = 1$) is SLS if $\bar{c} > \bar{a}$ (or $\bar{b} > \bar{d}$).

As an example, based on the stochastic local stability of the boundaries (or fixation states) in the stochastic imitation dynamics, the SPD game [18] is considered. We found that for a given value of β, a higher noise intensity should be more conducive to promoting the evolution of cooperation, and similarly, for a given noise intensity, a higher value of β should be more conducive to promoting the evolution of cooperation. This implies that for the evolution of cooperation in stochastic imitation dynamics, there may be a mutual compensation effect between the noise intensity and the parameter β. This result may provide a new perspective for understanding the evolution of cooperation in stochastic environments. Furthermore, based on the stochastic local stability of a constant interior equilibrium in the stochastic imitation dynamics, the effect of noise intensity on the nonlinear dynamical behavior of the system is also considered. For a special case with $\boldsymbol{\Pi}(t) = \begin{pmatrix} 0 & b(t) \\ b(t) & 0 \end{pmatrix}$, where $\langle b(t)\rangle = 4$ and $\beta = 6$, we found that an increase in noise intensity may have a profound impact on the nonlinear dynamic behavior of the system. This result implies that the nonlinear dynamic properties of imitation dynamics could be profoundly affected by environmental noise.

Just as the evolutionary stability and stochastic evolutionary stability are respectively the most fundamental theoretical concepts in evolutionary game dynamics and in stochastic evolutionary game dynamics [6–8,21], the stochastic evolutionary stability should be also the core theoretical concept of stochastic imitation dynamics. Based on the analysis of stochastic local stability of stochastic imitation dynamics, the stochastic evolutionary stability can also be equivalently described as follows: if a strategy is called a SES strategy, then its fixation state must be SLS for any one possible mutant strategy. According to this definition, we found that (i) if a strategy $\hat{\mathbf{u}}$ satisfies $\mathbb{P}\{[\boldsymbol{\Pi}(t)\hat{\mathbf{u}}]_1 = [\boldsymbol{\Pi}(t)\hat{\mathbf{u}}]_2\} = 1$, then it is SES if $\hat{\mathbf{u}} \cdot \bar{\boldsymbol{\Pi}}\mathbf{u} > \mathbf{u} \cdot \bar{\boldsymbol{\Pi}}\mathbf{u}$ for all possible $\mathbf{u} \neq \hat{\mathbf{u}}$, and (ii) if a strategy $\hat{\mathbf{u}}$ satisfies $\mathbb{P}\{[\boldsymbol{\Pi}(t)\hat{\mathbf{u}}]_1 = [\boldsymbol{\Pi}(t)\hat{\mathbf{u}}]_2\} < 1$, then the pure strategy S_1 (or S_2) is SES if $\frac{\bar{b}-\bar{d}}{\bar{b}-\bar{d}+\bar{c}-\bar{a}} \geqslant 1$ (or $\frac{\bar{b}-\bar{d}}{\bar{b}-\bar{d}+\bar{c}-\bar{a}} \leqslant 0$), and a completely mixed strategy $\hat{\mathbf{u}} = (\hat{u}, 1-\hat{u})$ with $\hat{u} = \frac{\bar{b}-\bar{d}}{\bar{b}-\bar{d}+\bar{c}-\bar{a}} \in (0, 1)$ is SES if $\bar{b} > \bar{d}$ and $\bar{c} > \bar{a}$, or $\bar{d} > \bar{b}$ and $\bar{a} > \bar{c}$. Therefore, for the stochastic imitation dynamics, the stochastic evolutionary stability depends only on the mean payoff matrix $\bar{\boldsymbol{\Pi}}$ of the random payoff matrix $\boldsymbol{\Pi}(t)$. This result should have significant theoretical value for understanding the effect of environmental noise on the evolutionary stability of imitation dynamics.

Finally, in this study, based on some previous studies [3,14], we still use the Fermi function to measure the probability of pairs of individuals imitating each other. However, a natural question is whether the different mathematical forms of the function measuring the strategy switching probability will have a significant impact on the results. This is a question that is well worth considering in future studies.

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APPENDIX: PROOF OF THE STOCHASTIC LOCAL STABILITY OF EQUATION (5)

Following Karlin and Liberman [15,16] and Zheng *et al.* [6], we can see that Eq. (5) can be written in the form

$$\frac{u_{t+1}}{u_t} = \frac{2}{1+e^{-\beta[b(t)-d(t)]}}, \tag{A1}$$

from which

$$\frac{1}{n}(\ln u_n - \ln u_0) = \frac{1}{n}\sum_{t=0}^{n-1} \ln \frac{2}{1+e^{-\beta[b(t)-d(t)]}} \tag{A2}$$

for $n \geqslant 1$. Let

$$\mu = \left\langle \ln \frac{2}{1+e^{-\beta[b(t)-d(t)]}} \right\rangle \tag{A3}$$

and define

$$E = \left\{ \frac{1}{n}\sum_{t=0}^{n-1} \ln \frac{2}{1+e^{-\beta[b(t)-d(t)]}} \to \mu \right\}. \tag{A4}$$

The strong law of large numbers guarantees that $\mathbb{P}(E) = 1$. Under these conditions, Eq. (A2) implies that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{t=0}^{n-1} \ln \frac{2}{1+e^{-\beta[b(t)-d(t)]}} \leqslant 0 \tag{A5}$$

if this limit exists. This is not possible in the set E if $\mu > 0$. In this case, we conclude that

$$\mathbb{P}(u_t \to 0) \leqslant \mathbb{P}(E^C) = 0. \tag{A6}$$

This means that $\hat{u} = 0$ is stochastically unstable if $\mu > 0$.

Now consider the case where $\mu < 0$. By the strong law of large numbers and Egorov's theorem, for any $\epsilon > 0$, there exists an integer $N \geqslant 1$ such that the probability of the event

$$F = \left\{ \frac{1}{n}\sum_{t=0}^{n-1} \ln \frac{2}{1+e^{-\beta[b(t)-d(t)]}} < \frac{\mu}{2}, \quad \forall\, n \geqslant N \right\} \tag{A7}$$

satisfies

$$\mathbb{P}(F) \geqslant 1-\epsilon. \tag{A8}$$

Therefore, there exists $0 < \delta_0 < \delta$ such that $u_t < \delta$ for $t = 0, 1, \ldots, N-1$ as soon as $u_0 < \delta_0$. As a consequence, Eq. (A2) for $n = N$ yields

$$\frac{1}{N}(\ln u_N - \ln u_0) < \frac{\mu}{2} < 0 \tag{A9}$$

in the set F as soon as $u_0 < \delta_0$, which implies that

$$u_N < u_0 < \delta, \tag{A10}$$

and by recurrence that $u_n < \delta$ for all $n \geqslant N$.

It remains to show that $u_n \to 0$ in F if $u_0 < \delta_0$ as claimed in Karlin and Liberman [15,16], since then

$$\mathbb{P}(u_n \to 0) \geqslant \mathbb{P}(F) \geqslant 1-\epsilon. \tag{A11}$$

It suffices to note that Eq. (A2) for all $n \geqslant N$ under the above conditions gives

$$\frac{1}{n}(\ln u_n - \ln u_0) < \frac{\mu}{2} < 0, \tag{A12}$$

from which

$$\ln u_n < \ln u_0 + \frac{n\mu}{2} \to -\infty. \tag{A13}$$

This means that $u_n \to 0$, which completes the proof.

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