Universal distribution of the number of minima for random walks and Lévy flights

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We compute exactly the full distribution of the number m of local minima in a one-dimensional landscape generated by a random walk or a Lévy flight. We consider two different ensembles of landscapes, one with a fixed number of steps N and the other till the first-passage time of the random walk to the origin. We show that the distribution of m is drastically different in the two ensembles (Gaussian in the former case, while having a power-law tail $m^{-3/2}$ in the latter case). However, the most striking aspect of our results is that, in each case, the distribution is completely universal for all m (and not just for large m), i.e., independent of the jump distribution in the random walk. This means that the distributions are exactly identical for Lévy flights and random walks with finite jump variance. Our analytical results are in excellent agreement with our numerical simulations.

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I. INTRODUCTION

Estimating the number of stationary points of a random manifold is a problem of fundamental importance across fields such as physics, chemistry, mathematics, and computer science [1–8]. Counting of such stationary points appears in many contexts, such as in liquids where they represent the maxima, minima and the saddles of the random potential energy landscape [5]. In glassy systems, the number of such stationary points provides a measure of the complexity (entropy) of metastable states [9–14]. In string theory one is often interested in estimating the number of local extrema in the moduli space that represent different possible vacua [15,16]. In this context, random matrix theory is often used as an important tool to count such stationary points [17–24]. The local maxima of the phenotypic fitness landscape representing optimal phenotypes play an important role in evolutionary biology [25–28]. In optics, the estimation of the number of specular points on a random reflecting surface and also the electric field intensity of speckle laser patterns require the knowledge of the number of stationary points of a Gaussian random field [6,7]. Another recent application concerns data science where such stationary points play an important role in the nonconvex optimization of large-dimensional data [29]. Most studies typically focus on the mean number of stationary points of a random landscape, using for instance the Kac-Rice formula [30]. However, computing the full distribution of the number of stationary points, and also the number of maxima, minima and saddles, remains a formidably challenging problem, except for uncorrelated random fields for which the distribution is Gaussian by the central limit theorem [12].

In this paper, we consider a one-dimensional non-Gaussian landscape generated by the trajectory of a one-dimensional random walk of N steps [see Fig. 1(a)]. We consider a discrete-time random walker (RW) on a line, starting at the origin. Its position x_n at step n evolves via

$$x_n = x_{n-1} + \eta_n, \;\; n \geqslant 1, \tag{1}$$

where η_n's are independent and identically distributed (i.i.d.) random jumps, each drawn from a symmetric and continuous distribution $\phi(\eta)$. This random walk model includes Lévy flights where $\phi(\eta) \sim 1/|\eta|^{1+\mu}$ has a power-law tail with Lévy exponent $0 < \mu < 2$.

This simple random walk landscape model, where the positions x_n's of the walker play the role of the heights of the random landscape and the time n marks the spatial coordinate of the one-dimensional landscape plays an important role in many contexts.

This includes the celebrated Sinai's model of the transport of a single particle in a random walk landscape [31–38], with applications to understanding slow dynamics in glassy disordered systems as well as in biology [39–41]. The number of maxima of such a random walk landscape is precisely the number of barriers that the particle has to cross and the statistics of this number plays an important role in the slow dynamics of the particle. Similarly, the total number of local minima corresponds to the number of troughs where the particle can get trapped.

Another well known system is $1+1$-dimensional discrete solid-on-solid (SOS) models, defined on a lattice of size N, that are known to converge in their stationary state to precisely a random walk trajectory of N steps given by Eq. (1). In this case the effective noise distribution is directly related to the nearest neighbor interaction in the SOS model [42]. Computing the statistics of the number of local maxima and minima for such stationary interfaces are important to characterize the roughness of surface fluctuations, with interesting applications in massively parallel algorithms for discrete-event simulations [43]. In this context, the mean number of such stationary points for different discrete interface models has

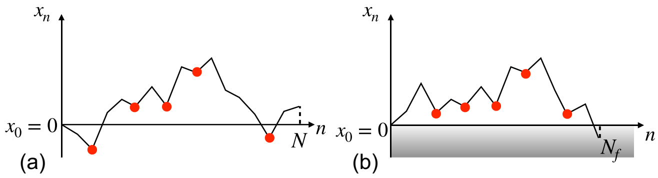

FIG. 1. (a) A typical trajectory of a random walk evolving via Eq. (1) up to N steps, starting at $x_0 = 0$. The solid red dots denote the local minima up to step N. (b) A typical trajectory of the same random walk as in Eq. (1) but up to step N_f where it crosses its initial value from above for the first time, with the red dots indicating the local minima in the trajectory.

been computed [43], but its full distribution still remains elusive.

Yet another application is the trajectory of a continuous-time run and tumble particle (RTP) in d-dimensions where a particle like *Escherichia coli* bacteria, starting from the origin, chooses a velocity $\mathbf{v}$ drawn from an arbitrary isotropic distribution $W(|\mathbf{v}|)$ and moves ballistically during a random time drawn from an exponential distribution (with rate γ) and then tumbles instantaneously [i.e., it chooses a new velocity from $W(|\mathbf{v}|)$]. The runs and tumblings alternate [44–49]. The x component of this d-dimensional continuous-time process can be mapped onto a discrete-time random walk of N steps, where the number of tumblings $N-1$ is a random variable, given the duration t [48,49]. It is natural to ask how many of these tumblings in time t result in a direction reversal of the particle. This is precisely the number of stationary points of the underlying random walk landscape.

In addition to the statistics of the number of maxima/minima of such a random walk landscape of fixed N steps, it is also interesting to study these questions for a random walk till its first-passage time to its starting point; see Fig. 1(b). This is a relevant question in finance where x_n may represent the price of a stock starting from its initial value x_0. The stock is deemed “active” till it crosses its initial value x_0 from above for the first time, and when this happens, the stock becomes “bad” and typical investors get rid of this stock from their portfolios. One can set $x_0 = 0$ without any loss of generality. The number of stationary points then represents the number of price reversals of this “active” stock. Such first-passage functionals, i.e., the statistical properties of observables till its first-passage time, have been well studied for the Brownian case with many applications ranging from queuing theory, directed polymers, all the way to astrophysics, e.g., in the study of the life-time of a comet in the solar system [37]. However, we are not aware of any study of such first-passage functionals for discrete-time random walks such as Lévy flights. Our exact results in this paper on the number of the maxima/minima till the first-passage time, valid for random walks with arbitrary jump distributions $\phi(\eta)$ including Lévy flights, thus provide such an example.

II. SUMMARY OF THE MAIN RESULTS

It is useful to summarize our main results. We compute the full distribution of the number of minima m both in the fixed N ensemble as well as up to the first-passage time N_f. Note that the starting and the end point of the walk are not part of the local minima/maxima. While the distributions in the two ensembles are different, we find the striking result that each of them is universal, i.e., independent of the jump distribution $\phi(\eta)$. For fixed N ensemble, this universality holds for any symmetric $\phi(\eta)$, not necessarily continuous. However, in the first-passage ensemble, the origin of universality is different and it requires $\phi(\eta)$ to be both symmetric and continuous. This includes standard random walk of finite variance jumps, as well as Lévy flights. Remarkably, this universality holds for all values of m and not just for large m. More precisely we find the following explicit results.

(1) For the fixed N ensemble [Fig. 1(a)], we show that the distribution of the number of minima $Q(m, N)$ vanishes for $m > N/2$, while it has a nonzero value for $0 \leqslant m \leqslant N/2$ given by

$$Q(m, N) = \frac{1}{2^N} \frac{(N+1)!}{(N-2m)!\,(2m+1)!}, \tag{2}$$

valid for arbitrary symmetric and continuous $\phi(\eta)$. The universality of this result can be traced back to the fact that, for the statistics of m, only the signs of the jumps matter and not the actual position of the walker. It is easy to see, using the symmetry of $\phi(\eta)$, that the distribution of the number of maxima M up to N steps has the same expression $Q(M, N)$ with $m \to M$ in Eq. (2). In the large N limit, $Q(m, N)$ converges to a Gaussian distribution centered at $N/4$ with a variance given by $N/16$, with non-Gaussian large deviation tails that we compute explicitly. Furthermore, we show that the joint distribution $\mathcal{Q}(m, M, N)$ of the number of minima m and maxima M up to step N is also universal, i.e., independent of $\phi(\eta)$, for m, M and N. This result also demonstrates nontrivial universal correlations between m and M. In particular the connected two-point correlation, for $N \geqslant 2$, is given by

$$C_c(N) = \langle mM \rangle - \langle m \rangle \langle M \rangle = \frac{N-3}{16}. \tag{3}$$

Interestingly they are anticorrelated for $N = 2$, uncorrelated for $N = 3$ and positively correlated for $N > 3$.

(2) For the first-passage ensemble, we show that the distribution $Q^{(\mathrm{fp})}(m)$ of the number of minima m till the first-passage time to the origin is also universal for all m and is given by

$$Q^{(\mathrm{fp})}(m) = \begin{cases} \frac{3}{4} & \text{for} \quad m = 0, \\ \frac{1}{2^{2m+2}} \, \frac{(2m)!}{m!(m+1)!} & \text{for} \quad m \geqslant 1. \end{cases} \tag{4}$$

It turns out that the mechanism responsible for the universality in the first-passage ensemble is completely different from that of the fixed N ensemble, because here the statistics of m actually depends on the position of the walk, since the position has to remain positive till the first crossing of the origin. We show that the universality in this case can be traced back, via a nontrivial mapping, to the Sparre Andersen theorem [50] for the survival probability of one-dimensional random walks starting at the origin [51–53]. Unlike in the fixed N ensemble, the distribution $Q^{(\mathrm{fp})}(m)$ has a power-law tail $\sim m^{-3/2}$ for large m, indicating that all moments of m, including its average $\langle m \rangle$, diverge.

(3) For both the ensembles we have computed the distribution of the total number K of stationary points (minima +

maxima). For the fixed N ensemble the corresponding distribution, denoted by $P(K, N)$, represents the probability of finding K stationary points till step N. For the first-passage ensemble, this distribution is denoted by $P^{(\mathrm{fp})}(x_0, K)$ and it represents the probability of having K stationary points till the RW makes a first passage to the origin starting from $x_0 > 0$. Similar to the distribution of the number of minima $Q(m, N)$, we find that $P(K, N)$ also has a universal form and is given by

$$P(K,N) = \frac{1}{2^{N-1}}\binom{N-1}{K}, \quad K = 0, 1, \cdots, N-1 \quad \text{and} \quad N \geqslant 2. \tag{5}$$

However, for the first-passage ensemble, we find that the distribution $P^{(\mathrm{fp})}(x_0, K)$ does not posses any universality for arbitrary x_0. However, for the special case $x_0 = 0$, we find that the distribution $P^{(\mathrm{fp})}(0, K)$ is universal across different choices of jump distributions $\phi(\eta)$ as long as they are symmetric and continuous. We find the following explicit expression for the distribution of stationary points till first passage:

$$\begin{aligned} P^{(\mathrm{fp})}(0, K=0) &= \frac{1}{2}, \\ P^{(\mathrm{fp})}(0, K=2m-1) &= \frac{2^{-2m}}{2(2m-1)}\binom{2m}{m}, \\ P^{(\mathrm{fp})}(0, K=2m) &= 0, \end{aligned} \tag{6}$$

with $m \geqslant 1$. Note that the number of stationary points K in this case is either 0 or an odd number. While the universality of $P(K, N)$ in Eq. (5) can be proved using simple combinatorics (as shown later), proving the universality for $P^{(\mathrm{fp})}(0, K)$ is nontrivial.

The paper is organized as follows. In the next two sections we provide derivations of the results for the two ensembles separately. We start with deriving the results for the fixed N ensemble in Sec. III. In the main part of this section we formulate the problem and demonstrate the universality of $Q(m, N)$ by deriving an explicit expression for it. The later part of this section is subdivided into Secs. III A–III C. In the first subsection we discuss correlation between the number of minima and maxima till step N of the walk. Next in Sec. III B we derive the distribution $P(K, N)$ of finding K stationary points till step N. We apply our study of stationary points on RW paths in fixed N ensemble to run-and-tumble motion in Sec. III C but for a fixed time t. Next we present our derivation of the results for first-passage ensemble in Sec. IV which is again subdivided into few subsections. In the beginning of this section we define the quantities of interest and formulate the problem. After that in Sec. IV A we present the proof of the universality of $Q^{(\mathrm{fp})}(0, m)$ for arbitrary choices of jump distribution $\phi(\eta)$. As an essential step, this proof requires a nontrivial mapping to the survival problem of an auxiliary random walk. In Sec. IV B we discuss the distribution of stationary points till first passage to the origin. Finally, in Sec. V we provide the conclusion of our paper. Some calculations are provided in detail in the Appendixes.

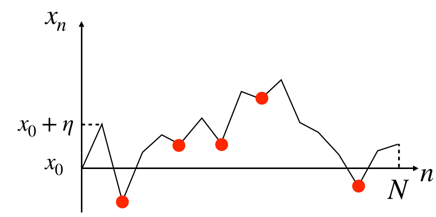

FIG. 2. A trajectory of a random walk of N steps starting at x_0. At the first step the walker jumps to $x_0 + \eta$ where η is a random jump drawn from $\phi(\eta)$ which is symmetric and continuous. The local minima are marked as filled red circles.

III. DERIVATION FOR THE FIXED N ENSEMBLE

We start with the fixed N ensemble of the random walk defined in Eq. (1). Let us first define the "spin" variables $s_i = \mathrm{sgn}(\eta_i) = \pm 1$, which are also independent. A stationary point (a maximum or a minimum) of the random walk landscape occurs at step i if $s_i s_{i+1} = -1$, irrespective of the starting point of the walk. Thus, the statistics of stationary points does not depend on the actual magnitude but rather only on the signs of the jump variables η_i's. Hence, for all symmetric jump distribution $\phi(\eta)$, one expects these statistics to be universal. For instance, the total number of stationary points K is just the number of "bonds" such that $s_i s_{i+1} = -1$. Given that there are $N-1$ bonds and each of them are equally likely to be ± 1, it follows that the distribution $P(K, N)$ is simply given by the binomial distribution $P(K, N) = \binom{N-1}{K}/2^{N-1}$ for $K = 0, 1, \cdots, N-1$ and $N \geqslant 2$. However, deriving the distribution of the number of minima m by such a simple combinatorial argument is less trivial.

We start with the discrete-time random walk $x_n = x_{n-1} + \eta_n$ [see Eq. (1)] evolving on a continuous line with the position x_n at step n updated via the Markov jump rule (see Fig. 2) where η_n's are independent and identically distributed (IID) random variables, each drawn from a continuous and symmetric distribution $\phi(\eta)$. The walker starts at x_0. To compute the distribution of the number of minima up to step N, it is convenient to introduce a pair of quantities $Q_\pm(x_0, m, N)$ denoting respectively the probability of having m minima in N steps, starting from x_0 with the first jump either in "+" or "−" direction. The idea is to write down an exact pair of recursion relations by observing what happens after the first jump. This is the analog of backward Fokker-Planck equations. The pair of recursion relations read

$$Q_+(x_0, m, N) = \int_0^\infty d\eta\, [Q_+(x_0+\eta, m, N-1) + Q_-(x_0+\eta, m, N-1)]\phi(\eta), \tag{7}$$

$$Q_-(x_0, m, N) = \int_{-\infty}^0 d\eta\, [Q_+(x_0+\eta, m-1, N-1) + Q_-(x_0+\eta, m, N-1)]\phi(\eta), \tag{8}$$

where η denotes the first random jump. If the walker starts with a positive (or negative) jump, then it arrives at the next step at $x_0 + \eta$ with $\eta \geqslant 0$ (respectively $\eta \leqslant 0$)—see Fig. 2. In the case when the first jump is positive, there is no new minimum generated by the second jump and hence in the recursion m remains the same in Eq. (7). In contrast, if the

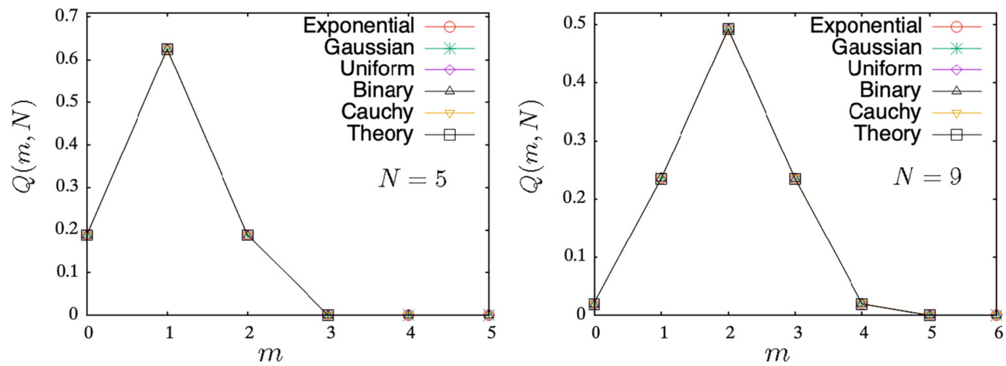

FIG. 3. The universal expression of $Q(m, N)$ in Eq. (2) is verified numerically for five choices of $\phi(\eta)$: (i) Exponential: $\phi(\eta) = \frac{a}{2}\exp(-a|\eta|)$, (ii) Gaussian: $\phi(\eta) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp(-\frac{\eta^2}{2\sigma^2})$, (iii) Uniform: $\phi(\eta) = \frac{1}{2\Delta}\Theta(z+\Delta)\Theta(\Delta - z)$, (iv) Binary: $\phi(\eta) = \frac{1}{2}[\delta(\eta-1) + \delta(\eta+1)]$, and (v) Cauchy: $\phi(\eta) = a/[\pi(\eta^2 + a^2)]$. Left figure is for $N = 5$ and the right one is for $N = 9$. The collapse of the data for the different jump distributions on a single curve (in both the plots) clearly demonstrate the universality of $Q(m, N)$.

first jump is negative and the second one is positive, then it creates a minimum at the end of the first step. This means that we need to have $m-1$ minima for the rest of the $N-1$ steps, starting at $x_0 + \eta$ with $\eta \leqslant 0$. This explains the first term in Eq. (8). Similarly, if both the first and the second jumps are negative, then there is no minimum at the end of the first step and this leads to the second term in Eq. (8). Fortunately, one can exploit the translational invariance with respect to the initial position x_0, i.e., the fact that $Q(x_0, m, N)$ is actually independent of x_0. This is because, no matter where the random walk starts, it is only the relative signs of the jumps that can create a minimum (a negative jump followed by a positive jump). Using the independence of $Q_\pm(x_0, m, N) \equiv Q_\pm(m, N)$ on x_0 and the symmetry of the jump distribution namely $\int_0^\infty \phi(\eta)d\eta = \int_{-\infty}^0 \phi(\eta)\, d\eta = 1/2$, the recursion relations (7) and (8) simplify to

$$Q_+(m, N) = \frac{Q_+(m, N-1) + Q_-(m, N-1)}{2}, \quad (9)$$

$$Q_-(m, N) = \frac{Q_+(m-1, N-1) + Q_-(m, N-1)}{2}, \quad (10)$$

valid for $N \geqslant 3$. Note that the above recursion relations can be written directly by observing what happens in the trajectory after the first jump (analogue of backward Fokker-Planck equations). If the first step is positive (which happens with probability 1/2) and the second step is either positive or negative, then no minimum is created. This explains Eq. (9). In contrast, if the first step is negative, then a minimum is created if the second step is positive and hence the number of minima in the rest of the trajectory must be $m-1$. This explains the first term of Eq. (10). However, if the second step is negative, then no new minimum is created, explaining the second term in Eq. (10). For $N = 2$, it is easy to show that $Q_+(m, 2) = \delta_{m,0}/2$ and $Q_-(m, 2) = (\delta_{m,0} + \delta_{m,1})/4$. The distribution $Q(m, N)$ of m is then given by $Q(m, N) = Q_+(m, N) + Q_-(m, N)$. These recursion relations (9) and (10) already demonstrate that the dependence on the noise distribution has dropped out, indicating that $Q_\pm(m, N)$, and consequently $Q(m, N)$, are universal for all m and N.

These recursion relations (9) and (10) can be solved using generating function techniques (see Appendix A 1 for details), which leads to the result in Eq. (2). Note that in deriving this result we only used the symmetry of $\phi(\eta)$ but it does not have to be continuous. Indeed, for the binary jump distribution $\phi(\eta) = (\delta_{\eta,1} + \delta_{\eta,-1})/2$, this result also holds. In Fig. 3, we verify this analytical result via numerical simulations for four additional different jump distributions. From the formula in Eq. (2) one can calculate all the moments of m. For example, the mean and the variance are given by

$$\langle m \rangle = \frac{N-1}{4}, \quad \mathrm{Var}(m) = \langle m^2 \rangle - \langle m \rangle^2 = \frac{N+1}{16}. \quad (11)$$

By expanding the factorials in Eq. (2) using Stirling formula, one can analyze the asymptotic scaling limit where both m and N are large but with their ratio $\alpha = m/N$ fixed. We find that $Q(m, N)$ takes a large deviation form

$$Q(m, N) \approx e^{-N\,\Phi(\alpha = m/N)}, \quad (12)$$

where the rate function

$$\Phi(\alpha) = \ln 2 + 2\alpha \ln(2\alpha) + (1-2\alpha)\ln(1-2\alpha), \quad (13)$$

has a unique minimum at $\alpha = 1/4$. Expanding $\Phi(\alpha)$ around $\alpha = 1/4$, one gets to leading order $\Phi(\alpha) \approx 8(\alpha - 1/4)^2$. Substituting this quadratic behavior in the large deviation form, we get a Gaussian distribution for the typical fluctuations of m, with mean $\langle m \rangle \approx N/4$ and variance $N/16$. Interestingly, this limiting Gaussian distribution was derived for the special case of Bernoulli random walk with $\phi(\eta) = (\delta_{\eta,1} + \delta_{\eta,-1})/2$ in the maths literature by a different method [54], but the issue of the universality of $Q(m, N)$ for all m and N was not noticed. The large deviation form of $Q(m, N)$ in Eqs. (12) and (13) is numerically verified in Fig. 4(a).

A. Correlation between number of minima and maxima

Since $\phi(\eta)$ is symmetric, we can reflect the trajectory $x_n \to -x_n$ such that the local minima become the local maxima. Hence, one finds that the distribution of the number of maxima M is again given by the same result in Eq. (2) with m replaced by M. What is however more interesting is to investigate if there are correlations between m and M. To characterize these correlations, we define $\mathcal{Q}_\pm(m, M, , N)$ as the joint distribution of m and M in N steps, starting with a positive or negative jump respectively. As in the case of

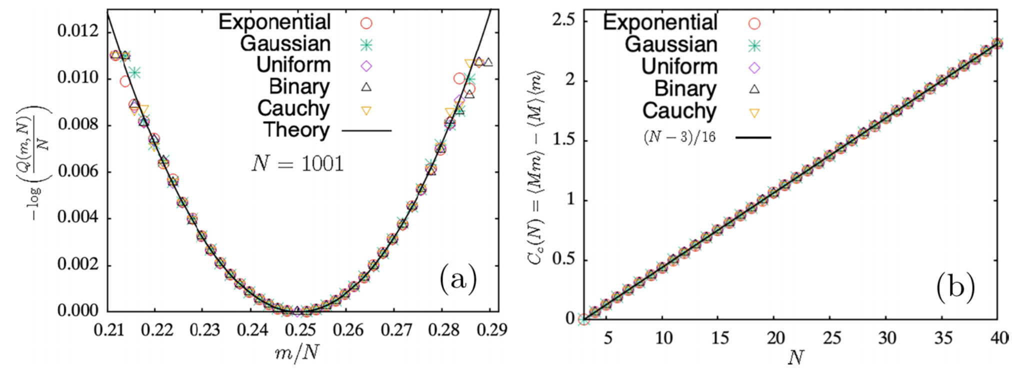

FIG. 4. (a) Numerical verification of the large deviation form of $Q(m, N)$ given in Eqs. (12) and (13) for the five choices of jump distributions mentioned in the caption of Fig. 3. (b) Connected correlation function $C_c(N) = \langle mM\rangle - \langle m\rangle\langle M\rangle$ between the number of minima m and the number of maxima M in a random walk landscape of N steps. The points represent numerical data for five different jump distributions and the solid line represents the analytical result in Eq. (17). AK: Fig. 4(a) replaces Fig. 2(a) of the prl version and is not clubbed with the correlation plot.

$Q(m, N)$, this joint distribution is also independent of the starting point x_0 for symmetric $\phi(\eta)$. Following the same steps as in Eqs. (9) and (10) by counting what happens after the first jump, we can write down a pair of exact recursion relations (see Appendix A 2 for details). The triple generating function of the joint distribution $\mathcal{Q}(m, M, , N) = \mathcal{Q}_+(m, M, , N) + \mathcal{Q}_-(m, M, , N)$ with respect to m, M, and N, defined by $S(u, v, z) = \sum_{m\geqslant 0, M\geqslant 0, N\geqslant 2} \mathcal{Q}(m, M, , N)u^m v^M z^N$, can be computed explicitly as [see Eq. (A29)]

$$S(u, v, z) = z^2 \frac{(2-z) + (u+v) + zuv}{(2-z)^2 - uvz^2}. \tag{14}$$

For details of the derivation see Appendix A 2. Clearly the result in Eq. (14) is again universal, i.e., independent of $\phi(\eta)$. From this explicit expression of the generating function $S(u, v, z)$, one can compute different moments and correlations of m and M. For example, let $C(N) = \langle mM\rangle$ denote the correlation between M and m then $\bar{C}(z) = \sum_{N=2}^{\infty} C(N)z^N$ can be obtained from $S(u, v, z)$ as

$$\begin{aligned}\bar{C}(z) &= \left(\frac{d^2}{du dv} S(u, v, z)\right)_{u=1, v=1} = \frac{z^3(2-z)}{8(1-z)^3}\\ &= \sum_{N=2}^{\infty} \frac{(N+1)(N-2)}{16} z^N.\end{aligned} \tag{15}$$

This gives

$$C(N) = \frac{(N+1)(N-2)}{16}, \quad \text{for} \quad N \geqslant 2. \tag{16}$$

So the connected correlation $C_c(N) = \langle mM\rangle - \langle m\rangle\langle M\rangle$ is given by

$$C_c(N) = \frac{N-3}{16}, \quad \text{for} \quad N \geqslant 3, \tag{17}$$

as announced in Eq. (3). To get this result we have used $\langle m\rangle = \langle M\rangle = (N-1)/4$. We compare the analytical prediction in Eq. (17) with numerical simulations in Fig. 4(b) for five different jump distributions and find excellent agreement.

B. Distribution $P(K, N)$ of the total number of stationary points till step N

Let $K = m + M$ denote the total number of stationary points (maxima and minima) for a random walk landscape of N steps. For the distribution $P(K, N)$ of K, we have earlier provided a very simple combinatorial proof of the exact formula $P(K, N) = \binom{N-1}{K}/2^{N-1}$ for $K = 0, 1, \cdots, N-1$ and $N \geqslant 2$ (see the first paragraph of Sec. III). Here, we provide an alternative derivation of this result using the backward recursion relations as in the derivation of $Q(m, N)$, i.e., the distribution of the number of minima up to N steps. Once again, the distribution $P(K, N)$ is independent of the starting position x_0. As usual, it is convenient to define the pair of probabilities $P_\pm(K, N)$ denoting the distributions starting with a positive or negative step. Investigating what happens in the first step, it is easy to see that they satisfy the recursion relations

$$P_+(K, N) = \frac{P_+(K, N-1) + P_-(K-1, N-1)}{2}, \tag{18}$$

$$P_-(K, N) = \frac{P_-(K, N-1) + P_+(K-1, N-1)}{2}, \tag{19}$$

valid for $N \geqslant 3$ and $K \geqslant 1$. Since $P(K, N) = P_+(K, N) + P_-(K, N)$, it follows by adding these two equations that the recursion relation for the sum is closed and reads

$$\begin{aligned} P(K, N) &= \tfrac{1}{2}P(K, N-1) + \tfrac{1}{2}P(K-1, N-1),\\ &N \geqslant 3 \quad \text{and} \quad K \geqslant 0, \end{aligned} \tag{20}$$

with the convention that $P(-1, N) = 0$ for all $N \geqslant 2$. For $N = 2$, direct inspection gives

$$P(K, 2) = \tfrac{1}{2}\delta_{K,0} + \tfrac{1}{2}\delta_{K,1}. \tag{21}$$

The recursion relation (20) is solved in Appendix A 3 using generating function techniques which leads to the result in Eq. (5). This analytical result is compared with numerical simulations for five different jump distributions in Fig. 5 where we observe a perfect agreement. From the formula in Eq. (5) it is easy to calculate all the moments of K. For example, the

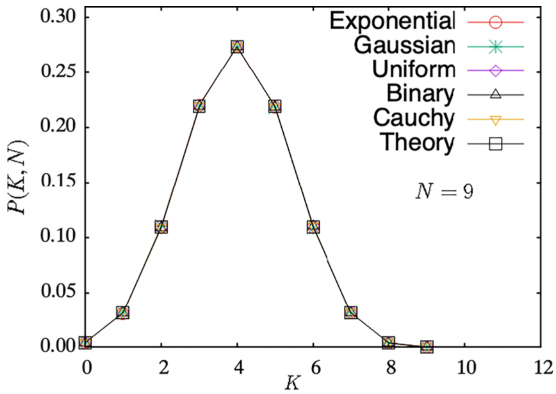

FIG. 5. Plot of the numerically obtained $P(K, N)$ for five different jump distributions (given in the caption of Fig. 3) and for $N = 9$. The collapse of the data for the different jump distributions on a single curve clearly demonstrates the universality of $P(K, N)$. Numerical results are compared with theoretical expression in Eq. (5) (square symbols) and we observe a perfect agreement.

mean and the variance are given by

$$\langle K\rangle = \frac{N-1}{2}, \quad \mathrm{Var}(K) = \langle K^2\rangle - \langle K\rangle^2 = \frac{N-1}{4}. \quad (22)$$

Since $K = m + M$, we have

$$\mathrm{Var}(K) = \mathrm{Var}(m) + \mathrm{Var}(M) + 2C_c(N), \quad (23)$$

where $C_c(N) = \langle mM\rangle - \langle m\rangle\langle M\rangle$ is the connected correlation function between the number of minima m and the number of maxima M. Since M has the same statistics as m by symmetry, it follows from Eq. (23) that

$$C_c(N) = \frac{\mathrm{Var}(K) - 2\mathrm{Var}(m)}{2} = \frac{N-3}{16}, \quad (24)$$

where we have used the results for $\mathrm{Var}(K)$ in Eq. (22) and for $\mathrm{Var}(m)$ in Eq. (11). This result matches perfectly with the result obtained from the joint distribution $\mathcal{Q}(m, M, N)$ in Eq. (17) and verified numerically in Fig. 4(b).

C. Application to the run-and-tumble particle

In this subsection, we apply our results obtained in the previous subsections for the distribution of the number of stationary points as well as that of the number of minima for a random walk of N steps, to the problem of a run-and-tumble particle (RTP) up to a total duration t. Let us recall the definition of the RTP model in d dimensions. A particle, such as an *E. coli* bacteria, starting at the origin, chooses a random velocity $\mathbf{v}$ drawn from an arbitrary isotropic distribution $W(|\mathbf{v}|)$ and moves ballistically with this velocity during a random time τ distributed as $p(\tau) = \gamma\, e^{-\gamma\tau}$. At the end of this run, the particle tumbles, i.e., it chooses a new velocity, again from $W(|\mathbf{v}|)$ and a new run time τ, from $p(\tau)$. The runs and tumblings alternate. We consider this process up to a final fixed time t. Now let us consider the x component of this d-dimensional process denoted by $x(t)$ and this projected trajectory of duration t constitutes a one-dimensional landscape, starting at the origin, and consisting of peaks and troughs (see Fig. 6). These peaks and troughs are the stationary points of this one-dimensional landscape and their number indicates

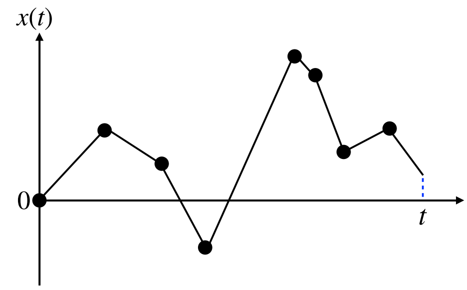

FIG. 6. A typical trajectory $x(t)$ denoting the x component of an RTP of duration t. The straight lines show the x projections of the successive runs and the filled circles denote the tumbles following each run. The last run before t is incomplete.

the number of direction reversals that the RTP undergoes in time t. We are interested in computing the distribution of the stationary points $P(K, t)$ and also the distribution $Q(m, t)$ of the number of minima (troughs) till time t.

For smooth presentation of the paper, we provide detailed calculations for these two quantities in Appendix A 4, where we obtain explicit analytical expressions for $P(K, t)$ and $Q(m, t)$. We find that the distribution $P(K, t)$ of the number of stationary points till time t is a Poisson distribution with mean $\gamma\, t/2$ and is given by

$$P(K, t) = \sum_{N=1}^{\infty} \frac{1}{2^{N-1}} \binom{N-1}{K} e^{-\gamma\, t} \frac{(\gamma\, t)^{N-1}}{(N-1)!}$$
$$= e^{-\frac{\gamma t}{2}} \frac{\left(\frac{\gamma t}{2}\right)^K}{K!}, \text{ for all } \quad K = 0, 1, \cdots. \quad (25)$$

In Fig. 7 we give a plot of $P(K, t)$ versus K for a fixed $\gamma t = 10$ and compare this result with numerical simulations.

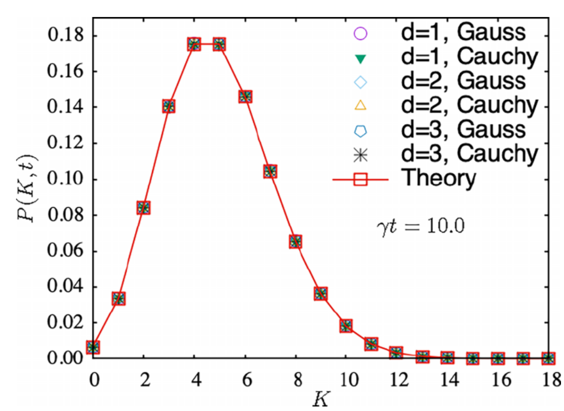

FIG. 7. Plot of the numerically obtained $P(K, t)$ vs K for an RTP in different dimensions $d = 1, 2$ and 3 and for different velocity distributions $W(|\mathbf{v}|)$ (Gaussian and Cauchy distributions) with $\gamma t = 10$, compared to the theoretical expression, which is a Poissonian distribution with mean $\gamma t/2$ given in Eq. (25), plotted with square symbols, showing a perfect agreement. The collapse of the data for different dimensions d and different velocity distributions $W(|\mathbf{v}|)$ on a single curve clearly demonstrates the universality of $P(K, t)$ for an RTP.

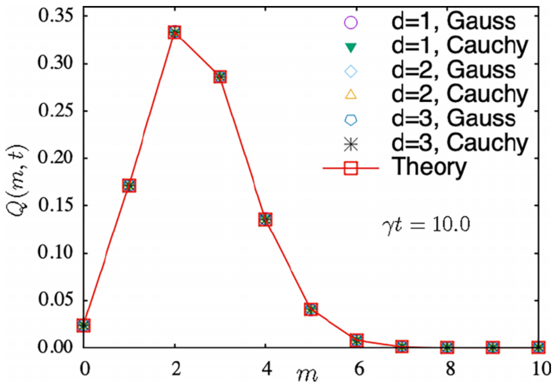

FIG. 8. Plot of the numerically obtained $Q(m, t)$ vs m for an RTP in different dimensions $d = 1, 2$ and 3 and for different velocity distributions $W(|\mathbf{v}|)$ (Gaussian and Cauchy distributions) with $\gamma t = 10$, compared to the theoretical expression in Eq. (26), plotted with square symbols, showing a perfect agreement. The collapse of the data for different dimensions d and different velocity distributions $W(|\mathbf{v}|)$ on a single curve clearly demonstrates the universality of $Q(m, t)$ for an RTP.

For the distribution $Q(m, t)$ of the number of minima till time t, we get the following expression:

$$Q(m,t) = e^{-\gamma t/2}\frac{(\gamma\, t/2)^{2m-1}}{2(2m+1)!} \times \left[(2m+1)(2m+\gamma t) + \left(\frac{\gamma\, t}{2}\right)^2\right], \quad \text{for all} \quad m = 0, 1, \cdots . \tag{26}$$

This is clearly a highly non-Poissonian distribution, unlike the distribution of the number of stationary points in Eq. (25). For example, the mean and the variance are not equal unlike in the Poisson distribution, but rather are given by

$$\langle m\rangle = \sum_{m=0}^{\infty} m\, Q(m,t) = \frac{\gamma t}{4},$$
$$\mathrm{Var}(m) = \langle m^2\rangle - \langle m\rangle^2 = \frac{1}{8}(1 - e^{-\gamma t} + \gamma t). \tag{27}$$

In Fig. (8) we give a plot of $Q(m, t)$ versus m for a fixed $\gamma t = 10$ and compare this result with numerical simulations.

Let us remark that the results for $P(K, t)$ in Eq. (25) and for $Q(m, t)$ in Eq. (26) are universal, i.e., independent of the spatial dimension d as well as the velocity distribution $W(|\mathbf{v}|)$ (see Figs. 7 and 8). The spatial dimension d and the velocity distribution $W(|\mathbf{v}|)$ do affect the jump distribution of the effective random walk consisting of the RTP runs as steps [48,49]. However, since $P(K, N)$ and $Q(m, N)$ are independent of the jump distributions, and since $P(N|t)$ is Poissonian it is clear these jump distributions do not enter in the formulas for $P(K, t)$ and $Q(m, t)$. Furthermore, let us also emphasize that these two quantities are universal for all time t, and not just for large t.

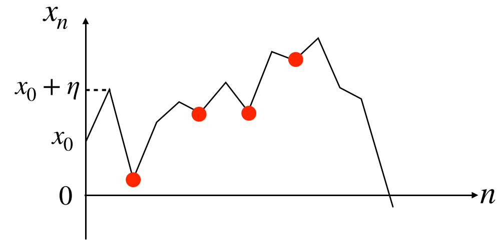

FIG. 9. A trajectory of a random walk, starting at $x_0 \geqslant 0$ till the first time it crosses the origin from above. At the first step the walker jumps to $x_0 + \eta$ where η is a random jump drawn from $\phi(\eta)$ which is symmetric and continuous. The local minima are marked as filled red circles.

IV. DERIVATION FOR THE FIRST-PASSAGE ENSEMBLE

In this section we study our random walk in Eq. (1), starting at initial position x_0 and the process stops when the walker crosses the origin for the first time. Without loss of generality, we take $x_0 \geqslant 0$ (see Fig. 9).We investigate the the distribution of the number of minima $Q^{(\mathrm{fp})}(m)$ on these first-passage trajectories. Unlike in the fixed N ensemble, here the translation invariance is lost since the process stops when it hits the origin for the first time. Here it is convenient to define the pair of probabilities

$$Q_{\pm}^{(\mathrm{fp})}(x_0, m) = \mathrm{Prob.}\begin{bmatrix}\text{that the RW has } m \text{ minima till} \\ \text{its first passage to the origin,} \\ \text{starting from } x_0 > 0 \text{ with first} \\ \text{jump in the } \pm \text{ direction}\end{bmatrix}. \tag{28}$$

One can write down the backward recursion relations satisfied by $Q_{\pm}^{(\mathrm{fp})}(x_0, m)$, again by observing what happens in the first step. However, unlike in the fixed N ensemble, where these equations were independent of the noise distribution [see Eqs. (9) and (10)], for the first-passage ensemble, the recursion relations are integral equations that explicitly involve $\phi(\eta)$. It is easy to show that the probabilities $Q_{\pm}^{(\mathrm{fp})}(x_0, m)$ satisfy the following coupled recursion relations

$$Q_{+}^{(\mathrm{fp})}(x_0, m) = \int_0^{\infty} d\eta\, \phi(\eta)\, [Q_{+}^{(\mathrm{fp})}(x_0 + \eta, m) + Q_{-}^{(\mathrm{fp})}(x_0 + \eta, m)], \tag{29}$$

$$Q_{-}^{(\mathrm{fp})}(x_0, m) = \int_{-x_0}^{0} d\eta\, \phi(\eta)\, [Q_{+}^{(\mathrm{fp})}(x_0 + \eta, m - 1) + Q_{-}^{(\mathrm{fp})}(x_0 + \eta, m)] + \delta_{m,0}\int_{-\infty}^{-x_0} \phi(\eta)\, d\eta. \tag{30}$$

If the first jump is positive and the particle arrives as $x_0 + \eta$ with $\eta > 0$, and if the second jump is either positive or negative, then no new minimum is created and integrating over all positive η gives Eq. (29). In contrast, if the first jump is negative, then there are two possibilities: (i) either $x_0 + \eta > 0$ and in this case, if the second jump is positive, then a minimum is created while no minimum occurs if the second jump is negative. This explains the first two terms in Eq. (30), and (ii) if $x_0 + \eta < 0$, then the position becomes negative after this

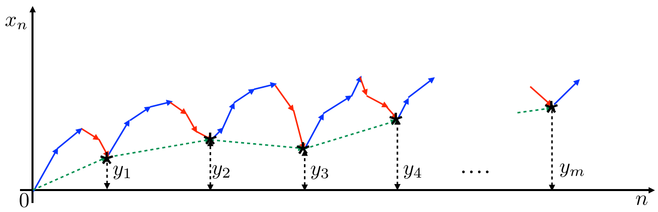

FIG. 10. Schematic trajectory of a random walk (discrete time and continuous space) that starts at the origin $x_0 = 0$, stays non-negative and has *at least* m local minima. The $*$'s denote the local minima of this configuration (not counting the starting position 0) and $\{y_1 \geqslant 0,\ y_2 \geqslant 0,\ y_3 \geqslant 0 \ldots,\ y_m \geqslant 0\}$ denote the heights of the successive local minimum. The configuration till the m-th minimum can then be broken into m blocks or segments separated by the dashed vertical lines. Each block contains only one peak (maxima). One can construct an effective auxiliary random walk that jumps from one minimum to the next minimum (of the original RW) with positions $\{y_1 \geqslant 0,\ y_2 \geqslant 0,\ \ldots\ y_m \geqslant 0\}$ (as shown by the green dashed lines). Thus, the step number of the auxiliary walk is identified with the label of the block. The number of steps of the auxiliary walk is exactly equal to the number of blocks in the configuration, i.e., the number of local minima. The y_k's constitute an auxiliary random walk, $y_k = y_{k-1} + \xi_k$ with an effective jump distribution $\Psi(\xi)$ that is continuous and symmetric.

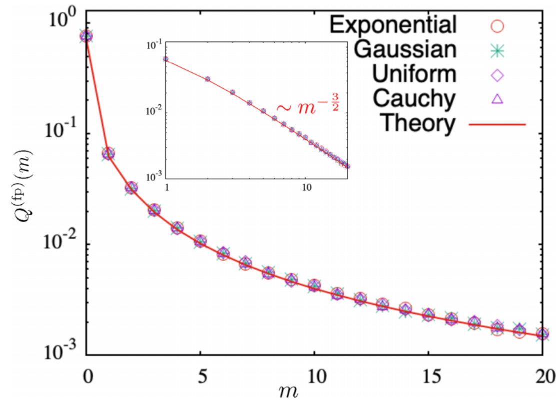

FIG. 11. Numerical results for the distribution $Q^{(\mathrm{fp})}(m)$ plotted as function for m for four different symmetric and continuous jump distributions. Also plotted the analytical result of $Q^{(\mathrm{fp})}(m)$ in Eq. (4) by a solid line. The fact that they coincide for all m indicates the strong universality of the result. The inset shows numerical verification of the $m^{-3/2}$ decay of $Q^{(\mathrm{fp})}(m)$ for large m.

first jump and hence the process ends. In this latter case, the number of minima is clearly zero and this explains the last term in Eq. (30). These equations are valid for $m \geqslant 0$ with the interpretation $Q_+(x_0, -1) = 0$.

To solve these equations we define the following generating functions

$$Z_{\pm}^{(\mathrm{fp})}(x_0, u) = \sum_{m=0}^{\infty} Q_{\pm}^{(\mathrm{fp})}(x_0, m)\, u^m, \tag{31}$$

which, from Eqs. (29) and (30), can be shown to satisfy the following equations:

$$Z_{+}^{(\mathrm{fp})}(x_0, u) = \int_{x_0}^{\infty} dy\, \phi(y - x_0)\, [Z_{+}^{(\mathrm{fp})}(y, u) + Z_{-}^{(\mathrm{fp})}(y, u)], \tag{32}$$

$$Z_{-}^{(\mathrm{fp})}(x_0, u) = \int_{0}^{x_0} dy\, \phi(y - x_0)\, [u Z_{+}^{(\mathrm{fp})}(y, u) + Z_{-}^{(\mathrm{fp})}(y, u)], + \int_{-\infty}^{-x_0} \phi(\eta)\, d\eta. \tag{33}$$

These equations are valid for general jump distributions $\phi(\eta)$. They are of the Wiener-Hopf types and are hard to solve for generic $\phi(\eta)$. However, for the special case $\phi(\eta) = (1/2)\, e^{-|\eta|}$, they are exactly solvable as shown in Appendix B 1 a. In this special case, setting $x_0 = 0$, we find the result for $Q^{(\mathrm{fp})}(m) = Q_{+}^{(\mathrm{fp})}(x_0 = 0, m) + Q_{-}^{(\mathrm{fp})}(x_0 = 0, m)$ given in Eq. (4) which is verified in Fig. 11. Then we performed numerical simulations for other continuous and symmetric jump distributions $\phi(\eta)$ (not necessarily double-exponential) and, amazingly, the simulation points fell exactly on top of the results (4) for the double-exponential jump distribution (see Fig. 11). This indicated that the result in Eq. (4) is also universal for all m. Such a strong universality (for all m) came as an unexpected surprise and the mechanism behind it is far from obvious. This universal result is hard to prove from the integral equations in Eqs. (32) and (33). However, we found an alternative method via an exact mapping to an auxiliary random walk that allows us to prove this universal result. We present this proof below for arbitrary symmetric and continuous jump distribution $\phi(\eta)$.

A. Proof of the universality of $Q^{(\mathrm{fp})}(m)$ for arbitrary $\phi(\eta)$

It turns out to be easier to consider the cumulative probability of the number of minima, i.e., the probability that the walk, till its first passage, has *at least* m local minima. To understand this universality, below we first map the minima counting problem to an auxiliary discrete-time random walk problem with an effective jump distribution. Under this mapping, the distribution $Q^{(\mathrm{fp})}(m)$ in the original problem is related exactly to the survival probability of this auxiliary walk up to step m. Then, using the universality of the latter quantity via the celebrated Sparre Andersen theorem [50], we prove this amazing universality. Thus, the mechanism behind this universality in the first-passage ensemble is much more subtle than the universality encountered before in the fixed time ensemble.

To proceed, let us consider a trajectory of the random walk that has not yet crossed the origin, i.e., still surviving (see Fig. 10) and has *at least* m local minima. We first locate the local minima in this configuration and denote them by $*$'s in Fig. 10. Let $\{y_1 \geqslant 0,\ y_2 \geqslant 0,\ y_3 \geqslant 0, \ldots,\ y_m \geqslant 0\}$ denote the heights (position) of the successive local minimum. In this configuration $y_i \geqslant 0$ for all $i = 1, 2, \ldots, m$, since the walk is surviving. The first crucial point to realize is that between any two successive minima, there can be only one global maximum with monotonically increasing jumps on its left (shown by blue arrows) and monotonically decreasing jumps on its right (red arrows). This is because if there is more than one peak between the two minima, that would automatically mean that there is an additional minimum between the two, which is ruled out by construction since we are considering successive minima. Thus, the configuration till the mth minimum can be broken into m blocks or segments separated by the dashed vertical lines showing the times of occurrences of the local minima. By construction, each block contains only one peak (in the interior of the block and not at its edges). If we can integrate out all one-peak configurations between two successive local mimima, then we can then construct a new "auxiliary" or effective random walk that jumps from one minimum to the next in "one" step (shown by the green dashed lines in Fig. 10) with positions $\{y_1, y_2, y_3 \ldots\}$. The height y_k of the kth local minimum can be expressed as

$$y_k = y_{k-1} + \xi_k, \tag{34}$$

where ξ_k represents the difference in heights between the $(k-1)$th and the kth minima. Thus, y_k represents the position of the auxiliary random walk at step k, starting from $y_0 = 0$.

The next step is to compute the distribution $\Psi(\xi)$ of the jump variable ξ_k, which can be computed explicitly in terms of the original jump distribution $\phi(\eta)$. Note that the jump distribution $\Psi(\xi)$ is essentially the transition probability density $G(y_1, y_2)$ of the original walk that, starting at y_1 it will arrive at y_2 (in arbitrary number of original steps) with only one peak (or turnaround) in between. For arbitrary $\phi(\eta)$ (symmetric and continuous), the jump distribution $\Psi(\xi) = G(y_1, y_2)$ is computed in Appendix B 2. where it is shown to be continuous and symmetric [see Eq. (B28)], and normalized to unity [see Eq. (B31)]. Hence, the transition probability $\Psi(\xi)$ can be viewed as an "effective" jump probability associated with the auxiliary random walk process (see Fig. 10) $y_k = y_{k-1} + \xi_k$ starting at $y_0 = 0$, where ξ_k's are i.i.d. jump variables each drawn from the symmetric and continuous jump density $\Psi(\xi)$ defined in Eq. (B28).

Having computed the jump probability $\Psi(\xi)$ of the auxiliary random walk y_k in Eq. (34), we now come back to the original question in Fig. 10, namely, what is the probability that the surviving walk has at least m local minima? The derivation in Appendix B 2 does not provide an explicit form for the jump density $\Psi(\xi)$. However, as we will see below that we do not need the explicit form of $\Psi(\xi)$ for the quantity of our interest. Using the mapping to the auxiliary process, this is equivalent to saying that the auxiliary process y_k, starting at $y_0 = 0$, stays positive up to step m (because to have at least m minima we must have m blocks, i.e., m steps for the auxiliary walk). But it is not just enough to have m blocks, we also have to ensure that the last position y_m at the end of the m-th block must be a local minimum. In other words, the step of the original walk immediately following y_m must be upward (only then y_m will be a local minimum). This last event occurs simply with probability $1/2$. Hence, the probability that the number of minima $N_{\min}$ in the original walk till its first-passage time exceeds m is simply

$$\text{Prob.}(N_{\min} \geqslant m) = \sum_{k=m}^{\infty} Q^{(\text{fp})}(0, k) = \frac{1}{2}\, q_m, \tag{35}$$

where q_m is the probability that the auxiliary walk stays nonnegative up to step m and the factor $1/2$ comes from the fact that the m-th position of the auxiliary walk must be a local minimum of the original walk. However, the celebrated Sparre Andersen theorem [50] tells us that the survival probability q_m of the auxiliary walk in Eq. (34) up to step m, starting at the origin, is universal, i.e., independent of the jump distribution $\Psi(\xi)$ as long as it is symmetric and continuous. Indeed, we have proved above that $\Psi(\xi)$ is symmetric, continuous and normalized to unity. Hence, we can apply the Sparre Andersen theorem to this auxiliary walk. The Sparre Andersen result says that the survival probability q_m up to step m for a random walk starting at the origin is given by [50]

$$q_m = \binom{2m}{m} 2^{-2m} \qquad m = 0, 1, 2, \ldots. \tag{36}$$

Hence, using Eq. (35), the probability of having exactly m minima up to the first-passage time, for $m \geqslant 1$, is given by

$$\begin{aligned} Q^{(\text{fp})}(m) \equiv Q^{(\text{fp})}(0, m) &= \frac{1}{2}\,[q_m - q_{m+1}] \\ &= \frac{1}{2^{2m+2}}\,\frac{(2m)!}{m!\,(m+1)!}, \end{aligned} \tag{37}$$

where we used the expression of q_m in Eq. (36). Note that one has to be a bit careful for the special case $m = 0$, i.e., configurations where there is no minimum. Indeed, extending this result to $m = 0$, it predicts that $Q^{(\text{fp})}(0, 0) = 1/4$. However, this does not include the case where the walker jumps to the negative side at the first step, which happens with probability $1/2$. Hence, adding this contribution, we get

$$Q^{(\text{fp})}(0, m = 0) = \frac{3}{4}. \tag{38}$$

Using these results in Eqs. (37) and (38), one can check that $Q^{(\text{fp})}(0, m)$ is normalized to unity, i.e.,

$$\sum_{m=0}^{\infty} Q^{(\text{fp})}(0, m) = 1. \tag{39}$$

This then completes the proof of the universal result announced in Eq. (4). The analytical expression in Eq. (4) is numerically verified for four choices of jump distributions in Fig. 11 demonstrating the universality of the result across different continuous and symmetric jump distributions $\phi(\eta)$. For large m, the distribution $Q^{(\text{fp})}(m)$ in Eq. (4) has a power-law tail

$$Q^{(\text{fp})}(m) \overset{m\to\infty}{\approx} \frac{1}{4\sqrt{2\pi}}\,\frac{1}{m^{3/2}}, \tag{40}$$

which is also numerically verified in the inset of Fig. 11. This power-law tail can be understood from the following scaling argument. In an N-step random walk, the number of minima m typically scales as $m \sim N$ for large N. However, the number of steps till the first-passage time has a power-law distribution $N^{-3/2}$ for large N, which follows from the Sparre Andersen theorem [50]. This shows that distribution of m will have the power-law decay with the same exponent $3/2$.

B. Distribution of the number of stationary points till the first-passage time to the origin

In this subsection, we first set up the integral equations for the distribution $P_{\pm}^{(\text{fp})}(x_0, K)$ denoting the distribution of the number of stationary points (minima and maxima) till the first-passage time to the origin, starting from $x_0 > 0$ with either a positive or a negative jump. Examining the different possibilities after the first jump, we can write down the exact recursion relations:

$$P_+^{(\mathrm{fp})}(x_0, K) = \int_0^\infty d\eta\, \phi(\eta)[P_+^{(\mathrm{fp})}(x_0+\eta, K) + P_-^{(\mathrm{fp})}(x_0+\eta, K-1)], \tag{41}$$

$$P_-^{(\mathrm{fp})}(x_0, K) = \int_{-x_0}^0 d\eta\, \phi(\eta)[P_+^{(\mathrm{fp})}(x_0+\eta, K-1) + P_-^{(\mathrm{fp})}(x_0+\eta, K)] + \delta_{K,0}\int_{-\infty}^{-x_0} \phi(\eta)\, d\eta. \tag{42}$$

If the first jump is positive and the particle arrives as $x_0+\eta$ with $\eta > 0$, and if the second jump is also positive, then no new stationary point is created. This explains the first term in Eq. (41). In contrast, if the second jump is negative, then a maximum is created and hence the rest of the trajectory, starting at $x_0+\eta$, must have $K-1$ stationary points, explaining the second term in Eq. (41). Similarly, if the first step is negative, such that $x_0+\eta > 0$, then, depending on the second step, we will either have either $K-1$ or K stationary points, starting at $x_0+\eta$. This explains the non-δ-function term in Eq. (42). However, if the first jump puts the walker on the negative side, then the process stops and we just have zero stationary point, explaining the Kronecker δ-function term in Eq. (42).

We define the generating functions

$$\tilde{Z}_\pm^{(\mathrm{fp})}(x_0, u) = \sum_{K=0}^\infty P_\pm^{(\mathrm{fp})}(x_0, K)\, u^K, \tag{43}$$

which, from Eqs. (41) and (42), satisfy the following equations:

$$\tilde{Z}_+^{(\mathrm{fp})}(x_0, u) = \int_{x_0}^\infty dy\, \phi(y-x_0)[\tilde{Z}_+^{(\mathrm{fp})}(y, u) + u\,\tilde{Z}_-^{(\mathrm{fp})}(y, u)], \tag{44}$$

$$\tilde{Z}_-^{(\mathrm{fp})}(x_0, u) = \int_0^{x_0} dy\, \phi(y-x_0)[u\,\tilde{Z}_+^{(\mathrm{fp})}(y, u) + \tilde{Z}_-^{(\mathrm{fp})}(y, u)] + \int_{-\infty}^{-x_0} \phi(\eta)\, d\eta. \tag{45}$$

Once again we find that, for generic $\phi(\eta)$, it is hard to solve the above integral equations. However, for the special choice $\phi(\eta) = (1/2)\, e^{-|\eta|}$, they are exactly solvable as shown in Appendix B 3. For $x_0 = 0$, we find an explicit expression for $P^{(\mathrm{fp})}(0, K) = P_+^{(\mathrm{fp})}(x_0 = 0, K) + P_-^{(\mathrm{fp})}(x_0 = 0, K)$ which is given by Eq. (6). This result is verified numerically in Fig. 12. In the same figure we also plot the distribution $P^{(\mathrm{fp})}(0, K)$ for other choices of jump distribution $\phi(\eta)$ different from the exponential one. Interestingly, we observe that the simulation data points for other choices of $\phi(\eta)$ also match with this theoretical curve implying universality of the distribution $P^{(\mathrm{fp})}(0, K)$ in Eq. (6). Below we prove this universality for arbitrary $\phi(\eta)$ which is continuous and symmetric.

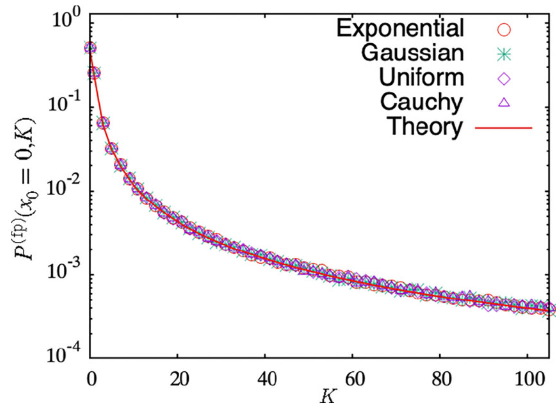

FIG. 12. Plot of the numerically obtained $P^{(\mathrm{fp})}(x_0 = 0, K)$ for four different jump distributions, compared to the theoretical expression in Eq. (6), plotted with a solid line, showing a perfect agreement. The collapse of the data for the different jump distributions on a single curve clearly demonstrates the universality of $P^{(\mathrm{fp})}(0, K)$.

C. Universality of the distribution of the number of stationary points $P^{(\mathrm{fp})}(0, K)$

The argument used above for the number of minima in Sec. IV A can be extended also to prove the universality of the number of stationary points K till the first-passage time. The argument proceeds as follows. It is convenient again to examine a typical trajectory as in Fig. 10. This figure provides a configuration where there are at least m number of minima. Next we observe that every local minimum in this configuration is preceded by a local maximum. Thus, this configuration has exactly m maxima. In other words, the number of stationary points N_{stat} in this configuration is at least $2m$. It then follows from Eq. (35) that

$$\mathrm{Prob.}(N_{\mathrm{stat}} \geqslant 2m) = \tfrac{1}{2} q_m, \tag{46}$$

where q_m is given in Eq. (36). Setting $m = \ell - 1$ with $\ell \geqslant 1$, we get

$$\mathrm{Prob.}(N_{\mathrm{stat}} \geqslant 2\ell - 2) = \tfrac{1}{2} q_{\ell-1}. \tag{47}$$

We now recall that the number of stationary points till the first-passage time is always an odd number. Hence, Eq. (47) shows that

$$\mathrm{Prob.}(N_{\mathrm{stat}} = 2\ell - 1) = P^{(\mathrm{fp})}(0, K = 2\ell - 1) = \tfrac{1}{2}(q_{\ell-1} - q_\ell), \quad \text{for} \quad \ell \geqslant 1. \tag{48}$$

Note that the result for $K = 0$ is different. The case $K = 0$, i.e., no stationary point till the first-passage time can happen only if the walk jumps to the negative side after the first step itself. Any other configuration will have $K > 0$. Since the probability that the walker crosses to the negative side after the first step is simply 1/2, we get

$$\mathrm{Prob.}(N_{\mathrm{stat}} = 0) = P^{(\mathrm{fp})}(0, K = 0) = \tfrac{1}{2}. \tag{49}$$

Note that these results for $P^{(\mathrm{fp})}(0, K)$ in Eqs. (49) and (48) match perfectly with the more direct exact results obtained from the solution of the integral equation for the special double-exponential jump distribution in Eqs. (B37) and (B38). Finally, in Fig. 12, we compare these theoretical predictions in Eqs. (48) and (49) with numerical simulations for four different jump distributions, showing a perfect agreement.

As a final remark, we note that these universal results for random walk landscapes in the first-passage ensemble can be directly transported to the RTP problem. Consider an RTP in d dimensions till its x component crosses the origin for the first time. In fact, the result for $P^{\rm fp}(0, K)$ in Eq. (48) and $Q^{\rm fp}(0, m)$ in Eq. (37) directly hold for this RTP problem. This is unlike the fixed t ensemble of RTP, discussed in Sec. III C, where we had to average over the distribution of the number N of runs. In the first-passage ensemble, since the crossing-time is summed over, we do not need any additional averaging and these results thus hold directly for the x component of the RTP till its first-passage time to the origin. Hence, for the RTP, these results are also universal, i.e., independent of the spatial dimension d and the velocity distribution $W(|\mathbf{v}|)$.

V. CONCLUSION

To conclude, we have shown that the distribution of the number of minima/maxima of a random walk landscape is universal, i.e., independent of the jump distribution. We have computed this distribution exactly both for a fixed number of steps as well as till the first-passage time. These universal results are valid even for long-ranged landscapes generated by Lévy flights. Indeed, for the Lévy flights, our result provides a rare exactly solvable example of a first-passage functional. Our results can be directly applied to the landscape generated by an RTP (see Sec. III C). For instance, one can show that the number of stationary points of the RTP landscape is simply a Poissonian with mean $\gamma t/2$, independent of the post-tumble velocity distribution. The distribution of the number of minima for an RTP of duration t is also universal, but highly non-Poissonian as shown in Eq. (26). Our work opens up many interesting directions. For example, it would be interesting to compute the distribution of minima/maxima for landscapes generated by anomalous subdiffusive processes [32,55,56]. In this paper, we focused on discrete-time processes, where local minima/maxima are well defined. For continuous-time processes, one needs appropriate regularization schemes to define them [57] and their statistics at short scales will clearly depend on the regularization scheme while on larger scales, we expect the discrete-time results to hold.

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APPENDIX A: DISTRIBUTION OF THE NUMBER OF MINIMA AND MAXIMA OF A RANDOM WALK LANDSCAPE OF FIXED NUMBER OF STEPS N

In this Appendix, we provide the derivation of the main results for the fixed N ensemble of the random walk landscape. In Appendix A 1, we provide the derivation of the universal result for $Q(m, N)$ denoting the distribution of the number of minima up to step N. In Appendix A 2 we study the joint distribution $\mathcal{Q}(m, M, N)$ of the number of minima m and the number of maxima M up to N steps. In Appendix A 3, we provide an independent derivation of the distribution of the total number of stationary points $P(K, N)$ up to step N where $K = m + M$. In Appendix A 4, we provide a direct application of our results for random walk to a landscape generated by a run-and-tumble particle (RTP).

1. Distribution $Q(m, N)$ of the number of minima m up to step N

Here we solve the recursion relations (9) and (10) and obtain the solution in Eq. (2) of the main text. To solve these coupled recursion relations, we define the following generating functions:

$$Z_\pm(m, z) = \sum_{N=2}^{\infty} Q_\pm(m, N) z^N, \quad |z| < 1. \tag{A1}$$

Multiplying both sides of Eqs. (9) and (10) by z^N and summing over $N \geqslant 3$, one finds

$$Z_+(m, z) = \frac{z}{2}[Z_+(m, z) + Z_-(m, z)] + \frac{z^2}{2}\delta_{m,0}, \quad \text{for } m \geqslant 0, \tag{A2}$$

$$Z_-(m, z) = \frac{z}{2}[Z_+(m-1, z) + Z_-(m, z)] + \frac{z^2}{4}[\delta_{m,0} + \delta_{m,1}], \quad \text{for } m \geqslant 0, \tag{A3}$$

with the convention $Z_+(-1, 0) = 0$. Setting $m = 0$ in the second equation gives

$$Z_-(0, z) = \frac{z^2}{2(2 - z)}. \tag{A4}$$

Substituting this in the first line with $m = 0$ then gives

$$Z_+(0, z) = \frac{z^2(4 - z)}{2(2 - z)^2}. \tag{A5}$$

To solve Eqs. (A2) and (A3) we further define the following generating functions with respect to m:

$$S_\pm(u, z) = \sum_{m=0}^{\infty} Z_\pm(m, z) u^m = \sum_{N=2}^{\infty}\sum_{m=0}^{\infty} u^m z^N Q_\pm(m, N). \tag{A6}$$

Multiplying both sides of Eqs. (A2) and (A3) by u^m and summing over $m \geqslant 0$, we get

$$S_+(u, z) = \frac{z}{2}[S_+(u, z) + S_-(u, z)] + \frac{z^2}{2}, \tag{A7}$$

$$S_-(u, z) = \frac{z}{2}[uS_+(u, z) + S_-(u, z)] + \frac{z^2(1 + u)}{4}. \tag{A8}$$

Solving this pair of equations gives

$$S_+(u,z) = \frac{z^2[4+(u-1)z]}{2[(2-z)^2-uz^2]}, \tag{A9}$$

$$S_-(u,z) = \frac{z^2[2(1+u)-z(1-u)]}{2[(2-z)^2-uz^2]}. \tag{A10}$$

The sum $S(u,z)=S_+(u,z)+S_-(u,z)$ is given by

$$S(u,z) = \frac{z^2(3+u-z+uz)}{(2-z)^2-uz^2}. \tag{A11}$$

Inverting Eq. (A9) using Cauchy's formula, one finds

$$Z_+(m,z) = \frac{1}{2\pi i}\oint_{C_0} du \frac{1}{u^{m+1}}\frac{1}{2}\frac{4+(u-1)z}{\left[\left(\frac{2-z}{z}\right)^2-u\right]}, \tag{A12}$$

where C_0 is a contour around $u=0$ in the complex u plane. Noting that the integrand has a simple pole at $u=[(2-z)/z]^2$, this integral can be trivially done to give

$$Z_+(m,z) = \begin{cases} \frac{2z^{2m+1}}{(2-z)^{2m+2}}, & \text{for} \quad m\geqslant 1,\\ \frac{z^2(4-z)}{2(2-z)^2}, & \text{for} \quad m=0. \end{cases} \tag{A13}$$

Similarly, for $Z_-(m,z)$ we get

$$Z_-(m,z) = \begin{cases} \frac{2\,z^{2m}}{(2-z)^{2m+1}}, & \text{for} \quad m\geqslant 1,\\ \frac{z^2}{2(2-z)}, & \text{for} \quad m=0. \end{cases} \tag{A14}$$

Expanding further in powers of z gives the desired results

$$Q_+(m,N) = \frac{1}{2^N}\frac{\Gamma(N+1)}{\Gamma(N-2m)\Gamma(2m+2)}, \quad \text{for} \quad m\geqslant 1 \tag{A15}$$

and

$$Q_-(m,N) = \frac{1}{2^N}\frac{\Gamma(N+1)}{\Gamma(N+1-2m)\Gamma(2m+2)}, \quad \text{for } 0\leqslant m\leqslant \frac{N}{2}. \tag{A16}$$

Finally, $Q(m,N)=Q_+(m,N)+Q_-(m,N)$ is given by

$$Q(m,N) = \frac{1}{2^N}\frac{\Gamma(N+2)}{\Gamma(N+1-2m)\Gamma(2m+2)}, \quad \text{for } 0\leqslant m\leqslant \frac{N}{2}, \tag{A17}$$

which indeed reduces to the result announced in Eq. (2). This result is universal for all m and N and is valid for any arbitrary jump distribution $\phi(\eta)$ as long as $\phi(\eta)$ is symmetric and continuous.

2. Joint distribution $\mathcal{Q}(m,M,N)$ of the number of minima m and the number of maxima M up to step N

In this subsection we study the joint probability distribution, denoted by $\mathcal{Q}(m,M,N)$, of having m minima and M maxima up to step N of the random walk in Eq. (1). As in the case of the distribution of the number of minima discussed in the previous subsection, the joint distribution $\mathcal{Q}(m,M,N)$ is also independent of the starting point x_0 of the walk. As in the previous section, it is convenient to define $\mathcal{Q}_\pm(m,M,N)$ denoting the joint distribution of m and M with the first step positive or negative. Then $\mathcal{Q}(m,M,N)=\mathcal{Q}_+(m,M,N)+\mathcal{Q}_-(m,M,N)$. By investigating what happens after the first jump, it is straightforward to write down the pair of backward recursion relations

$$\mathcal{Q}_+(m,M,N) = \tfrac{1}{2}[\mathcal{Q}_+(m,M,N-1) + \mathcal{Q}_-(m,M-1,N-1)], \tag{A18}$$

$$\mathcal{Q}_-(m,M,N) = \tfrac{1}{2}[\mathcal{Q}_+(m-1,M,N-1) + \mathcal{Q}_-(m,M,N-1)], \tag{A19}$$

which are valid for $N\geqslant 3, M\geqslant 0$ and $m\geqslant 0$ with the convention

$$\mathcal{Q}_+(m,-1,N)=0, \quad \mathcal{Q}_-(-1,M,N)=0, \tag{A20}$$

for $N\geqslant 0$. For $N=2$, it is easy to show that

$$\mathcal{Q}_+(m,M,2) = \tfrac{1}{4}[\delta_{M,1}+\delta_{m,0}], \tag{A21}$$

$$\mathcal{Q}_-(m,M,2) = \tfrac{1}{4}[\delta_{M,0}+\delta_{m,1}]. \tag{A22}$$

The generating functions

$$Z_\pm(m,M,z) = \sum_{N=2}^{\infty}\mathcal{Q}_\pm(m,M,N)\,z^N \tag{A23}$$

then satisfy

$$Z_+(m,M,N) = \frac{z}{2}[Z_+(m,M,z)+Z_-(m,M-1,z)] + \frac{z^2}{4}\delta_{m,0}[\delta_{M,1}+\delta_{M,0}], \tag{A24}$$

$$Z_-(m,M,N) = \frac{z}{2}[Z_+(m-1,M,z)+Z_-(m,M,z)] + \frac{z^2}{4}\delta_{M,0}[\delta_{m,1}+\delta_{m,0}], \tag{A25}$$

with $Z_+(-1,M,z)=0$ and $Z_-(m,-1,z)=0$.

To solve Eqs. (A24) and (A25) we further define the following generating functions:

$$S_\pm(u,v,z) = \sum_{M=0}^{\infty}\sum_{m=0}^{\infty} Z_\pm(m,M,z)\,u^m\,v^M, \tag{A26}$$

which simply satisfy the coupled equations

$$(2-z)S_+(u,v,z) - vzS_-(u,v,z) = \frac{z^2}{2}(1+v),$$
$$zuS_+(u,v,z) - (2-z)S_-(u,v,z) = -\frac{z^2}{2}(1+u). \tag{A27}$$

Solving this pair of linear equations we get

$$S_+(u,v,z) = \frac{z^2}{2}\frac{(2-z)(1+v)+zv(1+u)}{(2-z)^2-uvz^2},$$
$$S_-(u,v,z) = \frac{z^2}{2}\frac{(2-z)(1+u)+zu(1+v)}{(2-z)^2-uvz^2}. \tag{A28}$$

Hence, the sum $S(u, v, z) = S_+(u, v, z) + S_-(u, v, z)$ is given by

$$S(u, v, z) = z^2 \frac{(2 - z) + (u + v) + zuv}{(2 - z)^2 - uvz^2}. \quad \text{(A29)}$$

This provides the derivation of the result in Eq. (7) of the main text. Note that by setting $v = 1$ in (A29) one recovers the marginal generating function $S(u, z)$ in Eq. (A11).

3. Distribution of the number of stationary points $P(K, N)$ for a random walk landscape up to N steps

To solve the recursion relation (21), we define the double generating function

$$S(u, z) = \sum_{N=2}^{\infty} \sum_{K=0}^{\infty} P(K, N)\, u^K z^N. \quad \text{(A30)}$$

From Eq. (20) it is easy to show that $S(u, z)$ is given explicitly by

$$S(u, z) = \frac{\frac{z^2}{2}(1 + u)}{1 - \frac{z}{2}(1 + u)}. \quad \text{(A31)}$$

Expanding in powers of z we get

$$\sum_{K=0}^{\infty} P(K, N)\, u^K = \frac{1}{2^{N-1}}(1 + u)^{N-1}. \quad \text{(A32)}$$

Expanding in powers of u immediately gives the binomial result in Eq. (5).

4. Calculations on the run-and-tumble particle

To proceed, we consider the process $x(t)$ representing the x component of the d-dimensional RTP trajectory in continuous time as in Fig. 6. This can be viewed as a discrete-time random walk landscape but with the number of steps $N(t)$ as a fluctuating random variable, for fixed t. Note that the number of steps $N(t)$ is also precisely the number of runs in the RTP trajectory. Our first goal is to compute the distribution $P(N(t) = N|t)$ of the number of runs in time t. To compute this, it is convenient to first introduce the joint distribution $P(\tau_1, \tau_2, \cdots, \tau_N, N(t) = N|t)$ of the run times $\{\tau_1, \tau_2, \cdots, \tau_N\}$ and the number $N(t)$ of runs till time t, which we assume is fixed. This joint distribution can be written down very simply, since the successive run times are statistically independent, namely [48,49]

$$P(\tau_1, \tau_2, \cdots, \tau_N, N(t) = N|t) = \left[\prod_{i=1}^{N-1} p(\tau_i)\right] q(\tau_N)\, \delta\left(\sum_{i=1}^{N} \tau_i - t\right), \quad \text{(A33)}$$

where $p(\tau) = \gamma\, e^{-\gamma\tau}$ and $q(\tau) = \int_\tau^\infty p(\tau')\, d\tau'$. This can be understood as follows: the first $(N - 1)$ runs are complete and each is distributed independently via $p(\tau)$—this explains the product in Eq. (A33). The last run τ_N is yet to be complete and hence it is distributed via $q(\tau_N) = \int_{\tau_N}^\infty p(\tau')\, d\tau'$, which comes from the fact that the completion time has to occur after τ_N. Finally, the δ function ensures that the total time spent is t.

Taking the Laplace transform with respect to t and integrating over τ_i's one finds

$$\int_0^\infty P(N|t)\, e^{-st}\, dt = [\tilde{p}(s)]^{N-1} \tilde{q}(s), \quad \text{(A34)}$$

where $P(N|t) = \int_0^\infty d\tau_1 \cdots \int_0^\infty d\tau_N P(\tau_1, \tau_2, \cdots, \tau_N, N|t)$ is the marginal distribution of $N(t)$, given t and $\tilde{p}(s) = \int_0^\infty p(\tau)\, e^{-s\tau}\, d\tau$ and $\tilde{q}(s) = \int_0^\infty q(\tau)\, e^{-s\tau}\, d\tau$ are the Laplace transforms of $p(\tau)$ and $q(\tau)$. Using $p(\tau) = \gamma e^{-\gamma\tau}$ and $q(\tau) = e^{-\gamma\tau}$, one gets from Eq. (A34)

$$\int_0^\infty P(N|t)\, e^{-st}\, dt = \frac{\gamma^{N-1}}{(\gamma + s)^N}. \quad \text{(A35)}$$

Inverting this Laplace transform, one gets the Poisson distribution [48,49]

$$P(N|t) = e^{-\gamma t} \frac{(\gamma t)^{N-1}}{(N - 1)!}, \quad N = 1, 2, \cdots. \quad \text{(A36)}$$

We now consider the distribution $P(K, t)$ of the total number of stationary points K up to time t in the RTP. We have seen in the previous subsection that the distribution $P(K, N)$ of the number of stationary points in a discrete-time random walk of N steps is given exactly as in Eq. (5). Now for the RTP problem, N itself is a random variable distributed via Eq. (A36). Taking the product of the two and summing over all $N = 1, 2, \cdots$, we get the expression

$$P(K, t) = \sum_{N=1}^{\infty} \frac{1}{2^{N-1}} \binom{N - 1}{K} e^{-\gamma t} \frac{(\gamma t)^{N-1}}{(N - 1)!} = e^{-\frac{\gamma t}{2}} \frac{\left(\frac{\gamma t}{2}\right)^K}{K!}, \quad \text{for all} \quad K = 0, 1, \cdots, \quad \text{(A37)}$$

as mentioned earlier in Eq. (25).

We now turn to the distribution $Q(m, t)$ of the number of minima m up to time t in the RTP trajectory. We have seen in the previous subsections that the distribution $Q(m, N)$ for the number of minima m in a discrete-time random walk trajectory of N steps is given by the exact formula in Eq. (2). Using again the fact that, for fixed t, the number of runs N itself is a random variable distributed via Eq. (A36), we get

$$Q(m, t) = \sum_{N=1}^{\infty} Q(m, N)\, e^{-\gamma t} \frac{(\gamma t)^{N-1}}{(N - 1)!} = e^{-\gamma t} \sum_{N=1}^{\infty} \frac{1}{2^N} \frac{(N + 1)!}{(2m + 1)!(N - 2m)!} \frac{(\gamma t)^{N-1}}{(N - 1)!}. \quad \text{(A38)}$$

Simplifying and performing a shift $N \to N + 2m$ we get

$$Q(m, t) = e^{-\gamma t} \frac{(\gamma t/2)^{2m-1}}{(2m + 1)!} \times \sum_{N=0}^{\infty} \frac{(N + 2m)(N + 2m + 1)}{N!} \left(\frac{\gamma t}{2}\right)^N. \quad \text{(A39)}$$

Fortunately, this sum can be performed explicitly (using Mathematica) and we get the expression in Eq. (26).

APPENDIX B: FIRST-PASSAGE ENSEMBLE

In this Appendix, we present the details of the calculations presented in Sec. IV.

1. Distribution of the number of minima till the first-passage time to the origin

In this subsection, our goal is to compute the distribution of the number of minima $Q^{\rm fp}(x_0, m)$, for a random walk in Eq. (1) starting at $x_0 \geqslant 0$, till the first time the walk goes on the negative side. For this one requires to solve the integral Eqs. (32) and (33). These equations are hard to solve for arbitrary $\phi(\eta)$. Below, we consider the specific choice of the double-exponential jump distribution $\phi(\eta) = \frac{1}{2}\exp(-|\eta|)$ for which we show that these equations can be solved explicitly.

a. Double-exponential jump distribution $\phi(\eta) = \frac{1}{2}\exp(-|\eta|)$

For this case Eqs. (32) and (33) reduce to

$$e^{-x_0} Z_+^{\rm (fp)}(x_0,u) = \int_{x_0}^{\infty} dy\, e^{-y}\,\frac{[Z_+^{\rm (fp)}(y,u) + Z_-^{\rm (fp)}(y,u)]}{2}, \tag{B1}$$

$$e^{x_0} Z_-^{\rm (fp)}(x_0,u) = \frac{1}{2} + \int_0^{x_0} dy\, e^{y}\,\frac{[uZ_+^{\rm (fp)}(y,u) + Z_-^{\rm (fp)}(y,u)]}{2}. \tag{B2}$$

Taking derivatives on both sides of the above equations with respect to x_0 yields

$$e^{-x_0}\left[\frac{d}{dx_0} - 1\right] Z_+^{\rm (fp)}(x_0,u) = -\frac{1}{2}\,e^{-x_0}\,[Z_+^{\rm (fp)}(x_0,u) + Z_-^{\rm (fp)}(x_0,u)], \tag{B3}$$

$$e^{x_0}\left[\frac{d}{dx_0} + 1\right] Z_-^{\rm (fp)}(x_0,u) = \frac{1}{2}\,e^{x_0}\,[uZ_+^{\rm (fp)}(x_0,u) + Z_-^{\rm (fp)}(x_0,u)]. \tag{B4}$$

Simplifying one gets

$$\left[\frac{d}{dx_0} - \frac{1}{2}\right] Z_+^{\rm (fp)}(x_0,u) = -\frac{1}{2}\,Z_-^{\rm (fp)}(x_0,u), \tag{B5}$$

$$\left[\frac{d}{dx_0} + \frac{1}{2}\right] Z_-^{\rm (fp)}(x_0,u) = \frac{1}{2}\,uZ_+^{\rm (fp)}(x_0,u). \tag{B6}$$

It is easy to see that these two equations can be rewritten as

$$\left[\frac{d^2}{dx_0^2} - \frac{1}{4}\right] Z_\pm^{\rm (fp)}(x_0,u) = -\frac{u}{4} Z_\pm^{\rm fp}(x_0,u). \tag{B7}$$

It is clear from the definition of the generating functions $Z_\pm(x_0,u)$ in Eq. (31) that they can not diverge exponentially as the starting point $x_0 \to \infty$. Using this condition, we get the two solutions

$$Z_+^{\rm (fp)}(x_0,u) = A\,\exp\left(-\frac{\sqrt{1-u}}{2}x_0\right), \tag{B8}$$

$$Z_-^{\rm (fp)}(x_0,u) = B\,\exp\left(-\frac{\sqrt{1-u}}{2}x_0\right), \tag{B9}$$

where the two constants A and B are yet to be determined. Setting $x_0 = 0$ in Eq. (B2) one immediately gets $Z_-^{\rm fp}(0,u) = 1/2$. Using this condition in Eq. (B9) it follows that

$$B = \tfrac{1}{2}. \tag{B10}$$

To fix the other constant A, we proceed as follows. The integral Eqs. (B1) and (B2) actually contain more informations than the derived differential Eqs. (B5) and (B6). Hence, one has to additionally ensure that the solutions of the differential equation also satisfy the integral equations. Indeed, substituting Eqs. (B8) and (B9) into the integral equations, one sees that these are indeed the solutions provided:

$$A = \frac{1-\sqrt{1-u}}{2u}, \quad B = \frac{1}{2}. \tag{B11}$$

Hence, we have the following explicit solution of $Z_\pm^{\rm (fp)}(x_0,u)$:

$$Z_+^{\rm (fp)}(x_0,u) = \frac{1-\sqrt{1-u}}{2u}\,\exp\left(-\frac{\sqrt{1-u}}{2}x_0\right), \tag{B12}$$

$$Z_-^{\rm (fp)}(x_0,u) = \frac{1}{2}\,\exp\left(-\frac{\sqrt{1-u}}{2}x_0\right). \tag{B13}$$

Now to find $Q_\pm(x_0,m)$ one requires to perform the inverse transform given by the Cauchy formula

$$Q_\pm^{\rm (fp)}(x_0,m) = \frac{1}{2\pi i}\oint du\,\frac{1}{u^{m+1}}\,Z_\pm^{\rm (fp)}(x_0,u). \tag{B14}$$

The results become more explicit for the case $x_0 = 0$. In this case, the expressions of $Z_\pm^{\rm (fp)}(0,u)$ simplifies and for $Z^{\rm (fp)}(0,u) = Z_+^{\rm (fp)}(0,u) + Z_-^{\rm (fp)}(0,u)$ one has

$$\begin{aligned} Z^{\rm (fp)}(0,u) &= \frac{1}{2} + \frac{1}{2u}\left(1-\sqrt{1-u}\right), \\ &= \frac{3}{4} + \frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{2^{2k+1}}\,\frac{(2k)!}{k!(k+1)!}\,u^k. \end{aligned} \tag{B15}$$

From the coefficient of u^m it is easy to extract

$$Q^{\rm (fp)}(m) = Q^{\rm (fp)}(0,m) = \begin{cases} \frac{3}{4} & \text{for} \quad m = 0, \\ \frac{1}{2^{2m+2}}\,\frac{(2m)!}{m!(m+1)!} & \text{for} \quad m \geqslant 1. \end{cases} \tag{B16}$$

This analytical expression is numerically verified in Fig. 11 where the solid red line corresponds to the expression in Eq. (B16), whereas the cross symbols are obtained from simulation for the double-exponential jump distribution $\phi(\eta) = \frac{1}{2}\exp(-|\eta|)$.

For large m, the distribution $Q^{\rm (fp)}(m)$ in Eq. (B16) has a power-law tail $Q^{\rm (fp)}(m) \overset{m\to\infty}{\approx} \frac{1}{4\sqrt{2\pi}}\,\frac{1}{m^{3/2}}$. This power-law tail is similar to the decay of the first-passage distribution $N^{-3/2}$ which follows from the Sparre Andersen theorem [50]. To get a large number of minima on the path, the walker should have a very large first-passage time. It is expected that the tails of the distribution of these quantities should have power-law decay with the same exponent. Note that while this scaling argument predicts correctly the universal exponent 3/2, it can not be used to obtain the exact universal prefactor $1/(4\sqrt{2\pi})$ in Eq. (40). For that one needs to prove the result in

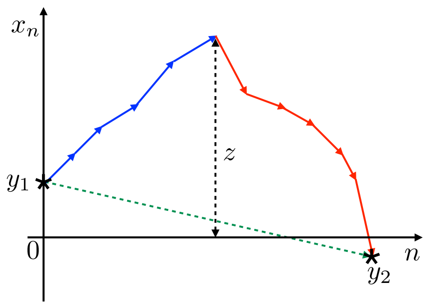

FIG. 13. Schematic trajectory of a random walk (discrete time and continuous space), that starts at y_1 and arrives at y_2 (in arbitrary number of steps), but with the constraint that there is only one peak (maximum) in the interior (and not at the edge). This constraint allows only configurations where the walker, starting at y_1 moves up to the peak height $z \geqslant y_1$ by consecutive upward steps and then from the peak comes down to $y_2 \leqslant z$ by consecutive downward moves. One then integrates over all possible heights z of the peak. The probability $G(y_1, y_2)$ of this event will provide the jump probability from y_1 to y_2 for the auxiliary walk (as indicated by the dashed green arrow).

Eq. (B16) for arbitrary jump distribution $\phi(\eta)$ which is done in Sec. IV A.

2. Computation of the jump distribution $\Psi(\xi)$ of the auxiliary walk

In this Appendix we compute the jump distribution $\Psi(\xi)$. As mentioned previously this distribution is basically the probability of transition $G(y_1, y_2)$ from y_1 to y_2 of the original random walk in arbitrary number of steps for general y_1 and y_2, not necessarily positive and with the constraint that there can be only one peak (maximum) in between. A typical configuration is shown in Fig. (13). Let z denote the height of this single peak. Since z is the maximum, we must have $z \geqslant \max(y_1, y_2)$. Thus, the walker, starting at y_1 moves up to the peak height z by consecutive upward steps and then from the peak comes down to y_2 by consecutive downward moves. One then integrates over all possible heights z of the peak to compute the transition probability $G(y_1, y_2)$ of the auxiliary walk that jumps from y_1 to y_2. Let the probability that the walker, starting at y_1, reaches the peak of height $z \geqslant y_1$ only by consecutive upward moves be denoted by $\mathscr{P}_{\text{left}}(z - y_1)$. This can be easily computed in terms of the jump distribution $\phi(\eta)$ as follows.

Suppose $\mathscr{P}_k(x)$ denotes the probability that the walk takes k consecutive upward steps to reach a point x, starting from 0. Then, $\mathscr{P}_{\text{left}}(x)$ is simply given by

$$\mathscr{P}_{\text{left}}(x) = \sum_{k=1}^{\infty} \mathscr{P}_k(x). \tag{B17}$$

Clearly, $\mathscr{P}_k(x)$ satisfies the recursion relation, for $k \geqslant 1$ and $x \geqslant 0$,

$$\mathscr{P}_k(x) = \int_0^x dx'\, \mathscr{P}_{k-1}(x')\, \phi(x - x'), \tag{B18}$$

starting from $\mathscr{P}_0(x) = \delta(x)$. Since $x \geqslant 0$ and Eq. (B18) has a convolution form, it is useful to define the Laplace transform

$$\tilde{\mathscr{P}}_k(\lambda) = \int_0^\infty \mathscr{P}_k(x)\, e^{-\lambda x}\, dx. \tag{B19}$$

Taking Laplace transform of Eq. (B18) and iterating (using the initial condition), one gets for $k \geqslant 1$

$$\tilde{\mathscr{P}}_k(\lambda) = [\tilde{\phi}(\lambda)]^k, \tag{B20}$$

where

$$\tilde{\phi}(\lambda) = \int_0^\infty \phi(\eta)\, e^{-\lambda\eta}\, d\eta. \tag{B21}$$

Note that

$$\tilde{\phi}(0) = \int_0^\infty \phi(\eta)\, d\eta = \frac{1}{2}, \tag{B22}$$

where we used the symmetry of $\phi(\eta)$. Finally, the Laplace transform of $\mathscr{P}_{\text{left}}(x)$ is then given by

$$\tilde{\mathscr{P}}_{\text{left}}(\lambda) = \int_0^\infty \mathscr{P}_{\text{left}}(x)\, e^{-\lambda x}\, dx = \sum_{k=1}^{\infty} [\tilde{\phi}(\lambda)]^k = \frac{\tilde{\phi}(\lambda)}{1 - \tilde{\phi}(\lambda)}. \tag{B23}$$

The distribution $\mathscr{P}_{\text{left}}(x)$ has a support over $x \geqslant 0$. Furthermore, using Eq. (B22), it follows from (B23) that $\tilde{\mathscr{P}}_{\text{left}}(\lambda = 0) = 1$, indicating that the distribution $\mathscr{P}_{\text{left}}(x)$ is normalized to unity,

$$\int_0^\infty \mathscr{P}_{\text{left}}(x)\, dx = 1. \tag{B24}$$

Thus, $\mathscr{P}_{\text{left}}(x)$ clearly depends explicitly on the jump distribution $\phi(\eta)$. However, we will see below that the detailed form of $\mathscr{P}_{\text{left}}(x)$ does not really matter for establishing the proof of universality. The only thing that matters is that $\mathscr{P}_{\text{left}}(x)$ has a support over $x \geqslant 0$ and is normalized to unity.

Getting back to Fig. 13, the probability of reaching $y_2 \leqslant z$, starting at z, is given by $\mathscr{P}_{\text{left}}(z - y_2)$, where we have simply reversed the steps to relate this probability to the function $\mathscr{P}_{\text{left}}(x)$. Finally, integrating over $z \geqslant \max(y_1, y_2)$ we get the transition probability

$$G(y_1, y_2) = \int_{\max(y_1, y_2)}^{\infty} dz\, \mathscr{P}_{\text{left}}(z - y_1)\, \mathscr{P}_{\text{left}}(z - y_2). \tag{B25}$$

When $y_1 > y_2$, the lower limit in the integral is y_1 and then making the shift $u = z - y_1$ one gets

$$G(y_1, y_2) = \int_0^\infty du\, \mathscr{P}_{\text{left}}(u)\, \mathscr{P}_{\text{left}}(u + y_1 - y_2). \tag{B26}$$

Conversely, when $y_2 > y_1$, one similarly gets

$$G(y_1, y_2) = \int_0^\infty du\, \mathscr{P}_{\text{left}}(u)\, \mathscr{P}_{\text{left}}(u + y_2 - y_1). \tag{B27}$$

Consequently, the transition probability $G(y_1, y_2)$ depends only on the difference $y_2 - y_1$ and is given by

$$\begin{aligned} G(y_1, y_2) &\equiv \psi(y_2 - y_1) \\ &= \int_0^\infty du\, \mathscr{P}_{\text{left}}(u)\, \mathscr{P}_{\text{left}}(u + |y_2 - y_1|). \end{aligned} \tag{B28}$$

Thus, the effective transition probability $\Psi(\xi)$ is continuous and symmetric around $\xi = 0$. To see that it is also normalized to unity, we consider the integral

$$\int_{-\infty}^{\infty} \Psi(\xi)\, d\xi = \int_{-\infty}^{\infty} d\xi \int_{0}^{\infty} du\, \mathscr{P}_{\rm left}(u)\, \mathscr{P}_{\rm left}(u+|\xi|) = \int_{0}^{\infty} du\, \mathscr{P}_{\rm left}(u) \int_{-\infty}^{\infty} d\xi\, \mathscr{P}_{\rm left}(u+|\xi|). \tag{B29}$$

Next, we write

$$\int_{-\infty}^{\infty} d\xi\, \mathscr{P}_{\rm left}(u+|\xi|) = \int_{-\infty}^{0} d\xi\, \mathscr{P}_{\rm left}(u-\xi) + \int_{0}^{\infty} d\xi\, \mathscr{P}_{\rm left}(u+\xi) = 2\int_{u}^{\infty} dv\, \mathscr{P}_{\rm left}(v), \tag{B30}$$

where we made the change of variable $u - \xi = v$ in the first integral and $u + \xi = v$ in the second one. Substituting (B30) in Eq. (B29) gives

$$\int_{-\infty}^{\infty} \Psi(\xi)\, d\xi = 2\int_{0}^{\infty} du\, \mathscr{P}_{\rm left}(u) \int_{u}^{\infty} dv\, \mathscr{P}_{\rm left}(v) = 1. \tag{B31}$$

The last equality is established by making the change of variable $z = \int_u^\infty dv\, \mathscr{P}_{\rm left}(v)$. One sees immediately that the detailed form of $\mathscr{P}_{\rm left}(x)$ is not important in establishing the normalization of the transition probability $\Psi(\xi)$.

3. Distribution of stationary points till first passage to origin for $\phi(\eta) = \frac{1}{2}\exp(-|\eta|)$

For this case Eqs. (44) and (45) reduce to

$$e^{-x_0}\tilde{Z}_+^{(\rm fp)}(x_0,u) = \frac{1}{2}\int_{x_0}^{\infty} dy\, e^{-y}[\tilde{Z}_+^{(\rm fp)}(y,u) + u\,\tilde{Z}_-^{(\rm fp)}(y,u)], \tag{B32}$$

$$e^{x_0}\tilde{Z}_-^{(\rm fp)}(x_0,u) = \frac{1}{2} + \frac{1}{2}\int_{0}^{x_0} dy\, e^{y}[u\tilde{Z}_+^{(\rm fp)}(y,u) + \tilde{Z}_-^{(\rm fp)}(y,u)]. \tag{B33}$$

These equations can be solved as in the previous subsection leading to the final result

$$\tilde{Z}_+^{(\rm fp)}(x_0,u) = \frac{1-\sqrt{1-u^2}}{2u}\, \exp\left(-\frac{\sqrt{1-u^2}}{2}x_0\right), \tag{B34}$$

$$\tilde{Z}_-^{(\rm fp)}(x_0,u) = \frac{1}{2}\, \exp\left(-\frac{\sqrt{1-u^2}}{2}x_0\right). \tag{B35}$$

The result becomes more explicit for the case $x_0 = 0$. In this case, we find that $\tilde{Z}^{(\rm fp)}(0,u) = \tilde{Z}_+^{(\rm fp)}(0,u) + \tilde{Z}_-^{(\rm fp)}(0,u)$ is simply

$$\tilde{Z}^{(\rm fp)}(0,u) = \sum_{K=0}^{\infty} P^{(\rm fp)}(0,K)\, u^K = \frac{1}{2} + \frac{1}{2u}(1-\sqrt{1-u^2}). \tag{B36}$$

Expanding in powers of u, we get the final result

$$P^{(\rm fp)}(0, K=0) = \tfrac{1}{2} \tag{B37}$$

$$P^{(\rm fp)}(0, K=2\ell-1) = \frac{q_{\ell-1} - q_\ell}{2}, \quad \text{for} \quad \ell \geqslant 1, \tag{B38}$$

where q_ℓ is given by

$$q_\ell = \frac{1}{2^{2\ell}}\binom{2\ell}{\ell}. \tag{B39}$$

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