Random matrix theory approach to quantum Fisher information in quantum ergodic systems

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We theoretically investigate quantum parameter estimation in quantum chaotic systems. Our analysis is based on an effective description of quantum ergodic systems in terms of a random matrix Hamiltonian. Based on this approach, we derive an analytical expression for the time evolution of the quantum Fisher information (QFI), which we find to have three distinct timescales. Initially, the QFI increase is quadratic in time, characterizing the timescale over which initial information is extractable from local measurements only. This quickly passes into linear increase with slope determined by the decay rate of the measured spin observable. When the information is fully spread among all degrees of freedom, a second quadratic timescale determines the long-time behavior of the QFI. We test our random matrix theory prediction with the exact diagonalization of a nonintegrable spin system, focusing on the estimation of a local magnetic field by measurements of the many-body state. Our numerical calculations agree with the effective random matrix theory approach and show that the information on the local Hamiltonian parameter is distributed throughout the quantum system during the quantum thermalization process.

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I. INTRODUCTION

Initially excited quantum systems typically equilibrate to states exhibiting thermal properties, a process known as quantum thermalization [1–6]. At the core of our current understanding of this intriguing phenomenon is the eigenstate thermalization hypothesis (ETH), which assumes that the many-body eigenstates of quantum ergodic systems yield the same expectation values of local observables as those calculated with a microcanonical ensemble [7–10]. The ETH can be formally expressed as a conjecture on the properties of matrix elements of local observables, which in turn can be derived from an effective description of quantum ergodic systems in terms of random matrix theory (RMT). The validity of the ETH has been confirmed for a broad range of many-body systems by means of exact diagonalizations [11–16]. Furthermore, experimental quantum optical systems have allowed for investigation of quantum thermalization and the emergence of statistical physics in isolated quantum systems. Examples include recent experiments with ultracold atoms [17,18], trapped ions [19–21], and superconducting qubits [22].

An important fundamental issue in quantum many-body theory is how information on local properties can be retrieved or estimated from observing the quantum system's dynamics. This problem is closely related to fundamental research on the connection between quantum chaos and scrambling of quantum information [23,24] and also to applications like quantum metrology [25–27]. For example, the exponential sensitivity to small perturbation in imperfect time-reversal quantum dynamics is a widely studied signal for irreversibility [28–31]. Another approach is the information gain in tomography, which can be used as a signature of quantum chaos [32,33]. Here we address this question by investigating the dynamics of the quantum Fisher information (QFI) in quantum ergodic systems. The QFI is a quantity of central importance in quantum metrology. It quantifies the sensitivity of a given input state to a unitary transformation, and provides the fundamental bound of the parameter estimation [34–37]. The QFI also provides a sufficient condition to recognize entanglement in multiparticle state [38–41].

In this paper, we study the time evolution of the QFI of quantum ergodic systems by a RMT approach. We model the Hamiltonian of the ergodic system as the sum of two contributions: a free, noninteracting, diagonal part and an interaction term modeled by a Gaussian orthogonal random matrix. This approach is valid as long as the coupling between a subsystem and the rest of the closed system can be modeled as random matrix. Such an approach was originally proposed by Deutsch as a toy model to describe the emergence of quantum thermalization in isolated quantum systems [7]. Recently, it was shown that this approach can be extended to predict the off-diagonal elements of observables, recovering the ETH [42]. The description on RMT relies on strong assumptions that are basically equivalent to the ETH itself, however, it allows us to make scaling predictions that can be tested in experiments or exact numerical diagonalizations.

We test the results predicted from RMT by using an exact diagonalization of a nonintegrable spin chain. The model consists of a system Hamiltonian describing one or a few noninteracting spins coupled with large spin system which plays the role of a finite quantum many-body bath. We show that three time regimes appear in the time evolution of the QFI. In the beginning of the time evolution, the QFI increases *quadratically*. In this short time period, the information of the parameter can be extracted by measuring the local spin observable

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with uncertainty bounded by the standard quantum limit (SQL). After this time period, the QFI quickly passes into a *linear* increase with slope defined by the width of the random wave functions. Essentially, this width is the decay rate of the observable to the microcanonical average. In this second stage, the information of the parameter propagates along the entire system in a sense that the other spin observables begin to depend on the local magnetic field. Remarkably, a second *quadratic* timescale appears and it determines the long-time behavior of the QFI, which occurs when the information is spread among all quantum states involved in the evolution. We show that in this third stage the QFI is inversely proportional to the effective dimension of the system, a measure which quantifies the ergodicity of the system. The transition from the linear to quadratic regime in time occurs at Heisenberg time, determined by the density of states of the system.

For N_S noninteracting spins, one can expect that the QFI scales as $\sim N_S$, which gives the standard shot-noise limit. Crucially, for a few spins coupled to the quantum many-body heat bath, the system-bath interaction gives rise to a spin-spin correlation within the small subsystem. Hence, we show the quantum correlation may increase the QFI in the sense that it can exceed the QFI corresponding to the SQL without any initial entangled state preparation.

The paper is organized as follows: In Sec. II, we provide the theoretical framework for quantum parameter estimation. Section III presents the main result of our paper. We model the interaction Hamiltonian in terms of random matrix. Based on this, we derive an analytical expression for the time evolution of the QFI. In Sec. IV, we test the prediction from RMT with the exact diagonalization of a spin chain. Finally, the conclusions are presented in Sec. V.

II. QUANTUM PARAMETER ESTIMATION

We consider a quantum system described by a Hamiltonian, $\hat{H}(\lambda) = \hat{H}_0 + \hat{H}_I$, consisting of a noninteracting Hamiltonian $\hat{H}_0 = \hat{H}_\mathrm{S} + \hat{H}_\mathrm{B}$, with $\hat{H}_\mathrm{S}$ and $\hat{H}_\mathrm{B}$ being the Hamiltonians for the subsystem and the many-body environment, and an interaction part $\hat{H}_I$ describing the system-bath interaction. The eigenvectors and eigenenergies of the total Hamiltonian are $|\psi_\mu\rangle$ and E_μ, such that $\hat{H}|\psi_\mu\rangle = E_\mu|\psi_\mu\rangle$. We also define noninteracting energy eigenbasis, $\hat{H}_0|\varphi_\alpha\rangle = E_\alpha|\varphi_\alpha\rangle$. The system is initially prepared in an out-of-equilibrium state $|\Psi_0\rangle = \sum_\mu a_\mu|\psi_\mu\rangle$ with $a_\mu = \langle\psi_\mu|\Psi_0\rangle$, which evolves under the action of the unitary propagator, $|\psi(\lambda)\rangle = e^{-i\hat{H}(\lambda)t}|\Psi_0\rangle$.

The classical Fisher information (CFI)

$$F_\mathrm{C}(\lambda) = \sum_n \frac{(\partial_\lambda p(n|\lambda))^2}{p(n|\lambda)}, \tag{1}$$

quantifies the amount of information on the parameter λ, which can be derived by performing discrete measurements with conditional probability $p(n|\lambda) = \mathrm{Tr}(\hat{\Pi}_n\hat{\rho}(\lambda))$ to obtain the value n when the parameter has a value λ. Here $\{\hat{\Pi}_n\}$, with $\sum_n \hat{\Pi}_n = 1$, are the elements of a positive operator-valued measure and $\hat{\rho}(\lambda) = |\psi(\lambda)\rangle\langle\psi(\lambda)|$ is the density operator. The optimal strategy to measure the value of λ is, however, associated with a privileged observable that maximizes the CFI. The CFI is upper bounded $F_C(\lambda) \leqslant F_Q(\lambda)$, where $F_Q(\lambda)$ is the QFI. The ultimate achievable precision of the parameter estimation is quantified via the quantum Cramér-Rao bound $\delta\lambda^2 \geqslant 1/(MF_Q(\lambda))$, where M is the repetition number. For a pure state, the QFI is given by [34]

$$F_Q(\lambda) = 4(\langle\partial_\lambda\psi(\lambda)|\partial_\lambda\psi(\lambda)\rangle - |\langle\psi(\lambda)|\partial_\lambda\psi(\lambda)\rangle|^2). \tag{2}$$

Furthermore, the QFI can be interpreted as a measure of distinguishability of two quantum states $|\psi(\lambda)\rangle$ and $|\psi(\lambda + d\lambda)\rangle$ with respect to the infinitesimal variation of the parameter of interest λ. Indeed, we can define the Bures distance between two infinitesimally close quantum states by $ds_\mathrm{B}^2 = 1 - \mathcal{F}(\lambda, \lambda + d\lambda)$, where $\mathcal{F}(\lambda, \lambda + d\lambda) = |\langle\psi(\lambda)|\psi(\lambda + d\lambda)\rangle|^2$ is the fidelity between the states. Therefore, it is straightforward to show that $ds_\mathrm{B}^2 = \frac{1}{4}F_Q(\lambda)d\lambda^2$ [34]. We note that the QFI is also related to the Loschmidt echo (LE), which is defined as a fidelity between a perturbed and unperturbed time-evolving states [29].

Hereafter, we assume that the dependence on λ comes only from the noninteracting Hamiltonian $\hat{H}_0(\lambda)$. This example corresponds to cases where λ represents spin frequency or external magnetic field strength. Furthermore, we may express the QFI (2) in the basis of $|\psi_\mu\rangle$ eigenvectors. We have (see Appendix A)

$$F_Q(\lambda) = 4t^2\left\{\sum_{\mu\nu\rho} a_\mu^* a_\nu (\partial_\lambda\hat{H}_0)_{\mu\rho}(\partial_\lambda\hat{H}_0)_{\rho\nu} e^{i\theta_{\mu\nu}t}\mathrm{sinc}(\theta_{\mu\rho}t)\mathrm{sinc}(\theta_{\rho\nu}t) - \left|\sum_{\mu\nu} a_\mu^* a_\nu e^{i\theta_{\mu\nu}t}(\partial_\lambda\hat{H}_0)_{\mu\nu}\mathrm{sinc}(\theta_{\mu\nu}t)\right|^2\right\}, \tag{3}$$

where $(\partial_\lambda\hat{H}_0)_{\mu\nu} = \langle\psi_\mu|\partial_\lambda\hat{H}_0|\psi_\nu\rangle$ are the matrix elements in the many-body interacting basis, $\theta_{\mu\nu} = (E_\mu - E_\nu)/2$, and $\mathrm{sinc}(x) = \sin(x)/x$. The expression (3) is convenient for our further consideration because we can apply a RMT approach to evaluate the respective matrix elements.

III. RANDOM MATRIX APPROACH

Our analysis of the QFI is based on the random matrix model in which the noninteracting Hamiltonian $\hat{H}_0$ is modeled by diagonal matrix of size N, with $\omega = 1/N$ being the constant spacing between the energy levels. The interaction term $\hat{H}_I$ is modeled by a random matrix, $(\hat{H}_I)_{\alpha\beta} = h_{\alpha\beta}$, where $h_{\alpha\beta}$ are independent random numbers selected from the Gaussian orthogonal ensemble with probability distribution $P(h) \propto e^{-\frac{N}{4g^2}\mathrm{Tr}h^2}$, giving average $\langle h_{\alpha\beta}\rangle = 0$, and variance $\langle h_{\alpha\beta}^2\rangle = g^2(1 + \delta_{\alpha\beta})/N$, where g is the coupling strength [7,42].

We expand the eigenstates of $\hat{H}$ in the noninteracting basis, $|\psi_\mu\rangle = \sum_\alpha c_\mu(\alpha)|\varphi_\alpha\rangle$, where $c_\mu(\alpha)$ are random variables whose statistical properties depend on the properties of the random matrix $\hat{H}_I$. The probability distribution of eigenstates then takes a Lorentzian form [7,42]

$$\langle |c_\mu(\alpha)|^2 \rangle_V = \Lambda(\mu,\alpha) = \frac{\omega\Gamma}{\pi}\frac{1}{(E_\mu - E_\alpha)^2 + \Gamma^2}, \tag{4}$$

where $\Gamma = \pi g^2$ is the width of the wave function distribution and $\langle \ldots \rangle_V$ denotes an average over realizations of the random matrix $(\hat{H}_I)_{\alpha\beta}$. The Lorentzian function is normalized such that $\sum_\mu \Lambda(\mu,\alpha) = \sum_\alpha \Lambda(\mu,\alpha) = 1$. Furthermore, we assume a self-averaging condition where sum over random wave functions are replaced with their ensemble average $\sum_{\alpha\ldots\beta} c_\mu(\alpha)\ldots c_\nu(\beta) = \sum_{\alpha\ldots\beta}\langle c_\mu(\alpha)\ldots c_\nu(\beta)\rangle_V$ (see Appendix B).

The self-averaging condition is essential for the evaluation of the QFI (3), and is shown to hold in the description of observables in Refs. [43,44]. Indeed, we can evaluate the sum of the matrix elements in (3) in terms of averages of products of random-wave functions $c_\mu(\alpha)$. The treatment of $c_\mu(\alpha)$ as an independent random Gaussian variable, however, is not sufficient to consistently determine the value of the off-diagonal matrix elements of an observable [42,43,45,46]. To yield consistent results, a non-Gaussian correction should be included, which arises as a result of the orthonormality condition. For the above random matrix model, calculations of observable quantities can be made under the following assumptions: (i) We assume *sparsity* of $\hat{H}'_0$, which implies that its matrix elements in the noninteracting basis is represented by a diagonal matrix, or at least by a matrix with only a few nondiagonal elements. (ii) We define *smoothness* of the matrix elements of an observable in the following way:

$$[(\hat{H}'_0)_{\alpha\alpha}]_\mu = \sum_\alpha \Lambda(\mu,\alpha)(\hat{H}'_0)_{\alpha\alpha}, \tag{5}$$

which represents the microcanonical average of the matrix elements $(\hat{H}'_0)_{\alpha\alpha}$ around the energy E_μ, namely, $(\hat{H}'_0)_{\rm mc} = [(\hat{H}'_0)_{\alpha\alpha}]_\mu$. Such an average is well-defined as long as $\omega/\Gamma \ll 1$, which ensures that a large number of matrix elements are averaged in $[(\hat{H}'_0)_{\alpha\alpha}]_\mu$. We also neglect the corrections due to self-averaging decoupling which are of order of $O(\omega^2/\Gamma^2)$, see Appendix D. These conditions are held for the random matrix model for large N, and large enough g such that eigenstates are spread over a significant number of noninteracting basis states. Conditions for the validity of these assumptions to realistic systems have been discussed in detail in Ref. [47].

Based on the above conditions, the QFI (3) is given by (see Appendix C for more details)

$$F_Q(\lambda) \approx 4t^2\left\{\frac{\omega}{\pi\Gamma}(\hat{H}'^2_0)_{\rm mc} + \frac{(\Delta\hat{H}'^2_0)_{\rm mc}}{2(\Gamma t)^2}(e^{-2\Gamma t} - 1 + 2\Gamma t)\right\}. \tag{6}$$

This is the main result of our paper. Here $\partial_\lambda \hat{H}_0 = \hat{H}'_0$ and $(\hat{H}'_0)_{\rm mc}$ is the microcanonical average of an observable $\hat{H}'_0$ and $(\Delta\hat{H}'^2_0)_{\rm mc} = (\hat{H}'^2_0)_{\rm mc} - (\hat{H}'_0)^2_{\rm mc}$ is the microcanonical average of the variance of $\hat{H}'_0$.

In Fig. 1, we show the microcanonical average of the observable $\hat{H}'_0 = \alpha\delta_{\alpha\beta}$ according to (5) compared with the diagonal average $\langle \bar{H}'_0 \rangle = {\rm Tr}(\hat{H}'_0\hat{\rho}_{\rm DE})$, where $\hat{\rho}_{\rm DE} = \sum_\mu |a_\mu|^2 |\psi_\mu\rangle\langle\psi_\mu|$ is the density matrix of the diagonal ensemble. The result is not sensitive to the choice of the initial state except for states which are at the edge of the energy spectrum. The numerical and the analytical results are averaged over ten realizations of the random Hamiltonian, but the result does not vary significantly even if we take smaller number of realizations.

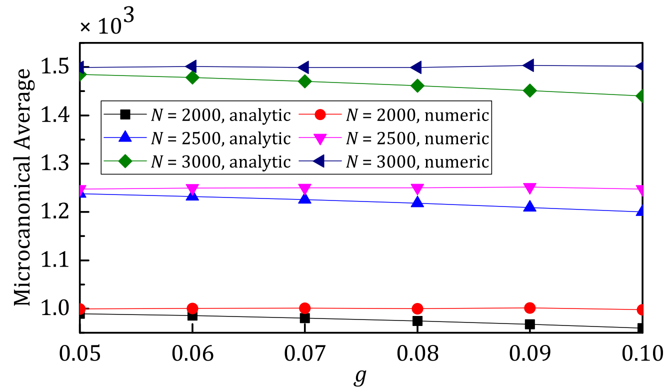

FIG. 1. Estimates for the microcanonical average for varying values of g and N. Analytic results (5) are compared with the predictions given by the diagonal ensemble (numerical results). Average over ten realizations of the random Hamiltonian is taken. The initial state is $|\Psi_0\rangle = |\varphi_\alpha\rangle$ with $\alpha = N/2$.

In Fig. 2, we show the time evolution of the QFI where we set $\lambda = \omega$. We compare the exact result based on Eq. (2) using the random matrix model and analytical expression (6). We set the initial state to be an eigenstate of the noninteracting Hamiltonian, $|\Psi_0\rangle = |\varphi_\alpha\rangle$ with α selected at the middle of the energy spectrum. We compare the results for various g and N. Crucially, we expect good agreement between the analytical

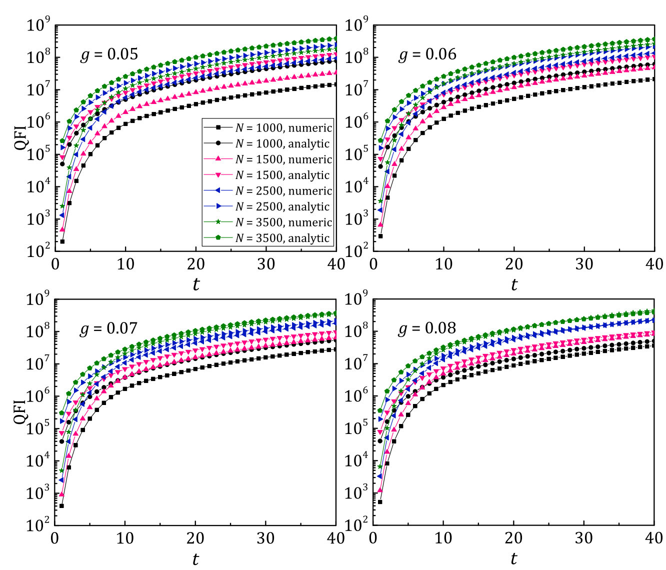

FIG. 2. Quantum Fisher information as a function of time for various g and N. We compare the results for the QFI derived from the random matrix Hamiltonian using Eq. (2) and the analytical result Eq. (6). Average over 10 realizations of the random Hamiltonian is taken. The initial state is $|\Psi_0\rangle = |\varphi_\alpha\rangle$ with $\alpha = N/2$ and $(\hat{H}'_0)_{\alpha\beta} = \alpha\delta_{\alpha\beta}$.

and numerical results for large N and small g, where the latter condition is required for accuracy of the Lorentzian form of $\Lambda(\mu,\alpha)$. On one hand, we observe that by lowering g for constant N the decay rate Γ decreases and the ratio ω/Γ increases such that high order terms in (6) become significant. This explains the deviation between the numerical and the analytical results for small g. On the other hand, for a given Γ by increasing N the frequency ω decreases, which improves the agreement between both results, as shown in Fig. 2. Finally, although for high coupling g the higher order terms in (6) are suppressed, the Lorentzian form of the probability distribution is no longer a good approximation, which would modify the result for the QFI. However, as we will see later, the same time dependence of the QFI still holds even in the strong coupling regime. We also test the result for other initial states $|\varphi_\alpha\rangle$. We find that the relative error converges faster for initial states at the lower half of the noninteracting energy spectrum and, respectively, slower at the upper half.

IV. EXACT DIAGONALIZATION

We now turn to the comparison of our main result (6) with exact diagonalization of a nonintegrable spin chain. We consider a 1D spin system with a Hamiltonian of the form

$$\hat{H} = \hat{H}_{\rm S} + \hat{H}_{\rm B} + \hat{H}_{\rm SB}. \tag{7}$$

The system Hamiltonian describes a single spin in a presence of a B field,

$$\hat{H}_{\rm S} = B\sigma_1^z, \tag{8}$$

where σ_j^q $(q = x, y, z)$ are the Pauli matrices acting on jth site and B is the parameter we wish to estimate, namely, $\lambda = B$. The bath Hamiltonian describes a spin chain with Ising interaction

$$\hat{H}_{\rm B} = \sum_{k>1}^{N} B_x^{(\rm B)}\sigma_k^x + \sum_{k>1}^{N-1} J_x(\sigma_k^+\sigma_{k+1}^- + \sigma_k^-\sigma_{k+1}^+), \tag{9}$$

where $B_x^{(B)}$ is the magnetic field along the x-axis and $J_x > 0$ is the spin-spin coupling. The interaction Hamiltonian describes a coupling between the system spin and a single bath spin of index r,

$$\hat{H}_{\rm SB} = J_z^{(\rm SB)}\sigma_1^z\sigma_r^z + J_x^{(\rm SB)}(\sigma_1^+\sigma_r^- + \sigma_1^-\sigma_r^+), \tag{10}$$

with coupling strengths $J_z^{(\rm SB)}$ and $J_x^{(\rm SB)}$.

As long as the quantum ergodic system is well described by RMT, we expect that expression Eq. (6) holds with the modification $\omega \to 1/D(E_0)$, where $D(E_0)$ is the density of states at the initial energy E_0 [42]. This assumption that $D(E)$ is constant over the width Γ, the energy width of the random wave functions, is what leads to the effective random matrix model with constant energy gap $\omega = 1/N$. This limits the above approach to a weak coupling regime. Outside of the weak coupling regime, the structure of the density of states modulates the eigenstate distribution, often yielding instead Gaussian-distributed eigenstates [50], as will be discussed below.

Then using Eq. (6) with $\hat{H}_0' = \sigma_1^z$ and thereby $(\hat{H}_0'^2)_{\rm mc} = 1$ and $(\Delta\hat{H}_0'^2)_{\rm mc} = 1 - (\sigma_1^z)^2_{\rm mc}$, the QFI becomes

$$F_Q(B) \approx 4t^2\left\{\frac{1}{\pi D(E_0)\Gamma} + \frac{1-(\sigma_1^z)^2_{\rm mc}}{2(\Gamma t)^2}(e^{-2\Gamma t} - 1 + 2\Gamma t)\right\}. \tag{11}$$

Note that in applying the QFI as above, we are describing a local observable of the spin system in terms of RMT. In the Appendix E we show that such local observables are indeed well described in terms of RMT as long as certain energy scales of the system are sufficiently separated. Notably, the sparsity condition above follows trivially for a local observable [47].

To find the value of Γ, we use that the time dependence of an observable $\hat{O}$ obeys [44,45]

$$\langle\hat{O}(t)\rangle = \langle\hat{O}(t)\rangle_0 e^{-2\Gamma t} + \langle\bar{O}\rangle(1 - e^{-2\Gamma t}), \tag{12}$$

where $\langle\hat{O}(t)\rangle_0$ is the evolution of the observable according the noninteraction Hamiltonian $\hat{H}_0$ and $\langle\bar{O}\rangle$ is the time-average value defined by $\langle\bar{O}\rangle = {\rm Tr}(\hat{O}\hat{\rho}_{\rm DE})$. Thermalization of a closed system implies the equality $\langle\bar{O}\rangle \approx (\hat{O})_{\rm mc}$.

In Fig. 3(a), we plot the density of states as a function of the energy. From here, we can extract the value of $D(E_0)$ at the initial energy E_0. We also fit the time evolution of the observable σ_1^z to obtain the value of Γ; see Fig. 3(b). In Fig. 3(c), we show the comparison between the exact result for the QFI using (2) and the analytical expression (11) for various initial states. Since the small spin subsystem thermalizes, the information for the parameter B is locally lost. What we see, however, is that because of the spin-spin interaction the information has not been lost, but spread among the other degrees of freedom. At the beginning of the time evolution for $t \lesssim (2\Gamma)^{-1}$, the time scaling of the QFI is quadratic, $F_Q(B) \approx 4t^2(\Delta\hat{H}_0'^2)_{\rm mc}$, as shown in Fig. 4(a). In this first stage, the information for the parameter is still not locally lost in a sense that it can be determined by measuring only the local spin observable. Since the QFI scales quadratically with time, the statistical uncertainty of the parameter estimation is bounded by the SQL. After this short time period, the growth of the QFI becomes linear in time with the slope determined by the decay rate Γ, namely, $F_Q(B) \approx (4t/\Gamma)(\Delta\hat{H}_0'^2)_{\rm mc}$. In this second stage, the information for the parameter B propagates along the entire system in a sense that global measurements of the spin observables are required in order to determine B. Remarkably, a second quadratic timescale defines the long-time behavior of the QFI that occurs when the information has fully spread between all degrees of freedom. In this third case, we have $F_Q(B) \approx (4t^2/\pi D(E_0)\Gamma)(\hat{H}_0'^2)_{\rm mc}$. In fact, we may connect the third time scaling of QFI with the effective dimension of the system defined by $d_{\rm eff} = (\sum_\mu |\langle\Psi(0)|\psi_\mu\rangle|^4)^{-1}$, which quantifies the ability of a quantum system to thermalize [48]. Also, the mean amplitude of time fluctuations of an observable is bounded by $d_{\rm eff}^{-1/2}$ [42]. The condition $d_{\rm eff} \gg 1$ implies that the initial state is composed of a large number of energy eigenstates which leads to suppression of temporal fluctuations of an observable and equilibration of the system. Using the RMT approach, it can be shown

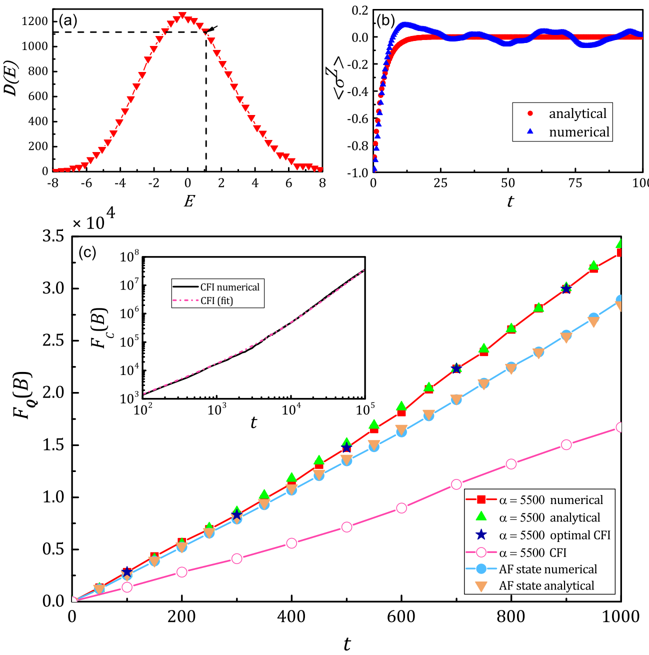

FIG. 3. (a) Density of states, as a function of energy for $N = 13$ and initial state $|\Psi_0\rangle = |\varphi_\alpha\rangle$ with $\alpha = 5500$. The dashed lines show the point $D(E_0)$, where $E_0 = \langle\Psi_0|\hat{H}|\Psi_0\rangle$. (b) Time evolution of the spin system observable σ_1^z both numerically and analytically using Eq. (12). (c) Quantum Fisher information as a function of time for initial state $|\Psi_0\rangle = |\varphi_\alpha\rangle$ with $\alpha = 5500$ for $N = 13$ and for antiferromagnetic initial state $|\Psi_0\rangle = |\uparrow\rangle_S|\downarrow\uparrow\downarrow\ldots\rangle_B$ for $N = 15$. We compare the exact result for the QFI (2) with Hamiltonian (7) and the analytical expression (11). The parameters are set to $B = 0.01$, $B_x^{(\mathrm{B})} = 0.3$, $J_z^{(\mathrm{SB})} = 0.2$, $J_x^{(\mathrm{SB})} = 0.4$, $J_x = 1$, and $r = 5$. The decay rate is $\Gamma = 0.15$ for both $N = 13$ and $N = 15$ is calculated by using Eq. (12) to fit the exact time evolution of the σ_1^z operator for the system spin. The density of states $D(E_0)$ is evaluated by interpolation of $D(E)$. Inset: Long-time evolution of the CFI for $J_x^{(\mathrm{SB})} = 0.4$. We compare the numerical result with the function $at^2 + bt$ with $a = 0.0035$ and $b = 14$.

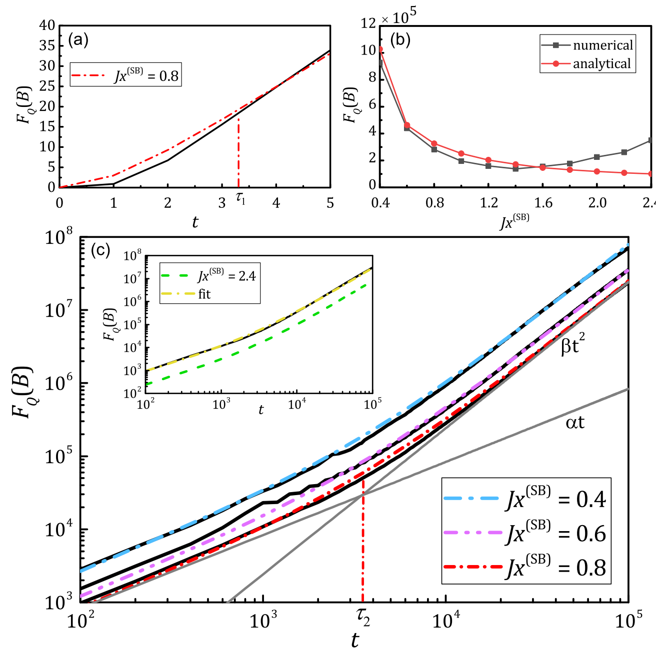

FIG. 4. (a) Short time evolution of QFI for chain with $N = 13$ spins. The vertical dashed-dot line indicates the transition time $\tau_1 = 1/2\Gamma$ between quadratic to linear time regime of QFI. (b) The QFI for various J_x^{SB} for $t = 10^4$. (c) Long time evolution of the QFI. The initial state is $|\Psi_0\rangle = |\varphi_\alpha\rangle$ for $\alpha = 5500$. We compare the numerical results (solid lines) with the analytical expression (11) The vertical dashed-dot line indicates the transition time $\tau_2 = \pi D(E_0)$ between linear to quadratic time regime of QFI. The grey solid lines show the asymptotic behavior of the QFI with $\alpha = 4/\Gamma$ and $\beta = 4/\pi D(E_0)\Gamma$. Inset: Long-time evolution of the QFI for $J_x^{\mathrm{SB}} = 2.4$. We compare the numerical result (solid line) with the analytical expression (green dashed line). We fit the numerical result for the QFI with the function $ct^2 + dt$ for $c = 0.0028$ and $d = 9$ (yellow dash-dotted line).

that $d_{\mathrm{eff}} = (2\pi/3)D(E_0)\Gamma$ [42]. Hence, the long-time behavior of the QFI becomes $F_Q(B) \approx (8t^2/3d_{\mathrm{eff}})(\hat{H}_0'^2)_{\mathrm{mc}}$. This relation indicates that the final quadratic behavior of the QFI occurs when the information of the parameter has been distributed over all quantum states involved in the evolution of the quantum system. The crossover between the linear-to-quadratic time regimes occurs at the Heisenberg time $\tau_2 \approx \pi D(E_0)((\Delta\hat{H}_0'^2)_{\mathrm{mc}}/(\hat{H}_0'^2)_{\mathrm{mc}})$, which is defined as the longest timescale for the system [49]. We point out that the density of states is related to the microcanonical entropy S via the relation $1/D(E_0) = e^{-S}$. Since the entropy is an extensive quantity, the transition time τ increases with the number of spins.

Finally, we note that the study of the time evolution of the quantum echo-dynamics can be used as a signature for quantum chaos. The LE is a measure for the sensitivity of the state vector evolution to small perturbations. Since the QFI and the LE are related, we may expect that similar behavior occurs in the time evolution of the LE [28]. Indeed, the dynamics of the LE for quantum chaotic systems is expected to have diffusive behavior where it scales linearly with time, whereas for regular dynamics one expects ballistic behavior with quadratic time scaling [28,29]. In this sense, our RMT approach provides a general framework for the dynamics of the QFI, which can be related with the properties of the LE. Furthermore, our approach allows one to express the transition times between the different time regimes in terms of decay rate, density of states, and microcanonical average of the observable.

In Fig. 4, we plot the short- and long-time behavior of the QFI. We observe good agreement between the exact and the analytical results. As we see, increasing the time the QFI makes a transition to quadratic time regime is shown in Fig. 4(c). Increasing the spin-bath coupling $J_x^{(\mathrm{SB})}$ leads to higher decay rate Γ, which lowers the QFI according to Eq (11). We note that for larger spin-bath couplings, we expect the same general phenomena, however, with differing functional forms of the random wave-function distribution. For example, at intermediate couplings it has been observed that the random wave function takes a Gaussian form [50], and for strong couplings where a full random matrix Hamiltonian is valid, the density of states dominates the energy dependence and hence leads to a decay in the form of a Bessel function [51,52]. In each case a RMT approach holds, however, the assumption here of Lorentzian wave functions is strictly valid

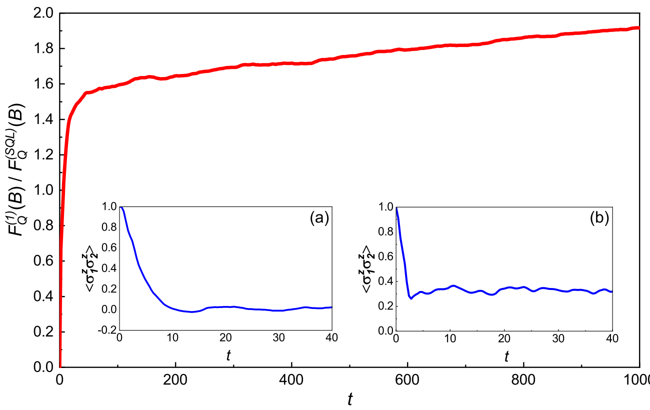

FIG. 5. Exact time evolution of the ratio $F_Q^{(1)}/F_Q^{(\mathrm{SQL})}$ for a spin chain consisting of two system spins. The initial state is $|\Psi_0\rangle = |\uparrow\uparrow\rangle_S|\uparrow\downarrow\uparrow\ldots\rangle_B$ for $N = 15$. Case 1: The system spins interact with different bath spins ($r_1 = 5$ and $r_2 = 8$) with QFI $F_Q^{(\mathrm{SQL})}$. Case 2: Both systems spins interact with the same bath spin ($r_1 = 5$) with corresponding QFI $F_Q^{(1)}$. Time evolution of the correlation $\langle\sigma_1^z\sigma_2^z\rangle$ for cases 1 (a) and 2 (b).

for weak couplings. We show in Fig. 4(c) (inset) the long-time behavior of the QFI for a strong spin-bath coupling. As we expect in this limit, the numerical result deviates quantitatively, however, not qualitatively from the analytical expression (11), as the predicted distinct timescales are evident. We fit the numerical result with the function $ct^2 + dt$, indicating that the same time dependency holds even in this case.

An important issue is whether we can recover the behavior of the QFI by measuring a suitable observable. An optimal measurement that provides equality between CFI and QFI is given by the eigenvectors of the so-called SLD operator $\hat{L}_B$. For a pure state, the SLD operator can be written as $\hat{L}_B = 2(|\partial_B\psi\rangle\langle\psi| + |\psi\rangle\langle\partial_B\psi|)$. We numerically diagonalize $\hat{L}_B$ and, respectively, calculate the CFI as is shown in Fig. 3(c). Such a basis, however, is composed by entangled states and is not suitable for measurement. A more convenient approach is to detect the spin populations $p_{s_1,\ldots,s_N} = \mathrm{Tr}(\hat{\rho}(t)\hat{\Pi}_{s_1,\ldots,s_N})$, where $\hat{\rho}(t) = |\psi(t)\rangle\langle\psi(t)|$ is the density operator and $\hat{\Pi}_{s_1,\ldots,s_N}$ is the projection operator with $s_l = \uparrow_l, \downarrow_l$. In Fig. 3(c) (inset), we plot the long-time behavior of the CFI for such spin observables. We see that although the CFI is lower than QFI, the associated timescales of the QFI are captured by detecting the spin populations.

We proceed with an application of our result (11) to a spin system Hamiltonian consisting of two spins, $\hat{H}_S = B(\sigma_1^z + \sigma_2^z)$. In that case, we have $(\hat{H}_0^{\prime 2})_{\mathrm{mc}} = (\Delta\hat{H}_0^{\prime 2})_{\mathrm{mc}} = 2 + 2(\sigma_1^z\sigma_2^z)_{\mathrm{mc}}$. The system-bath Hamiltonian is given by

$$\hat{H}_{\mathrm{SB}} = J_z^{(\mathrm{SB})}\sigma_1^z\sigma_{r_1}^z + J_x^{(\mathrm{SB})}(\sigma_1^+\sigma_{r_1}^- + \sigma_1^-\sigma_{r_1}^+) + J_z^{(\mathrm{SB})}\sigma_2^z\sigma_{r_2}^z + J_x^{(\mathrm{SB})}(\sigma_2^+\sigma_{r_2}^- + \sigma_2^-\sigma_{r_2}^+), \tag{13}$$

where r_k $(k = 1, 2)$ denotes the position of the bath spins. For two spins coupled to different bath spins $r_1 \neq r_2$, we find that no correlation is created between the system spins in a sense that $(\sigma_1^z\sigma_2^z)_{\mathrm{mc}} = (\sigma_1^z)_{\mathrm{mc}}(\sigma_2^z)_{\mathrm{mc}} \approx 0$, see Fig. 5(a). In that case, the QFI is twice the QFI for a single system spin, $F_Q^{(\mathrm{SQL})}(B) = 2F_Q(B)$, which corresponds to the SQL. Let us now consider the case $r_1 = r_2$, where the two system spins are coupled to a single bath spin. Then the spin-bath interaction creates a correlation between the two system spins in a sense that $(\sigma_1^z\sigma_2^z)_{\mathrm{mc}} \neq 0$; see Fig. 5(b). As long as $(\sigma_1^z\sigma_2^z)_{\mathrm{mc}} > 0$, we have $F_Q^{(1)}(B) > F_Q^{(\mathrm{SQL})}(B)$ and thus one can overcome the SQL. We plot in Fig. 5 the exact time evolution of the ratio $F_Q^{(1)}/F_Q^{(\mathrm{SQL})}$. We see that the positive quantum correlation between the system spins leads to enhancement of the QFI compared with the $F_Q^{(\mathrm{SQL})}(B)$.

V. CONCLUSION

We use a RMT approach to derive an analytical expression for the time evolution of the QFI in quantum ergodic systems. We find that the QFI obeys three different time regimes. At the beginning of the time evolution, the QFI grows quadratically, which quickly passes into a linear growth with a slope defined by the width of the random wave function. Furthermore, we find a second quadratic timescale which determines the long-time behavior of the QFI. This timescale is shown to correspond to the Heisenberg time, after which the information of the local observable has spread throughout all accessible degrees of freedom of the system.

We have compared our RMT result with an exact diagonalization of a nonintegrable spin chain, confirming the RMT prediction of the three separate timescales. We have shown that the information for a parameter describing a single spin system is locally lost but propagates among the other degrees of freedom of the spin system. The transition time between the linear and quadratic time regimes depends on the density of states, and increases with the number of spins.

Our analytical approach is based on the assumption that the probability distribution of the random variables is given by a Lorentzian functional form. Such a form is obtained in the perturbative regime of weak coupling, and thus we only expect good agreement with the RMT result when the interaction Hamiltonian is small. The only feature of our approach, which is sensitive to this assumption, however, is that the explicit expression for the time evolution of the QFI depends on the Lorentzian probability distribution derived from the weak coupling assumptions. In the strong coupling regime, we may expect that the probability distribution of the random variables will become Gaussian. This will modify our main result but we anticipate, and indeed numerically observe, that the asymptotic behavior of the QFI will be the same but with different prefactors. Furthermore, the result for the QFI in quantum ergodic systems can be used for a comparison with the evolution of the QFI in a system with disorder, where the ergodicity is broken and the systems can fail to thermalize. Thus, the observation of the time evolution of QFI can be used as a potential signature for many-body localization.

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APPENDIX A: QUANTUM FISHER INFORMATION IN THE MANY-BODY INTERACTING BASIS

The quantum Fisher information for a pure state is given by

$$F_Q(\lambda) = 4\{\langle\partial_\lambda\psi|\partial_\lambda\psi\rangle - \langle\psi|\partial_\lambda\psi\rangle\langle\partial_\lambda\psi|\psi\rangle\}, \tag{A1}$$

where the state vector is $|\psi\rangle = e^{-i\hat{H}t}|\Psi_0\rangle$ and $|\Psi_0\rangle$ is the initial state which is independent on parameter λ. Therefore, we have $|\partial_\lambda\psi\rangle = (\partial_\lambda e^{-i\hat{H}t})|\Psi_0\rangle$. The partial derivative can be written as

$$\partial_\lambda e^{-i\hat{H}t} = -it\int_0^1 ds e^{-i\hat{H}t}e^{i\hat{H}ts}(\partial_\lambda\hat{H})e^{-i\hat{H}ts}. \tag{A2}$$

The quantum Fisher information can be rewritten as

$$\begin{aligned}F_Q(\lambda) = 4\{&\langle\Psi_0|(\partial_\lambda e^{i\hat{H}t})(\partial_\lambda e^{-i\hat{H}t})|\Psi_0\rangle\\ &- |\langle\Psi_0|e^{i\hat{H}t}(\partial_\lambda e^{-i\hat{H}t})|\Psi_0\rangle|^2\}.\end{aligned} \tag{A3}$$

Now, let us consider the first term. We have

$$\begin{aligned}&\langle\Psi_0|(\partial_\lambda e^{i\hat{H}t})(\partial_\lambda e^{-i\hat{H}t})|\Psi_0\rangle\\ &= \sum_{\mu\nu\rho} a_\mu^* a_\nu\langle\psi_\mu|\partial_\lambda e^{i\hat{H}t}e^{-i\hat{H}t}|\psi_\rho\rangle\\ &\quad\times\langle\psi_\rho|e^{i\hat{H}t}\partial_\lambda e^{-i\hat{H}t}|\psi_\nu\rangle,\end{aligned} \tag{A4}$$

where we use that $|\Psi_0\rangle = \sum_\mu a_\mu|\psi_\mu\rangle$ and $\sum_\rho|\psi_\rho\rangle\langle\psi_\rho| = \mathbf{1}$. Using (A2), we obtain

$$\begin{aligned}\langle\psi_\rho|e^{i\hat{H}t}\partial_\lambda e^{-i\hat{H}t}|\psi_\nu\rangle &= -it\int_0^1 ds\langle\psi_\rho|e^{is\hat{H}t}\partial_\lambda\hat{H}e^{-is\hat{H}t}|\psi_\nu\rangle\\ &= -it\langle\psi_\rho|\partial_\lambda\hat{H}|\psi_\nu\rangle\int_0^1 ds e^{i(E_\rho-E_\nu)st}\\ &= -it\langle\psi_\rho|\partial_\lambda\hat{H}|\psi_\nu\rangle e^{i\theta_{\rho\nu}t}\mathrm{sinc}(\theta_{\rho\nu}t).\end{aligned} \tag{A5}$$

Here we have defined $\theta_{\mu\nu} = \frac{E_\mu - E_\nu}{2}$. Therefore, we get

$$\begin{aligned}&\langle\Psi_0|(\partial_\lambda e^{i\hat{H}t})(\partial_\lambda e^{-i\hat{H}t})|\Psi_0\rangle\\ &\quad= t^2\sum_{\mu\nu\rho} a_\mu^* a_\nu\langle\psi_\mu|\partial_\lambda\hat{H}|\psi_\rho\rangle\langle\psi_\rho|\partial_\lambda\hat{H}|\psi_\nu\rangle\\ &\qquad\times e^{i\theta_{\mu\nu}}\mathrm{sinc}(\theta_{\mu\rho}t)\mathrm{sinc}(\theta_{\rho\nu}t).\end{aligned} \tag{A6}$$

Similarly, for the second term, we obtain

$$\begin{aligned}&\langle\Psi_0|e^{i\hat{H}t}(\partial_\lambda e^{-i\hat{H}t})|\Psi_0\rangle\\ &\quad= -it\sum_{\mu\nu} a_\mu^* a_\nu\int_0^1 ds\langle\psi_\mu|e^{is\hat{H}t}\partial_\lambda\hat{H}e^{-is\hat{H}t}|\psi_\nu\rangle\\ &\quad= -it\sum_{\mu\nu} a_\mu^* a_\nu e^{i\theta_{\mu\nu}t}\langle\psi_\mu|\partial_\lambda\hat{H}|\psi_\nu\rangle\mathrm{sinc}(\theta_{\mu\nu}t).\end{aligned} \tag{A7}$$

Using (A1), (A6), and (A7) we obtain

$$F_Q(\lambda) = 4t^2\left\{\sum_{\mu\nu\rho} a_\mu^* a_\nu(\partial_\lambda\hat{H}_0)_{\mu\rho}(\partial_\lambda\hat{H}_0)_{\rho\nu}e^{i\theta_{\mu\nu}t}\mathrm{sinc}(\theta_{\mu\rho}t)\mathrm{sinc}(\theta_{\rho\nu}t) - \left|\sum_{\mu\nu} a_\mu^* a_\nu e^{i\theta_{\mu\nu}t}(\partial_\lambda\hat{H}_0)_{\mu\nu}\mathrm{sinc}(\theta_{\mu\nu}t)\right|^2\right\}. \tag{A8}$$

APPENDIX B: CORRELATION FUNCTIONS

In this Appendix, we outline the core RMT approach to eigenstate correlations formulated in Ref. [42]. We can calculate arbitrary correlation functions of the random wave functions $c_\mu(\alpha)$ by defining the respective generating function. For $\mu = \nu$, it reads

$$G_{\mu\mu}(\vec{\xi}_\mu) \propto e^{\frac{1}{2}\sum_\alpha \xi_{\mu,\alpha}^2\Lambda(\mu,\alpha)}, \tag{B1}$$

where $\vec{\xi}_\mu = (\xi_{\mu,1}, \xi_{\mu,2}\ldots\xi_{\mu,N})$ are ancillary fields. An arbitrary correlation function of the random wave functions can be obtained via

$$\langle c_\mu(\alpha)c_\mu(\alpha')\ldots c_\mu(\beta)c_\mu(\beta')\rangle_V = \frac{1}{G_{\mu\mu}}\partial_{\xi_{\mu,\alpha}}\partial_{\xi_{\mu,\alpha'}}\ldots\partial_{\xi_{\mu,\beta}}\partial_{\xi_{\mu,\beta'}}G_{\mu\mu}|_{\xi_{\mu,\alpha}=0}. \tag{B2}$$

Similarly, for $\mu \neq \nu$ the generating function is

$$G_{\mu\nu}(\vec{\xi}_\mu, \vec{\xi}_\nu) \propto e^{\frac{1}{2}\sum_\alpha \xi_{\mu,\alpha}^2\Lambda(\mu,\alpha) + \frac{1}{2}\sum_\alpha \xi_{\nu,\alpha}^2\Lambda(\nu,\alpha) - \frac{1}{2}\sum_{\alpha\beta}\xi_{\mu,\alpha}\xi_{\mu,\beta}\xi_{\nu,\alpha}\xi_{\nu,\beta}\frac{\Lambda(\mu,\alpha)\Lambda(\mu,\beta)\Lambda(\nu,\alpha)\Lambda(\nu,\beta)}{\sum_\gamma\Lambda(\mu,\gamma)\Lambda(\nu,\gamma)}} \tag{B3}$$

and the correlation function becomes

$$\langle c_\mu(\alpha)c_\nu(\alpha')\ldots c_\mu(\beta)c_\nu(\beta')\rangle_V = \frac{1}{G_{\mu\nu}}\partial_{\xi_{\mu,\alpha}}\partial_{\xi_{\nu,\alpha'}}\ldots\partial_{\xi_{\mu,\beta}}\partial_{\xi_{\nu,\beta'}}G_{\mu\nu}|_{\xi_{\mu,\alpha}=0,\xi_{\nu,\alpha}=0}. \tag{B4}$$

To evaluate the QFI, we focus on two sets of four-point correlation functions of interest: $\langle c_\mu(\alpha)c_\nu(\beta)c_\mu(\alpha')c_\nu(\beta')\rangle_V$ for $\mu = \nu$ and $\mu \neq \nu$.

For $\mu = \nu$, the random wave functions can be treated as independent random variables, namely,

$$\langle c_\mu(\alpha)c_\mu(\beta)c_\mu(\alpha')c_\mu(\beta')\rangle_V = \Lambda(\mu,\alpha)\Lambda(\mu,\beta)\delta_{\alpha\alpha'}\delta_{\beta\beta'} + \Lambda(\mu,\alpha)\Lambda(\mu,\alpha')(\delta_{\alpha'\beta'}\delta_{\alpha\beta} + \delta_{\alpha\beta'}\delta_{\alpha'\beta}). \tag{B5}$$

For $\mu \neq \nu$, we have

$$\langle c_\mu(\alpha)c_\nu(\beta)c_\mu(\alpha')c_\nu(\beta')\rangle_V = \Lambda(\mu,\alpha)\Lambda(\nu,\beta)\delta_{\alpha\alpha'}\delta_{\beta\beta'} - \frac{\Lambda(\mu,\alpha)\Lambda(\nu,\beta)\Lambda(\mu,\alpha')\Lambda(\nu,\beta')}{\sum_\gamma \Lambda(\mu,\gamma)\Lambda(\nu,\gamma)}(\delta_{\alpha\beta}\delta_{\alpha'\beta'} + \delta_{\alpha\beta'}\delta_{\alpha'\beta}). \tag{B6}$$

The first term in (B6) describes the four-point correlation function as an independent random Gaussian variables, while the last two terms correspond to the non-Gaussian correction which arises as a result of the orthogonality condition.

In fact, we may express graphically the correlation functions as a sum of products of two-point correlation functions. Consider first $\mu = \nu$. Then we have

$$\langle c_\mu(\alpha)c_\mu(\beta)c_\mu(\alpha')c_\mu(\beta')\rangle_V = \langle c_\mu(\overline{\alpha)c_\mu(\beta})c_\mu(\overline{\alpha')c_\mu(\beta'})\rangle_V + \langle c_\mu(\overline{\alpha)c_\mu(\underline{\beta)c_\mu(\alpha'})c_\mu(\beta'})\rangle_V + \langle c_\mu(\overline{\alpha)c_\mu(\underline{\beta)c_\mu(\alpha'})c_\mu(\beta'})\rangle_V. \tag{B7}$$

Each of the terms can be written as a product of two-point correlation functions. For example,

$$\langle c_\mu(\overline{\alpha)c_\mu(\beta})c_\mu(\overline{\alpha')c_\mu(\beta'})\rangle_V = \langle c_\mu(\alpha)c_\mu(\beta)\rangle_V\langle c_\mu(\alpha')c_\mu(\beta')\rangle_V = \Lambda(\mu,\alpha)\delta_{\alpha\beta}\Lambda(\mu,\alpha')\delta_{\alpha'\beta'}. \tag{B8}$$

For the case $\mu \neq \nu$, we also need to include the non-Gaussian corrections. We have

$$\langle c_\mu(\alpha)c_\nu(\beta)c_\mu(\alpha')c_\nu(\beta')\rangle_V = \langle c_\mu(\overline{\alpha)c_\nu(\underline{\beta)c_\mu(\alpha'})c_\nu(\beta'})\rangle_V + \langle c_\mu(\overline{\alpha)c_\nu(\beta})c_\mu(\overline{\alpha')c_\nu(\beta'})\rangle_V + \langle c_\mu(\overline{\alpha)c_\nu(\underline{\beta)c_\mu(\alpha'})c_\nu(\beta'})\rangle_V. \tag{B9}$$

The last two terms in (B9) arise as a result of the orthogonality condition between the many-body eigenstates. For example, the first non-Gaussian term is

$$\langle c_\mu(\overline{\alpha)c_\nu(\beta})c_\mu(\overline{\alpha')c_\nu(\beta'})\rangle_V = -\frac{\Lambda(\mu,\alpha)\Lambda(\nu,\beta)\Lambda(\mu,\alpha')\Lambda(\nu,\beta')}{\sum_\gamma \Lambda(\mu,\gamma)\Lambda(\nu,\gamma)}\delta_{\alpha\beta}\delta_{\alpha'\beta'}, \tag{B10}$$

and similarly for the second one.

APPENDIX C: CALCULATION OF THE QFI USING RANDOM MATRIX APPROACH

Here we provide the method which we use to evaluate the QFI (6). We set $\partial_\lambda \hat{H}_0 = \hat{H}_0'$ and assume that $\hat{H}_0'$ is a diagonal matrix in the noninteracting basis. Therefore, the QFI is

$$F_Q(\lambda) = 4t^2\left\{\sum_{\mu\nu\rho} a_\mu^* a_\nu (\hat{H}_0')_{\mu\rho}(\hat{H}_0')_{\rho\nu} e^{i\theta_{\mu\nu}t}\mathrm{sinc}(\theta_{\mu\rho}t)\mathrm{sinc}(\theta_{\rho\nu}t) - \left|\sum_{\mu\nu} a_\mu^* a_\nu e^{i\theta_{\mu\nu}t}(\hat{H}_0')_{\mu\nu}\mathrm{sinc}(\theta_{\mu\nu}t)\right|^2\right\}. \tag{C1}$$

Let us now consider separately the first term in (C1), namely,

$$\begin{aligned}
&\sum_{\mu\nu\rho} a_\mu^* a_\nu \langle\psi_\mu|\hat{H}_0'|\psi_\rho\rangle\langle\psi_\rho|\hat{H}_0'|\psi_\nu\rangle e^{i\theta_{\mu\nu}t}\mathrm{sinc}(\theta_{\mu\rho}t)\mathrm{sinc}(\theta_{\rho\nu}t)\\
&= \sum_\mu |a_\mu|^2|\langle\psi_\mu|\hat{H}_0'|\psi_\mu\rangle|^2 + \sum_{\substack{\mu\nu\\ \mu\neq\nu}} |a_\mu|^2|\langle\psi_\mu|\hat{H}_0'|\psi_\nu\rangle|^2\mathrm{sinc}^2(\theta_{\mu\nu}t)\\
&\quad + \sum_{\substack{\mu\nu\\ \mu\neq\nu}} a_\mu^* a_\nu\langle\psi_\mu|\hat{H}_0'|\psi_\nu\rangle\langle\psi_\nu|\hat{H}_0'|\psi_\nu\rangle e^{i\theta_{\mu\nu}t}\mathrm{sinc}(\theta_{\mu\nu}t) + \sum_{\substack{\mu\nu\\ \mu\neq\nu}} a_\mu^* a_\nu\langle\psi_\mu|\hat{H}_0'|\psi_\mu\rangle\langle\psi_\mu|\hat{H}_0'|\psi_\nu\rangle e^{i\theta_{\mu\nu}t}\mathrm{sinc}(\theta_{\mu\nu}t)\\
&\quad + \sum_{\substack{\mu\nu\rho\\ \mu\neq\nu\neq\rho}} a_\mu^* a_\nu\langle\psi_\mu|\hat{H}_0'|\psi_\rho\rangle\langle\psi_\rho|\hat{H}_0'|\psi_\nu\rangle e^{i\theta_{\mu\nu}t}\mathrm{sinc}(\theta_{\mu\rho}t)\mathrm{sinc}(\theta_{\rho\nu}t).
\end{aligned} \tag{C2}$$

Now we apply the self-averaging condition for each of the terms in (C2). For the first one, we have

$$\sum_\mu |a_\mu|^2|\langle\psi_\mu|\hat{H}_0'|\psi_\mu\rangle|^2 = \sum_\mu \langle|a_\mu|^2|\langle\psi_\mu|\hat{H}_0'|\psi_\mu\rangle|^2\rangle_V. \tag{C3}$$

To evaluate (C3), we may further decouple the coefficients a_μ describing the initial state part and observable in the sense that (see Appendix D for more details)

$$\sum_\mu \langle|a_\mu|^2|\langle\psi_\mu|\hat{H}_0'|\psi_\mu\rangle|^2\rangle_V = \sum_\mu \langle|a_\mu|^2\rangle_V\langle|\langle\psi_\mu|\hat{H}_0'|\psi_\mu\rangle|^2\rangle_V. \tag{C4}$$

Therefore, we have

$$\begin{aligned}\langle|\langle\psi_\mu|\hat{H}_0'|\psi_\mu\rangle|^2\rangle_V &= \sum_{\alpha\beta}\langle c_\mu(\alpha)c_\mu(\alpha)c_\mu(\beta)c_\mu(\beta)\rangle_V(\hat{H}_0')_{\alpha\alpha}(\hat{H}_0')_{\beta\beta}\\ &= \sum_{\alpha\beta}(2\Lambda^2(\mu,\alpha)\delta_{\alpha\beta}(\hat{H}_0')_{\alpha\alpha}(\hat{H}_0')_{\beta\beta} + \Lambda(\mu,\alpha)(\hat{H}_0')_{\alpha\alpha}\Lambda(\mu,\beta)(\hat{H}_0')_{\beta\beta}).\end{aligned} \tag{C5}$$

We define the average $[(\hat{H}_0')_{\alpha\alpha}]_\mu = \sum_\alpha \Lambda(\mu,\alpha)(\hat{H}_0')_{\alpha\alpha}$, which is essentially a microcanonical average centered on the energy E_μ. We also apply the smoothness condition, which implies that the variation of $[(\hat{H}_0')_{\alpha\alpha}]_\mu$ as a function of E_μ can be neglected. Using this, we obtain

$$\langle|\langle\psi_\mu|\hat{H}_0'|\psi_\mu\rangle|^2\rangle_V \approx 2\big[(\hat{H}_0')^2_{\alpha\alpha}\big]_\mu \sum_\alpha \Lambda^2(\mu,\alpha) + [(\hat{H}_0')_{\alpha\alpha}]^2_\mu. \tag{C6}$$

Further, we take the continuum limit, substituting $\sum_\alpha \to \int_{-\infty}^{\infty}\frac{dE_\alpha}{\omega}$, and thereby obtain

$$\sum_\alpha \Lambda^2(\mu,\alpha) = \frac{1}{\omega}\int_{-\infty}^{\infty}\Lambda^2(\mu,\alpha)dE_\alpha = \frac{\omega}{2\pi\Gamma}. \tag{C7}$$

Substituting in Eq. (C3), we have

$$\sum_\mu |a_\mu|^2|\langle\psi_\mu|\hat{H}_0'|\psi_\mu\rangle|^2 = \sum_\mu\langle|a_\mu|^2\rangle_V\left(\frac{\omega}{\pi\Gamma}\big[(\hat{H}_0')^2_{\alpha\alpha}\big]_\mu + [(\hat{H}_0')_{\alpha\alpha}]^2_\mu\right). \tag{C8}$$

As long as $[(\hat{H}_0')^2_{\alpha\alpha}]_\mu$ and $[(\hat{H}_0')_{\alpha\alpha}]^2_\mu$ are smooth functions of the energy E_μ and the probabilities $|a_\mu|^2$ take nonvanishing value close to the mean energy $E_0 = \langle\Psi_0|\hat{H}|\Psi_0\rangle$ with $|\Psi_0\rangle$ being the initial state, the ETH ensures that Eq. (C8) is equivalent to a microcanonical average:

$$\sum_\mu |a_\mu|^2|\langle\psi_\mu|\hat{H}_0'|\psi_\mu\rangle|^2 = \frac{\omega}{\pi\Gamma}\big(\hat{H}_0'^2\big)_{\rm mc} + (\hat{H}_0')^2_{\rm mc}. \tag{C9}$$

Consider the second term in (C2):

$$\sum_{\substack{\mu\nu\\ \mu\neq\nu}} |a_\mu|^2|\langle\psi_\mu|\hat{H}_0'|\psi_\nu\rangle|^2{\rm sinc}^2(\theta_{\mu\nu}t) = \sum_{\substack{\mu\nu\\ \mu\neq\nu}}\langle|a_\mu|^2\rangle_V\langle|\langle\psi_\mu|\hat{H}_0'|\psi_\nu\rangle|^2\rangle_V{\rm sinc}^2(\theta_{\mu\nu}t). \tag{C10}$$

For the matrix element, we have

$$\langle|\langle\psi_\mu|\hat{H}_0'|\psi_\nu\rangle|^2\rangle_V = \sum_{\alpha\beta}\langle c_\mu(\alpha)c_\nu(\alpha)c_\mu(\beta)c_\nu(\beta)\rangle_V(\hat{H}_0')_{\alpha\alpha}(\hat{H}_0')_{\beta\beta}. \tag{C11}$$

Now, using (B6) we obtain

$$\langle c_\mu(\alpha)c_\nu(\alpha)c_\mu(\beta)c_\nu(\beta)\rangle_V = \Lambda(\mu,\alpha)\Lambda(\nu,\alpha)\delta_{\alpha\beta} - \frac{\Lambda(\mu,\alpha)\Lambda(\nu,\alpha)\Lambda(\mu,\beta)\Lambda(\nu,\beta)}{\sum_\gamma\Lambda(\mu,\gamma)\Lambda(\nu,\gamma)} - \frac{\Lambda^2(\mu,\alpha)\Lambda^2(\nu,\alpha)}{\sum_\gamma\Lambda(\mu,\gamma)\Lambda(\nu,\gamma)}\delta_{\alpha\beta} \tag{C12}$$

and the matrix element becomes

$$\begin{aligned}\langle|\langle\psi_\mu|\hat{H}_0'|\psi_\nu\rangle|^2\rangle_V &= \sum_{\alpha\beta}\left(\Lambda(\mu,\alpha)\Lambda(\nu,\alpha)\delta_{\alpha\beta} - \frac{\Lambda(\mu,\alpha)\Lambda(\nu,\alpha)\Lambda(\mu,\beta)\Lambda(\nu,\beta)}{\sum_\gamma\Lambda(\mu,\gamma)\Lambda(\nu,\gamma)} - \frac{\Lambda^2(\mu,\alpha)\Lambda^2(\nu,\alpha)}{\sum_\gamma\Lambda(\mu,\gamma)\Lambda(\nu,\gamma)}\delta_{\alpha\beta}\right)(\hat{H}_0')_{\alpha\alpha}(\hat{H}_0')_{\beta\beta}\\ &\approx [(\Delta\hat{H}_0')_{\alpha\alpha}]^2_{\bar{\mu}}\sum_\alpha\Lambda(\mu,\alpha)\Lambda(\nu,\alpha),\end{aligned} \tag{C13}$$

where $[(\Delta\hat{H}_0')_{\alpha\alpha}]^2_{\bar{\mu}}$ is the variance, $\bar{\mu} = \frac{\mu+\nu}{2}$. Going in the continuum limit, we get

$$\sum_\alpha\Lambda(\mu,\alpha)\Lambda(\nu,\alpha) = \frac{2\omega\Gamma}{\pi}\frac{1}{(E_\mu-E_\nu)^2+4\Gamma^2}. \tag{C14}$$

Therefore, the second term becomes

$$\sum_{\substack{\mu\nu\\ \mu\neq\nu}}\langle|a_\mu|^2\rangle_V\langle|\langle\psi_\mu|\hat{H}_0'|\psi_\nu\rangle|^2\rangle_V{\rm sinc}^2(\theta_{\mu\nu}t) = \frac{2\omega\Gamma}{\pi}\sum_{\substack{\mu\nu\\ \mu\neq\nu}}\langle|a_\mu|^2\rangle_V[(\Delta\hat{H}_0')_{\alpha\alpha}]^2_{\bar{\mu}}\frac{{\rm sinc}^2(\theta_{\mu\nu}t)}{(E_\mu-E_\nu)^2+4\Gamma^2}. \tag{C15}$$

We now replace the sum over the index ν with integration, namely,

$$\sum_{\nu}\frac{\mathrm{sinc}^2(\theta_{\mu\nu}t)}{(E_\mu-E_\nu)^2+4\Gamma^2}\to\frac{1}{\omega}\int_{-\infty}^{\infty}\frac{\mathrm{sinc}^2(\theta_{\mu\nu}t)}{(E_\mu-E_\nu)^2+4\Gamma^2}dE_\nu=\frac{t}{2\omega}\int_{-\infty}^{\infty}\frac{\mathrm{sinc}^2(x)}{x^2+(\Gamma t)^2}dx=\frac{\pi t}{4\omega(\Gamma t)^3}(e^{-2\Gamma t}-1+2\Gamma t).\quad\text{(C16)}$$

Finally, we have

$$\sum_{\substack{\mu\nu\\ \mu\neq\nu}}|a_\mu|^2|\langle\psi_\mu|\hat{H}_0'|\psi_\nu\rangle|^2\mathrm{sinc}^2(\theta_{\mu\nu}t)=\frac{(\Delta\hat{H}_0'^2)_{\mathrm{mc}}}{2(\Gamma t)^2}(e^{-2\Gamma t}-1+2\Gamma t).\quad\text{(C17)}$$

We note that in Eq. (C13) we have neglected the contribution from the third term which is of order of $(\omega/\Gamma)^2$. Indeed, we have

$$\sum_{\alpha}\Lambda^2(\mu,\alpha)\Lambda^2(\nu,\alpha)\to\left(\frac{\omega\Gamma}{\pi}\right)^4\frac{1}{\omega}\int_{-\infty}^{\infty}\frac{dE_\alpha}{((E_\mu-E_\alpha)^2+\Gamma^2)^2((E_\nu-E_\alpha)^2+\Gamma^2)^2}$$
$$=\left(\frac{\omega\Gamma}{\pi}\right)^4\frac{\pi}{\omega\Gamma^3}\frac{(E_\mu-E_\nu)^2+20\Gamma^2}{((E_\mu-E_\nu)^2+4\Gamma^2)^3}.\quad\text{(C18)}$$

Therefore, using (C14) we conclude that the third term in (C13) is of the order of $(\omega/\Gamma)^2$. We also note that the contribution for $\mu=\nu$ in Eq. (C15) can be neglected.

Consider the third term in (C2). We have

$$\sum_{\substack{\mu\nu\\ \mu\neq\nu}}a_\mu^*a_\nu\langle\psi_\mu|\hat{H}_0'|\psi_\nu\rangle\langle\psi_\nu|\hat{H}_0'|\psi_\nu\rangle e^{i\theta_{\mu\nu}t}\mathrm{sinc}(\theta_{\mu\nu}t)=\sum_{\substack{\mu\nu\\ \mu\neq\nu}}\langle a_\mu^*a_\nu\rangle_V\langle\langle\psi_\mu|\hat{H}_0'|\psi_\nu\rangle\langle\psi_\nu|\hat{H}_0'|\psi_\nu\rangle\rangle_V e^{i\theta_{\mu\nu}t}\mathrm{sinc}(\theta_{\mu\nu}t).\quad\text{(C19)}$$

The matrix elements expressed in the noninteracting basis are

$$\langle\langle\psi_\mu|\hat{H}_0'|\psi_\nu\rangle\langle\psi_\nu|\hat{H}_0'|\psi_\nu\rangle\rangle_V=\sum_{\alpha\beta}\langle c_\mu(\alpha)c_\nu(\alpha)c_\nu(\beta)c_\nu(\beta)\rangle_V(\hat{H}_0')_{\alpha\alpha}(\hat{H}_0')_{\beta\beta}.\quad\text{(C20)}$$

Using the generating function Eq. (B3), the four-point correlation function is

$$\langle c_\mu(\alpha)c_\nu(\alpha)c_\nu(\beta)c_\nu(\beta)\rangle_V=0.\quad\text{(C21)}$$

Similarly, for the last term in (C2) we have

$$\langle\langle\psi_\mu|\hat{H}_0'|\psi_\rho\rangle\langle\psi_\rho|\hat{H}_0'|\psi_\nu\rangle\rangle_V=\sum_{\alpha\beta}\langle c_\mu(\alpha)c_\rho(\alpha)c_\rho(\beta)c_\nu(\beta)\rangle_V(\hat{H}_0')_{\alpha\alpha}(\hat{H}_0')_{\beta\beta}.\quad\text{(C22)}$$

Now, the four-point correlation function contains three different indices. To evaluate it, we need to introduce three auxiliary fields. Because the indexes μ and ν repeat only once, we have

$$\langle c_\mu(\alpha)c_\rho(\alpha)c_\rho(\beta)c_\nu(\beta)\rangle_V=0.\quad\text{(C23)}$$

Combining all averages in (C2), we obtain

$$\sum_{\mu\nu\rho}a_\mu^*a_\nu\langle\psi_\mu|\hat{H}_0'|\psi_\rho\rangle\langle\psi_\rho|\hat{H}_0'|\psi_\nu\rangle e^{i\theta_{\mu\nu}t}\mathrm{sinc}(\theta_{\mu\rho}t)\mathrm{sinc}(\theta_{\rho\nu}t)=\frac{\omega}{\pi\Gamma}(\hat{H}_0'^2)_{\mathrm{mc}}+(\hat{H}_0')^2_{\mathrm{mc}}+\frac{(\Delta\hat{H}_0'^2)_{\mathrm{mc}}}{2(\Gamma t)^2}(e^{-2\Gamma t}-1+2\Gamma t).\quad\text{(C24)}$$

Let us now consider separately the second term in (C1), namely

$$\sum_{\mu\nu}\sum_{\mu'\nu'}a_\mu^*a_\nu a_{\mu'}a_{\nu'}^*\langle\psi_\mu|\hat{H}_0'|\psi_\nu\rangle\langle\psi_{\nu'}|\hat{H}_0'|\psi_{\mu'}\rangle e^{i\theta_{\mu\nu}t}e^{-i\theta_{\mu'\nu'}t}\mathrm{sinc}(\theta_{\mu\nu}t)\mathrm{sinc}(\theta_{\mu'\nu'}t)$$
$$=\sum_{\mu}|a_\mu|^4|\langle\psi_\mu|\hat{H}_0'|\psi_\mu\rangle|^2+\sum_{\substack{\mu\nu\\ \mu\neq\nu}}|a_\mu|^2|a_\nu|^2\langle\psi_\mu|\hat{H}_0'|\psi_\mu\rangle\langle\psi_\nu|\hat{H}_0'|\psi_\nu\rangle+\sum_{\substack{\mu\nu\\ \mu\neq\nu}}|a_\mu|^2|a_\nu|^2\langle\psi_\mu|\hat{H}_0'|\psi_\nu\rangle\langle\psi_\nu|\hat{H}_0'|\psi_\mu\rangle\mathrm{sinc}^2(\theta_{\mu\nu}t)$$
$$+\sum_{\substack{\mu\nu\\ \mu\neq\nu}}a_\mu^{*2}a_\nu^2\langle\psi_\mu|\hat{H}_0'|\psi_\nu\rangle\langle\psi_\mu|\hat{H}_0'|\psi_\nu\rangle e^{2i\theta_{\mu\nu}t}\mathrm{sinc}^2(\theta_{\mu\nu}t)+\cdots.\quad\text{(C25)}$$

We have

$$\sum_{\mu}|a_\mu|^4|\langle\psi_\mu|\hat{H}_0'|\psi_\mu\rangle|^2=\sum_{\mu}\langle|a_\mu|^4\rangle_V\left(\frac{\omega}{\pi\Gamma}[(\hat{H}_0')^2_{\alpha\alpha}]_\mu+\left[(\hat{H}_0')_{\alpha\alpha}\right]_\mu^2\right)=\left(\frac{\omega}{\pi\Gamma}(\hat{H}_0'^2)_{\mathrm{mc}}+(\hat{H}_0')^2_{\mathrm{mc}}\right)\sum_{\mu}\langle|a_\mu|^4\rangle_V.\quad\text{(C26)}$$

Note that $(\frac{\omega}{\pi\Gamma})(\hat{H}_0'^2)_{\rm mc}\langle\sum_\mu |a_\mu|^4\rangle_V$ is of order of $(\omega/\Gamma)^2$ and thereby neglected. The second term in (C25) is

$$\sum_{\substack{\mu\nu\\ \mu\neq\nu}} |a_\mu|^2|a_\nu|^2\langle\psi_\mu|\hat{H}_0'|\psi_\mu\rangle\langle\psi_\nu|\hat{H}_0'|\psi_\nu\rangle = \sum_{\substack{\mu\nu\\ \mu\neq\nu}}\langle|a_\mu|^2|a_\nu|^2\rangle_V\langle\langle\psi_\mu|\hat{H}_0'|\psi_\mu\rangle\langle\psi_\nu|\hat{H}_0'|\psi_\nu\rangle\rangle_V. \tag{C27}$$

The matrix elements are

$$\begin{aligned}\langle\langle\psi_\mu|\hat{H}_0'|\psi_\mu\rangle\langle\psi_\nu|\hat{H}_0'|\psi_\nu\rangle\rangle_V &= \sum_{\alpha\beta}\langle c_\mu(\alpha)c_\mu(\alpha)c_\nu(\beta)c_\nu(\beta)\rangle_V(\hat{H}_0')_{\alpha\alpha}(\hat{H}_0')_{\beta\beta}\\ &= \sum_{\alpha\beta}\left(\Lambda(\mu,\alpha)\Lambda(\nu,\beta) - 2\frac{\Lambda^2(\mu,\alpha)\Lambda^2(\nu,\beta)}{\sum_\gamma \Lambda(\mu,\gamma)\Lambda(\nu,\gamma)}\delta_{\alpha\beta}\right)(\hat{H}_0')_{\alpha\alpha}(\hat{H}_0')_{\beta\beta}\\ &\approx [(\hat{H}_0')_{\alpha\alpha}]^2_{\bar{\mu}},\end{aligned} \tag{C28}$$

where we neglect the second term, which is of order of $(\omega/\Gamma)^2$. Therefore, we obtain

$$\sum_{\substack{\mu\nu\\ \mu\neq\nu}}\langle|a_\mu|^2|a_\nu|^2\rangle_V[(\hat{H}_0')_{\alpha\alpha}]^2_{\bar{\mu}} \approx (\hat{H}_0')^2_{\rm mc}\left(\sum_{\mu\nu}\langle|a_\mu|^2|a_\nu|^2\rangle_V - \sum_\mu\langle|a_\mu|^4\rangle_V\right) = (\hat{H}_0')^2_{\rm mc}\left(1-\sum_\mu\langle|a_\mu|^4\rangle_V\right). \tag{C29}$$

Consider the term

$$\sum_{\substack{\mu\nu\\ \mu\neq\nu}} |a_\mu|^2|a_\nu|^2\langle|\langle\psi_\mu|\hat{H}_0'|\psi_\nu\rangle|^2\rangle_V{\rm sinc}^2(\theta_{\mu\nu}t) = \sum_{\substack{\mu\nu\\ \mu\neq\nu}}\langle|a_\mu|^2|a_\nu|^2\rangle_V\langle|\langle\psi_\mu|\hat{H}_0'|\psi_\nu\rangle|^2\rangle_V{\rm sinc}^2(\theta_{\mu\nu}t). \tag{C30}$$

Using (C13), we obtain

$$\sum_{\substack{\mu\nu\\ \mu\neq\nu}}\langle|a_\mu|^2|a_\nu|^2\rangle_V\langle|\langle\psi_\mu|\hat{H}_0'|\psi_\nu\rangle|^2\rangle_V{\rm sinc}^2(\theta_{\mu\nu}t) \approx \left(\Delta\hat{H}_0'^2\right)_{\rm mc}\frac{2\Gamma\omega}{\pi}\sum_{\substack{\mu\nu\\ \mu\neq\nu}}\langle|a_\mu|^2|a_\nu|^2\rangle_V\frac{{\rm sinc}^2(\theta_{\mu\nu}t)}{(E_\mu-E_\nu)^2+4\Gamma^2}. \tag{C31}$$

Let us now assume that the initial state is an eigenstate of the noninteraction Hamiltonian $\hat{H}_0$, namely, $|\psi(0)\rangle = |\varphi_{\alpha_0}\rangle$. Then we have

$$\langle|a_\mu|^2|a_\nu|^2\rangle_V = \Lambda(\mu,\alpha_0)\Lambda(\nu,\alpha_0) \tag{C32}$$

and we get

$$\left(\Delta\hat{H}_0'^2\right)_{\rm mc}\frac{2\Gamma\omega}{\pi}\sum_{\substack{\mu\nu\\ \mu\neq\nu}}\langle|a_\mu|^2|a_\nu|^2\rangle_V\frac{{\rm sinc}^2(\theta_{\mu\nu}t)}{(E_\mu-E_\nu)^2+4\Gamma^2} = \left(\Delta\hat{H}_0'^2\right)_{\rm mc}\frac{2\Gamma\omega}{\pi}\sum_{\substack{\mu\nu\\ \mu\neq\nu}}\Lambda(\mu,\alpha_0)\Lambda(\nu,\alpha_0)\frac{{\rm sinc}^2(\theta_{\mu\nu}t)}{(E_\mu-E_\nu)^2+4\Gamma^2}. \tag{C33}$$

We replace the sum with the integration, such that we have

$$\frac{2\Gamma\omega}{\pi}\frac{1}{\omega}\int_{-\infty}^{\infty}\Lambda(\mu,\alpha_0)\frac{{\rm sinc}^2(\theta_{\mu\nu}t)}{(E_\mu-E_\nu)^2+4\Gamma^2}dE_\mu \leqslant \frac{2\Gamma}{\pi}\frac{\omega}{\pi\Gamma}\int_{-\infty}^{\infty}\frac{{\rm sinc}^2(\theta_{\mu\nu}t)}{(E_\mu-E_\nu)^2+4\Gamma^2}dE_\mu = \frac{\omega}{\pi\Gamma}\frac{1}{2(\Gamma t)^2}(e^{-2\Gamma t}-1+2\Gamma t). \tag{C34}$$

In the above equation, we have used that for any two functions $f(x)>0$ and $g(x)>0$, which obey $f(x)g(x)\leqslant f_{\max}g(x)$, it follows that $\int_{-\infty}^{\infty}f(x)g(x)dx \leqslant f_{\max}\int_{-\infty}^{\infty}g(x)dx$. Therefore, we obtain

$$\sum_{\substack{\mu\nu\\ \mu\neq\nu}}\langle|a_\mu|^2|a_\nu|^2\rangle_V\langle|\langle\psi_\mu|\hat{H}_0'|\psi_\nu\rangle|^2\rangle_V{\rm sinc}^2(\theta_{\mu\nu}t) \leqslant \left(\Delta\hat{H}_0'^2\right)_{\rm mc}\frac{\omega}{\pi\Gamma}\frac{1}{2(\Gamma t)^2}(e^{-2\Gamma t}-1+2\Gamma t). \tag{C35}$$

As long as $\Gamma \gg \omega$, we neglect this term.

Similarly, we have

$$2\sum_{\substack{\mu\nu\\ \mu\neq\nu}}a_\mu^{*2}a_\nu^2|\langle\psi_\mu|\hat{H}_0'|\psi_\nu\rangle|^2\cos(2\theta_{\mu\nu}t){\rm sinc}^2(\theta_{\mu\nu}t) \approx 2\left(\Delta\hat{H}_0'^2\right)_{\rm mc}\frac{2\Gamma\omega}{\pi}\sum_{\substack{\mu\nu\\ \mu\neq\nu}}\Lambda(\mu,\alpha_0)\Lambda(\nu,\alpha_0)\frac{\cos(2\theta_{\mu\nu}t){\rm sinc}^2(\theta_{\mu\nu}t)}{(E_\mu-E_\nu)^2+4\Gamma^2}. \tag{C36}$$

Replacing the sum with integration, we get

$$\frac{2\Gamma\omega}{\pi}\int_{-\infty}^{\infty}\Lambda(\mu,\alpha_0)\frac{\cos(2\theta_{\mu\nu}t){\rm sinc}^2(\theta_{\mu\nu}t)}{(E_\mu-E_\nu)^2+4\Gamma^2}dE_\mu \leqslant \frac{2\Gamma}{\pi}\frac{\omega}{\pi\Gamma}\int_{-\infty}^{\infty}\frac{\cos(2\theta_{\mu\nu}t){\rm sinc}^2(\theta_{\mu\nu}t)}{(E_\mu-E_\nu)^2+4\Gamma^2}dE_\mu. \tag{C37}$$

The integral is given by

$$\frac{2\Gamma}{\pi}\int_{-\infty}^{\infty}\frac{\cos(2\theta_{\mu\nu}t)\mathrm{sinc}^2(\theta_{\mu\nu}t)}{(E_\mu-E_\nu)^2+4\Gamma^2}dE_\mu=\frac{(e^{-2\Gamma t}-1)^2}{4(\Gamma t)^2}. \quad \text{(C38)}$$

Therefore, we obtain

$$2\sum_{\substack{\mu\nu\\ \mu\neq\nu}} a_\mu^{*2}a_\nu^2\langle\psi_\mu|\hat{H}_0'|\psi_\nu\rangle\langle\psi_\mu|\hat{H}_0'|\psi_\nu\rangle\cos(2\theta_{\mu\nu}t)\mathrm{sinc}^2(\theta_{\mu\nu}t)$$
$$\leqslant (\Delta\hat{H}_0'^2)_{\mathrm{mc}}\left(\frac{\omega}{\pi\Gamma}\right)\frac{(e^{-2\Gamma t}-1)^2}{4(\Gamma t)^2}, \quad \text{(C39)}$$

which we neglect in the limit $\Gamma \gg \omega$. All other terms in (C25) contain matrix elements with two and three equal indexes and, respectively, four different indexes, and their ensemble average is zero.

Combining all averages in the second term (C25), we get

$$\sum_{\mu\nu}\sum_{\mu'\nu'} a_\mu^* a_\nu a_{\mu'} a_{\nu'}^*\langle\psi_\mu|\hat{H}_0'|\psi_\nu\rangle\langle\psi_{\nu'}|\hat{H}_0'|\psi_{\mu'}\rangle e^{i\theta_{\mu\nu}t}e^{-i\theta_{\mu'\nu'}t}$$
$$\times\,\mathrm{sinc}(\theta_{\mu\nu}t)\mathrm{sinc}(\theta_{\mu'\nu'}t)\approx(\hat{H}_0')^2_{\mathrm{mc}}. \quad \text{(C40)}$$

Finally, using Eqs. (C1), (C24), and (C40), we obtain

$$F_Q(\lambda)=4t^2\left\{\frac{\omega}{\pi\Gamma}(\hat{H}_0'^2)_{\mathrm{mc}}+\frac{(\Delta\hat{H}_0'^2)_{\mathrm{mc}}}{2(\Gamma t)^2}(e^{-2\Gamma t}-1+2\Gamma t)\right\}. \quad \text{(C41)}$$

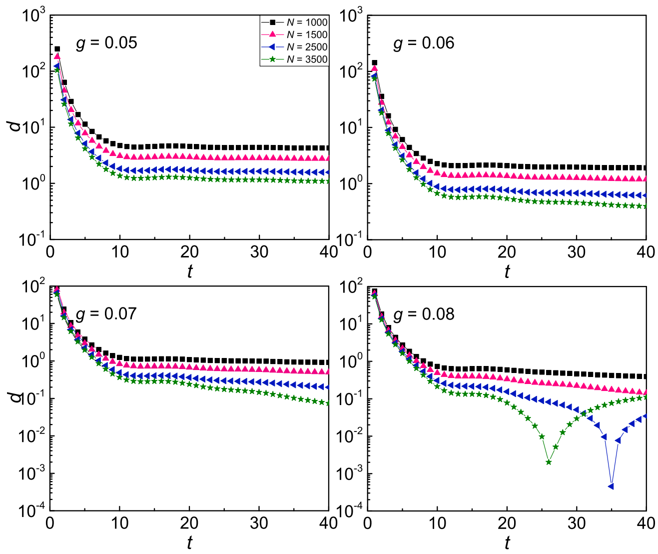

FIG. 6. Relative error $d=|1-(F_Q(\omega))_{\mathrm{RMT}}/(F_Q(\omega))|$ between results for the QFI derived from Eqs. (6) and (2).

In Fig. 6, we show the relative error between the exact result derived from Eq. (A1) and the analytical formula Eq. (C41) for various g and N.

We see that increasing the time scaling of the QFI passes from linear to quadratic. In fact, we can obtain the long time scaling of QFI using that $\lim_{t\to\infty}\mathrm{sinc}(x)=0$. Using (C1), we obtain

$$\lim_{t\to\infty}\frac{F_Q(\lambda)}{t^2}=4\left\{\sum_\mu|a_\mu|^2|\langle\psi_\mu|\hat{H}_0'|\psi_\mu\rangle|^2-\sum_{\mu\nu}|a_\mu|^2|a_\nu|^2\langle\psi_\mu|\hat{H}_0'|\psi_\mu\rangle\langle\psi_\nu|\hat{H}_0'|\psi_\nu\rangle\right\}$$
$$=4\left\{\sum_\mu|a_\mu|^2|\langle\psi_\mu|\hat{H}_0'|\psi_\mu\rangle|^2-\sum_\mu|a_\mu|^4|\langle\psi_\mu|\hat{H}_0'|\psi_\mu\rangle|^2-\sum_{\substack{\mu\nu\\ \mu\neq\nu}}|a_\mu|^2|a_\nu|^2\langle\psi_\mu|\hat{H}_0'|\psi_\mu\rangle\langle\psi_\nu|\hat{H}_0'|\psi_\nu\rangle\right\}$$
$$=4\left\{\frac{\omega}{\pi\Gamma}(\hat{H}_0'^2)_{\mathrm{mc}}+(\hat{H}_0')^2_{\mathrm{mc}}-(\hat{H}_0')^2_{\mathrm{mc}}\sum_\mu\langle|a_\mu|^4\rangle_V-(\hat{H}_0')^2_{\mathrm{mc}}\left(1-\sum_\mu\langle|a_\mu|^4\rangle_V\right)\right\}. \quad \text{(C42)}$$

Therefore, neglecting terms of order of $(\omega/\Gamma)^2$ we obtain the long time limit of QFI as $F_Q(\lambda)\approx(\hat{H}_0'^2)_{\mathrm{mc}}(4\omega/\pi\Gamma)t^2$. Similarly, we may consider the short-time limit of the QFI by using that $\lim_{t\to 0}\mathrm{sinc}(x)=1$. Then we have

$$\lim_{t\to 0}\frac{F_Q(\lambda)}{t^2}=4\left\{\sum_\mu|a_\mu|^2|\langle\psi_\mu|\hat{H}_0'|\psi_\mu\rangle|^2+\sum_{\substack{\mu\nu\\ \mu\neq\nu}}|a_\mu|^2|\langle\psi_\mu|\hat{H}_0'|\psi_\nu\rangle|^2\right.$$
$$\left.-\sum_\mu|a_\mu|^4|\langle\psi_\mu|\hat{H}_0'|\psi_\mu\rangle|^2-\sum_{\substack{\mu\nu\\ \mu\neq\nu}}|a_\mu|^2|a_\nu|^2\langle\psi_\mu|\hat{H}_0'|\psi_\mu\rangle\langle\psi_\nu|\hat{H}_0'|\psi_\nu\rangle\right\}$$
$$=4\left\{\frac{\omega}{\pi\Gamma}(\hat{H}_0'^2)_{\mathrm{mc}}+\frac{2\omega\Gamma}{\pi}(\Delta\hat{H}_0'^2)_{\mathrm{mc}}\sum_{\substack{\mu\nu\\ \mu\neq\nu}}\frac{|a_\mu|^2}{(E_\mu-E_\nu)^2+4\Gamma^2}\right\}. \quad \text{(C43)}$$

Furthermore, we replace the sum with integration such that we have

$$\sum_{\nu}\frac{1}{(E_\mu-E_\nu)^2+4\Gamma^2}\rightarrow\frac{1}{\omega}\int_{-\infty}^{\infty}\frac{dE_\nu}{(E_\mu-E_\nu)^2+4\Gamma^2}=\frac{\pi}{2\omega\Gamma}. \tag{C44}$$

Therefore, neglecting the terms of order of ω/Γ, the short time scaling of the QFI is $F_Q(\lambda)\approx 4t^2(\Delta\hat{H}_0'^2)_{\rm mc}$.

APPENDIX D: CORRECTIONS DUE TO SELF-AVERAGING DECOUPLING

Let us assume that the initial state is an eigenstate of non-interaction Hamiltonian $\hat{H}_0$, namely, $|\Psi_0\rangle=|\varphi_{\alpha_0}\rangle$. Therefore, we have

$$\sum_{\mu}\langle|a_\mu|^2|\langle\psi_\mu|\hat{H}_0'|\psi_\mu\rangle|^2\rangle_V=\sum_{\mu}\sum_{\alpha\beta}\langle c_\mu(\alpha_0)c_\mu(\alpha_0)c_\mu(\alpha)c_\mu(\alpha)c_\mu(\beta)c_\mu(\beta)\rangle_V(\hat{H}_0')_{\alpha\alpha}(\hat{H}_0')_{\beta\beta}. \tag{D1}$$

To evaluate the average, we use the generating function (B1). There are in total 15 terms. Consider the term

$$\langle c_\mu(\alpha_0)c_\mu(\alpha_0)c_\mu(\alpha)c_\mu(\alpha)c_\mu(\beta)c_\mu(\beta)\rangle_V=\Lambda^2(\mu,\alpha)\Lambda(\mu,\alpha_0)\delta_{\alpha\beta}. \tag{D2}$$

Hence we get

$$\begin{aligned}\sum_{\mu}\sum_{\alpha\beta}\langle c_\mu(\alpha_0)c_\mu(\alpha_0)c_\mu(\alpha)c_\mu(\alpha)c_\mu(\beta)c_\mu(\beta)\rangle_V(\hat{H}_0')_{\alpha\alpha}(\hat{H}_0')_{\beta\beta}&=\sum_{\mu}\left(\sum_{\alpha}\Lambda^2(\mu,\alpha)(\hat{H}_0')^2_{\alpha\alpha}\right)\Lambda(\mu,\alpha_0)\\ \approx\left[(\hat{H}_0')^2_{\alpha\alpha}\right]_\mu\sum_{\mu}\sum_{\alpha}\Lambda^2(\mu,\alpha)\Lambda(\mu,\alpha_0)=\frac{\omega}{2\pi\Gamma}\left[(\hat{H}_0')^2_{\alpha\alpha}\right]_\mu.\end{aligned} \tag{D3}$$

Similarly, we have

$$\sum_{\mu}\sum_{\alpha\beta}\langle c_\mu(\alpha_0)c_\mu(\alpha_0)c_\mu(\alpha)c_\mu(\alpha)c_\mu(\beta)c_\mu(\beta)\rangle_V(\hat{H}_0')_{\alpha\alpha}(\hat{H}_0')_{\beta\beta}\approx\frac{\omega}{2\pi\Gamma}\left[(\hat{H}_0')^2_{\alpha\alpha}\right]_\mu. \tag{D4}$$

Another six-point correlation term is

$$\begin{aligned}\sum_{\mu}\sum_{\alpha\beta}\langle c_\mu(\alpha_0)c_\mu(\alpha_0)c_\mu(\alpha)c_\mu(\alpha)c_\mu(\beta)c_\mu(\beta)\rangle_V(\hat{H}_0')_{\alpha\alpha}(\hat{H}_0')_{\beta\beta}&=\sum_{\mu}\sum_{\alpha\beta}\Lambda(\mu,\alpha)\Lambda(\mu,\beta)\Lambda(\mu,\alpha_0)(\hat{H}_0')_{\alpha\alpha}(\hat{H}_0')_{\beta\beta}\\ \approx[(\hat{H}_0')_{\alpha\alpha}]^2_\mu\sum_{\mu}\sum_{\alpha\beta}\Lambda(\mu,\alpha)\Lambda(\mu,\beta)\Lambda(\mu,\alpha_0)=[(\hat{H}_0')_{\alpha\alpha}]^2_\mu.\end{aligned} \tag{D5}$$

Combining Eqs. (D3), (D4), and (D5) we obtain Eq. (C9). Now, let us consider the corrections. We have

$$\langle c_\mu(\alpha_0)c_\mu(\alpha_0)c_\mu(\alpha)c_\mu(\alpha)c_\mu(\beta)c_\mu(\beta)\rangle_V=\Lambda(\mu,\beta)\delta_{\alpha_0,\beta}\Lambda(\mu,\alpha_0)\delta_{\alpha,\alpha_0}\Lambda(\mu,\alpha)\delta_{\alpha\beta}. \tag{D6}$$

Therefore, we obtain

$$\sum_{\mu}\sum_{\alpha\beta}\langle c_\mu(\alpha_0)c_\mu(\alpha_0)c_\mu(\alpha)c_\mu(\alpha)c_\mu(\beta)c_\mu(\beta)\rangle_V(\hat{H}_0')_{\alpha\alpha}(\hat{H}_0')_{\beta\beta}=(\hat{H}_0')^2_{\alpha_0\alpha_0}\sum_{\mu}\Lambda^3(\mu,\alpha_0). \tag{D7}$$

Such a term gives correction of order of $(\omega/\Gamma)^2$. In fact, there are in total eight terms which give corrections of such order.

Consider now the term

$$\langle c_\mu(\alpha_0)c_\mu(\alpha_0)c_\mu(\alpha)c_\mu(\alpha)c_\mu(\beta)c_\mu(\beta)\rangle_V=\Lambda^2(\mu,\alpha)\delta_{\alpha,\alpha_0}\Lambda(\mu,\beta). \tag{D8}$$

Hence we get

$$\begin{aligned}&\sum_{\mu}\sum_{\alpha\beta}\langle c_\mu(\alpha_0)c_\mu(\alpha_0)c_\mu(\alpha)c_\mu(\alpha)c_\mu(\beta)c_\mu(\beta)\rangle_V(\hat{H}_0')_{\alpha\alpha}(\hat{H}_0')_{\beta\beta}\\ &=\sum_{\mu}\sum_{\beta}\Lambda(\mu,\beta)(\hat{H}_0')_{\beta\beta}\Lambda^2(\mu,\alpha_0)(\hat{H}_0')_{\alpha_0\alpha_0}\\ &\approx[(\hat{H}_0')_{\alpha\alpha}]_\mu(\hat{H}_0')_{\alpha_0\alpha_0}\sum_{\mu}\sum_{\beta}\Lambda(\mu,\beta)\Lambda^2(\mu,\alpha_0)=\frac{\omega}{2\pi\Gamma}[(\hat{H}_0')_{\alpha\alpha}]_\mu(\hat{H}_0')_{\alpha_0\alpha_0}.\end{aligned} \tag{D9}$$

There are in total four terms with the same contribution. Note that for spin chains that we consider, the microcanonical average of the spin observable is zero and thus these terms can be neglected.

APPENDIX E: DEFINING A LOCAL OBSERVABLE IN RMT

In the main text, we analyzed the QFI for a *local* observable of the system interacting with a bath. In this Appendix, we will see that such an approach can be formalized within our RMT approach, and indeed gives way to a crucial condition—observable sparsity—of the application of RMT. We will see that an additional condition is required on the system and bath parts of the total system for RMT to apply to such local observables, namely, that the system energy is much smaller than that of the bath.

We begin by separating the system into system and bath components via $\hat{H}_0 = \hat{H}_S(\lambda) \otimes \mathbf{1}_B + \mathbf{1}_S \otimes \hat{H}_B$, with $\mathbf{1}_{S(B)}$ the identity on the system (bath) Hilbert space. Crucially, here the system part of the Hamiltonian is assumed to depend on some parameter λ. The eigenstates of $\hat{H}_0$ are then

$$|\phi_\alpha\rangle = |s(\alpha)\rangle_S \otimes \big|\phi^{(B)}_{\alpha_B(\alpha)}\big\rangle_B, \tag{E1}$$

with energies

$$\begin{aligned} E_\alpha &= {}_S\langle s(\alpha)|_B\big\langle\phi^{(B)}_{\alpha_B(\alpha)}\big|\hat{H}_S(\lambda) + \hat{H}_B|s(\alpha)\rangle_S \otimes \big|\phi^{(B)}_{\alpha_B(\alpha)}\big\rangle_B \\ &= \epsilon_{s(\alpha)}(\lambda) + E^{(B)}_{\alpha_B(\alpha)}, \end{aligned} \tag{E2}$$

where we have denoted eigenenergies of the system and bath Hamiltonians by $\epsilon_{s(\alpha)}(\lambda)$ and $E^{(B)}_{\alpha_B(\alpha)}$ respectively.

Relevant observables in our approach act on the system Hilbert space as $\hat{O} = \hat{O}_S \otimes \mathbf{1}_B$, which have matrix elements

$$\begin{aligned} O_{\alpha\beta} &= {}_S\langle s(\alpha)|_B\big\langle\phi^{(B)}_{\alpha_B(\alpha)}\big|\hat{O}|s(\beta)\rangle_S \otimes \big|\phi^{(B)}_{\alpha_B(\beta)}\big\rangle_B \\ &= (O_S)_{s(\alpha)s(\beta)}\delta_{\alpha_B(\alpha)\alpha_B(\beta)}, \end{aligned} \tag{E3}$$

where $(O_S)_{s(\alpha)s(\beta)} = {}_S\langle s(\alpha)|\hat{O}_S|s(\beta)\rangle_S$. We see here that the local observable $\hat{O}$ is guaranteed to be sparse if the dimension of the system Hilbert space d_S is much lower than that of the bath, d_B, as there are a maximum of $d_S(d_S-1)$ independent off diagonal matrix elements of $O_{\alpha\beta}|_{\alpha\neq\beta}$, corresponding to the possible system state transitions, plus d_S possible diagonal matrix elements.

These possible transitions that the local operator $\hat{O}$ may induce must obey

$$E_\alpha - E_\beta = \epsilon_{s(\alpha)}(\lambda) - \epsilon_{s(\beta)}(\lambda) := \Delta^{(S)}_{\alpha\beta}(\lambda), \tag{E4}$$

and, more generally, we have

$$E_\alpha - E_\beta = \Delta^{(S)}_{\alpha\beta}(\lambda) + E^{(B)}_{\alpha_B(\alpha)} - E^{(B)}_{\alpha_B(\beta)}. \tag{E5}$$

For the random matrix model, we have $E_\alpha = \alpha\omega$, so we require for the RMT to hold that $E_\alpha - E_\beta \approx (\alpha-\beta)\omega$ can be approximated by an equidistant spacing of energies that does not depend on λ. This is understood to hold if $\hat{H}_B$ is itself a nonintegrable Hamiltonian, and if $\Delta^{(S)}_{\alpha\beta}(\lambda) \ll E^{(B)}_{\alpha_B(\alpha)} - E^{(B)}_{\alpha_B(\beta)}$, indicating that the possible transitions induced by the local observable are negligible in energy in comparison to the bath energy for the state $|\phi_\alpha\rangle$.

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