Performance at maximum figure of merit for a Brownian Carnot refrigerator

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This paper focuses on the coefficient of performance (COP) at maximum χ^R figure of merit for a Brownian Carnot-like refrigerator, within the context of the low-dissipation approach. Our proposal is based on the Langevin equation for a Brownian particle bounded to a harmonic potential trap, which can perform Carnot-like cycles at finite time. The theoretical approach is related to the equilibrium ensemble average of $\langle x^2 \rangle_{\text{eq}}$ which plays the role of a *statelike* equation, x being the Brownian particle position. This statelike equation comes from the macroscopic version of the corresponding Langevin equation for a Brownian particle. We show that under quasistatic conditions the COP has the same expression as the macroscopic Carnot refrigerator, while for irreversible cycles at finite time and under symmetric dissipation the optimal COP is the counterpart of Curzon-Ahlborn efficiency as also obtained for irreversible macroscopic refrigerators.

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I. INTRODUCTION

In his 1824 celebrated paper, Carnot proposed that the efficiency of any heat engine has an upper bound [1,2], known as the Carnot efficiency, given by $\eta_C = 1 - T_c/T_h$, where T_c and T_h are respectively the temperatures of the cold and hot thermal baths with which the thermal engine can operate. The value of η_C is achieved when the engine operates quasistatically, resulting in zero power output. To provide a framework for the operation of thermal engines capable of producing power, an extension of thermodynamics known as finite time thermodynamics (FTT) emerged. This theory incorporates sources of irreversibility and finite operation time into the models, consequently resulting in power production. Chambadal [3] and Novikov [4] proposed an initial engine model equivalent to the one later introduced by Curzon and Ahlborn (CA) [5]. This model relies upon an internal Carnot cycle but includes entropy production during the isothermal processes. CA optimized their model using power output as the objective function. By applying a linear law to the heat transfers, the authors concluded that the efficiency of a heat engine operating at maximum power satisfies $\eta_{\text{CA}} = 1 - \sqrt{T_c/T_h} \leqslant \eta_C$. Subsequently, other objective functions, such as the ecological function, the omega function, and efficient power [6,7], were proposed.

One of the tasks within FTT has been the introduction of an objective function for optimizing refrigerator performance. Specifically, Yan and Chen [8] proposed an objective function called the χ^R figure of merit to optimize the operation of an endoreversible refrigerator. When there is a linear heat transfer between the thermal reservoirs and the working fluid, the COP of a refrigerator at maximum χ^R fulfills $\varepsilon_{\text{CA}} = [1/\sqrt{1 - T_c/T_h}] - 1 \leqslant \varepsilon_c$, where ε_C is the Carnot COP for refrigerators, given by $\varepsilon_c = [1/(1 - T_c/T_h)] - 1$.

In 2010, Esposito *et al.* [9] proposed an idea within the context of finite time energy conversion called the low dissipation (LD) approach. The authors start with a Carnot-like heat engine that operates reversibly and then consider that during the isothermal branches the working fluid is in contact with the cold (hot) reservoir for a finite time t_c (t_h). Hence, the amount of heat exchanged (per cycle) between the system and the cold (hot) reservoir is given by $Q_c = T_c(-\Delta S - \Sigma_c/t_c)$ ($Q_h = T_h(\Delta S - \Sigma_h/t_h)$), with $Q_\infty^c = -T_c\Delta S$ ($Q_\infty^h = T_h\Delta S$), which is the heat exchanged with the cold (hot) reservoir under reversible conditions ($t_c,\ t_h \to \infty$).

The quantity Σ_c/t_c (Σ_h/t_h) represents the entropy production for the cold (hot) isothermal branch. The parameter Σ accounts for irreversibilities present along the isothermal branches due to coupling with the thermal baths over finite time. In this approach, the adiabatic processes are considered instantaneous, and their entropy production is zero. Esposito *et al.* [9] found that the efficiency at maximum power output depends not only on T_c and T_h, but also on Σ_c and Σ_h. In the particular case where $\Sigma_c = \Sigma_h$, known as symmetric dissipation, the CA efficiency, η_{CA}, is recovered. The symmetric LD condition means that the irreversibilities are the same in both isothermal processes. The equivalence between the LD and CA methodologies has been studied in [10–12].

Two years after the paper by Esposito *et al.* [9], the LD model was successfully extended to the study of macroscopic Carnot-like refrigerators under both symmetric [13] and asymmetric [14] conditions. In [13], the authors proposed a unified optimization criterion for both Carnot-like

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heat engines and refrigerators. This criterion defines a figure of merit as $\chi = zQ_{\text{in}}/t_{\text{cycle}}$, where z represents the performance quality of the converter, Q_{in} denotes the heat absorbed by the working fluid, and $t_{\text{cycle}} = t_c + t_h$ represents the total cycle time, following a similar concept to Yan and Chen [8]. When the thermal engine operates as a refrigerator, $Q_{\text{in}} = Q_c$ is the heat exchanged with the cold reservoir, $z = \varepsilon = Q_c/W$ represents the coefficient of performance (COP), and W denotes the amount of input work. Thus, the figure of merit becomes $\chi^R = \varepsilon Q_c/t_{\text{cycle}}$. The main conclusion in [13] is that under symmetric LD conditions the optimal COP at maximum χ^R aligns with the Curzon-Ahlborn efficiency for an endoreversible refrigerator model.

Shortly after, in [14], the study of optimal COP was extended to the case of asymmetric LD conditions. It was also shown that under symmetric LD conditions the CA efficiency is recovered. The optimal COP, denoted by ε_*, is bounded as $0 \leqslant \varepsilon_* \leqslant (\sqrt{9+8\varepsilon_c} - 3)/2$. Due to these reasons, the efficiency and performance of the Curzon-Ahlborn model turn out to be exact properties for Carnot devices operating under conditions of low symmetric dissipation.

Following a similar strategy to [15], in this paper, we study the asymmetric low-dissipation approach for Brownian Carnot-like refrigerators. Our system consists of a Brownian particle confined in an optical trap (represented by a harmonic potential), performing finite time Carnot-like cycles between two thermal baths at time-dependent temperature $T_h(t)$ and $T_c(t)$, externally controlled. The adiabatic processes are assumed to be instantaneous, and the irreversible effects are present only in the two isothermal branches. This means that the Brownian particle's relaxation time is much faster than the quenching time of the internal temperature [15,16]. While Brownian refrigerator models have been optimized before [17–21], the LD approach has not been applied until now.

Our theoretical analysis relies on the overdamped Langevin equation associated with a Brownian particle bound to a harmonic trap. The strategy also involves transforming the Langevin equation into a macroscopic one for the average value $\langle x^2(t)\rangle$, which, in the long-time limit, plays the role of a statelike equation. This allows us to obtain all the thermodynamic quantities under quasistatic conditions, with irreversible effects accounted for using the LD approach. Both the potential stiffness $\kappa(t)$ and the temperature $T(t)$ of the surroundings are time-dependent parameters. For a stochastic Carnot-like refrigerator, we are interested in calculating the corresponding energetic quantities, such as the average COP defined as $\langle\varepsilon\rangle = \langle Q_c\rangle/\langle W\rangle$, where $\langle W\rangle = \langle Q_h\rangle - \langle Q_c\rangle$ is the total amount of input work. The χ^R figure of merit is given by $\langle\chi^R\rangle = \langle\varepsilon\rangle\langle Q_c\rangle/t_{\text{cycle}}$, with $t_{\text{cycle}} = t_c + t_h$. In this paper, we study the COP at maximum χ^R figure of merit [8] under both asymmetric and symmetric low-dissipation conditions. Notably, an analysis under symmetric LD conditions allows us to obtain the same expressions for the optimal CA's COP as for macroscopic cases [8,13,14]. As mentioned above, the LD approach starts by describing the reversible model of the device, in this case, a refrigerator. Then, the irreversible model is established by accounting for additional entropy production in the isothermal branches of the cycle.

Our paper is structured as follows: In Sec. II, we provide a review of the harmonic oscillator Langevin equation to establish the statelike equation for $\langle x^2\rangle$. Additionally, we propose a brief study on how to obtain the efficiency of a quasistatic Carnot-like refrigerator. The main content of our paper is presented in Sec. III, where the optimal COP at maximum χ^R figure of merit for a Brownian Carnot-like refrigerator is obtained. This COP is calculated under asymmetric LD conditions but reduces to the CA efficiency under symmetric conditions. We also examine the behavior of the cooling power and dissipation function, both of which have typically been studied within the context of FTT. Conclusions are given in Sec. IV, and at the end of our paper two Appendixes are included for explicit calculations.

II. REVIEW OF LANGEVIN DYNAMICS

To characterize the dynamics of a stochastic heat engine we considered a Brownian particle bounded to a harmonic potential trap $U(x) = \frac{1}{2}\kappa x^2$, with stiffness κ, in contact with a thermal bath at temperature T. In the overdamped regime the Langevin equation associated is

$$\gamma\frac{dx}{dt} = -\kappa x + \xi(t), \tag{1}$$

where $\gamma = 6\pi\zeta a$ is the friction coefficient, ζ the viscosity, and a the radius of the particle assumed to be a sphere. The thermal noise $\xi(t)$ satisfies the properties of a Gaussian white noise, that is, $\langle\xi(t)\rangle = 0$ and $\langle\xi(t)\xi(t')\rangle = 2\gamma k_B T\delta(t-t')$. The macroscopic equation associated with $\langle x^2\rangle$ is straightly obtained from Eq. (1), where

$$\gamma\frac{d\langle x^2\rangle}{dt} = -2\kappa\langle x^2\rangle + 2\langle x\xi(t)\rangle. \tag{2}$$

From (1), we also obtained that $\langle x\xi(t)\rangle = k_B T$, and thus

$$\gamma\frac{d\langle x^2\rangle}{dt} = -2\kappa\langle x^2\rangle + 2k_B T. \tag{3}$$

From this, it is clear that in the equilibrium stationary state $\langle x^2\rangle_{\text{eq}} = k_B T/\kappa$. From the experimental point of view, the stiffness as well as the temperature are externally controllable time-dependent parameters [16,22]. In this case, we considered that stiffness depends on time, and then the macroscopic equation for $\langle x^2\rangle$ can be obtained from

$$\gamma\frac{dx}{dt} = -\kappa(t)x + \xi(t). \tag{4}$$

From Eq. (4), we obtain the next macroscopic equation:

$$\gamma\frac{d\langle x^2\rangle}{dt} = -2\kappa(t)\langle x^2\rangle + 2k_{_B}T(t). \tag{5}$$

Also in the equilibrium stationary state, $\kappa(t)$ and $T(t)$ both become constants, and therefore $\langle x^2\rangle_{\text{eq}} = k_B T/\kappa$. In principle, given a particular expression for $\kappa(t)$ and $T(t)$, one can solve Eq. (5) and calculate the energetics [23–26]. Instead of following this route, we take advantage of LD strategy to calculate the COP at maximum χ^R figure of merit of a refrigerator operating with a stochastic system, in a similar way as proposed in [15].

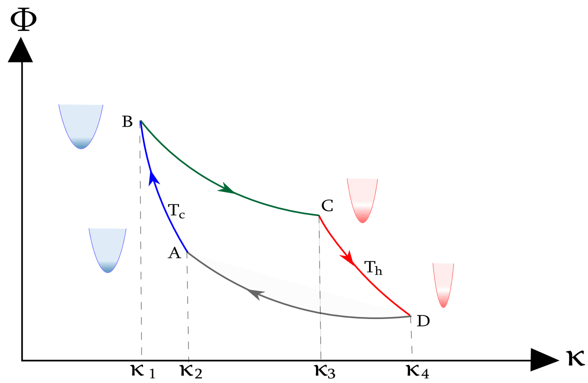

FIG. 1. Φ-κ cycle of a Brownian Carnot-like refrigerator with (i) an isothermal expansion (A-B path), (ii) an adiabatic compression (B-C path), (iii) an isothermal compression (C-D path), and (iv) an adiabatic expansion (D-A path).

A. Quasistatic Brownian Carnot-like refrigerator

As is well known, a macroscopic Carnot refrigerator operates following a Carnot cycle, but in the opposite direction its objective is to extract heat from a cold thermal bath, using an amount of work, and deliver an amount of heat into a hot thermal bath. In this regard, the quasistatic version of the Brownian refrigerator cycle is shown in Fig. 1, once the Brownian Carnot engine was previously studied [15]. To obtain all the thermodynamic quantities along the cycle's paths, the total amount of energy E is required. This energy is related to the amount of work and heat exchanged by a Brownian particle with its surroundings (heat bath), and can be calculated from the law of energy balance along a single stochastic trajectory. According to Sekimoto [27] (chapter 4) this energy balance reads $dE = d'Q + d'W$. Also in Sec. 4.1.3.2 of Sekimoto's book it is shown that, in the overdamped regime of the Langevin equation, the average of heat $d'Q$ between t and $t + dt$ for a harmonic potential is given by

$$\langle d'Q\rangle = \left[-\frac{\kappa^2}{\gamma}\langle x^2\rangle + \frac{\kappa}{\gamma}k_BT\right]dt. \tag{6}$$

Upon substitution of the solution of Eq. (3) into Eq. (6), it is shown that the average of the total heat exchanged with the surroundings reads $\langle Q\rangle = \frac{1}{2}k_BT$, and therefore in the overdamped regime the total thermodynamic energy becomes $E = \langle U\rangle + \langle Q\rangle = \frac{1}{2}\kappa\langle x^2\rangle + \frac{1}{2}k_BT$, or $E = k_BT$ (from now on we will identify any thermodynamic quantity as $y \equiv \langle y\rangle$). From the definitions of the total work W_{ab} and heat Q_{ab}, the amounts exchanged with the surroundings along a quasistatic trajectory from a state a to another state b are given by

$$W_{ab} = \frac{1}{2}\int_a^b \langle x^2\rangle\, d\kappa, \quad Q_{ab} = \frac{1}{2}\int_a^b \kappa\, d\langle x^2\rangle + \frac{1}{2}k_B(T_b - T_a). \tag{7}$$

The variable Φ has been introduced as an auxiliary variable in the differential form of the free energy $dF = -SdT + \Phi dk$, analogous to the one given for an ideal gas in thermodynamics [15]. From this equation it is easy to show that $\Phi = (\frac{\partial F}{\partial k})_T = k_BT/2 = \langle x^2\rangle/2$, which is related to the statelike equation.

It is straightforward to show that the COP of a Brownian Carnot-like refrigerator, defined as $\varepsilon_c = Q_c/(Q_h - Q_c)$, becomes

$$\varepsilon_c = \frac{k_BT_c\ln\left(\frac{\kappa_2}{\kappa_1}\right)}{k_B\left[T_h\ln\left(\frac{\kappa_4}{\kappa_3}\right) - T_c\ln\left(\frac{\kappa_2}{\kappa_1}\right)\right]} = \frac{T_c}{T_h - T_c}, \tag{8}$$

where $\kappa_2/\kappa_1 = \kappa_4/\kappa_3$ due to the adiabatic equation. Equation (8) is the same obtained for a macroscopic Carnot refrigerator, where it only depends on the temperatures of the thermal baths.

B. Irreversible Brownian Carnot-like refrigerator

If the cycle is no longer reversible but irreversible at finite time, then the dissipative processes play an important role. In this case, it has been shown that a suitable theoretical approach used to characterize out of equilibrium macroscopic heat engines and refrigerators is the low-dissipation approach [9,13,14]. To study the model of a Carnot-like cycle at finite time for a Brownian refrigerator, we adopt a similar idea as in [14] for an engine, as enumerated in the following.

1. Isothermal expansion

The cycle stars when the Brownian particle (working fluid) is in contact with a cold thermal bath at temperature T_c; during the interval $0 < t < t_c$, the expansion process means that the control parameter decreases from κ_2 to $\kappa_1(< \kappa_2)$, while $T(t) = T_c$. In this finite process an amount of heat Q_c is absorbed by the particle (it is assumed that $Q_c > 0$ and $Q_h < 0$). Here, the variation of entropy can be written as

$$\Delta S_c = \frac{Q_c}{T_c} + \Delta S_c^{\mathrm{ir}}, \tag{9}$$

where $\Delta S_c^{\mathrm{ir}} \geqslant 0$ is one part of the inner entropy change and fulfills $\Delta S_c \geqslant \Delta S_c^{\mathrm{ir}}$.

2. Adiabatic compression

In a similar way as done with the heat engine, this adiabatic process occurs instantaneously, that is, the particle suddenly decouples from the cold thermal bath at T_c and then comes into contact with the hot one at T_h. The compression means that during this transition process the potential stiffness suddenly is increased from κ_1 to $\kappa_3(> \kappa_1)$. This physically means that the relaxation time of Brownian particles is much faster with respect to the quenching time of the temperature. In this path $Q_2 = 0$ and thus the entropy change $\Delta S_2 = 0$.

3. Isothermal compression

In this process, the Brownian particle is in contact with the hot thermal bath at temperature T_h, and the potential stiffness is increased from κ_3 to κ_4 for $t_c < t < t_c + t_h$, while $T(t) = T_h$. In this finite process the heat Q_h is released by the particle to the hot bath. The variation of entropy is now

$$\Delta S_h = -\frac{Q_h}{T_h} + \Delta S_h^{\mathrm{ir}}, \tag{10}$$

and $\Delta S_h^{\mathrm{ir}} \geqslant 0$ is the other part of the inner entropy change.

4. Adiabatic expansion

In this last branch, the Brownian particle again suddenly decouples from the hot thermal bath at T_h and then comes into contact with the cold one at T_c. During this transition, the potential stiffness is decreased from κ_4 to $\kappa_2(<\kappa_4)$. In this branch, $Q_4 = 0$ and $\Delta S_4 = 0$.

III. OPTIMIZATION AT MAXIMUM χ^R FIGURE OF MERIT

As stated above, in this paper the χ^R figure of merit is used to study the optimal COP of a Brownian refrigerator model. This criterion was previously applied to macroscopic refrigerators both endoreversible [8] and within the LD scheme [14]. Using Eqs. (9) and (10) the figure of merit, $\chi^R = \varepsilon Q_c/t_{\rm cycle}$, reads

$$\chi^R = \frac{T_c^2\left(\Delta S - \Delta S_c^{\rm ir}\right)^2}{\left[(T_h - T_c)\Delta S + T_c\Delta S_c^{\rm ir} + T_h\Delta S_h^{\rm ir}\right](t_c + t_h)}. \tag{11}$$

χ^R reaches its maximum value when $\Delta S_c^{\rm ir}$ and $\Delta S_h^{\rm ir}$ fulfill a minimum value with respect to the protocols $\kappa_c(t)$ and $\kappa_h(t)$. We express the $\min\{\Delta S_c^{\rm ir}\} \propto L_c(t_c)$ and $\min\{\Delta S_h^{\rm ir}\} \propto L_h(t_h)$, which are expected to be monotonous decreasing functions of t_c and t_h, respectively, because the larger the time for completing the isothermal steps the closer these steps are to quasistatic processes so that the irreversible entropy production $\Delta S_c^{\rm ir}$ and $\Delta S_h^{\rm ir}$ becomes smaller. In this case, $L_c(t_c) \approx 1/t_c \equiv x_c$ and $L_h(t_h) \approx 1/t_h \equiv x_h$, or $L_c = \Sigma_c x_c$ and $L_h = \Sigma_h x_h$, Σ_c and Σ_h being two quantities related to the irreversibilities. Thus, when times $t_c \to \infty$ and $t_h \to \infty$, the entropy productions $\Delta S_c^{\rm ir}$ and $\Delta S_h^{\rm ir}$ should vanish, and therefore

$$Q_c = T_c[\Delta S - L_c], \tag{12}$$

$$Q_h = T_h[\Delta S + L_h], \tag{13}$$

and thus the COP for the Brownian refrigerator becomes

$$\varepsilon = \frac{Q_c}{Q_h - Q_c} = \frac{T_c(\Delta S - L_c)}{(T_h - T_c)\Delta S + T_c L_c + T_h L_h}. \tag{14}$$

We now proceed to calculate the optimum COP at maximum χ^R. Equation (11) can be written as

$$\chi^R = \frac{Q_c^2 x_c x_h}{Q_h x_h + Q_h x_c - Q_c x_h - Q_c x_c}. \tag{15}$$

The optimization criterion leads us to calculate $\frac{\partial \chi^R}{\partial x_{h,c}} = 0$. And so, with respect to variables x_c and x_h, the following two equations arise:

$$x_h(Q_h - Q_c) = \left(\frac{2Q_h}{Q_c} - 1\right)x_c T_c L_c'(x_h + x_c), \tag{16}$$

$$x_c(Q_h - Q_c) = T_h L_h' x_h (x_h + x_c), \tag{17}$$

where L_c' and L_h' are the derivatives of L_c and L_h associated with x_c and x_h, respectively. Dividing Eqs. (16) and (17), we show that the COP ε_* at maximum χ^R figure of merit fulfills

$$T_h L_h' x_h^2 = \left(\frac{2Q_h}{Q_c} - 1\right)T_c L_c' x_c^2. \tag{18}$$

The optimal COP also reads as $\epsilon_* = Q_c/(Q_h - Q_c)$, and thus

$$\varepsilon_* T_h L_h' x_h^2 = (\varepsilon_* + 2)T_c L_c' x_c^2. \tag{19}$$

On the other hand, adding Eqs. (16) and (17), it is possible to show that

$$\frac{1}{\varepsilon_*} = \frac{T_h - T_c}{T_c} + \frac{T_h(L_h + L_c)}{2T_c L_c' x_c + \varepsilon_* T_h L_h' x_h + \varepsilon_* T_c L_c' x_c}, \tag{20}$$

which can be rewritten as

$$\frac{1}{\varepsilon_*} = \frac{1}{\varepsilon_c} + \frac{1 + \varepsilon_c}{N\varepsilon_*(1 + \varepsilon_c) + (2\varepsilon_c - \varepsilon_*)M}, \tag{21}$$

where $M = \frac{L_c' x_c}{L_c + L_h} = \frac{\Sigma_c x_c}{\Sigma_h x_h + \Sigma_c x_c}$ and $N = \frac{L_c' x_c + L_h' x_h}{L_c + L_h} = 1$.

A. Optimum performance: Symmetric case

When $\Sigma_c = \Sigma_h \equiv \Sigma$, we have the symmetric case, that is, equal dissipation occurs in both isothermal branches. In this case the M parameter reads $M = x_c/(x_h + x_c)$, and after some algebra we show that the optimal COP at maximum figure of merit becomes (see Appendix A 1)

$$\varepsilon_* \equiv \varepsilon_{\rm CA} = \sqrt{1 + \varepsilon_c} - 1 = \frac{1}{\sqrt{1 - \theta}} - 1, \tag{22}$$

where $\theta = T_c/T_h$. This COP is the counterpart of Curzon-Ahlborn efficiency for refrigerators. This result was first derived by Yan and Chen for the particular case of an endoreversible Carnot-like refrigerator [8]. Also, it can be shown that the critical values of times t_c^* and t_h^* are given by (see Appendix A 2)

$$t_c^* = \frac{2\Sigma}{\Delta S}\left(1 + \frac{1}{\sqrt{1 - \theta}}\right) = \frac{4\Sigma}{k_B \ln(\kappa_1/\kappa_2)}\left(1 + \frac{1}{\sqrt{1 - \theta}}\right), \tag{23}$$

$$t_h^* = \frac{2\Sigma}{\Delta S}\left(\frac{1}{\sqrt{1 - \theta}}\right) = \frac{4\Sigma}{k_B \ln(\kappa_1/\kappa_2)}\left(\frac{1}{\sqrt{1 - \theta}}\right), \tag{24}$$

both of which were obtained in [13] for macroscopic refrigerators. However, in the particular case of this Brownian refrigerator, the entropy change at equilibrium reads $\Delta S = (k_B/2)\ln(\kappa_1/\kappa_2)$. In addition to the COP, it is useful to observe the evolution of other energetic quantities linked to this device as cooling power ($R \equiv \dot{Q}_c$) and dissipation function ($\Phi_R \equiv T_h \Delta \dot{S}_{\rm tot}$); at maximum χ^R, from Eqs. (23) and (24), it follows that the absorbed and transferred heat fluxes are, respectively,

$$\dot{Q}_c^* = \frac{Q_c^*}{t_h^* + t_c^*} = \frac{T_h[\ln(\kappa_1/\kappa_2)]^2}{16\Sigma}\left(\frac{\theta\sqrt{1 - \theta}}{1 + \sqrt{1 - \theta}}\right) \tag{25}$$

and

$$\dot{Q}_h^* = \frac{Q_h^*}{t_h^* + t_c^*} = \left(\frac{T_h[\ln(\kappa_1/\kappa_2)]^2}{16\Sigma}\right)\sqrt{1 - \theta}. \tag{26}$$

According to the second law of thermodynamics, if the total entropy change ($\Delta S_{\rm tot}^*$) under this regime can be written as

$$\Delta S_{\rm tot}^* = \frac{Q_h^*}{T_h} - \frac{Q_c^*}{T_c} = \Sigma\left(\frac{1}{t_h^*} + \frac{1}{t_c^*}\right) > 0, \tag{27}$$

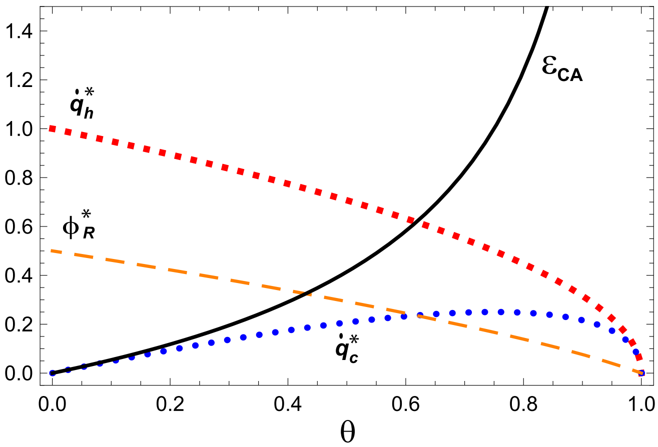

FIG. 2. Graphs of dimensionless functions $\dot{q}_h^*$ (squares), $\dot{q}_c^*$ (dots), ϕ_R^* (dashed line), and $\varepsilon_{\rm CA}$ (solid line) at maximum χ^R figure of merit as a function of θ.

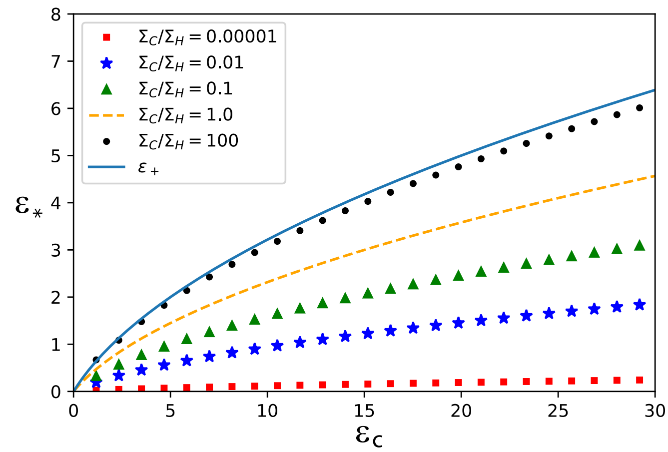

FIG. 3. The COP ε_* given by Eq. (29), as a function of ε_C, given a specific value of the ratio Σ_c/Σ_h. This plot validates the consequences derived from Eq. (29), under asymmetric LD conditions, between lower $\varepsilon_- = 0$ and upper $\varepsilon_+ = (\sqrt{9+8\varepsilon_c}-3)/2$ bounds.

then the total entropy production of the cooling process is $\Delta\dot{S}_{\rm tot}^* = (\dot{Q}_h^*/T_h) - (\dot{Q}_c^*/T_c)$. From the dissipation function definition, Φ_R^* reads

$$\Phi_R^* = \dot{Q}_h^* - \frac{1}{\theta}\dot{Q}_c^* = \frac{T_h[\ln(\kappa_1/\kappa_2)]^2}{16\Sigma}\left(\frac{1-\theta}{1+\sqrt{1-\theta}}\right). \quad (28)$$

Figure 2 shows the dimensionless functions $\dot{q}_c^*$, $\dot{q}_h^*$, and ϕ_R^*, where $\dot{q}_h^* \equiv 4\dot{Q}_h^*/(T_h\Delta S^2)$, $\dot{q}_c^* \equiv 4\dot{Q}_c^*/(T_h\Delta S^2)$, and $\phi_R^* \equiv 4\Phi_R^*/(T_h\Delta S^2)$. It can be observed that, as θ increases, the optimum COP of the refrigerator increases but the dissipated energy decreases. From Eqs. (23) and (24), the ratio t_c^*/t_h^*, for critical values of the times in the regime of optimal function χ^R, can be expressed as $t_c^*/t_h^* = 1+\sqrt{1-\theta}$, which only depends on the temperatures of the thermal baths.

B. Optimum performance: Asymmetric case

In the asymmetric case $\Sigma_h \neq \Sigma_c$ (there are different amounts of dissipation in the isothermal branches); there is no specific expression for ε_*, as the one obtained in the symmetric case. However, Eq. (21) can be reduced into an appropriate expression to see its behavior as a function of ε_C, given a value of the ratio Σ_c/Σ_h. This can be done if we notice that the M parameter can be written as $M = \frac{1}{1+(\Sigma_h x_h/\Sigma_c x_c)}$, and therefore ε_* given by Eq. (21) also depends on the ratio $(\Sigma_h x_h/\Sigma_c x_c)$. To achieve the goal we eliminate the ratio (x_h/x_c) from Eqs. (19) to show that the optimal COP at maximum χ^R figure of merit under asymmetric LD condition reads (see Appendix B)

$$2\varepsilon_c z^2 - 3z - 1 = \alpha\sqrt{1+2z}, \quad (29)$$

where $\alpha = \sqrt{T_h\Sigma_h/T_c\Sigma_c}$ and $z = 1/\varepsilon_*$. In this case, it can be seen that $\alpha \to 0$ when $\Sigma_c/\Sigma_h \to \infty$, and that $\alpha \to \infty$ when $\Sigma_c/\Sigma_h \to 0$. For $\alpha \to 0$ the solution of Eq. (29) becomes $z_0 = (\sqrt{9+8\varepsilon_c})/4\varepsilon_c$. Then, $\varepsilon_* \to 0$ when $\Sigma_c/\Sigma_h \to 0$, and $\varepsilon_* \to 1/z_0 = (\sqrt{9+8\varepsilon_c}-3)/2$ when $\Sigma_c/\Sigma_h \to \infty$. Therefore the lower bound is $\varepsilon_- = 0$ and the upper one $\varepsilon_+ = (\sqrt{9+8\varepsilon_c}-3)/2$.

In Fig. 3, the optimal COP given by Eq. (29) is plotted as a function of ε_C for different values of the ratio Σ_c/Σ_h. As can be seen, the plot shows that, when $\Sigma_c/\Sigma_h \to 0$, ε_* tends to lower bound ε_-, which means that the dissipation is greater along the hot isothermal branch than that generated along the cold one. In contrast, when $\Sigma_c/\Sigma_h \to \infty$, ε_* tends to upper bound ε_+; now, it is on the cold isotherm where dissipation dominates over that of the hot branch. And, when $\Sigma_c/\Sigma_h = 1$, the irreversibilities are the same in both isothermals, and thus $\varepsilon_* = \varepsilon_{\rm CA}$ is recovered, as expected.

In Fig. 4 the upper and lower bounds of the COP are plotted as a function of $\theta = T_c/T_h$. The black line is CA's COP and red diamonds correspond to the numerical calculations of the COP obtained in [21], for a Brownian refrigerator. As can be seen, the results follow within the asymmetric region of

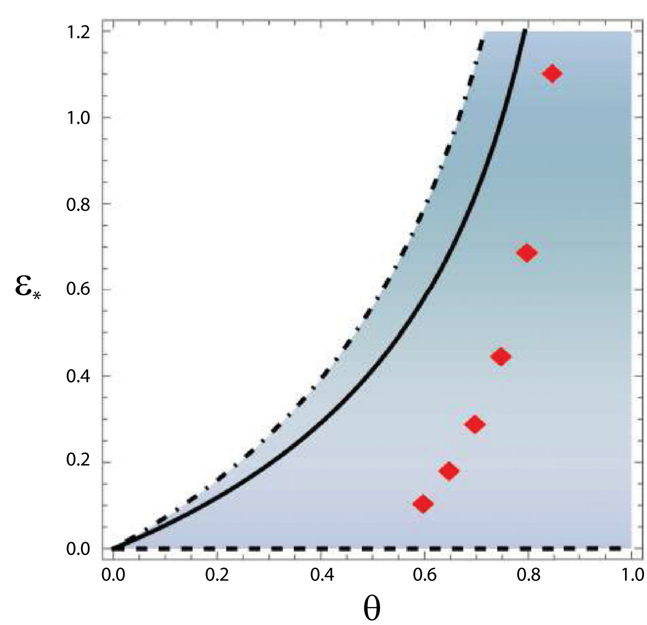

FIG. 4. Plot of optimum COP (ε_*) as a function of θ. The CA's COP is denoted by the solid line. Diamonds represent the numerical evaluations in [21]. The upper and lower bounds for the asymmetric case are marked by a dot dashed line and a dashed line, respectively.

LD, and also as θ increases it tends to the CA result. This comparison shows the robustness of the LD approach.

IV. CONCLUSIONS

In this paper, a Carnot-like Brownian refrigerator is presented by performing a cycle in the reverse direction of a previously published stochastic Carnot engine. For the Brownian Carnot-like refrigerator, we have calculated the optimal COP, ε_*, at maximum χ^R figure of merit when the LD asymmetry conditions are considered. It has been shown that the optimal COP is bounded, such that $0 \leqslant \varepsilon_* \leqslant (\sqrt{9+8\varepsilon_c}-3)/2$, with $\varepsilon_- = 0$ and $\varepsilon_+ = (\sqrt{9+8\varepsilon_c}-3)/2$ being the lower and upper bounds, respectively, similar to the macroscopic refrigerator model. In the particular case of symmetric dissipation, ε_* reduces to the Curzon-Ahlborn efficiency, as given by Eq. (22). Figure 3 shows the consequences derived from Eq. (29) for different values of the ratio Σ_c/Σ_h. With this contribution, the special status of CA performance is clarified: it turns out to be an exact property of Brownian Carnot-like refrigerators operating under symmetric low-dissipation conditions.

The key of our proposal relies on the statelike equation associated with the average $\langle x^2\rangle$, similar to what was done in [15] for three stochastic thermal engine models. Our paper suggests the construction of a Carnot-like refrigerator at the microscopic level, in a manner similar to the implementations by Blickle and Bechinger [16] and Martínez *et al.* [22] for stochastic heat engines.

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APPENDIX A: ε_* AT MAXIMUM FIGURE OF MERIT

1. Symmetric case

To obtain the COP ε_* given in Eq. (21) we proceed as follows: we define $X = \frac{x_h}{x_c}$, which according to Eq. (19) can be written as

$$X = \frac{x_h}{x_c} = \sqrt{\left(\frac{1}{1+\varepsilon_c}\right)\left(\frac{2\varepsilon_c}{\varepsilon_*}+\varepsilon_c\right)}, \tag{A1}$$

where we have used the identity $\varepsilon_c = T_c/(T_h - T_c)$. However, because $M = 1/(X+1)$, thus Eq. (21) is the same as

$$\frac{\varepsilon_c}{\varepsilon_*} = 1 + \frac{\varepsilon_c(1+\varepsilon_c)(X+1)}{\varepsilon_*(1+\varepsilon_c)(X+1)+2\varepsilon_c-\varepsilon_*}. \tag{A2}$$

If we define $y = \frac{\varepsilon_c}{\varepsilon_*}$, we get

$$y = 1 + \frac{y(1+\varepsilon_c)(X+1)}{(1+\varepsilon_c)(X+1)+2y-1}. \tag{A3}$$

After some manipulations and using the expression of X, it now reads

$$(y-1)(2y-1)-(1+\varepsilon_c) = \sqrt{(1+\varepsilon_c)(2y+\varepsilon_c)}. \tag{A4}$$

Simplifying, we arrive to

$$(y-1)^2-(1+\varepsilon_c) = 0, \tag{A5}$$

and finally

$$\varepsilon_* \equiv \varepsilon_{\text{CA}} = \sqrt{1+\varepsilon_c}-1. \tag{A6}$$

2. Calculation of t_c^* and t_h^*

Summing Eqs. (16) and (17), we obtain

$$Q_h - Q_c = \left(\frac{2Q_h}{Q_c}-1\right)T_cL_c'x_c + T_hL_h'x_h. \tag{A7}$$

Using Eq. (18) and symmetric dissipation $L_c' = L_h' = \Sigma$, we show that

$$Q_h - Q_c = \left(\frac{2Q_h}{Q_c}-1\right)T_c\Sigma x_c + T_h\Sigma x_h, \tag{A8}$$

$$T_hx_h^2 = \left(\frac{2Q_h}{Q_c}-1\right)T_cx_c^2. \tag{A9}$$

From Eqs. (12) and (13), it is easy to show that

$$\frac{Q_h-Q_c}{T_h\Sigma} = \frac{\Delta S}{\Sigma}(1-\theta)+x_h+\theta x_c. \tag{A10}$$

On the other side, Eq. (A8) is also written as

$$\frac{Q_h-Q_c}{T_h\Sigma} = \left(\frac{2Q_h}{Q_c}-1\right)\theta x_c + x_h. \tag{A11}$$

By equating both equations we arrive to

$$t_c = \frac{2\Sigma}{\Delta S}\left(\frac{Q_h}{Q_c}-1\right)\frac{\theta}{1-\theta}. \tag{A12}$$

From Eq. (A9) we get

$$t_c^2 = \left(\frac{2Q_h}{Q_c}-1\right)\theta\, t_h^2. \tag{A13}$$

Combining Eqs. (A12) and (A13), and taking into account when $\varepsilon = \varepsilon_* = Q_c/(Q_h - Q_c)$, as well as the expression of ε_* given by Eq. (A6), we obtain the same optimal time t_h^* as given by Eq. (24), and t_c^* from Eq. (A13), which will be the same as Eq. (23).

APPENDIX B: ASYMMETRIC CASE $\Sigma_c \neq \Sigma_h$

In a similar way, upon the definition of $Y = \frac{\Sigma_h x_h}{\Sigma_c x_c}$, and $z = \frac{1}{\varepsilon_*}$, it can be shown that Eq. (21) reduces to

$$(2\varepsilon_c z-1)(\varepsilon_c z-1) = (1+\varepsilon_c)(Y+1) \tag{B1}$$

or

$$2(\varepsilon_c z)^2 - 3(\varepsilon_c z) - \varepsilon_C = (1+\varepsilon_c)Y. \tag{B2}$$

Using now Eq. (19) we get

$$\left(\frac{1}{\varepsilon_c}+1\right)Y = \sqrt{\frac{T_h\Sigma_h}{T_c\Sigma_c}}\sqrt{1+2z}, \tag{B3}$$

and finally we obtain Eq. (29).

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