Spatial dependence of microscopic percolation conduction

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In two dimensions, the average electrical conductance from a point in a percolating network to the network boundary can be related by a conformal transformation to the conductance from one point to another in an unbounded network. We verify that this works at the percolation threshold for the square.

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I. MICROSCOPIC PERCOLATION

The percolation model studies the properties of networks derived from a lattice by random deletions of bonds [1]. It comprises several different questions. The earliest studies established that, as the fraction p of bonds that are present is increased, there is a kind of phase transition: when p is small, likely configurations only contain finite connected clusters; but above a threshold value p_c it is highly likely that there is an "infinite" connected cluster (or one that spans a large finite network from side to side) [2]. In two dimensions this "connectedness" model can be understood as the limiting case $Q \to 1$ of the Q-component Potts models; the exponents characterizing the correlation length and probability of a lattice site belonging to the infinite cluster are given by the den Nijs [3–5] relationships that describe all exponents of Potts models. Relevant to this study are the correlation length ξ,

$$\xi \approx |p - p_c|^{-\nu}, \tag{1}$$

and the probability P that an arbitrary site belongs to the infinite cluster,

$$P \approx |p - p_c|^{\beta}, \tag{2}$$

where $\nu = 4/3$ and $\beta = 5/36$. These basic properties of a percolating system can be readily studied in computer simulations.

Connectedness is not a very useful property of a real-world experimental system. This has led to the study of the properties of a percolating system that can be measured by established experimental methods. A paradigm model is to assign unit conductance to the bonds and measure the "macroscopic" electrical conductivity: the average current density produced by a uniform electric field [6]. In two dimensions the macroscopic electrical conductance is well described for $p > p_c$ by

$$G = G_0(p - p_c)^t \tag{3}$$

and at $p = p_c$ by

$$G = G_0 L^{-t/\nu}. \tag{4}$$

Simulations [7] indicate $t/\nu = 0.973$, independent of details of the model (e.g., periodic boundary conditions vs assigned potentials at the boundaries, as well as the details of how the sites are connected). The relationship between this exponent and those for the connectedness problem is unclear; unusual for a two-dimensional model, the exponent does not appear to be a rational number.

Another model problem, the subject of the present paper, is the "microscopic" conductance [8,9]: the conductance from a small source to a small sink at distance D, or the current from a small source to a grounded boundary at distance $D/2$. This seems to be a simpler and more prospective model to study than the macroscopic conductance, because for a chosen boundary shape it depends on just p and the distance D.

When the current is injected at the center of a square of edge L, the probability at p_c that the site is connected to the boundary is expected to vary as $L^{-\tilde{\beta}}$, where $\tilde{\beta} = \beta/\nu = 5/48 = 0.1048$. The average conductance of the connected configurations vary as L^{-u}, where u is the corresponding exponent. With all boundaries conducting, we made 20 000 random realizations of the system for $L = 21$, 41, 81, and 161 and averaged the resulting conductance from the center to the edge, finding $\tilde{\beta} = 0.11 \pm .01$ (in agreement with the expected result [10]) and $u = 1.01 \pm 0.01$. The latter is rather similar to $\tilde{t} = t/\nu = 0.972$, but it is not clear that they are, in fact, the same; for the Cayley tree (which represents the behavior at the upper critical dimension $D = 6$), they are not [9].

In studying the microscopic conductance we found it useful to ignore configurations for which the chosen site does not have a connected path to the boundary, and only average the conductance over the remainder, because the variance of the distribution of conductances is smaller than the variance of connectedness itself (the distribution of connectedness is binary), and the connectedness problem is well studied. Thus, for conductance from one point to another at distance D in an infinite system, we define

$$g_{\text{connected}} = g_0 D^{-u}. \tag{5}$$

The conductance defined in terms of all configurations would include the statistics of connectedness, so that

$$g_{\text{all}} = D^{-\beta/\nu} g_{\text{connected}}. \tag{6}$$

The variance of the conductance is not negligible. The values obtained from a large sample of realizations of the random lattice of finite size form a distribution whose width is comparable to its average, with the consequence that the

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arithmetic, geometric, and harmonic means differ and lead to slightly different estimates for the exponent. Lobb and Frank [7] showed that these estimates converge in the limit of large lattice size and that the geometric mean (the exponential of the average of the logarithms) is the least subject to this finite-size effect. We note that the geometric mean of the conductance is the reciprocal of the geometric mean of the resistance (this property does not hold for other central measures). For these reasons the analysis that follows is for the geometrical mean [11].

II. CONFORMAL INVARIANCE

Theoretical understanding of the percolation conduction models has been hindered by a lack of understanding how to incorporate these effects into the theoretical framework, since what is measured is not the derivative of a free energy with respect to an applied field, nor the correlation function of known operators. As a step towards gaining a better understanding of these problems, we studied how the microscopic conductance from an interior point to the boundary depends on the position of the interior point.

At the critical point, two-dimensional critical models often have conformal invariance [12,13], which is a symmetry that contains translational, rotational, and scale invariance. Representing the (X, Y) coordinates of a point by the complex number $Z = X + \mathrm{i}Y$, all analytic functions give conformal transformations. Cardy [14] has used ideas from conformal field theory to derive a formula for the probability of the presence of a percolating path joining opposite sides of a rectangle containing a percolating network at the percolation threshold. This suggests there may be a conformal field theory that governs other aspects of a percolating system, such as the microscopic percolation conductance.

Conformal invariance implies a relationship between the critical behavior of correlation functions for systems of different shapes; in the present case, the average conductance $C(Z)$ from the point Z to the boundary of a square can be related to the simpler problem of the resistance from the point W in an infinite half-plane to its boundary and thereby predicts a specific form for the positional dependence.

The rectangle $[-K < X < K, 0 < Y < K']$ is mapped into the upper half plane by the Jacobi elliptic function $\mathrm{sn}(X + \mathrm{i}Y, m)$, where $K(m)$ is the complete elliptic integral and $K'(m) = K(1 - m)$. The rectangle becomes a square when $2K(m) = K'$, which occurs for $m = [(\sqrt{2} - 1)/(\sqrt{2} + 1)]^2 = 0.029\,437$ (and then $K = 1.582\,55$). The square $-L/2 < X < L/2, 0 < Y < L$ is then mapped into the upper half plane by the transformation $W(Z) = \mathrm{sn}(2KZ/L, m)$ for those choices of parameters.

On the half plane, the only special distance is the imaginary part of W; scaling theory says the conductance should be a power law of this distance [$C(W) \approx |\mathrm{Im}\, W|^{-u}$]; and for the most common case of conformal field theory (the "quasiprimary field of spin zero"), this implies the conductance to the boundary of the square to be

$$C(Z) \approx \left|\frac{\mathrm{d}W(Z)/\mathrm{d}Z}{\mathrm{Im}\, W}\right|^u \approx \left|\frac{\mathrm{cn}\,\mathrm{dn}}{\mathrm{Im}\,\mathrm{sn}}\right|^u, \tag{7}$$

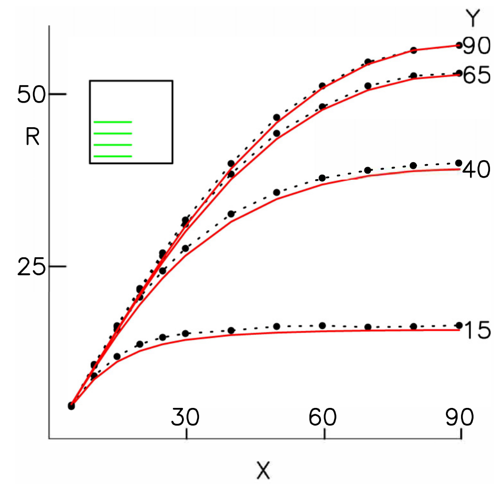

FIG. 1. Resistance to the boundary along paths through a square. Comparison of numerical experiment (black dots) and Eq. (7) (solid red line) for $L = 180$. The curves represent the geometrical average resistance of the connected configurations for various $5 \leqslant X \leqslant 90$, for $Y = 15, 40, 65$, and 90. The scaling exponent is 1.05 ± 0.03. The inset is a cartoon indicating the paths across the square where data were taken. The data for this figure are given in the Supplemental Material [16].

where the argument list is $(2KZ/L, m)$ for all of the elliptic functions.

Despite the asymmetrical appearance, this function has the symmetry of the square; for points near one of the edges of the square, it is a power law of the distance to the edge and smoothly interpolates this rule for points that are nearly equidistant to more than one edge.

III. NUMERICAL STUDY

We calculated the microscopic conductance for a source point in the square in a computer simulation. For a chosen realization and chosen interior point, we first verified that the interior point belongs to a cluster that touches a "conducting" boundary (that is, one that completes a circuit from the interior point) by means of the Hoshen-Kopelman algorithm [15]. Sites that had only a single connecting link (and thus were not part of the current-carrying circuit) were removed. Kirchhoff's equations give relationships between the voltages of neighboring points; these were solved by a pivoted Gaussian elimination.

Figure 1 compares the numerical results with Eq. (7) with $u = 1.03 \pm 0.02$, where the confidence range was determined by the condition that changing u by 0.02 doubles the mean square variation of the ratio of theory to experiment. We believe the discrepancy between theory and numerical experiment to be due to finite lattice spacing; the statistical errors are smaller. We observed that the discrepancy is larger for smaller lattice sizes.

IV. CONCLUSION

The agreement between the theoretical representation of the dependence of the conductance on position given by

Eq. (7) and the results of simulations shown in Fig. 1 supports the idea that the microscopic conductance can be regarded to be a kind of conformally invariant correlation function and thus hints at the existence of a continuum field theory for this and other percolation conduction problems.

See the Supplemental Material [16] for the numerical values used in constructing the figure in this paper.

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