Dynamical large deviations for long-range interacting inhomogeneous systems without collective effects

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We consider the long-term evolution of a spatially inhomogeneous long-range interacting N-body system. Placing ourselves in the dynamically hot limit, i.e., assuming that the system only weakly amplifies perturbations, we derive a large deviation principle for the system's empirical angle-averaged distribution function. This result extends the classical ensemble-averaged kinetic theory given by the so-called inhomogeneous Landau equation, as it specifies the probability of typical and large dynamical fluctuations. We detail the main properties of the associated large deviation Hamiltonian, particularly focusing on how it complies with the system's conservation laws and possesses a gradient structure.

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I. INTRODUCTION

As a result of violent relaxation [1], long-range interacting N-body systems generically find themselves on nonequilibrium quasistationary states. Finite-N fluctuations can then continue driving these systems closer to their thermodynamical equilibrium. Such a long-range dynamics covers quite a wide class of systems like plasmas [2], self-gravitating clusters [3], or even more generic systems [4], such as point vortices in two-dimensional hydrodynamics, see, e.g., [5] or classical Heisenberg spins on the unit sphere, see, e.g., [6]. In the present work, we focus our interest on (integrable) spatially inhomogeneous systems, as exemplified by globular clusters [7].

To describe the long-term evolution of these systems, a classical starting point is to consider the ensemble-averaged evolution of the system's distribution function (DF), where, here, the average is over a set of initial conditions. In the context of long-range interacting inhomogeneous systems, this is described by the inhomogeneous Balescu-Lenard (BL) equation [8,9]. In the limit of a dynamically hot system, i.e., a system which only weakly amplifies perturbations, this kinetic equation reduces to the (simpler) inhomogeneous Landau equation (see, e.g., [10] and references therein). Both kinetic equations satisfy an H theorem hence highlighting the irreversibility of the ensemble-averaged dynamics.

Yet such frameworks, because they solely focus on ensemble-averaged dynamics, cannot predict the detailed probabilities of typical and large dynamical fluctuations away from this mean evolution. Such an extension is the realm of large deviation theory (see, e.g., [11] and the detailed review therein) which describes the entire statistics of the system's empirical DF. This is the focus of the present paper. In particular, we build upon [12] and derive the large deviation Hamiltonian in the case of a dynamically hot long-range interacting inhomogeneous system. This calculation, therefore, generalizes the inhomogeneous Landau equation, which is immediately recovered from the large deviation theory through an ensemble average. As the upcoming sections will highlight, up to a few additional complications stemming from our accounting of the intricate orbital structure, these calculations share a lot of similarities with the ones presented in [12].

The paper is organized as follows. In Sec. II, we detail our system, the quasilinear expansion, and the inhomogeneous Landau equation. In Sec. III, we derive the system's large deviation Hamiltonian while neglecting collective amplification. In Sec. IV, we discuss the main properties of this Hamiltonian. We conclude in Sec. V. Technical details in the main text are kept to a minimum and deferred to the Appendices.

II. DYNAMICS OF LONG-RANGE INTERACTING SYSTEMS

A. System

We are interested in the long-term evolution of a long-range interacting Hamiltonian system in $2d$ dimensions. We denote phase space with $\mathbf{w} := (\mathbf{q}, \mathbf{p})$. The system is composed of $N \gg 1$ particles of individual mass, $m := M_{\mathrm{tot}}/N$, with M_{tot} the system's fixed total active mass. At any given time, the state of the system can be described by its empirical DF

$$F_{\mathrm{d}}(\mathbf{w}, t) := \sum_{i=1}^{N} m\, \delta_{\mathrm{D}}[\mathbf{w} - \mathbf{w}_i(t)], \tag{1}$$

with $\mathbf{w}_i(t)$ the location in phase space at time t of particle i. Here, F_{d} stands for the empirical ("discrete") DF, while δ_{D} is the usual Dirac delta function. We assume that particles are embedded within some given external potential, $U_{\mathrm{ext}}(\mathbf{w})$ (e.g., the kinetic energy) and coupled to one another via a long-range pairwise interaction, $U(\mathbf{w}, \mathbf{w}')$. We denote the typical amplitude of $U(\mathbf{w}, \mathbf{w}')$ with G. The limit $G \to 0$ corresponds to the dynamically hot limit. In that regime, the system only

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weakly amplifies perturbations, i.e., the collective effects are negligible. This will be the limit of interest in this work.

The instantaneous specific empirical Hamiltonian is

$$H_{\mathrm{d}} := U_{\mathrm{ext}}(\mathbf{w}) + \Phi_{\mathrm{d}}(\mathbf{w},t), \tag{2}$$

with the empirical ("discrete") potential

$$\Phi_{\mathrm{d}}(\mathbf{w},t) := \int \mathrm{d}\mathbf{w}'\, U(\mathbf{w},\mathbf{w}')\, F_{\mathrm{d}}(\mathbf{w}',t). \tag{3}$$

The dynamics of F_{d} is given by the Klimontovich equation [13]. It reads

$$\frac{\partial F_{\mathrm{d}}}{\partial t} + [F_{\mathrm{d}}, H_{\mathrm{d}}] = 0, \tag{4}$$

with the Poisson bracket

$$[f(\mathbf{w}), h(\mathbf{w})] := \frac{\partial f}{\partial \mathbf{q}}\cdot\frac{\partial h}{\partial \mathbf{p}} - \frac{\partial f}{\partial \mathbf{p}}\cdot\frac{\partial h}{\partial \mathbf{q}}. \tag{5}$$

In the present setup, stochasticity is said to be purely extrinsic since Eq. (4) is itself fully deterministic, and only the particles' initial conditions vary from one realization to the other. In the following, we will therefore consider ensemble averages over the initial conditions of the N particles. Denoting the ensemble average as $\langle\cdot\rangle$, we assume then that $F(\mathbf{w},t) := \langle F_{\mathrm{d}}(\mathbf{w},t)\rangle$ is a smooth function, and we introduce $H := \langle H_{\mathrm{d}}\rangle$ the associated smooth Hamiltonian. The mean system is assumed to be in an integrable stable quasistationnary equilibrium. There exist then canonical angle-action coordinates, $(\boldsymbol{\theta}, \mathbf{J})$ [3], so that

$$F(\mathbf{w},t) = F(\mathbf{J},t); \quad H(\mathbf{w},t) = H(\mathbf{J},t), \tag{6}$$

hence defining the orbital frequencies, $\boldsymbol{\Omega}(\mathbf{J}) := \partial H/\partial \mathbf{J}$. Such a system is said to be in a quasistationary equilibrium since $[F(\mathbf{J},t), H(\mathbf{J},t)] = 0$.

B. Quasilinear expansion

For a given realization, we define

$$F_N(\mathbf{J},t) := \int \frac{\mathrm{d}\boldsymbol{\theta}}{(2\pi)^d}\, F_{\mathrm{d}}(\mathbf{w},t), \tag{7}$$

by averaging over the angles. Following Eq. (3), we assume that $H_N := H_N[F_N]$, only depends on the actions. Since we have $\langle F_N\rangle = F$, it is natural to build the decomposition

$$F_{\mathrm{d}}(\mathbf{w},t) = F_N(\mathbf{J},t) + \frac{1}{\sqrt{N}}\,\delta F(\mathbf{w},t), \tag{8a}$$

$$H_{\mathrm{d}}(\mathbf{w},t) = H_N(\mathbf{J},t) + \frac{1}{\sqrt{N}}\,\delta\Phi(\mathbf{w},t), \tag{8b}$$

where the prefactor $1/\sqrt{N}$ ensures that the fluctuations, δF and $\delta\Phi$, are of order unity w.r.t. N.

It is crucial to note that Eq. (8) differs from the usual quasilinear decomposition, see, e.g., [9], $F_{\mathrm{d}} = F + \delta F/\sqrt{N}$, where one replaces F_N in Eq. (8a) with the (nonstochastic) mean DF, $F(\mathbf{J},t)$. Indeed, in the present approach F_N since it depends on the particles' initial conditions, remains a stochastic quantity that varies from one realization to another. In this work, our goal is to characterize the statistics of the dynamics of $F_N(\mathbf{J},t)$ and its deviation away from the mean evolution, i.e., the evolution of $F(\mathbf{J},t)$.

We can now inject Eqs. (8) into Eq. (4) to obtain evolution equations for δF and F_N. We get

$$\frac{\partial \delta F}{\partial \tau} + N\{[\delta F, H_N] + [F_N, \delta\Phi]\} = 0, \tag{9a}$$

$$\frac{\partial F_N(\mathbf{J},\tau)}{\partial \tau} + \int \frac{\mathrm{d}\boldsymbol{\theta}}{(2\pi)^d}\,[\delta F, \delta\Phi] = 0, \tag{9b}$$

where we truncated Eq. (9a) at first order in $1/\sqrt{N}$, performed an angle average in Eq. (9b), and introduced the (slow) time $\tau := t/N$. On the one hand, for $N \gg 1$, Eq. (9a) for δF is a fast process, with a timescale for τ of order $1/N$: it describes the fast dynamics of fluctuations. On the other hand, Eq. (9b) is associated with a slow process, with a timescale for τ of order 1: it describes the slow relaxation of orbits.

C. Kinetic equations

To describe the ensemble average of Eq. (9b) for the asymptotic process of δF for fixed F_N, one typically proceeds as follows: (i) Because the angles $\boldsymbol{\theta}$ are 2π-periodic, fluctuations can be decomposed in Fourier space, hence introducing the associated resonance vectors, $\mathbf{k}, \mathbf{k}' \in \mathbb{Z}^d$; (ii) Assuming a separation between the fast and slow timescales, one solves Eq. (9a) for $\delta F(\mathbf{w},t)$, assuming a fixed $F_N(\mathbf{J},\tau)$, see Appendix A; (iii) One finally injects these asymptotic expressions in the right-hand side (r.h.s.) of Eq. (9b) to obtain the kinetic collision operator, see [9] for details.

Placing oneself in the dynamically hot limit, i.e., $G \to 0$, one finds that the ensemble-averaged long-term evolution of the system is described by the inhomogeneous Landau equation [10] reading

$$\frac{\partial F(\mathbf{J},\tau)}{\partial \tau} = \frac{\partial}{\partial \mathbf{J}}\cdot\Bigg[\sum_{\mathbf{k},\mathbf{k}'}\mathbf{k}\int \mathrm{d}\mathbf{J}'\, B_{\mathbf{k}\mathbf{k}'}(\mathbf{J},\mathbf{J}') \times \left\{\mathbf{k}\cdot\frac{\partial F}{\partial \mathbf{J}}F(\mathbf{J}') - \mathbf{k}'\cdot\frac{\partial F}{\partial \mathbf{J}'}F(\mathbf{J})\right\}\Bigg] + o(G^2), \tag{10}$$

where we wrote $F(\mathbf{J}) = F(\mathbf{J},\tau)$ to shorten the notations. We also introduced

$$B_{\mathbf{k}\mathbf{k}'}(\mathbf{J},\mathbf{J}') := \pi(2\pi)^d M_{\mathrm{tot}}\,|\psi_{\mathbf{k}\mathbf{k}'}(\mathbf{J},\mathbf{J}')|^2 \times \delta_{\mathrm{D}}[\mathbf{k}\cdot\boldsymbol{\Omega}(\mathbf{J}) - \mathbf{k}'\cdot\boldsymbol{\Omega}(\mathbf{J}')], \tag{11}$$

with δ_{D} the Dirac delta. Here, $\psi_{kk'}(\mathbf{J},\mathbf{J}')$ stands for the bare coupling coefficients [see Eq. (A3)], which account for the system's inhomogeneity. When accounting for collective effects, i.e., going beyond the limit $G \to 0$, Eq. (10) becomes the BL equation [8,9]. It is obtained from Eq. (11) by replacing $|\psi_{\mathbf{k}\mathbf{k}'}(\mathbf{J},\mathbf{J}')|^2$ with their dressed analogs, $|\psi^{\mathrm{d}}_{\mathbf{k}\mathbf{k}'}(\mathbf{J},\mathbf{J}',\mathbf{k}\cdot\boldsymbol{\Omega}(\mathbf{J}))|^2$, as defined in Eq. (A10).

III. LARGE DEVIATION PRINCIPLE

We now want to go beyond the classical computation presented in Eq. (10) by estimating not only the ensemble average of Eq. (9b), but the entire scaled cumulant generating function. This allows one to retrieve not only the average evolution path for the angle-averaged DF, F_N, but the entire probability distribution function for any evolution path.

Following the same approach as in [12], we can generically estimate the probability that $F_N(\tau)$ follows a given time evolution $\{F(\tau)\}_{0\leqslant\tau\leqslant T}$ through[1]

$$\mathbb{P}(\{F_N(\tau)\}_{0\leqslant\tau\leqslant T} = \{F(\tau)\}_{0\leqslant\tau\leqslant T}) \underset{N\to+\infty}{\asymp} \exp\left[-\frac{N(2\pi)^d}{M_{\text{tot}}}\sup_P \int_0^T \mathrm{d}\tau \left\{\left(\int \mathrm{d}\mathbf{J}\dot{F}P\right) - \mathcal{H}[F,P]\right\}\right], \tag{12}$$

with $\dot{F} := \partial_\tau F$, and the prescription that $F_N(\tau=0)$ converges to $F(\tau=0)$ for $N\to+\infty$. Equation (12) also involves the conjugate field $\{P(\mathbf{J},\tau)\}_{0\leqslant\tau\leqslant T}$, over which the maximization must be performed. Finally, in Eq. (12), we introduce the large deviation Hamiltonian (i.e., the scaled cumulant generating function)

$$\mathcal{H}[F,P] := \lim_{\Delta\to+\infty} \frac{M_{\text{tot}}}{\Delta(2\pi)^d} \ln\left[\left\langle \exp\left(\frac{(2\pi)^d}{M_{\text{tot}}} \times \int_0^\Delta \mathrm{d}t \int \mathrm{d}\mathbf{J}\, P(\mathbf{J})\, \partial_\tau F_N[\delta F]\right)\right\rangle_F\right], \tag{13}$$

where both $F(\mathbf{J})$ and $P(\mathbf{J})$ are evaluated at time τ, and $\partial_\tau F_N[\delta F]$ follows from Eq. (9b). Here, $\langle\cdot\rangle_F$ denotes an expectation over the fast process δF with $F_N = F$ fixed. Equations (12) and (13) are generic results describing the large deviations for the slow evolution of a process driven by a random fast process. We refer to Appendix B for a brief heuristic derivation of Eqs. (12) and (13).

Naturally, here the difficulty lies in the computation of the average in Eq. (13). Following the same approach as in [12], we start this calculation by expanding Eq. (13) w.r.t. G. We can write

$$\mathcal{H}[F,P] = \mathcal{H}^{(1)}[F,P] + \mathcal{H}^{(2)}[F,P] + o(G^2), \tag{14}$$

where $\mathcal{H}^{(1)}$ (or $\mathcal{H}^{(2)}$) is the first (or second cumulant). Computing these cumulants is a cumbersome calculation. As a first step, this requires the computation of the asymptotic time evolution of the DF and potential fluctuations. This is presented in Appendix A. Then, in Appendix C, we compute explicitly the various first terms of Eq. (14).

The first cumulant reads (Appendix C 1)

$$\mathcal{H}^{(1)}[F,P] := -\sum_{\mathbf{k},\mathbf{k}'}\int \mathrm{d}\mathbf{J}\mathrm{d}\mathbf{J}'\, B_{\mathbf{k}\mathbf{k}'}(\mathbf{J},\mathbf{J}')\,\mathbf{k}\cdot\frac{\partial P}{\partial \mathbf{J}} \times \left[\mathbf{k}\cdot\frac{\partial F}{\partial \mathbf{J}}F(\mathbf{J}') - \mathbf{k}'\cdot\frac{\partial F}{\partial \mathbf{J}'}F(\mathbf{J})\right] + o(G^2), \tag{15}$$

while the second cumulant reads (Appendix C 2)

$$\mathcal{H}^{(2)}[F,P] := \sum_{\mathbf{k},\mathbf{k}'}\int \mathrm{d}\mathbf{J}\mathrm{d}\mathbf{J}'\, B_{\mathbf{k}\mathbf{k}'}(\mathbf{J},\mathbf{J}')\,\mathbf{k}\cdot\frac{\partial P}{\partial \mathbf{J}} \times \left[\mathbf{k}\cdot\frac{\partial P}{\partial \mathbf{J}} - \mathbf{k}'\cdot\frac{\partial P}{\partial \mathbf{J}'}\right]F(\mathbf{J})\,F(\mathbf{J}') + o(G^2). \tag{16}$$

Finally, in Appendix C 3, we justify why all cumulants beyond the two first ones can be neglected in the dynamically hot limit, i.e., they are all of order $o(G^2)$.

Equation (12) in conjunction with Eq. (14) is the main result of this section. These two equations characterize the dynamical large deviations of the angle-averaged DF, $F_N(\mathbf{J},\tau)$. Phrased differently, they describe simultaneously the system's mean evolution and the dispersion around it. We discuss their main properties in Sec. IV.

Unfortunately, the expansion from Eq. (14) stops being effective when accounting for collective effects, i.e., in the dynamically cold limit. In that case, one must resort to an explicit computation of the exponential average from Eq. (13), as all cumulants in Eq. (13) contribute to the large deviation Hamiltonian. The authors of [14] succeeded in performing this calculation in the case of a homogeneous system. Unfortunately, the theorems and methods used therein do not lend themselves straightforwardly to the inhomogeneous case. One reason of these additional difficulties lies in the fact that in homogeneous systems, the resonance condition, $\delta_{\mathrm{D}}(\mathbf{k}\cdot[\mathbf{v}-\mathbf{v}'])$, is diagonal w.r.t. the resonance number $\mathbf{k}$. This is not the case anymore in inhomogeneous systems where the resonance condition becomes $\delta_{\mathrm{D}}[\mathbf{k}\cdot\mathbf{\Omega}(\mathbf{J}) - \mathbf{k}'\cdot\mathbf{\Omega}(\mathbf{J}')]$, as in Eq. (11). Such a development will be the topic of a future work.

IV. PROPERTIES

In this section, we briefly present the main properties of the large deviation Hamiltonian from Eq. (14). We refer to Appendix D for technical details and to Sec. 2.1.1 of [12] for thorough discussions.

A. Most probable path

The evolution path that minimizes the large deviation action is the most probable evolution path for the empirical DF (see Sec. 7.2.2 in [11]). The associated Hamilton equation reads

$$\frac{\partial F(\mathbf{J},\tau)}{\partial\tau} = \frac{\delta\mathcal{H}}{\delta P(\mathbf{J})}[F, P=0], \tag{17}$$

and yields exactly the inhomogeneous Landau equation (10). As could have been expected *a priori*, the most likely evolution path is the one given by the usual ensemble-averaged kinetic theory.

B. Conservation laws

If $C[F]$ is a conserved quantity for the N-body dynamics, the large deviation Hamiltonian must satisfy

$$\int \mathrm{d}\mathbf{J}\,\frac{\delta C[F]}{\delta F(\mathbf{J})}\,\frac{\delta\mathcal{H}}{\delta P(\mathbf{J})} = 0. \tag{18}$$

[1]In Eq. (12), we introduce the logarithmic equivalence defined via $a_N \underset{N\to\infty}{\asymp} \mathrm{e}^{Na} \iff \lim_{N\to+\infty} \ln(a_N)/N = a$.

This symmetry implies that the large deviation action [i.e., the exponential argument in the r.h.s. of Eq. (12)] is infinite for evolution paths that do not satisfy $\int \mathrm{d}\mathbf{J}\dot{F}\,\delta C[F]/\delta F(\mathbf{J}) = 0$. Phrased differently, any evolution path for the DF violating the conservation laws has zero probability (see Sec. 7.2.6 in [11]). In Appendix D 2, we explicitly show that Eq. (14) is consistent with the mass and energy conservation, defined via

$$M[F] \;:=\; \int \mathrm{d}\mathbf{J}\, F(\mathbf{J}) \;\; \text{(mass conservation)}, \tag{19a}$$

$$E[F] \;:=\; \int \mathrm{d}\mathbf{J}\, H(\mathbf{J})\, F(\mathbf{J}) \;\; \text{(energy conservation)}. \tag{19b}$$

This is an important self-consistency property. It ensures that individual (stochastic) realizations only explore states leaving the system's global invariants unchanged.

C. Hamilton-Jacobi equation

We define the system's entropy as

$$S[F] := -\int \mathrm{d}\mathbf{J}\, F(\mathbf{J})\, \ln[F(\mathbf{J})]. \tag{20}$$

As detailed in Appendix D 3, one can show that it solves the stationary Hamilton-Jacobi equation

$$\mathcal{H}[F, -\delta S/\delta F] = 0, \tag{21}$$

where $\delta S/\delta F$ stands for the function $J \mapsto \delta S[F]/\delta F(\mathbf{J})$. This is a reassuring sanity check. Yet Eq. (21) is not sufficient to prove that the negative of the entropy is the quasipotential associated with the present large deviation principle (see Sec. 7.2.3 in [11]). Indeed, one also has to prove that $S[F]$ has an unique maximum: this is typically not true for self-gravitating systems (see, e.g., [15,16]).

D. Time-reversal symmetry

The large deviation Hamiltonian complies with the generalized time-reversal symmetry (see Appendix D 4)

$$\mathcal{H}[F, -P] = \mathcal{H}[F, P - \delta S/\delta F]. \tag{22}$$

Such a relation corresponds to a detailed balance condition at the level of large deviations, associated with the symmetry $\tau \to T - \tau$ in Eq. (12) (see Sec. 7.3.1 in [11]). This is the imprint, within the framework of large deviations, that the individual Hamiltonian equations of motions are time-reversible.

E. Gradient structure

The large deviation Hamiltonian generically induces a gradient flow [17]. Indeed, as detailed in Appendix D 5, the second cumulant from Eq. (16) can be written as

$$\mathcal{H}^{(2)}[F, P] = \int \mathrm{d}\mathbf{J}\mathrm{d}\mathbf{J}'\, P(\mathbf{J})\, P(\mathbf{J}')\, Q[F](\mathbf{J}, \mathbf{J}'), \tag{23}$$

where $Q[F]$ reads

$$Q[F](\mathbf{J}, \mathbf{J}') := \sum_{\mathbf{k},\mathbf{k}'} \mathbf{k} \cdot \frac{\partial}{\partial \mathbf{J}} \bigg\{ \frac{\partial}{\partial \mathbf{J}'} \cdot \bigg[-\mathbf{k}' F(\mathbf{J}) F(\mathbf{J}') B_{\mathbf{k}\mathbf{k}'}(\mathbf{J}, \mathbf{J}') + \mathbf{k}\, \delta_{\mathrm{D}}(\mathbf{J} - \mathbf{J}') F(\mathbf{J}) \int \mathrm{d}\mathbf{J}'' F(\mathbf{J}'') B_{\mathbf{k}\mathbf{k}'}(\mathbf{J}, \mathbf{J}'') \bigg] \bigg\}. \tag{24}$$

The inhomogeneous Landau Eq. (10) can then be rewritten as

$$\frac{\partial F(\mathbf{J}, \tau)}{\partial \tau} = \int \mathrm{d}\mathbf{J}'\, Q[F](\mathbf{J}, \mathbf{J}')\, \frac{\delta S[F]}{\delta F(\mathbf{J}')}, \tag{25}$$

with the entropy, $S[F]$, defined in Eq. (20). Given that $\mathcal{H}[F, P]$ is convex w.r.t. P, $Q[F]$ is a positive operator on the space of DFs. Therefore, Eq. (25) illustrates the increase of entropy along solutions of the Landau equation (see Sec. 5 in [11]). This is the celebrated H theorem recovered here through the characterization of dynamical large deviations.

F. Stochastic Landau equation

We note that the large deviation Hamiltonian from Eq. (14) is quadratic in $P(\mathbf{J}, t)$. This implies that large deviations are Gaussian (see, e.g., Sec. 4.2 in [12]). As a result, as detailed in Appendix D 6, one can setup a stochastic partial differential equation which obeys the large deviation principle from Eq. (12). It reads

$$\frac{\partial F_N(\mathbf{J}, \tau)}{\partial \tau} = \left[\frac{\partial F_N(\mathbf{J}, \tau)}{\partial \tau}\right]_{\mathrm{Landau}} + \zeta[F_N](\mathbf{J}, \tau), \tag{26}$$

where $[\partial F_N(\mathbf{J})/\partial \tau]_{\mathrm{Landau}}$ is the inhomogeneous Landau collision operator, i.e., the r.h.s. of Eq. (10). We also introduced the Gaussian random field, $\zeta[F](\mathbf{J}, \tau)$. Following Sec. 4.2 in [12], it obeys

$$\langle \zeta[F](\mathbf{J}, \tau) \rangle = 0, \tag{27a}$$

$$\langle \zeta[F](\mathbf{J}, \tau)\, \zeta[F](\mathbf{J}', \tau') \rangle = \frac{2m}{(2\pi)^d}\, Q[F](\mathbf{J}, \mathbf{J}')\, \delta_{\mathrm{D}}(\tau - \tau'), \tag{27b}$$

where averages are taken at fixed F.

Although intricate, Eq. (26) is an important result as it allows one to "mimic" directly the stochastic evolution of F_N and its large deviations, without ever integrating the N-body equations of motion. Indeed, in the same way that the usual Fokker-Planck equation can be recovered from the average of independent Langevin random walks [18], the statistics of the large deviations from Eq. (12) can be recovered from the average of independent realizations of the stochastic Eq. (26).

Following Sec. IV B, Eq. (26) exactly conserves the total mass and total energy. Finally, in Appendix D 6 [see Eq. (D21)], we present an alternative writing of $\zeta[F](\mathbf{J}, \tau)$ which (i) eases the effective sampling of the stochastic noise and (ii) highlights explicitly the compliance with the conservation laws. Such a rewriting should prove particularly useful to make numerical realizations of self-gravitating systems.

G. Homogeneous limit

It is straightforward to recover the results from [12] in the limit of a (multiperiodic) homogeneous system. In that case, (i) the angle-action coordinates become $(\boldsymbol{\theta}, \mathbf{J}) \to (\mathbf{x}, \mathbf{v})$, the volume elements $(2\pi)^d \to L^3$, with L the size of the box, and N/M_{tot} plays the role of the plasma parameter Λ; (ii) the bare coupling coefficients are constrained by symmetry and independent of the orbits, namely, $\psi_{\mathbf{k}\mathbf{k}'}(\mathbf{J}, \mathbf{J}') \to \delta_{\mathbf{k}\mathbf{k}'}/|\mathbf{k}|^2$, up to a prefactor. Equation (14) then falls back on the result from [12].

V. CONCLUSION

In this paper, we investigated dynamical large deviations in systems with long-range interactions. In the view of generalizing [12], we focused here on the case of (integrable) spatially inhomogeneous systems. Starting from the generic large deviation Hamiltonian of a slow-fast system and placing ourselves in the dynamically hot limit, i.e., neglecting collective effects, we computed the two first cumulants of the associated large deviation Hamiltonian. Equation (14) is the main result of this work, and encodes the likelihood of any given evolution path for the system's angle-averaged DF. In Sec. IV, we highlighted that Eq. (14) complies with all the expected properties, such as (i) recovering the inhomogeneous Landau equation [10] in the ensemble-averaged limit; (ii) ensuring the conservation laws; (iii) and possessing a natural gradient structure. Finally, we emphasized that the quadratic dependence of the large deviation Hamiltonian w.r.t. the conjugate field allows one to construct an effective stochastic partial differential equation with the expected dynamical large deviations.

The present work is only one step toward an ever more detailed description of the statistical properties of dynamical large deviations in long-range interacting inhomogeneous systems. We conclude by mentioning a few possible venues for future explorations.

Naturally, it would be useful to generalize Eq. (14) and lift the assumption $G \to 0$, so as to generalize [14] to the inhomogeneous case in the presence of collective effects. As emphasised in [14], this is a delicate calculation that requires computing, explicitly, all the cumulants of a quadratic form of a Gaussian random field. In inhomogeneous systems, this becomes even more challenging since both the resonance condition and the coupling coefficients get more intricate. Nonetheless, this would be of astrophysical relevance since typical self-gravitating systems are generically only weakly stable, e.g., as visible through spiral arms in galactic discs (see, e.g., [19]) or radial orbit instability in globular clusters (see, e.g., [20]).

Here, we started our calculation from the description of the dynamics using the Klimontovich equation (4). Yet, it is known that the inhomogeneous BL and Landau equations can also be derived from the Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy (see, e.g., [2]) through a different route [8]. Tailoring the framework of large deviations to the BBGKY's viewpoint could prove insightful.

The generic probability distribution function from Eq. (12) is challenging to evaluate in practice. Though, it would be enlightening to compute it numerically, starting with some simple long-range interacting systems. One could begin with the Hamiltonian mean-field model [21] in an inhomogeneous configuration, for example, tracking the statistics of the large deviations of the system's magnetization. Similarly, one could extend the present work to long-range interacting systems submitted to a stochastic forcing (see, e.g., [22–26]). These systems exhibit phase transitions whose most likely transition path could be recovered via a large deviation principle, that is, through the resolution of a system of coupled partial differential equations for $F(\mathbf{J}, \tau)$ and $P(\mathbf{J}, \tau)$. Such preliminary explorations are mandatory first steps before applying the present statistical approach to more realistic systems.

In Sec. IV F, we emphasized that the large deviations of a (dynamically hot) self-gravitating system can be directly mimicked from a stochastic partial differential equation, as given by Eq. (26). We believe that this should prove a powerful rewriting within the astrophysical context. Indeed, it is significantly simpler to realize Eq. (26) numerically than it is to effectively compute the supremum in the large deviation function from Eq. (12). As a result, the various evolution paths that would be obtained from Eq. (26) should offer new insights on the galactic diversity routinely observed in large hydrodynamical simulations (see, e.g., [27]).

Along the same line, we point out that the "diagonal" rewriting [see Eq. (D21)] of the stochastic partial differential equation shares a structure strikingly similar with the ones encountered when describing the long-term relaxation of globular clusters via the Monte Carlo method [28] or when deriving the BL equation from Rostoker principle [29]. This stems from the fact that all these approaches exactly conserve global invariants like the total energy. Ultimately, one can hope to benefit from this insight to generalize the Monte Carlo method from [28] so that it would explicitly account for the inhomogeneity of self-gravitating systems as well as their deviations around the mean evolution. This will be the topic of future investigation.

Here, the system was always assumed to be (strongly) linearly stable. This allowed us to neglect any possible contributions from the system's damped modes in the linear dynamics of fluctuations (see Appendix A 3). In the case of weakly stable systems, it would be interesting to extend the present calculation to account for the effects of slowly decaying modes and wave-particle interactions (see, e.g., [30]), as well as possible dynamical phase transitions toward linear instability, for example, occurring in spiral galaxies (see, e.g., [31]). Additionally, one could investigate the connections between the present dynamical large deviations and the thermal (van Kampen) fluctuations considered in [32,33].

Finally, we limited ourselves to diffusion sourced by two-body correlations, i.e., $1/N$ effects. Yet, in the case of one-dimensional systems, such a relaxation can identically vanish. This is a kinetic blocking (see, e.g., [6,34–37]) and these systems can only relax via the weaker $1/N^2$ effects, i.e., three-body correlations (see, e.g., [38]). It would be informative to investigate the properties of dynamical large deviations in such contrived systems. On the astrophysical front, such developments would be useful to describe the the long-term evolution of orbital orientations around supermassive black holes (see, e.g., [39]).

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APPENDIX A: LINEAR DYNAMICS OF FLUCTUATIONS

In this Appendix, we solve for the linear dynamics of the fluctuations δF, as driven by Eq. (9a). The calculations below follow closely [9,40]. In this Appendix, the angle-averaged DF, F_N, has simply been denoted with F to shorten the notations.

1. Laplace-Fourier transforms

We define the Fourier transform w.r.t. the angle $\boldsymbol{\theta}$ via

$$f(\mathbf{w},t) = \sum_{\mathbf{k}\in\mathbb{Z}^d} f_{\mathbf{k}}(\mathbf{J},t)\,\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\boldsymbol{\theta}}, \tag{A1a}$$

$$f_{\mathbf{k}}(\mathbf{J},t) := \int \frac{\mathrm{d}\boldsymbol{\theta}}{(2\pi)^d} f(\mathbf{w},t)\,\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\boldsymbol{\theta}}. \tag{A1b}$$

The self-consistency relation from Eq. (3) when used for the fluctuations then reads

$$\delta\Phi_{\mathbf{k}}(\mathbf{J},t) = (2\pi)^d \sum_{\mathbf{k}'} \int \mathrm{d}\mathbf{J}'\, \psi_{\mathbf{k}\mathbf{k}'}(\mathbf{J},\mathbf{J}')\,\delta F_{\mathbf{k}'}(\mathbf{J}',t). \tag{A2}$$

In that expression, $\psi_{\mathbf{k}\mathbf{k}'}(\mathbf{J},\mathbf{J}')$ are the bare coupling coefficients. They follow from the expansion of the pairwise interaction potential as

$$U(\mathbf{w},\mathbf{w}') = \sum_{\mathbf{k},\mathbf{k}'} \psi_{\mathbf{k}\mathbf{k}'}(\mathbf{J},\mathbf{J}')\,\mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\boldsymbol{\theta}-\mathbf{k}'\cdot\boldsymbol{\theta}')}. \tag{A3}$$

These coefficients satisfy the two symmetries

$$\psi_{\mathbf{k}'\mathbf{k}}(\mathbf{J}',\mathbf{J}) = \psi^*_{\mathbf{k}\mathbf{k}'}(\mathbf{J},\mathbf{J}'), \tag{A4a}$$

$$\psi_{-\mathbf{k}-\mathbf{k}'}(\mathbf{J},\mathbf{J}') = \psi^*_{\mathbf{k}\mathbf{k}'}(\mathbf{J},\mathbf{J}'). \tag{A4b}$$

We define the Laplace transform with the convention

$$\widetilde{f}(\omega) := \int_0^{+\infty} \mathrm{d}t\, f(t)\,\mathrm{e}^{\mathrm{i}\omega t}; \quad f(t) = \int_{\mathcal{B}} \frac{\mathrm{d}\omega}{2\pi}\,\widetilde{f}(\omega)\,\mathrm{e}^{-\mathrm{i}\omega t}, \tag{A5}$$

where the Bromwich contour $\mathcal{B}$ has to pass above all the poles of the integrand, i.e., $\mathrm{Im}[\omega]$ has to be large enough. When expressed in Fourier-Laplace space, Eq. (9a) becomes

$$\delta\widetilde{F}_{\mathbf{k}}(\mathbf{J},\omega) = -\frac{\mathbf{k}\cdot\partial F/\partial\mathbf{J}}{\omega-\mathbf{k}\cdot\boldsymbol{\Omega}}\,\delta\widetilde{\Phi}_{\mathbf{k}}(\mathbf{J},\omega) - \frac{\delta F_{\mathbf{k}}(\mathbf{J},0)}{\mathrm{i}(\omega-\mathbf{k}\cdot\boldsymbol{\Omega})}, \tag{A6}$$

with $\boldsymbol{\Omega} = \boldsymbol{\Omega}(\mathbf{J})$ and $\delta F_{\mathbf{k}}(\mathbf{J},0)$ describing the fluctuations in the DF at the initial time. Once again, we recall that for the sake of shorter notations, we wrote the angle-averaged DF, F_N, as F.

2. Self-consistency

We now act on both sides of Eq. (A6) with the same operator as in the r.h.s. of Eq. (A2). We get

$$\begin{aligned}\delta\widetilde{\Phi}_{\mathbf{k}}(\mathbf{J},\omega) = &-(2\pi)^d \sum_{\mathbf{k}'} \int \mathrm{d}\mathbf{J}'\, \frac{\psi_{\mathbf{k}\mathbf{k}'}(\mathbf{J},\mathbf{J}')}{\omega-\mathbf{k}'\cdot\boldsymbol{\Omega}'} \\ &\times \mathbf{k}'\cdot\frac{\partial F}{\partial\mathbf{J}'}\,\delta\widetilde{\Phi}_{\mathbf{k}'}(\mathbf{J}',\omega) \\ &-(2\pi)^d \sum_{\mathbf{k}'} \int \mathrm{d}\mathbf{J}'\, \frac{\psi_{\mathbf{k}\mathbf{k}'}(\mathbf{J},\mathbf{J}')}{\mathrm{i}(\omega-\mathbf{k}'\cdot\boldsymbol{\Omega}')}\,\delta F_{\mathbf{k}'}(\mathbf{J}',0),\end{aligned} \tag{A7}$$

with $\boldsymbol{\Omega}' = \boldsymbol{\Omega}(\mathbf{J}')$.

In the absence of collective effects (i.e., $G\to 0$), the first term in the r.h.s. of Eq. (A7) can be neglected to get

$$\delta\widetilde{\Phi}^{\mathrm{bare}}_{\mathbf{k}}(\mathbf{J},\omega) := -(2\pi)^d \sum_{\mathbf{k}'} \int \mathrm{d}\mathbf{J}'\, \frac{\delta F_{\mathbf{k}'}(\mathbf{J}',0)\,\psi_{\mathbf{k}\mathbf{k}'}(\mathbf{J},\mathbf{J}')}{\mathrm{i}(\omega-\mathbf{k}'\cdot\boldsymbol{\Omega}')}. \tag{A8}$$

When collective effects are accounted for, we may assume that the dressed potential perturbations follow the ansatz

$$\delta\widetilde{\Phi}^{\mathrm{dress}}_{\mathbf{k}}(\mathbf{J},\omega) := -(2\pi)^d \sum_{\mathbf{k}'} \int \mathrm{d}\mathbf{J}'\, \frac{\delta F_{\mathbf{k}'}(\mathbf{J}',0)\,\psi^{\mathrm{d}}_{\mathbf{k}\mathbf{k}'}(\mathbf{J},\mathbf{J}',\omega)}{\mathrm{i}(\omega-\mathbf{k}'\cdot\boldsymbol{\Omega}')}, \tag{A9}$$

where the frequency-dependent dressed coupling coefficients $\psi^{\mathrm{d}}_{\mathbf{k}\mathbf{k}'}(\mathbf{J},\mathbf{J}',\omega)$ remain to be determined. When injected into Eq. (A7), we find that the dressed coupling coefficients satisfy the self-consistent relation

$$\begin{aligned}\psi^{\mathrm{d}}_{\mathbf{k}\mathbf{k}'}(\mathbf{J},\mathbf{J}',\omega) = &\,\psi_{\mathbf{k}\mathbf{k}'}(\mathbf{J},\mathbf{J}') - (2\pi)^d \sum_{\mathbf{k}''} \int \mathrm{d}\mathbf{J}'' \\ &\times \frac{\psi_{\mathbf{k}\mathbf{k}''}(\mathbf{J},\mathbf{J}'')\,\mathbf{k}''\cdot\partial F/\partial\mathbf{J}''}{\omega-\mathbf{k}''\cdot\boldsymbol{\Omega}''}\,\psi^{\mathrm{d}}_{\mathbf{k}''\mathbf{k}'}(\mathbf{J}'',\mathbf{J}',\omega),\end{aligned} \tag{A10}$$

with $\boldsymbol{\Omega}'' = \boldsymbol{\Omega}(\mathbf{J}'')$. Since Eq. (A6) was derived assuming $\mathrm{Im}[\omega] > 0$ large enough, for $\omega = \omega_{\mathrm{R}} \in \mathbb{R}$, the resonant denominator in Eq. (A10) has to be interpreted as

$$\frac{1}{\omega_{\mathrm{R}}-\mathbf{k}''\cdot\boldsymbol{\Omega}''} \to \frac{1}{\omega_{\mathrm{R}}-\mathbf{k}''\cdot\boldsymbol{\Omega}''+\mathrm{i}\gamma}, \tag{A11}$$

with $\gamma\to 0^+$. We refer to [41] (and references therein) for a discussion of the associated Landau's prescription. We note that an explicit expression for $\psi^{\mathrm{d}}_{\mathbf{k}\mathbf{k}'}$ can be obtained using the basis method (see, e.g., [40] and references therein), but this will not be needed here.

Ultimately, we find that the coupling coefficients asymptotically scale w.r.t. G, the amplitude of the pairwise interaction, like

$$\psi_{\mathbf{k}\mathbf{k}'} \propto G, \tag{A12a}$$

$$\psi^{\mathrm{d}}_{\mathbf{k}\mathbf{k}'} \propto \frac{\psi_{\mathbf{k}\mathbf{k}'}}{1-\psi_{\mathbf{k}\mathbf{k}'}} \propto \frac{G}{1-G}. \tag{A12b}$$

3. Time evolution

Taking the inverse Laplace transform of Eq. (A9), the dressed fluctuations generically evolve according to

$$\delta\Phi_{\mathbf{k}}(\mathbf{J},t) = -(2\pi)^d \sum_{\mathbf{k}'} \int \mathrm{d}\mathbf{J}'\, \delta F_{\mathbf{k}'}(\mathbf{J}',0) \times \int_{\mathcal{B}} \frac{\mathrm{d}\omega}{2\pi} \frac{\psi^{\mathrm{d}}_{\mathbf{k}\mathbf{k}'}(\mathbf{J},\mathbf{J}',\omega)}{\mathrm{i}(\omega - \mathbf{k}'\cdot\mathbf{\Omega}')} \mathrm{e}^{-\mathrm{i}\omega t}, \tag{A13}$$

where the Bromwich contour $\mathcal{B}$ has to pass above all the poles of the integrand in the complex ω plane. Because we assumed that the system is linearly stable, the function $\omega \mapsto \psi^{\mathrm{d}}_{\mathbf{k}\mathbf{k}'}(\mathbf{J},\mathbf{J}',\omega)$ only has poles in the lower half of the complex plane. These are associated with Landau damped modes and correspond to frequencies ω_{M} with $\mathrm{Im}[\omega_{\mathrm{M}}] < 0$.

As usual, we proceed by distorting the contour $\mathcal{B}$ to the lower half of the complex plane so that $|\mathrm{e}^{-\mathrm{i}\omega t}| \to 0$, snagging onto the poles of the integrand. In Eq. (A13), there is a single pole along the real axis, namely, in $\omega = \mathbf{k}'\cdot\mathbf{\Omega}'$. Paying attention to the direction of integration, each pole contributes a $-2\mathrm{i}\pi\mathrm{Res}[\cdots]$. Placing ourselves in the limit $t \gg |1/\mathrm{Im}[\omega_{\mathrm{M}}]|$, we neglect the contributions from the damped modes (see, e.g., [30]). Once these have faded away, Eq. (A13) becomes

$$\delta\Phi_{\mathbf{k}}(\mathbf{J},t) = (2\pi)^d \sum_{\mathbf{k}'} \int \mathrm{d}\mathbf{J}'\, \mathrm{e}^{-\mathrm{i}\mathbf{k}'\cdot\mathbf{\Omega}' t}\, \delta F_{\mathbf{k}'}(\mathbf{J}',0) \times \psi^{\mathrm{d}}_{\mathbf{k}\mathbf{k}'}(\mathbf{J},\mathbf{J}',\mathbf{k}'\cdot\mathbf{\Omega}'). \tag{A14}$$

Having determined the time evolution of the potential fluctuations, we now set out to determine the time evolution of the DF fluctuations themselves. An efficient approach to perform this calculation is to rely on the self-consistency relation from Eq. (A2). We start with Eq. (A14) in which we replace $\psi^{\mathrm{d}}_{\mathbf{k}\mathbf{k}'}$ with the r.h.s. from Eq. (A10). We get

$$\begin{aligned}\delta\Phi_{\mathbf{k}}(\mathbf{J},t) &= (2\pi)^d \sum_{\mathbf{k}'} \int \mathrm{d}\mathbf{J}' \mathrm{e}^{-\mathrm{i}\mathbf{k}'\cdot\mathbf{\Omega}' t} \delta F_{\mathbf{k}'}(\mathbf{J}',0)\, \psi_{\mathbf{k}\mathbf{k}'}(\mathbf{J},\mathbf{J}') \\ &\quad - (2\pi)^{2d} \sum_{\mathbf{k}',\mathbf{k}''} \int \mathrm{d}\mathbf{J}'\mathrm{d}\mathbf{J}'' \mathrm{e}^{-\mathrm{i}\mathbf{k}'\cdot\mathbf{\Omega}' t} \delta F_{\mathbf{k}'}(\mathbf{J}',0) \\ &\quad \times \psi_{\mathbf{k}\mathbf{k}''}(\mathbf{J},\mathbf{J}'') \\ &\quad \times \frac{\psi^{\mathrm{d}}_{\mathbf{k}''\mathbf{k}'}(\mathbf{J}'',\mathbf{J}',\mathbf{k}'\cdot\mathbf{\Omega}')}{\mathbf{k}'\cdot\mathbf{\Omega}' - \mathbf{k}''\cdot\mathbf{\Omega}'' + \mathrm{i}\gamma} \mathbf{k}'' \cdot \frac{\partial F}{\partial \mathbf{J}''}.\end{aligned} \tag{A15}$$

We now perform the switch $(\mathbf{k}',\mathbf{J}') \leftrightarrow (\mathbf{k}'',\mathbf{J}'')$ in the second term. Factorizing the integrand with $\psi_{\mathbf{k}\mathbf{k}'}(\mathbf{J},\mathbf{J}')$, we finally rely on Eq. (A2) to identify the remainder of the integrand with $\delta F_{\mathbf{k}'}(\mathbf{J}',t)$. Ultimately, we get

$$\begin{aligned}\delta F_{\mathbf{k}}(\mathbf{J},t) &= \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{\Omega} t}\, \delta F_{\mathbf{k}}(\mathbf{J},0) \\ &\quad + (2\pi)^d \mathbf{k}\cdot\frac{\partial F}{\partial \mathbf{J}} \sum_{\mathbf{k}'} \int \mathrm{d}\mathbf{J}'\, \mathrm{e}^{-\mathrm{i}\mathbf{k}'\cdot\mathbf{\Omega}' t}\, \delta F_{\mathbf{k}'}(\mathbf{J}',0) \\ &\quad \times \frac{\psi^{\mathrm{d}}_{\mathbf{k}\mathbf{k}'}(\mathbf{J},\mathbf{J}',\mathbf{k}'\cdot\mathbf{\Omega}')}{\mathbf{k}\cdot\mathbf{\Omega} - \mathbf{k}'\cdot\mathbf{\Omega}' - \mathrm{i}\gamma},\end{aligned} \tag{A16}$$

with $\gamma \to 0^+$.

Glancing back at Eq. (A12), we find from Eqs. (A14) and (A16) that the potential and DF fluctuations scale asymptotically w.r.t. G like

$$\delta\Phi(t) \propto \delta F(0)\, \frac{G}{1-G}, \tag{A17a}$$

$$\delta F(t) \propto \delta F(0) + \delta F(0)\, \frac{G}{1-G}. \tag{A17b}$$

These scalings play an important role in easing the computation of the cumulants, as detailed in Appendix C.

APPENDIX B: LARGE DEVIATION PRINCIPLE

In this Appendix, we present a brief (very) heuristic derivation of the generic large deviation principle from Eqs. (12) and (13). We refer to Sec. 2.2 in [12] and references therein for a much more thorough presentation.

1. Gärtner-Ellis theorem

This section is inspired from Sec. 3.3 of [42]. As a first step, let us mimic the empirical DF from Eq. (1) and consider a real random variable F_N, parameterized by N. From it, we define the scaled cumulant generating function

$$\mathcal{H}[P] := \frac{1}{N} \ln[\langle \mathrm{e}^{N F_N P}\rangle], \tag{B1}$$

also called the large deviation Hamiltonian, which is assumed to be finite.

In the limit $N \gg 1$, we assume that F_N follows a large deviation principle of the form

$$\mathbb{P}(F_N = F) \underset{N\to+\infty}{\asymp} \mathrm{e}^{-N I[F]}, \tag{B2}$$

where N is the large deviation rate and $I[F]$ the large deviation function. For a given P, we can compute

$$\langle \mathrm{e}^{N F_N P}\rangle = \int \mathrm{d}F\, \mathrm{e}^{NFP}\, \mathbb{P}(F_N = F) \underset{N\to+\infty}{\asymp} \int \mathrm{d}F\, \mathrm{e}^{N(FP - I[F])}. \tag{B3}$$

Since $N \gg 1$, we estimate this integral using the saddle-point method. The dominating contribution comes from the maximum of the argument of the exponential to give

$$\langle \mathrm{e}^{N F_N P}\rangle \underset{N\to+\infty}{\asymp} \exp[N \sup_F \{FP - I[F]\}]. \tag{B4}$$

Comparing Eqs. (B4) and (B1), we asymptotically obtain the relation $\mathcal{H}[P] = \sup_F\{FP - I[F]\}$. For a differentiable $\mathcal{H}[P]$, this Legendre transform is involutive, and we finally obtain the large deviation function as

$$I[F] := \sup_P \{FP - \mathcal{H}[P]\}. \tag{B5}$$

This is the Gärtner-Ellis theorem (see, e.g., [42] and references therein), and constitutes the foundation on which Eqs. (12) and (13) lie.

2. Slow-fast systems

Let us now mimic the evolution equations at play, namely, Eqs. (9), by considering two coupled stochastic variables $(F_N, \delta F)$, of order unity, and evolving according to

$$\frac{\partial \delta F}{\partial \tau} = N\,\frac{\partial \delta F}{\partial \tau}[F_N, \delta F], \tag{B6a}$$

$$\frac{\partial F_N}{\partial \tau} = \frac{\partial F_N}{\partial \tau}[F_N, \delta F], \tag{B6b}$$

with the slow time $\tau := t/N$. Owing to the presence of the factor N in Eq. (B6a), this constitutes a slow-fast system: δF evolves on a (fast) timescale of order $\tau \simeq 1/N$, while F_N evolves on a (slow) timescale of order $\tau \simeq 1$.

It is essential to note that Eqs. (B6) are deterministic, but their initial conditions are random. In the following, we assume that the dynamics of the fast process is mixing, i.e., it forgets about the initial conditions rapidly enough [43]. Though we cannot rigorously prove this mixing hypothesis for the present system, this seems like a natural assumption.

Benefiting from this separation of timescales, let us introduce a discrete timestep $\Delta\tau$, satisfying

$$1/N \ll \Delta\tau \ll 1. \tag{B7}$$

We subsequently introduce the random variable G_N, associated with a finite-difference rate of change during a time $\Delta\tau$. It reads

$$G_N(\tau) := \frac{F_N(\tau + \Delta\tau) - F_N(\tau)}{\Delta\tau}. \tag{B8}$$

The probability of a given time evolution, $\{F_N(\tau)\}_{0\leqslant\tau\leqslant T}$, can then be estimated through the product of conditional probabilities

$$\mathbb{P}[\{F_N(\tau)\}_{0\leqslant\tau\leqslant T} = \{F(\tau)\}_{0\leqslant\tau\leqslant T}] \underset{N\to+\infty}{\asymp} \prod_n \mathbb{P}[G_N(n\Delta\tau) = \dot{F}(n\Delta\tau)\,|\,F_N(n\Delta\tau) = F(n\Delta\tau)], \tag{B9}$$

with $\dot{F} = \partial_\tau F$. Importantly, to obtain this expression, we used a Markovian decomposition of the path of evolution, assuming that increments of F_N over timescales of order $\Delta\tau \gg 1/N$, are independent from one another.

Now, let us compute the probability of a given increment $\mathbb{P}(G_N = \dot{F}\,|\,F_N = F)$. We use Eq. (B2) with $N\Delta\tau \gg 1$ as the large deviation rate. We obtain

$$\mathbb{P}(G_N = \dot{F}\,|\,F_N = F) \underset{N\to+\infty}{\asymp} \mathrm{e}^{-N\Delta\tau I[F,\dot{F}]}, \tag{B10}$$

where the dependence w.r.t. F emphasises that F can be taken as constant on a time interval of duration $\Delta\tau$. In Eq. (B10), the large deviation function follows from Eq. (B5) and reads

$$I[F, \dot{F}] := \sup_P\{\dot{F}P - \mathcal{H}[F, P]\}. \tag{B11}$$

Here, $\mathcal{H}[F, P]$, is the large deviation Hamiltonian. It follows from Eq. (B1), with the large deviation rate $\Delta t := N\Delta\tau$ and reads

$$\mathcal{H}[F, P] := \frac{1}{\Delta t}\ln[\langle \mathrm{e}^{\Delta t G_N P}\rangle_F], \tag{B12}$$

where the average is performed over the fast process from Eq. (B6a) with $F_N = F$. Starting from the definition of Eq. (B8), we can write

$$G_N = \frac{1}{\Delta\tau}\int_0^{\Delta\tau} \mathrm{d}\tau\,\partial_\tau F_N = \frac{1}{\Delta t}\int_0^{\Delta t} \mathrm{d}t\,\partial_\tau F_N, \tag{B13}$$

where, for simplicity, we started the time integral in $\tau = 0$ and $\partial_\tau F_N$ follows from Eq. (B6b). Since $\Delta t \gg 1$ [Eq. (B7)], we can approximate Eq. (B12) with

$$\mathcal{H}[F, P] = \lim_{\Delta\to+\infty} \frac{1}{\Delta}\ln\left[\left\langle \exp\left(P\int_0^{\Delta} \mathrm{d}t\,\partial_\tau F_N\right)\right\rangle_F\right]. \tag{B14}$$

The final step of the computation is to use the assumption $\Delta\tau \ll 1$ [Eq. (B7)] so that the product of probabilities in Eq. (B9) can be replaced by the exponential of an integral. More precisely, using Eq. (B10), we obtain

$$\mathbb{P}[\{F_N(\tau)\}_{0\leqslant\tau\leqslant T} = \{F(\tau)\}_{0\leqslant\tau\leqslant T}] \underset{N\to+\infty}{\asymp} \times \exp\left[-N\sup_P \int_0^T \mathrm{d}\tau\{\dot{F}P - \mathcal{H}[F, P]\}\right], \tag{B15}$$

where $P = P(\tau)$ is now a field w.r.t. the slow time τ. Naturally, Eqs. (B15) and (B14) bear a lot of similarities with the generic results from Eqs. (12) and (13). To fully recover these expressions, it only remains to (i) add the additional dependence of F_N w.r.t. $\mathbf{J}$, which also transmits to $P = P(\mathbf{J}, \tau)$; (ii) account correctly for the various normalization prefactors. Let us conclude by pointing out that the present heuristic proof does not check the differentiability of the large derivation Hamiltonian, $\mathcal{H}[F, P]$, and hence the possible non-convexity of the large deviation function, $I[F]$.

APPENDIX C: COMPUTING THE CUMULANTS

In this Appendix, we compute the first cumulants of Eq. (13). For a random variable X, we recall that

$$\ln[\langle \mathrm{e}^X\rangle] \simeq \langle X\rangle + \tfrac{1}{2}(\langle X^2\rangle - \langle X\rangle^2) + \cdots. \tag{C1}$$

Following the convention from Eq. (A1), we note that Eq. (9b) can be written as

$$\frac{\partial F_N(\mathbf{J})}{\partial \tau} = -\mathrm{i}\sum_{\mathbf{k}} \mathbf{k}\cdot\frac{\partial}{\partial \mathbf{J}}[\delta F_{\mathbf{k}}(\mathbf{J})\,\delta\Phi_{-\mathbf{k}}(\mathbf{J})]. \tag{C2}$$

Hence, we will apply Eq. (C1) using $X \propto \delta F\,\delta\Phi$.

1. First cumulant

Owing to Eq. (C2), the first cumulant of Eq. (13) can be written as

$$\mathcal{H}^{(1)}[F, P] = \sum_{\mathbf{k}} \int \mathrm{d}\mathbf{J}\,\mathbf{k}\cdot\frac{\partial P}{\partial \mathbf{J}}\,C^{(1)}_{\mathbf{k}}(\mathbf{J}), \tag{C3}$$

with the correlation function

$$C^{(1)}_{\mathbf{k}}(\mathbf{J}) := \lim_{\Delta\to+\infty}\frac{\mathrm{i}}{\Delta}\int_0^{\Delta} \mathrm{d}t\,\langle \delta F_{\mathbf{k}}(\mathbf{J}, t)\,\delta\Phi_{-\mathbf{k}}(\mathbf{J}, t)\rangle_F. \tag{C4}$$

The next step of the calculation is to inject the time-dependent expression of the DF and potential fluctuations, from Eqs. (A14) and (A16). These fluctuations are expressed as a function of the initial conditions $\delta F_{\mathbf{k}}(\mathbf{J}, 0)$. We assume that the initial fluctuations stem from some uncorrelated Poisson shot noise, hence making them Gaussian. Following Eq. (41) of [9], and paying attention to the prefactor $1/\sqrt{N}$ in Eq. (8), we have

$$\langle\delta F_{\mathbf{k}}(\mathbf{J},0)\delta F_{\mathbf{k}'}(\mathbf{J}',0)\rangle_F = M_{\mathrm{tot}}\frac{\delta_{\mathbf{k},-\mathbf{k}'}}{(2\pi)^d}\delta_{\mathrm{D}}(\mathbf{J}-\mathbf{J}')F(\mathbf{J}). \tag{C5}$$

Using this statistics, Eq. (C4) becomes

$$\begin{aligned} C^{(1)}_{\mathbf{k}}(\mathbf{J}) &= \mathrm{i}M_{\mathrm{tot}}F(\mathbf{J})\psi^{\mathrm{d}*}_{\mathbf{kk}}(\mathbf{J},\mathbf{J},\mathbf{k}\cdot\mathbf{\Omega}) \\ &\quad + \mathrm{i}(2\pi)^d M_{\mathrm{tot}}\mathbf{k}\cdot\frac{\partial F}{\partial \mathbf{J}}\sum_{\mathbf{k}'}\int \mathrm{d}\mathbf{J}'F(\mathbf{J}') \\ &\quad \times \frac{\left|\psi^{\mathrm{d}}_{\mathbf{kk}'}(\mathbf{J},\mathbf{J}',\mathbf{k}'\cdot\mathbf{\Omega}')\right|^2}{\mathbf{k}\cdot\mathbf{\Omega}-\mathbf{k}'\cdot\mathbf{\Omega}'-\mathrm{i}\gamma}, \end{aligned} \tag{C6}$$

where we used the symmetry

$$\psi^{\mathrm{d}}_{-\mathbf{k}-\mathbf{k}'}(\mathbf{J},\mathbf{J}',-\omega_{\mathrm{R}}) = \psi^{\mathrm{d}*}_{\mathbf{kk}'}(\mathbf{J},\mathbf{J}',\omega_{\mathrm{R}}), \tag{C7}$$

for $\omega_{\mathrm{R}}\in\mathbb{R}$.

At this stage, following the truncation from Eq. (10), we must compute Eq. (C6) at order $o(G^2)$. We can generically expand the dressed coupling coefficient as

$$\psi^{\mathrm{d}}_{\mathbf{kk}'}(\mathbf{J},\mathbf{J}',\omega) \simeq \psi_{\mathbf{kk}'}(\mathbf{J},\mathbf{J}') + \psi^{(2)}_{\mathbf{kk}'}(\mathbf{J},\mathbf{J}',\omega) + o(G^2), \tag{C8}$$

with the scaling $\psi^{(2)}_{\mathbf{kk}'}\propto G^2$, for $G\to 0$. Relying on the self-consistent relation from Eq. (A10), we get

$$\begin{aligned} \psi^{(2)}_{\mathbf{kk}}(\mathbf{J},\mathbf{J},\mathbf{k}\cdot\mathbf{\Omega}) &= -(2\pi)^d\sum_{\mathbf{k}'}\int \mathrm{d}\mathbf{J}' \\ &\quad \times \frac{|\psi_{\mathbf{kk}'}(\mathbf{J},\mathbf{J}')|^2\,\mathbf{k}'\cdot\partial F/\partial\mathbf{J}'}{\mathbf{k}\cdot\mathbf{\Omega}-\mathbf{k}'\cdot\mathbf{\Omega}'+\mathrm{i}\gamma}, \end{aligned} \tag{C9}$$

where we used the symmetry from Eq. (A4a) and the prescription from Eq. (A11).

At order $o(G^2)$, Eq. (C6) then becomes

$$\begin{aligned} C^{(1)}_{\mathbf{k}}(\mathbf{J}) &= \mathrm{i}M_{\mathrm{tot}}F(\mathbf{J})\psi^{*}_{\mathbf{kk}}(\mathbf{J},\mathbf{J}) \\ &\quad + \mathrm{i}(2\pi)^d M_{\mathrm{tot}}\sum_{\mathbf{k}}\int \mathrm{d}\mathbf{J}'\frac{|\psi_{\mathbf{kk}'}(\mathbf{J},\mathbf{J}')|^2}{\mathbf{k}\cdot\mathbf{\Omega}-\mathbf{k}'\cdot\mathbf{\Omega}'-\mathrm{i}\gamma} \\ &\quad \times \left\{\mathbf{k}\cdot\frac{\partial F}{\partial\mathbf{J}}F(\mathbf{J}')-\mathbf{k}'\cdot\frac{\partial F}{\partial\mathbf{J}'}F(\mathbf{J})\right\}. \end{aligned} \tag{C10}$$

Performing the symmetrization $\mathbf{k}\to-\mathbf{k}$ in Eq. (C3), and using the symmetry from Eq. (A4), we find that the term in $\psi^{*}_{\mathbf{kk}}(\mathbf{J},\mathbf{J})$ in Eq. (C10) does not contribute to $\mathcal{H}^{(1)}$. The final step of the calculation is to expand the resonant denominator using Plemelj formula

$$\frac{1}{\omega_{\mathrm{R}}-\mathrm{i}\gamma} = \mathcal{P}\left(\frac{1}{\omega_{\mathrm{R}}}\right) + \mathrm{i}\pi\,\delta_{\mathrm{D}}(\omega_{\mathrm{R}}), \tag{C11}$$

with $\mathcal{P}$ the Cauchy principal value. Performing the symmetrization $(\mathbf{k},\mathbf{k}')\to(-\mathbf{k},-\mathbf{k}')$ in Eqs. (C3) and (C10), and using once again the symmetries from Eq. (A4), we find that the principal value does not contribute to $\mathcal{H}^{(1)}$. Ultimately, at order $o(G^2)$, we are left with

$$\begin{aligned} \mathcal{H}^{(1)}[F,P] &= -\sum_{\mathbf{k},\mathbf{k}'}\int \mathrm{d}\mathbf{J}\mathrm{d}\mathbf{J}'\,\mathbf{k}\cdot\frac{\partial P}{\partial\mathbf{J}}B_{\mathbf{kk}'}(\mathbf{J},\mathbf{J}') \\ &\quad \times \left\{\mathbf{k}\cdot\frac{\partial F}{\partial\mathbf{J}}F(\mathbf{J}')-\mathbf{k}'\cdot\frac{\partial F}{\partial\mathbf{J}'}F(\mathbf{J})\right\}, \end{aligned} \tag{C12}$$

hence recovering Eq. (15).

2. Second cumulant

Let us now compute the second cumulant of Eq. (13). In that case, following Eq. (C1), we must compute averages of order X^2, with $X\propto\delta F\,\delta\Phi$. Since we are computing expressions at order $o(G^2)$, we follow the scalings from Eq. (A17) to simplify the expressions of $\delta\Phi(t)$ and $\delta F(t)$ to be used in this computation. More precisely, at order $o(G^2)$, we may replace Eq. (A14) with

$$\delta\Phi_{\mathbf{k}}(\mathbf{J},t) = (2\pi)^d\sum_{\mathbf{k}'}\int \mathrm{d}\mathbf{J}'\,\mathrm{e}^{-\mathrm{i}\mathbf{k}'\cdot\mathbf{\Omega}'t}\delta F_{\mathbf{k}'}(\mathbf{J}',0)\psi_{\mathbf{kk}'}(\mathbf{J},\mathbf{J}'), \tag{C13}$$

and Eq. (A16) with

$$\delta F_{\mathbf{k}}(\mathbf{J},t) = \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{\Omega}t}\,\delta F_{\mathbf{k}}(\mathbf{J},0). \tag{C14}$$

This is a key step to simplify the upcoming calculations.

The second cumulant of Eq. (13) then reads

$$\mathcal{H}^{(2)}[F,P] = \sum_{\mathbf{k},\mathbf{k}'}\int \mathrm{d}\mathbf{J}\mathrm{d}\mathbf{J}'\,\mathbf{k}\cdot\frac{\partial P}{\partial\mathbf{J}}\,\mathbf{k}'\cdot\frac{\partial P}{\partial\mathbf{J}'}C^{(2)}_{\mathbf{kk}'}(\mathbf{J},\mathbf{J}'), \tag{C15}$$

where we introduced the correlation function

$$\begin{aligned} C^{(2)}_{\mathbf{kk}'}(\mathbf{J},\mathbf{J}') &:= -\lim_{\Delta\to+\infty}\frac{1}{2}\frac{(2\pi)^d}{\Delta M_{\mathrm{tot}}}\int_0^\Delta \mathrm{d}t\int_0^\Delta \mathrm{d}t' \\ &\quad \times \langle\langle[\delta F_{\mathbf{k}}(\mathbf{J},t)\,\delta\Phi_{-\mathbf{k}}(\mathbf{J},t)][\delta F_{\mathbf{k}'}(\mathbf{J}',t') \\ &\quad \times \delta\Phi_{-\mathbf{k}'}(\mathbf{J}',t')]\rangle\rangle_F, \end{aligned} \tag{C16}$$

with the notation $\langle\langle XY\rangle\rangle_F := \langle XY\rangle_F - \langle X\rangle_F\langle Y\rangle_F$.

Injecting the dependence from Eqs. (C13) and (C14), we can rewrite Eq. (C16) as

$$\begin{aligned} C^{(2)}_{\mathbf{kk}'}(\mathbf{J},\mathbf{J}') &= -\lim_{\Delta\to+\infty}\frac{1}{2}\frac{(2\pi)^{3d}}{\Delta M_{\mathrm{tot}}}\int_0^\Delta \mathrm{d}t\int_0^\Delta \mathrm{d}t'\sum_{\mathbf{k}_1,\mathbf{k}_1'}\int \mathrm{d}\mathbf{J}_1\mathrm{d}\mathbf{J}_1'\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{\Omega}t}\mathrm{e}^{-\mathrm{i}\mathbf{k}_1\cdot\mathbf{\Omega}_1t}\psi_{-\mathbf{k}\mathbf{k}_1}(\mathbf{J},\mathbf{J}_1) \\ &\quad \times \mathrm{e}^{-\mathrm{i}\mathbf{k}'\cdot\mathbf{\Omega}'t'}\mathrm{e}^{-\mathrm{i}\mathbf{k}_1'\cdot\mathbf{\Omega}_1't'}\psi_{-\mathbf{k}'\mathbf{k}_1'}(\mathbf{J}',\mathbf{J}_1')\langle\langle[\delta F_{\mathbf{k}}(\mathbf{J},0)\,\delta F_{\mathbf{k}_1}(\mathbf{J}_1,0)][\delta F_{\mathbf{k}'}(\mathbf{J}',0)\,\delta F_{\mathbf{k}_1'}(\mathbf{J}_1',0)]\rangle\rangle_F. \end{aligned} \tag{C17}$$

Computing the average from Eq. (C16) requires the computation of the four-point correlation of $\delta F_{\mathbf{k}}(\mathbf{J},0)$. We assume that the initial fluctuations are Gaussian so that writing, $\delta F(1)=\delta F_{\mathbf{k}_1}(\mathbf{J}_1,0)$, we have

$$\langle\langle[\delta F(1)\,\delta F(2)][\delta F(3)\,\delta F(4)]\rangle\rangle_F=\langle\delta F(1)\,\delta F(3)\rangle_F\,\langle\delta F(2)\delta F(4)\rangle_F+\langle\delta F(1)\,\delta F(4)\rangle_F\,\langle\delta F(2)\delta F(3)\rangle_F. \tag{C18}$$

Following Eq. (C5), we can then rewrite Eq. (C17) as

$$C^{(2)}_{\mathbf{k}\mathbf{k}'}(\mathbf{J},\mathbf{J}')=-\lim_{\Delta\to+\infty}\frac{(2\pi)^d M_{\mathrm{tot}}}{2\Delta}\int_0^\Delta \mathrm{d}t\int_0^\Delta \mathrm{d}t'\bigg\{\mathrm{e}^{-\mathrm{i}(\mathbf{k}\cdot\boldsymbol{\Omega}-\mathbf{k}'\cdot\boldsymbol{\Omega}')(t-t')}\,|\psi_{\mathbf{k}\mathbf{k}'}(\mathbf{J},\mathbf{J}')|^2\,F(\mathbf{J})\,F(\mathbf{J}')+\delta_{\mathbf{k},-\mathbf{k}'}\delta_{\mathrm{D}}(\mathbf{J}-\mathbf{J}')\,F(\mathbf{J})$$
$$\times\sum_{\mathbf{k}''}\int \mathrm{d}\mathbf{J}''\,\mathrm{e}^{-\mathrm{i}(\mathbf{k}\cdot\boldsymbol{\Omega}-\mathbf{k}''\cdot\boldsymbol{\Omega}'')(t-t')}\,|\psi_{\mathbf{k}\mathbf{k}''}(\mathbf{J},\mathbf{J}'')|^2F(\mathbf{J}'')\bigg\}, \tag{C19}$$

where we used the symmetries from Eq. (A4).

We now use the identity

$$\lim_{\Delta\to+\infty}\frac{1}{\Delta}\int_0^\Delta \mathrm{d}t\int_0^\Delta \mathrm{d}t'\,\mathrm{e}^{-\mathrm{i}\omega_{\mathrm{R}}(t-t')}=\int_{-\infty}^{+\infty}\mathrm{d}t\,\mathrm{e}^{-\mathrm{i}\omega_{\mathrm{R}}t}=2\pi\,\delta_{\mathrm{D}}(\omega_{\mathrm{R}}), \tag{C20}$$

and Eq. (C19) becomes

$$C^{(2)}_{\mathbf{k}\mathbf{k}'}(\mathbf{J},\mathbf{J}')=-B_{\mathbf{k}\mathbf{k}'}(\mathbf{J},\mathbf{J}')F(\mathbf{J})F(\mathbf{J}')-\delta_{\mathbf{k},-\mathbf{k}'}\delta_{\mathrm{D}}(\mathbf{J}-\mathbf{J}')\times\sum_{\mathbf{k}''}\int \mathrm{d}\mathbf{J}''B_{\mathbf{k}\mathbf{k}''}(\mathbf{J},\mathbf{J}'')F(\mathbf{J})F(\mathbf{J}''), \tag{C21}$$

where we used the definition of $B_{\mathbf{k}\mathbf{k}'}(\mathbf{J},\mathbf{J}')$ from Eq. (11).

The last step of the calculation is to inject Eq. (C21) into Eq. (C15). We obtain

$$\mathcal{H}^{(2)}[F,P]=\sum_{\mathbf{k},\mathbf{k}'}\int \mathrm{d}\mathbf{J}\mathrm{d}\mathbf{J}'\,B_{\mathbf{k}\mathbf{k}'}(\mathbf{J},\mathbf{J}')\,\mathbf{k}\cdot\frac{\partial P}{\partial\mathbf{J}}\times\bigg[\mathbf{k}\cdot\frac{\partial P}{\partial\mathbf{J}}-\mathbf{k}'\cdot\frac{\partial P}{\partial\mathbf{J}'}\bigg]F(\mathbf{J})\,F(\mathbf{J}'), \tag{C22}$$

hence recovering Eq. (16).

3. High-order cumulants

Following Eq. (C1), cumulants of order higher than two involve averages of order X^k, with $k\geqslant 3$ and $X\propto\delta F\delta\Phi$. Following Eq. (A17), we have $\delta\Phi=\mathcal{O}(G)$ for $G\to 0$. Therefore, one has $X^k=\mathcal{O}(G^k)=o(G^2)$, for $k\geqslant 3$. As a conclusion, in the dynamically hot limit, cumulants of order higher than two do not contribute to the large deviation Hamiltonian from Eq. (14).

APPENDIX D: PROPERTIES

In this Appendix, we briefly justify the various properties of the large deviation Hamiltonian from Eq. (14).

1. Most probable path

As required by Eq. (17), we need to compute the functional gradient $\delta\mathcal{H}/\delta P(\mathbf{J})$. To do so, we rely on the fundamental identity

$$\frac{\delta P(\mathbf{J}')}{\delta P(\mathbf{J})}=\delta_{\mathrm{D}}(\mathbf{J}-\mathbf{J}'). \tag{D1}$$

Since $\mathcal{H}^{(2)}$ is quadratic in $P(\mathbf{J})$ [Eq. (16)], one has $\partial\mathcal{H}^{(2)}/\delta P(\mathbf{J})=0$ in $P=0$. As a consequence, only $\mathcal{H}^{(1)}$ contributes to the most probable path. Integrating by parts w.r.t. $\mathrm{d}\mathbf{J}$ the term $\mathbf{k}\cdot\partial P/\partial\mathbf{J}$ in Eq. (15) readily recovers the inhomogeneous Landau equation (10).

2. Conservation laws

Mass conservation. Following Eq. (19a), we have $\delta M/\delta F(\mathbf{J})=1$. We note from Eqs. (15) and (16) that the large deviation Hamiltonian only depends on derivatives of the conjugate field, $P(\mathbf{J})$. To check for mass conservation, we compute terms of the form

$$\begin{aligned}\int \mathrm{d}\mathbf{J}\,\frac{\delta M}{\delta F(\mathbf{J})}\frac{\delta\mathcal{H}}{\delta P(\mathbf{J})}&=\int \mathrm{d}\mathbf{J}\mathrm{d}\mathbf{J}'\,\frac{\delta}{\delta P(\mathbf{J})}\bigg[\frac{\partial P}{\partial\mathbf{J}'}\bigg]\cdots\\&=\int \mathrm{d}\mathbf{J}\mathrm{d}\mathbf{J}'\,\frac{\partial}{\partial\mathbf{J}'}\bigg[\delta_{\mathrm{D}}(\mathbf{J}-\mathbf{J}')\bigg]\cdots\\&=-\int \mathrm{d}\mathbf{J}\,\frac{\partial}{\partial\mathbf{J}}\bigg[\cdots\bigg]\\&=0.\end{aligned} \tag{D2}$$

This ensures consistence w.r.t. mass conservation.

Energy conservation. Following Eq. (19b), we have $\delta E/\delta F(\mathbf{J})=H(\mathbf{J})$. Following some integration by parts and manipulations, we get from Eq. (15)

$$\int \mathrm{d}\mathbf{J}\,\frac{\delta E}{\delta F(\mathbf{J})}\frac{\delta\mathcal{H}^{(1)}}{\delta P(\mathbf{J})}=-\sum_{\mathbf{k},\mathbf{k}'}\int \mathrm{d}\mathbf{J}\mathrm{d}\mathbf{J}'\,B_{\mathbf{k}\mathbf{k}'}(\mathbf{J},\mathbf{J}')\,\mathbf{k}\cdot\boldsymbol{\Omega}\times\bigg[\mathbf{k}\cdot\frac{\partial F}{\partial\mathbf{J}}F(\mathbf{J}')-\mathbf{k}'\cdot\frac{\partial F}{\partial\mathbf{J}'}F(\mathbf{J})\bigg]. \tag{D3}$$

Similarly, we get from Eq. (16)

$$\int \mathrm{d}\mathbf{J}\,\frac{\delta E}{\delta F(\mathbf{J})}\frac{\delta\mathcal{H}^{(2)}}{\delta P(\mathbf{J})}=\sum_{\mathbf{k},\mathbf{k}'}\int \mathrm{d}\mathbf{J}\mathrm{d}\mathbf{J}'\,B_{\mathbf{k}\mathbf{k}'}(\mathbf{J},\mathbf{J}')\,\mathbf{k}\cdot\boldsymbol{\Omega}\times 2F(\mathbf{J})F(\mathbf{J}')\bigg[\mathbf{k}\cdot\frac{\partial P}{\partial\mathbf{J}}-\mathbf{k}'\cdot\frac{\partial P}{\partial\mathbf{J}'}\bigg]. \tag{D4}$$

We now perform the symmetrization $(\mathbf{k},\mathbf{J})\leftrightarrow(\mathbf{k}',\mathbf{J}')$ in Eqs. (D3) and (D4) and rely on the symmetry from Eq. (A4).

Both equations then involve

$$B_{\mathbf{k}\mathbf{k}'}(\mathbf{J},\mathbf{J}')[\mathbf{k}\cdot\mathbf{\Omega}-\mathbf{k}'\cdot\mathbf{\Omega}']=0, \tag{D5}$$

which vanishes owing to the resonance condition from Eq. (11). As a conclusion, we therefore have

$$\int \mathrm{d}\mathbf{J}\,\frac{\delta E}{\delta F(\mathbf{J})}\frac{\partial \mathcal{H}}{\partial P(\mathbf{J})}=0. \tag{D6}$$

This ensures consistence w.r.t. energy conservation.

3. Hamilton-Jacobi equation

Following Eq. (20), we have

$$\frac{\delta S}{\delta F(\mathbf{J})}=-\ln[F(\mathbf{J})]+\mathrm{cst}. \tag{D7}$$

Since $\mathcal{H}[F,P]$ only depends on gradients of P (see Appendix D 2), the constant term in Eq. (D7) does not contribute to $\mathcal{H}[F,-\delta S/\delta F]$. Following Eq. (15), the first cumulant contributes

$$\mathcal{H}^{(1)}[F,-\delta S/\delta F]=-\sum_{\mathbf{k},\mathbf{k}'}\int \mathrm{d}\mathbf{J}\mathrm{d}\mathbf{J}' B_{\mathbf{k}\mathbf{k}'}(\mathbf{J},\mathbf{J}')\mathbf{k}\cdot\frac{\partial \ln[F(\mathbf{J})]}{\partial \mathbf{J}} \times\left[\mathbf{k}\cdot\frac{\partial F}{\partial \mathbf{J}}F(\mathbf{J}')-\mathbf{k}'\cdot\frac{\partial F}{\partial \mathbf{J}}F(\mathbf{J})\right]. \tag{D8}$$

Noting that $B_{\mathbf{k}'\mathbf{k}}(\mathbf{J}',\mathbf{J})=B_{\mathbf{k}\mathbf{k}'}(\mathbf{J},\mathbf{J}')$ [Eq. (11)], we symmetrise Eq. (D8) with $(\mathbf{k},\mathbf{J})\leftrightarrow(\mathbf{k}',\mathbf{J}')$ to get

$$\mathcal{H}^{(1)}[F,-\delta S/\delta F]=-\sum_{\mathbf{k},\mathbf{k}'}\int \mathrm{d}\mathbf{J}\mathrm{d}\mathbf{J}'\,\frac{B_{\mathbf{k}\mathbf{k}'}(\mathbf{J},\mathbf{J}')}{F(\mathbf{J})F(\mathbf{J}')} \times\left[\mathbf{k}\cdot\frac{\partial F}{\partial \mathbf{J}}F(\mathbf{J}')-\mathbf{k}'\cdot\frac{\partial F}{\partial \mathbf{J}'}F(\mathbf{J})\right]^2. \tag{D9}$$

Following Eq. (16), the second cumulant contributes

$$\mathcal{H}^{(2)}[F,-\delta S/\delta F]=\sum_{\mathbf{k},\mathbf{k}'}\int \mathrm{d}\mathbf{J}\mathrm{d}\mathbf{J}'\,B_{\mathbf{k}\mathbf{k}'}(\mathbf{J},\mathbf{J}')\,\mathbf{k}\cdot\frac{\partial \ln[F(\mathbf{J})]}{\partial \mathbf{J}} \times\left[\mathbf{k}\cdot\frac{\partial \ln[F(\mathbf{J})]}{\partial \mathbf{J}}-\mathbf{k}'\cdot\frac{\partial \ln[F(\mathbf{J}')]}{\partial \mathbf{J}'}\right] \times F(\mathbf{J})F(\mathbf{J}'). \tag{D10}$$

Performing the same symmetrization as in Eq. (D9), we are left with

$$\mathcal{H}^{(2)}[F,-\delta S/\delta F]=\sum_{\mathbf{k},\mathbf{k}'}\int \mathrm{d}\mathbf{J}\mathrm{d}\mathbf{J}'\,\frac{B_{\mathbf{k}\mathbf{k}'}(\mathbf{J},\mathbf{J}')}{F(\mathbf{J})F(\mathbf{J}')} \times\left[\mathbf{k}\cdot\frac{\partial F}{\partial \mathbf{J}}F(\mathbf{J}')-\mathbf{k}'\cdot\frac{\partial F}{\partial \mathbf{J}'}F(\mathbf{J})\right]^2. \tag{D11}$$

Combining Eqs. (D9) and (D11), we ultimately obtain the expected relation

$$\mathcal{H}[F,-\delta S/\delta F]=\mathcal{H}^{(1)}[F,-\delta S/\delta F]+\mathcal{H}^{(2)}[F,-\delta S/\delta F]=0. \tag{D12}$$

4. Time-reversal symmetry

Given that $\mathcal{H}^{(1)}[F,P]$ (or $\mathcal{H}^{(2)}[F,P]$) is linear (or quadratic) w.r.t. P, we immediately have

$$\mathcal{H}[F,-P]=-\mathcal{H}^{(1)}[F,P]+\mathcal{H}^{(2)}[F,P]. \tag{D13}$$

Similarly, by linearity we have

$$\mathcal{H}^{(1)}[F,P-\delta S/\delta F]=\mathcal{H}^{(1)}[F,P]+\mathcal{H}^{(1)}[F,-\delta S/\delta F]. \tag{D14}$$

As for the second cumulant, it reads

$$\mathcal{H}^{(2)}[F,P-\delta S/\delta F]=\mathcal{H}^{(2)}[F,P]+\mathcal{H}^{(2)}[F,-\delta S/\delta F]+\widetilde{\mathcal{H}}^{(2)}[F,P,-\delta S/\delta F]+\widetilde{\mathcal{H}}^{(2)}[F,-\delta S/\delta F,P], \tag{D15}$$

where, following Eq. (16), we introduced

$$\widetilde{\mathcal{H}}^{(2)}[F,P,Q]:=\sum_{\mathbf{k},\mathbf{k}'}\int \mathrm{d}\mathbf{J}\mathrm{d}\mathbf{J}'\,B_{\mathbf{k}\mathbf{k}'}(\mathbf{J},\mathbf{J}')\,\mathbf{k}\cdot\frac{\partial P}{\partial \mathbf{J}} \times\left[\mathbf{k}\cdot\frac{\partial Q}{\partial \mathbf{J}}-\mathbf{k}'\cdot\frac{\partial Q}{\partial \mathbf{J}'}\right]F(\mathbf{J})F(\mathbf{J}'). \tag{D16}$$

With the same symmetrisation as in Eq. (D9), one gets

$$\widetilde{\mathcal{H}}^{(2)}[F,P,-\delta S/\delta F]=-\mathcal{H}^{(1)}[F,P], \tag{D17a}$$

$$\widetilde{\mathcal{H}}^{(2)}[F,-\delta S/\delta F,P]=-\mathcal{H}^{(1)}[F,P]. \tag{D17b}$$

Recalling the result from Eq. (D12), all the previous relations lead to the needed result, namely,

$$\mathcal{H}[F,-P]=\mathcal{H}[F,P-\delta S/\delta F]. \tag{D18}$$

5. Gradient structure

The expression of $Q[F]$ in Eq. (24) follows from integration by parts of Eq. (16). We now compute the r.h.s. of Eq. (25). Following Eq. (D7), we face the term

$$\int \mathrm{d}\mathbf{J}'\,Q[F](\mathbf{J},\mathbf{J}')\,\frac{\partial S[F]}{\partial F(\mathbf{J}')}=\int \mathrm{d}\mathbf{J}'\,\mathbf{k}\cdot\frac{\partial}{\partial \mathbf{J}}\left\{\frac{\partial}{\partial \mathbf{J}'}\cdot\mathbf{A}(\mathbf{J},\mathbf{J}')\right\} \times\{-\ln[F(\mathbf{J}')]+\mathrm{cst}\}, \tag{D19}$$

where $\mathbf{A}(\mathbf{J},\mathbf{J}')$ follows from Eq. (24). Discarding boundary terms, we find that the constant terms do not contribute. Integrating by parts w.r.t. $\mathrm{d}\mathbf{J}'$, we get

$$(D19)=\mathbf{k}\cdot\frac{\partial}{\partial \mathbf{J}}\left\{\int \mathrm{d}\mathbf{J}'\,\frac{1}{F(\mathbf{J}')}\mathbf{A}(\mathbf{J},\mathbf{J}')\cdot\frac{\partial F}{\partial \mathbf{J}'}\right\}. \tag{D20}$$

Injecting $\mathbf{A}(\mathbf{J},\mathbf{J}')$ from Eq. (24), one recovers Eq. (25).

6. Stochastic Landau equation

As defined in Eq. (27), the correlation function of the stochastic noise, $\zeta[F](\mathbf{J},\tau)$, involves Dirac deltas. This makes the sampling of effective realisations challenging. In addition, although guaranteed by Eq. (18), it is not strikingly obvious that Eq. (26), indeed, complies with the system's conservation laws. In this Appendix, we tackle these two issues and devise a "diagonal" rewriting of $\zeta[F](\mathbf{J},\tau)$: it is sourced by a normal random Gaussian field and makes the conservation laws obvious.

For any given resonance vector $\mathbf{k}$, we assume that we can perform the change of variables $\mathbf{J}\mapsto(\omega=\mathbf{k}\cdot\mathbf{\Omega},\mathbf{z})$ (see, e.g., [41] for an explicit example) with a nondegenerate Jacobian $\mathcal{J}:=|\partial(\omega,\mathbf{z})/\partial\mathbf{J}|$. Physically, ω is the resonance frequency, and $\mathbf{z}$ covers the sets of all orbits that resonate with it. The existence of the present mapping is mandatory for the resonance condition $\delta_{\mathrm{D}}(\mathbf{k}\cdot\mathbf{\Omega}-\mathbf{k}'\cdot\mathbf{\Omega}')$ to be generically well posed in the Landau Eq. (10).

Let us consider the following ansatz for the noise

$$\zeta[F](\mathbf{J},\tau):=\frac{\sqrt{\pi}M_{\mathrm{tot}}}{\sqrt{N}}\frac{\partial}{\partial\mathbf{J}}\cdot\left[\sum_{\mathbf{k},\mathbf{k}'}\mathbf{k}\int\mathrm{d}\mathbf{J}'[\mathcal{J}\mathcal{J}']^{1/2}|\psi_{\mathbf{k}\mathbf{k}'}(\mathbf{J},\mathbf{J}')|[F(\mathbf{J})F(\mathbf{J}')]^{1/2}\delta_{\mathrm{D}}(\omega-\omega')\{\eta_{\mathbf{k}\mathbf{k}'}(\mathbf{z},\mathbf{z}',\omega,\tau)-\eta_{\mathbf{k}'\mathbf{k}}(\mathbf{z}',\mathbf{z},\omega',\tau)\}\right], \tag{D21}$$

where we used shortened notations for the change of variables, $\mathbf{J}\leftrightarrow_{\mathbf{k}}(\omega,\mathbf{z})$ and its Jacobian $\mathcal{J}$, and similarly for $\mathbf{J}'$. In Eq. (D21), we also introduced the normal Gaussian random field $\eta_{\mathbf{k}\mathbf{k}'}(\mathbf{z},\mathbf{z}',\omega,\tau)$ obeying

$$\langle\eta_{\mathbf{k}\mathbf{k}'}(\mathbf{z},\mathbf{z}',\omega,\tau)\rangle=0, \tag{D22a}$$

$$\langle\eta_{\mathbf{k}\mathbf{k}_1}(\mathbf{z},\mathbf{z}_1,\omega,\tau)\,\eta_{\mathbf{k}'\mathbf{k}'_1}(\mathbf{z}',\mathbf{z}'_1,\omega',\tau')\rangle=\delta_{\mathbf{k}\mathbf{k}'}\delta_{\mathbf{k}_1\mathbf{k}'_1}\delta_{\mathrm{D}}(\mathbf{z}-\mathbf{z}')\,\delta_{\mathrm{D}}(\mathbf{z}_1-\mathbf{z}'_1)\,\delta_{\mathrm{D}}(\omega-\omega')\,\delta_{\mathrm{D}}(\tau-\tau'). \tag{D22b}$$

Equation (D21) easily complies with the conservation laws from Eqs. (19). Indeed, Eq. (D21) is the divergence of a flux in action space, hence the total mass is conserved. As for the total energy, starting from Eq. (19b), up to prefactors, one must compute

$$\frac{\mathrm{d}E}{\mathrm{d}t}\propto\sum_{\mathbf{k},\mathbf{k}'}\int\mathrm{d}\mathbf{J}\mathrm{d}\mathbf{J}'\,\omega\,\delta_{\mathrm{D}}(\omega-\omega')[\mathcal{J}\mathcal{J}']^{1/2}|\psi_{\mathbf{k}\mathbf{k}'}(\mathbf{J},\mathbf{J}')|[F(\mathbf{J})F(\mathbf{J}')]^{1/2}\{\eta_{\mathbf{k}\mathbf{k}'}(\mathbf{z},\mathbf{z}',\omega,\tau)-\eta_{\mathbf{k}'\mathbf{k}}(\mathbf{z}',\mathbf{z},\omega',\tau)\}. \tag{D23}$$

Performing the symmetrization $(\mathbf{k},\mathbf{J})\leftrightarrow(\mathbf{k}',\mathbf{J}')$ leaves us with $(\omega-\omega')\,\delta_{\mathrm{D}}(\omega-\omega')=0$, so that $\mathrm{d}E/\mathrm{d}t=0$.

Let us now check Eqs. (27). Since $\zeta[F]$ is linear w.r.t. η, Eq. (D22a) naturally imposes $\langle\zeta[F]\rangle=0$, as required by Eq. (27a). As for the correlation function, we write

$$\begin{aligned}\langle\zeta[F](\mathbf{J},\tau)\,\zeta[F](\mathbf{J}',\tau')\rangle=\pi mM_{\mathrm{tot}}\sum_{\substack{\mathbf{k},\mathbf{k}_1\\ \mathbf{k}',\mathbf{k}'_1}}\int\mathrm{d}\mathbf{J}_1\mathrm{d}\mathbf{J}'_1\mathbf{k}\cdot\frac{\partial}{\partial\mathbf{J}}\Bigg\{&\mathbf{k}'\cdot\frac{\partial}{\partial\mathbf{J}'}[[\mathcal{J}\mathcal{J}_1\mathcal{J}'\mathcal{J}'_1]^{1/2}|\psi_{\mathbf{k}\mathbf{k}_1}(\mathbf{J},\mathbf{J}_1)||\psi_{\mathbf{k}'\mathbf{k}'_1}(\mathbf{J}',\mathbf{J}'_1)|\\ &\times\,\delta_{\mathrm{D}}(\omega-\omega_1)\delta_{\mathrm{D}}(\omega'-\omega'_1)[F(\mathbf{J})F(\mathbf{J}_1)F(\mathbf{J}')F(\mathbf{J}'_1)]^{1/2}\langle\{\eta_{\mathbf{k}\mathbf{k}_1}(\mathbf{z},\mathbf{z}_1,\omega,\tau)-\eta_{\mathbf{k}_1\mathbf{k}}(\mathbf{z}_1,\mathbf{z},\omega_1,\tau)\}\\ &\times\,\{\eta_{\mathbf{k}'\mathbf{k}'_1}(\mathbf{z}',\mathbf{z}'_1,\omega',\tau')-\eta_{\mathbf{k}'_1\mathbf{k}'}(\mathbf{z}'_1,\mathbf{z}',\omega'_1,\tau')\}\rangle]\Bigg\},\end{aligned} \tag{D24}$$

with $\mathbf{J}\leftrightarrow_{\mathbf{k}}(\omega,\mathbf{z})$ and similarly for $\mathbf{J}',\mathbf{J}_1,\mathbf{J}'_1$.

In Eq. (D24), to compute a given crossed term, say $\langle\eta_{\mathbf{k}\mathbf{k}_1}\eta_{\mathbf{k}'\mathbf{k}'_1}\rangle$, we use Eq. (D22b) and face a product of three resonant Dirac deltas in frequencies. We write it as

$$\delta_{\mathrm{D}}(\omega-\omega_1)\delta_{\mathrm{D}}(\omega'-\omega'_1)\,\delta_{\mathrm{D}}(\omega-\omega')=\delta_{\mathrm{D}}(\omega-\omega')\,\delta_{\mathrm{D}}(\omega_1-\omega'_1)\,\delta_{\mathrm{D}}(\omega-\omega_1). \tag{D25}$$

For the other crossed terms, we pick the appropriate set of differences of frequencies. Finally, we also use

$$[\mathcal{J}\mathcal{J}']^{1/2}\delta_{\mathbf{k}\mathbf{k}'}\delta_{\mathrm{D}}(\omega-\omega')\delta_{\mathrm{D}}(\mathbf{z}-\mathbf{z}')=\delta_{\mathbf{k}\mathbf{k}'}\delta_{\mathrm{D}}(\mathbf{J}-\mathbf{J}'), \tag{D26}$$

and similar variations. Following simple manipulations, one ultimately recovers Eq. (27b).

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