Virtual walks and phase transitions in the two-dimensional Biswas-Chatterjee-Sen model with extreme switches

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In kinetic exchange models of opinion formation, one considers pairwise interactions to update the states of the agents. We have studied a kinetic exchange model with three opinion states 0, ±1, by considering a walk in a one-dimensional virtual space which evolves according to the dynamical states of the agents. The model involves two noise parameters p and q; p represents the fraction of negative interactions and q corresponds to the probability of a stronger interaction with the other agent, which may result in extreme switches (change of state from +1 to −1 or vice versa). The nature of the walks can indicate where the phase transitions occur in the p-q plane; these results are corroborated with those obtained using finite-size scaling method. The criticality is found to be Ising-like, even when extreme switches are allowed. A new critical exponent associated with the probability distribution of the displacements in the virtual space is also obtained independent of the values of the critical parameters. The nature of the walks is compared to similar virtual walks studied earlier.

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I. INTRODUCTION

In the past few decades, extensive research has been made to study the problem of opinion formation in a society, using the tools of statistical physics [1–3]. Several models of opinion dynamics have been proposed in the past. One class of models, namely, the kinetic exchange (KE) models, involves an interaction between two agents at any instant [4]. A particular KE model, introduced in [5] and later popularly called by the last initials of the authors, such as the BChS (Biswas-Chatterjee-Sen) model, incorporates negative interactions as well.

In models mimicking social phenomena in general, an order-disorder phase transition driven by some appropriate noise can usually be observed. In particular, in opinion dynamics models, the phase transition occurs from a state where consensus or a majority opinion has been reached to a disordered state [6]. Apart from the fact that the BChS model manifests an order-disorder phase transition, it has also been possible to obtain some results from this model that correspond to realistic scenarios [7–9].

It is often convenient to study a model in statistical physics by mapping it into another model and studying the latter. In recent times, various models of statistical physics involving dynamics of spins, opinions, and financial status have been mapped to walks in a virtual space [10–21]. In these mappings, a walker is associated with each spin or agent. The position of the walker is updated according to the state of the spin or agent it is associated with, following either a Markovian or a non-Markovian dynamics. The general scheme is to update the ith walker's positions $X_i(t)$ as

$$X_i(t+1) = X_i(t) + \xi_i(t+1), \quad (1)$$

where the displacements take place in a virtual one-dimensional space and $\xi_i(t+1)$ is determined by the state of the spin or agent. For Ising spin models with spin values $s = \pm 1$, if one takes $\xi_i(t+1)$ equal to the spin value at $t+1$, then $X_i(t)$ is simply related to the *local* time averaged magnetization $m_i(t)$; $m_i(t) = X_i(t)/t$ [14]. In fact, such walks can be defined for any dynamical process in general, with ξ_i assuming values related to the state of the variable (not necessarily equal).

The studies of these walks made so far have been shown to impart a lot of information e.g., regarding the behavior of persistence probability [14–19]. The nature of the walks usually undergoes a change at the phase transition points, if any. This is indicated by the existence of a Gaussian distribution of the displacements of the walkers above the critical point while it has a double-peaked non-Gaussian behavior below it. Since for finite sizes obtaining the critical point is challenging, this method is useful and simpler in locating it. An interesting crossover behavior involving diverging timescales, not obtained directly from the model, has been observed previously also [19,20].

In this paper we have studied a KE model of opinion formation, an extended version of the so-called BChS model. In the present study, the agents of the opinion dynamics model are embedded on a two-dimensional (2D) lattice while the walks are defined in a virtual one-dimensional (1D) space.

The BChS model is taken with three opinion states denoted by ±1 and 0. The interactions here can be negative with probability p. In the original version [5], switches between the extreme states (i.e., +1 and −1) were not possible according to the dynamical rules. However, because of the complexity

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and inherent unpredictability of human behavior, the possibility of sudden changes in loyalty cannot be entirely dismissed. In fact, real-world examples of political figures switching allegiances to parties with opposing ideologies [22,23] indeed exist. These shifts are likely to be driven by personal gain, self-interest, or a lack of core convictions. Another way of justifying the extreme switch is, an agent in reality transits to the intermediate state for timescales much smaller than the observation time before making the switch to the opposite extreme, so that it appears as an extreme switch. The feature of extreme switches is incorporated by taking the possibility of a stronger influence made by the interacting agent with probability q. In some recent works, the present authors introduced the BChS model with extreme switches [24,25]. The mean-field case was studied to show that a phase diagram can be obtained in the $p-q$ plane showing the presence of ordered and disordered regions [25].

Our primary aim is to find the role of the extreme switches on the order-disorder transition. The critical behavior of the BChS model on two-dimensional square lattices without extreme switch (i.e., $q=0$) has been studied earlier using numerical simulations [26,27]. It was found that the model belongs to the two-dimensional Ising criticality class. In this paper we have simulated the model in 2D with extreme switches, i.e., $q\neq 0$, and obtained the corresponding walk from which the phase diagram can be estimated. In addition, for comparison, a few phase transition points have been found directly by analyzing the relevant physical quantities and using finite-size scaling. From the results obtained in the mean-field case [25], we expect that q will effectively impart an additional noise. It is also interesting to see whether the critical behavior is affected by q. In the mean-field case, it is not.

The walks are also analyzed quantitatively by fitting appropriate curves to the distributions and studying the fitting parameters. This leads to some further understanding of the system both quantitatively and qualitatively.

In Sec. II we describe the model and the methods used. The results are presented in Sec. III, and in the last section, discussions and conclusive statements are made.

II. MODEL AND METHOD

A. Simulating the BChS model on a two-dimensional lattice with the parameters *p* and *q*

We have simulated an agent-based BChS model where agents are located on the sites of a 2D square lattice, with N sites ($N = L\times L$) having periodic boundary conditions. o_i is considered as the opinion of the ith agent. Each opinion ($o_i=\pm 1, 0$) is updated upon interaction with one of the nearest neighbors denoted by k, with opinion o_k, following the expression

$$o_i(t+1) = o_i(t) + \mu o_k(t). \quad (2)$$

Here μ is a random variable representing the interaction and can have four possible values: $\mu=\pm 1$ with probability $1-q$ and equal to ± 2 with probability q, and $\mu<0$ with probability p and positive otherwise. If after an interaction the opinion exceeds 1 or becomes less than -1, it is adjusted to ± 1 respectively $[|o_i(t+1)|\leqslant 1$ at any time step $(t+1)]$. In the simulations a homogeneous disordered state is considered as the initial configuration, i.e., equal proportion of the population has opinion ± 1 and zero.

B. Mapping the BChS model into a virtual walk

Mapping of the opinion dynamics model to a virtual walk is done by associating a virtual walker with each agent on the 2D lattice. The virtual walk takes place in one dimension. Hence we have a scenario of N walkers performing walks on a one-dimensional lattice, which is, strictly speaking, unbounded. The walks are not independent, as they are generated from the interaction of the agents.

In the mapping scheme, the initial position of a walker is taken to be 0. The walks are implemented according to the opinions of the agents, which are updated asynchronously. The total distance traveled from the starting point (at $t=0$) by the ith walker at the $(t+1)$th Monte Carlo (MC) step is $X_i(t+1)$, given by

$$X_i(t+1) = X_i(t) + o_i(t+1), \quad (3)$$

i.e., here ξ_i in Eq. (1) is simply equal to the opinion state o_i. Only Markovian walks have been considered in the present work. The study of $X(t)$, which is the average value of $X_i(t)$ over all agents, and its related features are relevant in what we term as the *walk picture,* where the walk actually occurs in a virtual space.

As mentioned in the Introduction, $X_i(t)/t$ lying between -1 and $+1$, corresponds to the average local order, which is the average opinion of an individual over time, while in the walk picture, $X_i(t)$ is the important variable with $|X_i(t)|\leqslant t$. If $X_i(t)=t$, it implies that the agent has not changed opinion at all, i.e., it indicates persistent behavior. Lower values of $X_i(t)$ signify increasing tendency to change.

We have simulated the system on $L\times L$ lattices with various values of L. For the virtual walk, the results for $L=64$ (maximum size simulated) only are presented. The walk is incremented by the value of the opinion after the completion of one Monte Carlo step (MCS). One agent is selected randomly and allowed to interact with any of her four nearest neighbors, after which her opinion is updated; one MCS step comprises N such updates.

C. Finite-size scaling analysis

For a system manifesting a continuous phase transition driven by a certain parameter, the critical point and the exponents can be obtained using a finite-size scaling method [28] in a numerical simulation. Consider a physical quantity Φ, which either goes to zero or diverges in the manner $\Phi\propto\epsilon^{\phi}$, where ϵ is the small deviation from the critical point. In a finite system of size L, Φ can be expressed as

$$\Phi = L^{-\phi/\nu} f(\epsilon L^{1/\nu}),$$

where ν is the correlation length exponent. $f(z)$ is expected to vary as z^{ϕ} as $z\to\infty$ such that one recovers the behavior $\Phi\propto\epsilon^{\phi}$ for $L\to\infty$.

A data collapse can be obtained (i.e., all data for different system sizes fall on the same curve) when properly rescaled

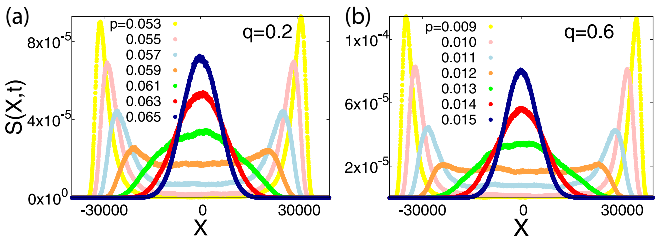

FIG. 1. The probability distribution of X at time step $t = 60\,000$, when the model has reached a steady state, has been plotted for two different values of q [0.2 in (a) and 0.6 in (b)] for various p values. These plots are presented for a system size of $N = 64 \times 64$. The peaks spread further as p is decreased.

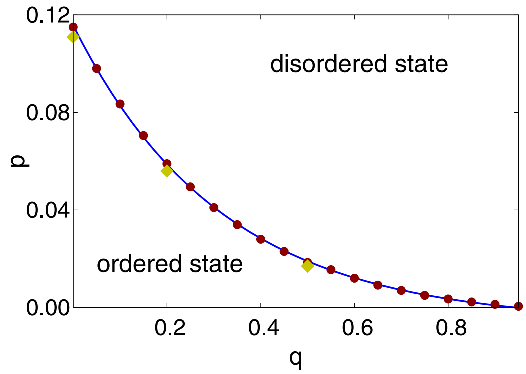

FIG. 2. The red colored dots in the p, q parameter space represent threshold values separating the ordered and disordered phases. The phase boundary can be fitted to the form $p_T = C_0 \exp^{-\lambda q_T} - C_1$, with $\lambda = 3.149\,57 \pm 0.026\,97$, $C_1 = 0.006\,26 \pm 0.000\,33$, and $C_0 = 0.122\,03 \pm 0.000\,36$. Three critical points directly obtained from the simulation for $q = 0, 0.2, 0.5$ using finite-size scaling are also shown by the yellow-colored points in the figure.

quantities are plotted with accurate choices of the values of the critical point and the exponents.

III. RESULTS

We have determined the probability $S(X, t)$ associated with the distance X covered by a walker at a specific time t by averaging over many configurations. In the Ising and the generalized voter model in 2D and also in the mean-field KE model [15,19,20], the nature of $S(X, t)$ has been observed to change at the phase transition points. Here we have observed a similar change beyond threshold values of the parameters p, q. We investigated the characteristics of the distribution both below and above the threshold points. Additionally, by performing data collapse analysis of the scaled data from the simulations conducted on 2D lattices, we identified the critical points and estimated the critical exponents. The latter study has been made at larger times such that the system definitely reaches equilibrium.

A. Features studied from the virtual walk

In the (p, q) parameter space, the threshold points, denoted as (p_T, q_T), have been determined using the virtual walk analysis. In general, we have kept $q = q_T$ fixed and varied p to study the behavior of the walk. It is observed that when p is less than p_T, the distribution $S(X, t)$ exhibits a non-Gaussian, double-peaked behavior. Above the threshold point, the distribution assumes a centrally peaked Gaussian shape. The data are shown in Fig. 1 for two different q_T values.

The determination of the threshold points p_T has been carried out for several values of q within the range of [0, 1), and as displayed in Fig. 2, these points can be fitted to the form

$$p_T \approx C_0 \exp^{-\lambda q_T} - C_1, \tag{4}$$

where $\lambda \approx 3.14$, $C_1 \approx 0.0063$, and $C_0 \approx 0.1220$. Specifically, one gets $p_T \to 0$ for $q_T = 1$, exactly as in the case of the mean-field version of the present model [25].

To determine the nature of the walk, $S(X, t)$ is fitted to the scaling form

$$S(X, t) \approx t^{-\alpha} F(X/t^{\alpha}). \tag{5}$$

When $p < p_T$, an approximate data collapse has been obtained with $\alpha = 1.0$ (shown in Fig. 3). $\alpha = 1$ implies a ballistic walk. For an absorbing phase, a perfectly ballistic behavior is expected at late times; however, such is not the case here. This indicates that for $p < p_T$, where the ordered phase exists as is concluded from the nature of $S(X, t)$, we have an active state. Indeed, the snapshots shown in Fig. 4 taken at different times show that the system is still active. It is understandable why an active phase is present; in the presence of noise, some agents will change opinion even at large times. In previous work, only for the noiseless case, it was shown that an absorbing phase is reached, although for a considerable number of configurations, it becomes a very slow process [9]. We will address this issue again later in this section.

On the other hand, the data collapse observed for $p > p_T$ yields $\alpha = 0.5$ (as shown in Fig. 5), which is consistent with the behavior expected in an unbiased random walk. The Gaussian function is written as $F(z) \propto e^{-\frac{z^2}{2\sigma^2}}$, where $z = X/t^{\frac{1}{2}}$. The distribution width σ, is observed to diverge as $(p - p_T)^{-\delta}$, plotted in Fig. 6(a) for different q values. δ shows no systematic dependence on the exact location on the phase boundary where it is calculated and lies within a small range; $\delta \approx 0.84$. Similar behavior can be observed when the variation of σ is plotted as a function of $(q - q_T)$ [shown in Fig. 6(b)]. Snapshots in the disordered phase taken at different times show that the states of the spins are changing considerably over time, which will show typical oscillations of the order parameter

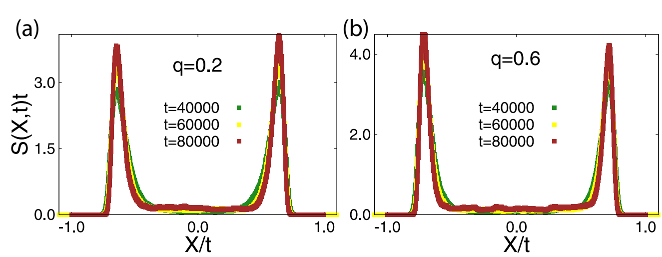

FIG. 3. The data collapse of $S(X, t)t$ using the scaling variable X/t is illustrated for two different points: (a) $q = 0.2$, $p = 0.055$ and (b) $q = 0.6$, $p = 0.010$, both of which lie below the phase boundary [Eq. (4)].

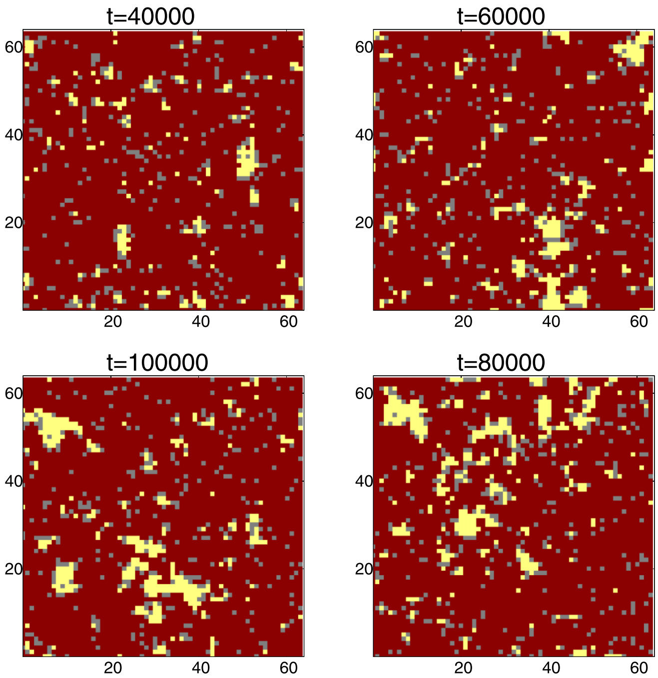

FIG. 4. Snapshots of the 2D lattice space at different time steps below the phase boundary. Red, yellow, and gray dots denote ±1 and 0 states, respectively.

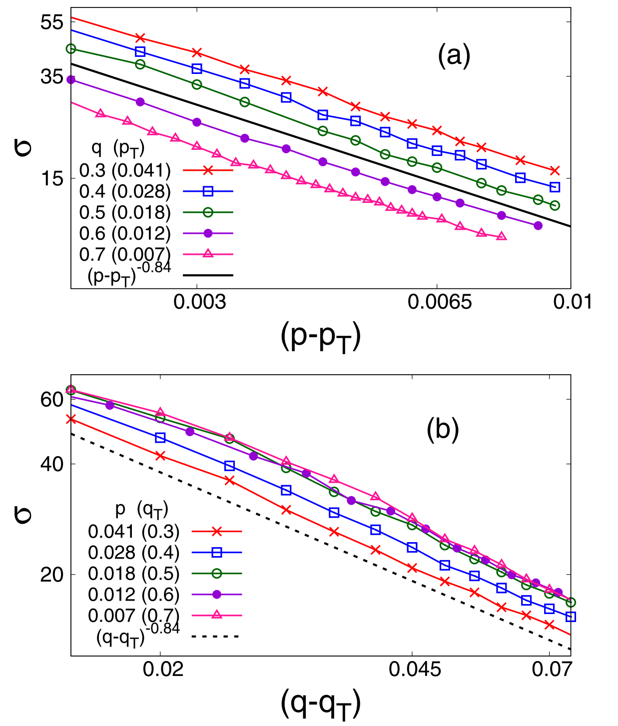

FIG. 6. Variation of the width σ of the Gaussian distributions as a function of p above p_T is shown in (a) for several q values. In (b), the variation of σ is shown as a function of q above q_T for different values of p. Both (a) and (b) show that the nature of the curves is compatible with a variation $\sigma \sim (x - x_T)^{-\delta}$ with $\delta \approx 0.84$ [where x symbolizes the parameters p and q in (a) and (b), respectively].

about zero for a single configuration (Fig. 7). It is interesting that both below and above the phase boundary, opinions equal to zero are much less in number in comparison to the others.

B. Exploring phase transitions by the finite-size scaling method

Up to this point, we have discussed the results of the virtual walk generated from the BChS model with extreme switches. In this section we report the results for the critical points in the $p - q$ space obtained directly from the simulation of the BChS model. The most prevalent approach for investigating order-disorder phase transitions involves the finite-size scaling of quantities such as the Binder cumulant [29], the order parameter, etc. In this particular model, the critical points can be expressed as (p_c, q_c). The order parameter for the system is the average of all opinions,

$$O = \frac{1}{N} \sum_i o_i, \tag{6}$$

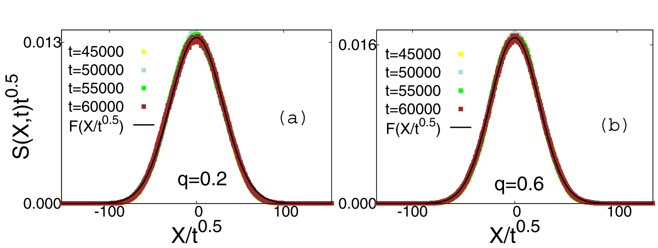

FIG. 5. The data collapse of $S(X, t)t^{0.5}$ using the scaling variable $X/t^{0.5}$ is illustrated for two cases (a) $q = 0.2$, $p = 0.064$ and (b) $q = 0.6$, $p = 0.015$, both of which lie above the phase boundary (i.e., $p < p_T$ in each case). Both sets of data can be fit using a Gaussian function denoted by $F(z)$ in the figures, where $z = \frac{X}{t^{1/2}}$.

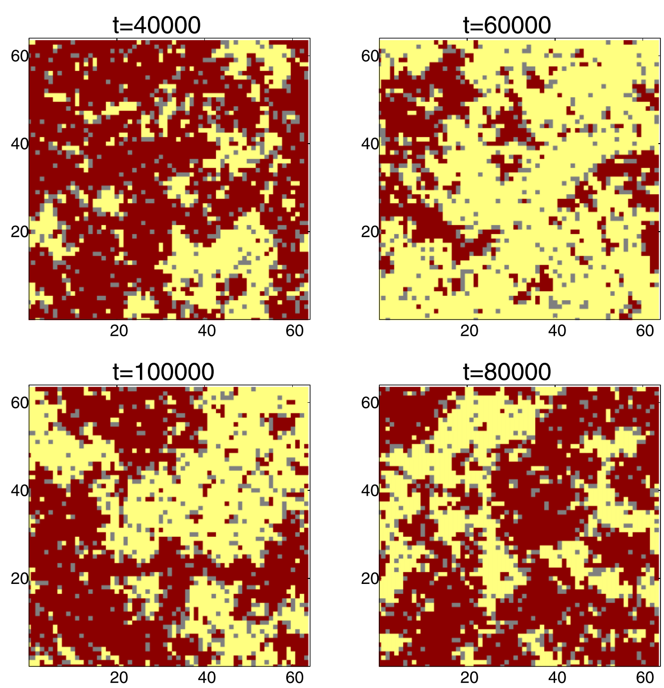

FIG. 7. Snapshots of the 2D lattice space at different time step above the phase boundary. Red, yellow, gray dots denote ±1 and 0 states, respectively.

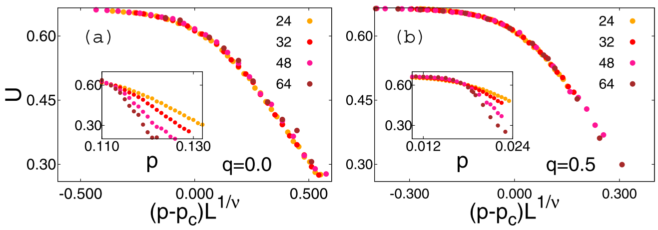

FIG. 8. Finite-size scaling behavior of the Binder cumulant U as a function of p is illustrated for two cases: (a) $q = 0.0$ and (b) $q = 0.5$. Notably, data collapses are observed with critical values of $p_c = 0.1110$ and $p_c = 0.0170$ in (a) and (b), respectively. In both cases the critical exponent ν is estimated to be approximately 1.0 ± 0.05. The inset displays the unscaled raw data for U vs p.

and the fourth-order Binder cumulant is

$$U = 1 - \frac{\langle O^4 \rangle}{3\langle O^2 \rangle^2}, \tag{7}$$

where the angular brackets indicate the ensemble average.

We conducted Monte Carlo simulations for various system sizes L varying between 12 and 64. The simulations were run with a sufficient number of time steps to allow measurable quantities to reach a steady value. Subsequently, we calculated the ensemble averages of these values. The number of configurations ranged from 2000 to 1000 with an increase in system size. The scaling behavior of the Binder cumulant and the order parameter are given by $U = f_1[(x - x_c)L^{\frac{1}{\nu}}]$ and $\langle |O| \rangle = L^{-\frac{\beta}{\nu}} f_2[(x - x_c)L^{\frac{1}{\nu}}]$, where the phase transition is driven by the parameter x and occurs at $x = x_c$. ν and β are critical exponents associated with the correlation length and order parameter, respectively.

Our aim is to check whether (p_T, q_T) and (p_c, q_c) are close enough so that it can be concluded that the virtual walks bear the signature of the phase transition. Here we have compared the values for three particular q values. Fixing the value of q as q_c, we determined p_c by identifying the crossing points of the Binder cumulant for different system sizes. Additionally, by employing the data collapse technique, we estimated the critical exponent ν to be very close to 1 (see Fig. 8). Using this value of ν, we estimated the critical exponent $\beta \approx 0.125$ from the data collapse of the scaled order parameter shown in Fig. 9. Both these exponent values

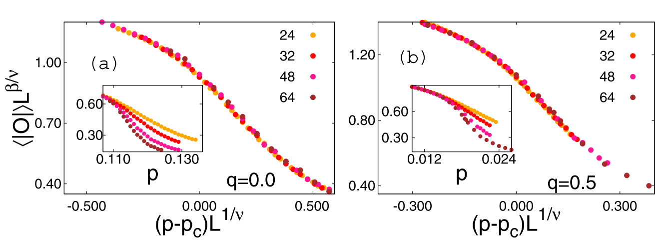

FIG. 9. Data collapse of the scaled order parameter $\langle |O| \rangle$ for different system sizes is shown for two cases, (a) $q = 0.0$ and (b) $q = 0.5$, using the critical values of p_c and ν obtained from the analysis of the Binder cumulant (Fig. 8). In both instances the critical exponent β is estimated to be 0.125 ± 0.001. The inset displays the unscaled raw data $\langle O \rangle$ vs p.

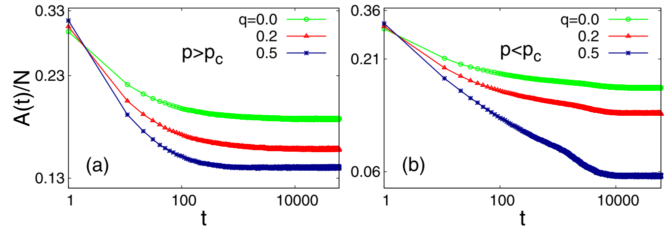

FIG. 10. Variation of the fraction of the number of active agents A among the total number of agents N is plotted against time for different values of q, above (a) and below (b) the phase boundary.

are very close to the exact values known for the 2D Ising model.

For $q = 0.0$ and 0.5, we obtain $p_c \approx 0.1110$ and ≈ 0.0170, respectively. From the walk picture, the corresponding threshold values are $p_T \approx 0.1150$ and ≈ 0.0185, respectively, which are fairly close to the results obtained using finite-size scaling.

Study of active agents- In order to investigate whether the ordered and disordered phases are active or absorbing, we have estimated the number of active agents $A(t)$ as a function of time t. $A(t)$ is defined as the number of agents whose opinion changed at time t, and the fraction of the active agents $A(t)/N$ is plotted in Fig. 10. This preliminary study of the fraction of active agents shows that indeed it decreases as q is made larger in both the disordered and ordered states. That the active agents fraction remains nonzero at long times also supports the fact that both the phases are active.

IV. DISCUSSIONS AND CONCLUSIONS

In the present work we have obtained a phase diagram in a two-parameter kinetic exchange model of opinion dynamics with three opinion states. The two parameters represent the fraction of negative interactions and the probability of extreme switches of opinions. By simulating the agent-based model on a 2D lattice, we generated the walks corresponding to each agent's opinion state in a 1D virtual space. Analysis of the distribution $S(X, t)$ of the displacements shows a change in the nature of the walk above threshold values of the parameters. Below these values the distribution is double peaked and the data for different times can be approximately collapsed using a scaling variable X/t, which reveals the nearly ballistic nature of the walk. This nature and the double-peaked structure with peaks occurring at nonzero values of X indicate that most of the opinions continue in a state of either $+1$ or -1. This is then a partially ordered state. It may be mentioned here that for the opinion equal to zero, no displacement is occurring in the walk and it does not significantly affect the walk's nature. Above the threshold values, we get Gaussian distributions centered at zero with the scaling variable $X/t^{1/2}$ indicating a diffusive walk. This means that the opinions are changing continuously and randomly in time and the system is disordered. Hence the two regions below and above the threshold values are identified as ordered and disordered regions, as had been observed earlier for such virtual walks corresponding to other dynamical systems [15,19,20].

Clearly, extreme switches act as a noise and make the society more disordered. However, the exponents are in general universal, i.e., not dependent on the exact location on the phase boundary. From Fig. 6 we note that as one increases q or p in the disordered phase, σ decreases, implying the average root-mean-square displacement of the agents decreases. This indicates rapid oscillations between ± 1, and one can conclude that the opinions of the society become more volatile as the disorders are increased.

To establish that indeed the walk changes its nature at the critical points, we have also located the critical points for three q values using finite-size scaling. The results agree fairly well: the small discrepancy may be due to finite-size effects. The critical exponents ν and β are estimated and found to be close to the Ising exponents in two dimensions, observed already for the one-parameter model without extreme switches [26]. One can also compare the results with the mean-field case [25], where the phase boundary was obtained as a straight line. Except for $q = 1$, the mean-field phase boundary lies above that of the 2D one, which is a logical result. In both the mean-field and two-dimensional versions, the role of q is to provide additional noise without changing the universality class. Interestingly, in both cases we find that the system becomes totally disordered when the probability of extreme switches is unity, i.e., when $q = 1$. In the mean-field case it was shown that for $q = 1$, the model becomes identical to a voter model with binary opinion values [24,25]; we conjecture that the same happens for the finite-dimensional case, as essentially the presence of q decreases the probability of having an opinion equal to zero. This is aptly reflected in the snapshots for a nonzero value of q.

From the walk picture, a critical exponent δ related to the diverging width of the Gaussian distribution is obtained. Like the static critical exponents, δ, characterizing the distribution of the walk in the disordered phase, is also universal as manifested in Fig. 6. Hence the signature of the phase transition is contained in the distribution in this manner as well. As δ apparently cannot be related to any static critical exponent, we claim it is a new exponent.

We also obtained the result that the ordered phase is not an absorbing one in general from the results obtained so far. It is known that for $p = 0, q = 0$, an absorbing state can be reached but may take a very long time [9]. In this case, the presence of opinion equal to zero states at the boundary of $+1$ and -1 domains was responsible for the slow dynamics. With a nonzero value of q, the population of agents with opinion equal to zero decreases and such states could be less significant for the dynamics. A detailed study of this and dynamics in general could be interesting; however, at the moment we restrict to presenting the walk features and the resulting phase diagram only.

One more point needs to be mentioned. In some earlier studies of the virtual walks, a crossover behavior in time had been observed in the ordered phase [19,20]. Here, however, no such tendency is noted. Interestingly, walks corresponding to the two-dimensional Ising model also do not show such a feature [30].

In conclusion, we find that a phase diagram for the two-dimensional BChS model with extreme switches can be obtained using the walk picture, showing that the transition to the disordered state is enhanced by the extreme switches. A usual finite-size analysis has also been done to confirm the results and to show that the criticality is of Ising class. One more exponent entirely related to the virtual walks has been obtained from the divergence of the width of the distribution above the critical points. For the two-dimensional voter model, a similar divergence was found [19]; however, for the mean-field BChS model, the width was found to be independent of p above p_c [20]. This feature therefore needs to investigated more by studying other models.

We end with the remark that it would be interesting to compare the present results with other similar models as far as the walk features are concerned. In particular, the behavior on complex topologies is expected to be different, which can also be investigated in the future.

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