Fractional heterogeneous telegraph processes: Interplay between heterogeneity, memory, and stochastic resetting

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Fractional heterogeneous telegraph processes are considered in the framework of telegrapher's equations accompanied by memory effects. The integral decomposition method is developed for the rigorous treating of the problem. Exact solutions for the probability density functions and the mean squared displacements are obtained. A relation between the fractional heterogeneous telegrapher's equation and the corresponding Langevin equation has been established in the framework of the developed subordination approach. The telegraph process in the presence of stochastic resetting has been studied, as well. An exact expression for both the nonequilibrium stationary distributions/states and the mean squared displacements are obtained.

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I. INTRODUCTION

A telegrapher's equation proposed by Kelvin and Heaviside in electrodynamics [1–3] is essentially employed in heat transfer theory and a persistent random walk [4–7]. It has also a particular interest as a master equation of a dichotomous Markov noise [8–11], to mention a few. Seminal results on the telegraph process relate to the detailed treatment of turbulent diffusion [12], and later to studies of a stochastic model [13], see also Ref. [14]. The standard telegrapher's equation reads [15]

$$\tau\frac{\partial^2}{\partial t^2}P_0(x,t)+\frac{\partial}{\partial t}P_0(x,t)=D\frac{\partial^2}{\partial x^2}P_0(x,t), \tag{1}$$

where τ is a time parameter, and D is a diffusion coefficient, which relates to a finite propagation velocity $v=\sqrt{D/\tau}$. Equation (1) combines properties of a hyperbolic (due to the second time derivative) and parabolic equation.[1] In present consideration of random walks of a particle to be at position x at time t, $P_0(x,t)$ is the probability density function (PDF) with the zero boundary conditions at infinity, and the following initial conditions:

$$P_0(x,t=0)=\delta(x-x_0),\qquad \left.\frac{\partial P_0(x,t)}{\partial t}\right|_{t=0}=0. \tag{2}$$

The corresponding Langevin equation takes the form[2]

$$\dot{x}(t)=v\,\zeta(t), \tag{3}$$

where v relates to τ and D and is the same as in Eq. (1), and $\zeta(t)$ is a stationary dichotomic Markov process that jumps between two states ± 1 with the mean rate ν, i.e., the inverse mean sojourn time for each state. The corresponding equation for the PDF $P_0(x,t)$ of such a process is the telegrapher's equation (1), where $\tau=\frac{1}{2\nu}$ and $D=v^2\tau$.

One way of generalization of the standard telegrapher's equation is within the persistent random walk theory yielding the following time fractional telegrapher's equation [3,6,7]

$$\tau^{\mu-1}\frac{\partial^{2\mu}}{\partial t^{2\mu}}P_{0,1}(x,t)+\tau^{-1}\frac{\partial^{\mu}}{\partial t^{\mu}}P_{0,1}(x,t)=D\tau^{-1}\frac{\partial^2}{\partial x^2}P_{0,1}(x,t), \tag{4}$$

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[1]When $\tau\to\infty$ with the fixed velocity $v=\sqrt{D/\tau}$, Eq. (1) becomes a wave equation. In the opposite case when $\tau\to 0$, while $v\to\infty$, Eq. (1) reduces to the diffusion equation with fixed diffusion coefficient $D=v^2\tau$.

[2]The telegrapher's equation (1) can be obtained from the Langevin equation (3) in the framework of the characteristic function approach [10], considering $\zeta(t)$ as a dichotomous Poisson process, and a stochastic differentiation technique [16].

with $0 < \mu < 1$, where $\frac{\partial^\nu}{\partial t^\nu}$ is the Caputo fractional derivative of order ν, defined by [17]

$$\frac{\partial^\nu}{\partial t^\nu} f(t) = \frac{1}{\Gamma(n-\nu)} \int_0^t (t-t')^{n-\nu-1} \frac{d^n}{dt'^n} f(t')\, dt', \qquad (5)$$

where $n-1 < \nu < n$, $n \in N$, and which Laplace transform[3] reads

$$\mathcal{L}\left[\frac{\partial^\nu}{\partial t^\nu} f(t)\right] = s^\nu \hat{f}(s) - \sum_{k=0}^{n-1} s^{\nu-k-1}\left[\lim_{t\to 0} \frac{d^k}{dt^k} f(t)\right]. \qquad (6)$$

It is worth noting that the fractional telegrapher's equation (4) has also been derived from the standard telegrapher's equation (1) by subordination with the Lévy stable process [18,19], and as a special case of a generalized Cattaneo equation with the power-law memory kernels [20].

Another time fractional generalization can be obtained by combining the fractional generalization of the continuity equation and the standard constitutive equation. This yields the following (fractional telegrapher's) equation [20,21]:

$$\frac{\partial^{1+\mu}}{\partial t^{1+\mu}} P_{0,2}(x,t) + \tau^{-1} \frac{\partial^\mu}{\partial t^\mu} P_{0,2}(x,t) = D\tau^{-1} \frac{\partial^2}{\partial x^2} P_{0,2}(x,t), \qquad (7)$$

with $0 < \mu < 1$. Note by passing that the fractional generalization of the continuity equation is connected to the fractional stationarity and a fractional Liouville equation causing a decreasing phase space in statistical systems, as discussed by Hilfer [22,23].

Moreover, the standard telegraph process can be generalized if one introduces a nonlinear Langevin equation with multiplicative dichotomic noise

$$\dot{x}(t) = v(x)\zeta(t), \qquad (8)$$

where $v(x) > 0$ is a position-dependent speed, and $\zeta(t)$ is the same dichotomic process as in Eq. (3), but here it is a multiplicative noise. The corresponding equation for the PDF is a so-called heterogeneous telegrapher's equation (HTE), and for such a heterogeneous telegraph process, the HTP, reads[4]

$$\frac{\partial^2}{\partial t^2} P(x,t) + \tau^{-1} \frac{\partial}{\partial t} P(x,t) = \frac{\partial}{\partial x}\left\{v(x)\frac{\partial}{\partial x}[v(x)P(x,t)]\right\}, \qquad (9)$$

where $v(x) = \sqrt{D(x)/\tau}$ [15], see also [24,25]. In Refs. [25,26], the inhomogeneous diffusivity/advection has been chosen in the power law form, when $D(x) = D_\alpha |x|^\alpha$, $\alpha < 2$.

Such heterogeneous telegrapher's equations have been used in cosmic-ray transport [27] and in the description of turbulent diffusion [28–30], and turbulent relative dispersion of particle pairs [31,32]. They are also employed for generalizations of the Richardson model [33] by taking into consideration the long-time velocity correlations in turbulent flows [34,35].

In Ref. [25], by using the integral decomposition method [18,36–39], it has been shown that the formal solution of the HTE (9) is given by

$$P(x,t) = \int_0^\infty h(u,t) p(x,u) du, \qquad (10)$$

where the decomposition function $h(u,t)$ in Laplace space reads

$$\hat{h}(u,s) = \frac{1}{\hat{K}_0(s)} e^{-\frac{us}{\hat{K}_0(s)}}, \quad \hat{K}_0(s) = \frac{1}{\tau s + 1}. \qquad (11)$$

Here $p(x,t)$ is the solution of the corresponding Fokker-Planck equation for the heterogeneous diffusion equation (with $\tau = 0$) [17,26,40–42],

$$\frac{\partial}{\partial t} p(x,t) = \frac{\partial}{\partial x}\left\{\sqrt{D(x)} \frac{\partial}{\partial x}[\sqrt{D(x)} p(x,t)]\right\}. \qquad (12)$$

By this approach the heterogeneous telegrapher's equation can be rewritten in the form

$$\frac{\partial}{\partial t} P(x,t) = \int_0^t K_0(t-t') \times \frac{\partial}{\partial x}\left\{\sqrt{D(x)} \frac{\partial}{\partial x}[\sqrt{D(x)} p(x,t')]\right\} dt', \qquad (13)$$

where $K_0(t) = \tau^{-1} e^{-t/\tau}$. This method was used by Sokolov [36] to obtain the standard telegrapher's equation from the diffusion equation by using the exponential memory kernel. Such exponential kernel was classified as "dangerous" since the decomposition function $\hat{h}(u,s)$ is not a completely monotone function. Moreover, in Ref. [36], the memory kernels for which $\hat{h}(u,s)$ is completely monotone were named "safe" kernels, and the decomposition method is known as subordination, while the decomposition function $h(u,t)$ as a subordination function.

The solution is obtained in Ref. [26] and it reads

$$p(x,t) = \frac{|x|^{1/\rho - 1}}{\sqrt{4\pi D_\alpha t}} \times \exp\left(-\frac{\rho^2 \left|\mathrm{sgn}(x)\,|x|^{1/\rho} - \mathrm{sgn}(x_0)\,|x_0|^{1/\rho}\right|^2}{4D_\alpha t}\right), \qquad (14)$$

where $\rho = \frac{2}{2-\alpha}$, and when $x_0 = 0$, it reduces to the solution [40]

$$p(x,t) = \frac{|x|^{1/\rho - 1}}{\sqrt{4\pi D_\alpha t}} \exp\left(-\frac{\rho^2 |x|^{2/\rho}}{4D_\alpha t}\right). \qquad (15)$$

Therefore, the PDF in Laplace space becomes

$$\hat{P}(x,s) = \int_0^\infty \hat{h}(u,s) p(x,u) du = \int_0^\infty p(x,u) \frac{1}{\hat{K}_0(s)} e^{-\frac{us}{\hat{K}_0(s)}} du = \frac{1}{\hat{K}_0(s)} \hat{p}\left(x, \frac{s}{\hat{K}_0(s)}\right), \qquad (16)$$

[3]The Laplace transform of a given function $f(t)$ reads $\hat{f}(s) = \int_0^\infty e^{-st} f(t)\, dt$.

[4]A rigorous analysis has been performed in Ref. [15].

where

$$\hat{p}(x,s)=\frac{|x|^{1/\rho-1}}{\sqrt{4D_\alpha}}s^{-1/2}\times\exp\left(-\frac{\rho\left|\mathrm{sgn}(x)\,|x|^{1/\rho}-\mathrm{sgn}(x_0)\,|x_0|^{1/\rho}\right|}{\sqrt{D_\alpha}}s^{1/2}\right). \tag{17}$$

From the general solution (16) for the PDF, we can find a general form of the mean squared displacement (MSD), which in Laplace space is

$$\langle\hat{x}^2(s)\rangle_0=\frac{1}{\hat{K}_0(s)}\left\langle\hat{x}^2\left(\frac{s}{\hat{K}_0(s)}\right)\right\rangle, \tag{18}$$

where $\langle x^2(t)\rangle$ is the MSD of the heterogeneous diffusion process, described by the Fokker-Planck equation (12), see [26],

$$\langle x^2(t)\rangle=\frac{\Gamma(1+2\rho)}{\rho^{2\rho}}\frac{(D_\alpha t)^\rho}{\Gamma(1+\rho)}\,{}_1F_1\left(-\rho,\frac{1}{2},-\rho^2\frac{|x_0|^{2/\rho}}{4\mathcal{D}_\alpha t}\right), \tag{19}$$

where ${}_1F_1(a,b,z)$ is the confluent hypergeometric function of the first kind [43] for $x_0\neq 0$. For the long time limit, the asymptotic expansion of (19), yields

$$\langle x^2(t)\rangle\sim C_1(\rho)(\mathcal{D}_\alpha t)^\rho+C_2(\rho)|x_0|^{2/\rho}(\mathcal{D}_\alpha t)^{\rho-1}, \tag{20}$$

where $C_1(\rho)=\frac{\Gamma(1+2\rho)}{\rho^{2\rho}\Gamma(1+\rho)}$ and $C_2(\rho)=\frac{\rho^3}{2}C_1(\rho)$. Since $\rho>0$, the MSD reads

$$\langle x^2(t)\rangle\sim\frac{\Gamma(1+2\rho)}{\rho^{2\rho}}\frac{(\mathcal{D}_\alpha t)^\rho}{\Gamma(1+\rho)}$$

for large times. For the short time limit, the MSD corresponds to diffusion

$$\langle x^2(t)\rangle\sim|x_0|^2+2(2\rho-1)/\rho\,|x_0|^{2(\rho-1)/\rho}\,\mathcal{D}_\alpha t.$$

For $\alpha=0$ ($\rho=1$) we recover the result for ordinary Brownian motion $\langle x^2(t)\rangle=x_0^2+2\mathcal{D}_0 t$.

For the limiting case $x_0=0$ [and thus ${}_1F_1(a,b,0)=1$], the MSD (19) is the known result [40]

$$\langle x^2(t)\rangle=\frac{\Gamma(1+2\rho)}{\rho^{2\rho}}\frac{(D_\alpha t)^\rho}{\Gamma(1+\rho)}. \tag{21}$$

In Laplace space, one has

$$\langle\hat{x}^2(s)\rangle=\frac{\Gamma(1+2\rho)}{\rho^{2\rho}}D_\alpha^\rho\,s^{-\rho-1}, \tag{22}$$

and thus

$$\langle x^2(t)\rangle_0=\frac{\Gamma(1+2\rho)}{\rho^{2\rho}}\left(\frac{D_\alpha}{\tau}\right)^\rho\mathcal{L}^{-1}\left[\frac{s^{-\rho-1}}{(s+\tau^{-1})^\rho}\right]=\frac{\Gamma(1+2\rho)}{\rho^{2\rho}}(D_\alpha\tau)^\rho\left(\frac{t}{\tau}\right)^{2\rho}E^\rho_{1,2\rho+1}\left(-\frac{t}{\tau}\right), \tag{23}$$

where $E^\delta_{\alpha,\beta}(z)$ is the three parameter Mittag-Leffler function defined by [44]

$$E^\gamma_{\alpha,\beta}(z)=\sum_{k=0}^\infty\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{z^k}{k!}, \tag{24}$$

with $\beta,\gamma,z\in C$, $\Re(\alpha)>0$, $(\gamma)_k=\frac{\Gamma(\gamma+k)}{\Gamma(\gamma)}$ is the Pochhammer symbol. Its Laplace transform reads

$$\mathcal{L}\left[t^{\beta-1}E^\gamma_{\alpha,\beta}(\pm\lambda t^\alpha)\right]=\frac{s^{\alpha\gamma-\beta}}{(s^\alpha\mp\lambda)^\gamma}, \tag{25}$$

where $|\lambda/s^\alpha|<1$. Its asymptotic behavior for large z can be found from the following formula for $0<\alpha<2$ [17,45]:

$$E^\gamma_{\alpha,\beta}(-z)=\frac{z^{-\gamma}}{\Gamma(\gamma)}\sum_{n=0}^\infty\frac{\Gamma(\gamma+n)}{\Gamma(\beta-\alpha(\gamma+n))}\frac{(-z)^{-n}}{n!},\quad z>1, \tag{26}$$

which gives

$$E^\gamma_{\alpha,\beta}(-z)\sim\frac{z^{-\gamma}}{\Gamma(\beta-\alpha\gamma)}-\gamma\frac{z^{-(\gamma+1)}}{\Gamma(\beta-\alpha(\gamma+1))},\quad z\gg 1. \tag{27}$$

II. FRACTIONAL HTE

We consider two different generalized forms of the HTE. The first one is a generalization of the fractional telegrapher's equation (4) and the second one is a generalization of the fractional telegrapher's equation (7), in which we introduce a position dependent velocity.

A. Case 1

The first case corresponds to the generalized HTE[5] for the PDF $P_1(x,t)$,

$$\tau^{\mu-1}\frac{\partial^{2\mu}}{\partial t^{2\mu}}P_1(x,t)+\tau^{-1}\frac{\partial^\mu}{\partial t^\mu}P_1(x,t)=\frac{\partial}{\partial x}\left\{\sqrt{D(x)/\tau}\frac{\partial}{\partial x}\left[\sqrt{D(x)/\tau}P_1(x,t)\right]\right\}. \tag{28}$$

We consider zero boundary conditions at infinity, and the initial conditions are defined in Eq. (2). By first using the Laplace and then inverse Laplace transform, we can rewrite this equation in the form

$$\frac{\partial}{\partial t}P_1(x,t)=\int_0^t K_1(t-t')\times\frac{\partial}{\partial x}\left\{\sqrt{D(x)}\frac{\partial}{\partial x}[\sqrt{D(x)}P_1(x,t)]\right\}dt', \tag{29}$$

where

$$\hat{K}_1(s)=\frac{1}{\tau^\mu}\frac{s^{1-\mu}}{s^\mu+\tau^{-\mu}}, \tag{30}$$

and thus

$$K_1(t)=\frac{t^{2\mu-2}}{\tau^\mu}E_{\mu,2\mu-1}\left(-\left[\frac{t}{\tau}\right]^\mu\right). \tag{31}$$

The next step of the analysis is convenient to perform in the framework of the integral decomposition, according to integration (10), when the PDF $P_1(x,t)$, is decomposed through

[5]We justify the introduction of such heterogeneous time fractional telegrapher's equation in Sec. III, by using the subordination approach.

TABLE I. Characteristic crossover regimes obtained by asymptotic analysis of the MSD (35).

	MSD – short time behavior	MSD – long time behavior
	$\langle x^2(t)\rangle \sim t^{\beta_1}$, $\beta_1 = 4\mu/(2-\alpha)$	$\langle x^2(t)\rangle \sim t^{\beta_2}$, $\beta_2 = \beta_1/2$
$2-2\mu < \alpha < 2$	$\beta_1 > 2$ – hyperdiffusion	$\beta_2 > 1$ – superdiffusion
$\alpha = 2-2\mu$	$\beta_1 = 2$ – ballistic motion	$\beta_2 = 1$ – normal diffusion
$2-4\mu < \alpha < 2-2\mu$	$1 < \beta_1 < 2$ – superdiffusion	$1/2 < \beta_2 < 1$ – subdiffusion
$\alpha = 2-4\mu$	$\beta_1 = 1$ – normal diffusion	$\beta_2 = 1/2$ – subdiffusion
$\alpha < 2-4\mu$	$0 < \beta_1 < 1$ – subdiffusion	$0 < \beta_2 < 1/2$ – subdiffusion

the PDF $p(x,t)$. The latter is a solution to the conventional Fokker-Planck equation (12). In this case, the function $h(u,t)$, describes a relation between the time variable t, and the new variable u. Thus, we have

$$\hat{P}_1(x,s) = \frac{1}{\hat{K}_1(s)}\hat{p}\left(x, \frac{s}{\hat{K}_1(s)}\right), \tag{32}$$

where $p(x,t)$ is the solution of the corresponding Fokker-Planck equation (12) for the heterogeneous diffusion process.

From the general solution for the PDF, we can find a general form of the MSD, i.e.,

$$\langle \hat{x}^2(s)\rangle_1 = \frac{1}{\hat{K}_1(s)}\left\langle \hat{x}^2\left(\frac{s}{\hat{K}_1(s)}\right)\right\rangle, \tag{33}$$

where $\langle x^2(t)\rangle$ is the MSD of the heterogeneous diffusion process.

From Eqs. (33) and (22), the MSD reads

$$\langle \hat{x}^2(s)\rangle_1 = \frac{\Gamma(1+2\rho)}{\rho^{2\rho}}\left(\frac{D_\alpha}{\tau^\mu}\right)^\rho \frac{s^{-\rho\mu-1}}{(s^\mu+\tau^{-\mu})^\rho}, \tag{34}$$

from where we obtain

$$\langle x^2(t)\rangle_1 = \frac{\Gamma(1+2\rho)(D_\alpha\tau^\mu)^\rho}{\rho^{2\rho}} \times \left(\frac{t}{\tau}\right)^{2\rho\mu} E^{\rho}_{\mu,2\rho\mu+1}\left(-\left[\frac{t}{\tau}\right]^\mu\right). \tag{35}$$

Note that for $\mu = 1$ we recover the MSD obtained in [25], while for $\rho = 1$ ($\alpha = 0$) the result has been obtained in [20]. The MSD for the standard telegrapher's equation is obtained for $\mu = 1$ and $\alpha = 0$, i.e., $\rho = 1$, [46]

$$\langle x^2(t)\rangle_1 = 2D_0\tau\left(\frac{t}{\tau}\right)^2 E_{1,3}\left(-\frac{t}{\tau}\right) = 2D_0\tau\left(\frac{t}{\tau} + e^{-t/\tau} - 1\right), \tag{36}$$

where $E_{\alpha,\beta}(z) = E^1_{\alpha,\beta}(z)$ is the two parameter Mittag-Leffler function.

By asymptotic analysis of the MSD, we obtain that there is a characteristic crossover dynamics from behavior $\langle x^2(t)\rangle_1 \sim t^{4\mu/(2-\alpha)}$ in the short time limit to $\langle x^2(t)\rangle_1 \sim t^{2\mu/(2-\alpha)}$ in the long time limit. Therefore, one observes a very rich behavior of the system and the resulting behavior depends on parameters μ and α, i.e., there is an interplay between the heterogeneity and memory. We note that, as a special case, for $2\mu = 2-\alpha$, there is a characteristic crossover from ballistic motion to normal diffusion, as in the case of the standard telegrapher's equation. For $4\mu = 2-\alpha$, there is normal diffusion in the short time limit, while subdiffusion with the anomalous diffusion exponent 1/2 takes place in the long time limit. For $\alpha = 0$, we recover the known result for the fractional telegrapher's equation, where the MSD in the short time limit behaves as $\sim t^{2\mu}$ and it turns to $\sim t^\mu$ in the long time limit [20]. That is, there is a crossover from superdiffusion to subdiffusion for $1/2 < \mu < 1$ and from subifusion to subdiffusion for $0 < \mu < 1/2$. For $\mu = 1$ the crossover from ballistic motion to normal diffusion is obtained, as expected for the standard telegrapher's equation. The characteristic crossover regimes are summarized in Table I. Noticeable change of the transport exponents is observed in the MSD plots in Fig. 1.

Note that, for $\tau = 0$, Eq. (28) reduces to the generalized (fractional) heterogeneous diffusion equation of the form

$$\frac{\partial^\mu}{\partial t^\mu}P_1(x,t) = \frac{\partial}{\partial x}\left\{\sqrt{D(x)}\frac{\partial}{\partial x}[\sqrt{D(x)}P_1(x,t)]\right\}, \tag{37}$$

with solution

$$\hat{P}_1(x,s) = s^{\mu-1}\hat{p}(x,s^\mu), \tag{38}$$

and MSD

$$\langle x^2(t)\rangle_1 = \frac{\Gamma(1+2\rho)}{\rho^{2\rho}}D^\rho_\alpha\frac{t^{\rho\mu}}{\Gamma(1+\rho\mu)}. \tag{39}$$

For $\mu = 1$, we recover the results for the heterogeneous diffusion process.

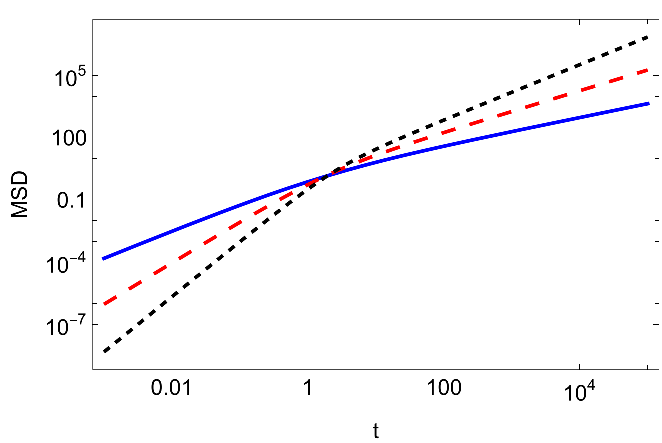

FIG. 1. MSD (35) for $\tau = 1$, $D_\alpha = 1$, $\alpha = 1/2$ and $\mu = 1/2$ (blue solid line), $\mu = 3/4$ (red dashed line), $\mu = 1$ (black dotted line). Red dashed line corresponds to the crossover from ballistic motion to normal diffusion.

B. Case 2

The second case is described by the following generalized HTE for the PDF $P_2(x,t)$,

$$\frac{\partial^{1+\mu}}{\partial t^{1+\mu}}P_2(x,t)+\tau^{-1}\frac{\partial^{\mu}}{\partial t^{\mu}}P_2(x,t) = \frac{\partial}{\partial x}\left\{\sqrt{D(x)/\tau}\frac{\partial}{\partial x}\left[\sqrt{D(x)/\tau}P_2(x,t)\right]\right\}. \quad (40)$$

We again consider zero boundary conditions at infinity, and initial conditions of the form (2). We note that for $\tau\to 0$ it transforms to the generalized heterogeneous diffusion equation (37). By means of the Laplace and inverse Laplace transform, we can rewrite this equation as follows

$$\frac{\partial}{\partial t}P_2(x,t)=\int_0^t K_2(t-t') \times\frac{\partial}{\partial x}\left\{\sqrt{D(x)}\frac{\partial}{\partial x}[\sqrt{D(x)}P_2(x,t)]\right\}dt', \quad (41)$$

where

$$\hat{K}_2(s)=\frac{1}{\tau}\frac{s^{1-\mu}}{s+\tau^{-1}} \quad\rightarrow\quad K_2(t)=\frac{t^{\mu-1}}{\tau}E_{1,\mu}\left(-\frac{t}{\tau}\right). \quad (42)$$

Following the same procedure of the integral decomposition, one finds

$$\hat{P}_2(x,s)=\frac{1}{\hat{K}_2(s)}\hat{p}\left(x,\frac{s}{\hat{K}_2(s)}\right), \quad (43)$$

where $p(x,t)$ is the solution of the Fokker-Planck equation (12). Therefore, for the MSD we find

$$\langle\hat{x}^2(s)\rangle_2=\frac{1}{\hat{K}_2(s)}\left\langle\hat{x}^2\left(\frac{s}{\hat{K}_2(s)}\right)\right\rangle, \quad (44)$$

where $\langle x^2(t)\rangle$ is the MSD (19) or (21) of the heterogeneous diffusion process with $x_0\neq 0$ or $x_0=0$, respectively. From Eqs. (44) and (22), for the MSD in Laplace space we have

$$\langle\hat{x}^2(s)\rangle_2=\frac{\Gamma(1+2\rho)}{\rho^{2\rho}}\left(\frac{D_\alpha}{\tau}\right)^{\rho}\frac{s^{-\rho\mu-1}}{(s+\tau^{-1})^{\rho}}, \quad (45)$$

from where, by the inverse Laplace transform, we obtain

$$\langle x^2(t)\rangle_2=\frac{\Gamma(1+2\rho)(D_\alpha\tau^{\mu})^{\rho}}{\rho^{2\rho}} \times\left(\frac{t}{\tau}\right)^{(\mu+1)\rho}E^{\rho}_{1,(\mu+1)\rho+1}\left(-\frac{t}{\tau}\right). \quad (46)$$

Note that for $\mu=1$ we recover the MSD obtained in [25], while for $\rho=1$ ($\alpha=0$) the result obtained in [20]. The MSD for the standard telegrapher's equation is obtained for $\mu=1$ and $\alpha=0$ ($\rho=1$). According to the asymptotic analysis of the MSD (46), there is a characteristic crossover dynamics from the MSD $\langle x^2(t)\rangle_2\sim t^{\beta_1}$ with the transport exponent $\beta_1=2(\mu+1)/(2-\alpha)$ in the short time limit to the MSD $\langle x^2(t)\rangle_2\sim t^{\beta_2}$ with the transport exponent $\beta_2=2\mu/(2-\alpha)=\beta_1-2/(2-\alpha)$ for the long time limit. Note that this crossover dynamics is more sophisticated than the one observed for the MSD in Eq. (35). Therefore, very reach diffusive behavior emerges here as well. However, there is no crossover from ballistic motion to normal diffusion. For example, if $2(\mu+1)/(2-\alpha)=2$ in the short time limit we have ballistic motion, but for large times it changes to subdiffusive behavior $\langle x^2(t)\rangle_2\sim t^{2\mu/(1+\mu)}$, while for $2\mu/(2-\alpha)=1$, it changes from superdiffusive behavior $\langle x^2(t)\rangle_2\sim t^{1+1/\mu}$ for small times to normal diffusion for large times. This situation is reflected in Fig. 2.

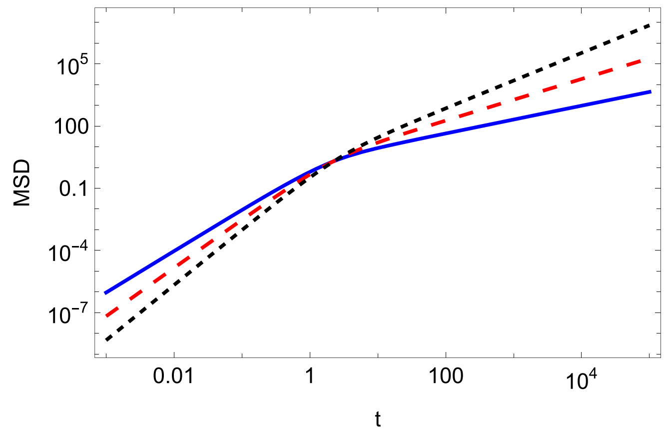

FIG. 2. MSD (46) for $\tau=1$, $D_\alpha=1$, $\alpha=1/2$ and $\mu=1/2$ (blue solid line), $\mu=3/4$ (red dashed line), $\mu=1$ (black dotted line). Red dashed line describes the crossover from hyperdiffusion with the slope $=7/3$ to normal diffusion with the slope $=1$.

III. GENERALIZED HTP AS A SUBORDINATED HTP

In this section, we show that the first form of the generalized HTP can be obtained as a subordinated HTP. Let us consider the following coupled Langevin equations:

$$\frac{dx(u)}{du}=v(u)\zeta(u), \qquad \frac{dt(u)}{du}=\xi(u), \quad (47)$$

where $\zeta(u)$ represents the same multiplicative dichotomic noise as in Eq. (8), and $\xi(u)$ is a Lévy stable noise with Lévy index in Laplace space given by $\hat{\Psi}(s)=\tau^{\mu-1}s^{\mu}$, $0<\mu<1$. Therefore, the process $t(u)=\int_0^u\xi(u')\,du'$ is a stable Lévy motion with characteristic function $\hat{L}(u,s)=e^{-u\tau^{\mu-1}s^{\mu}}$. The time t is known as physical time, while u as operational time. The corresponding PDF $P_s(x,t)$ of this subordinated HTP can be found from the subordination integral [18,19,47–50]

$$P_s(x,t)=\int_0^\infty P(x,u)h(u,t)\,du, \quad (48)$$

where $P(x,u)$ is the solution of Eq. (9), and $h(u,t)$ is the subodination function [37,49][6]

$$h(u,t)=-\frac{\partial}{\partial u}\langle\Theta(t-t(u))\rangle, \quad (49)$$

[6]As we mentioned before, when the function $h(u,t)$ is a PDF of the variable u (in such a case u is called operational time) at physical time t then the integral decomposition is known as subordination [18,36].

which in Laplace space reads

$$\hat{h}(u,s) = -\frac{1}{s}\frac{\partial}{\partial u}\left\langle \int_0^\infty \delta(t-t(u))e^{-st}\,dt \right\rangle = -\frac{1}{s}\frac{\partial}{\partial u}\langle e^{-st(u)}\rangle = -\frac{1}{s}\frac{\partial}{\partial u}\hat{L}(u,s) = \tau^{\mu-1}s^{\mu-1}e^{-u\tau^{\mu-1}s^{\mu}}. \tag{50}$$

Therefore, from Eqs. (48) and (50), for the PDF we have

$$\hat{P}_s(x,s) = \tau^{\mu-1}s^{\mu-1}\hat{P}(x,\tau^{\mu-1}s^{\mu}). \tag{51}$$

By the Laplace transform of Eq. (9), one obtains

$$s^2\hat{P}(x,s) - s\delta(x-x_0) + \frac{1}{\tau}[s\hat{P}(x,s) - \delta(x-x_0)] = \frac{\partial}{\partial x}\left\{\sqrt{D(x)/\tau}\frac{\partial}{\partial x}\left[\sqrt{D(x)/\tau}\hat{P}(x,s)\right]\right\}, \tag{52}$$

where the initial conditions are of the form (2). Introducing the change of the variables $s \to \tau^{\mu-1}s^{\mu}$ and $D(x)\tau^{1-\mu} \to D(x)$, Eq. (52) becomes

$$(\tau^{\mu}s^{\mu}+1)[\tau^{\mu-1}s^{\mu}\hat{P}(x,\tau^{\mu-1}s^{\mu}) - \delta(x-x_0)] = \frac{\partial}{\partial x}\left\{\sqrt{D(x)}\frac{\partial}{\partial x}[\sqrt{D(x)}\hat{P}(x,\tau^{\mu-1}s^{\mu})]\right\}. \tag{53}$$

By substituting Eqs. (51) in (53), we obtain

$$s^{\mu-1}(\tau^{\mu}s^{\mu}+1)[s\hat{P}_s(x,s) - \delta(x-x_0)] = \frac{\partial}{\partial x}\left\{\sqrt{D(x)}\frac{\partial}{\partial x}[\sqrt{D(x)}\hat{P}_s(x,s)]\right\}. \tag{54}$$

Then by the inverse Laplace transform we arrive at Eq. (28) for the PDF $P_1(x,t)$, i.e., $P_s(x,t) = P_1(x,t)$.

From Eq. (51), it follows that the MSD reads

$$\langle \hat{x}^2(s)\rangle_s = \tau^{\mu-1}s^{\mu-1}\langle \hat{x}^2(\tau^{\mu-1}s^{\mu})\rangle_0, \tag{55}$$

where $\langle x^2(t)\rangle_0$ is the MSD (23) for the HTP. Therefore, we obtain the same MSD as in Eq. (35), as expected.

IV. EFFECTS OF STOCHASTIC RESETTING

The recent experimental realizations of diffusion and first-passage under exponential resetting, using holographic optical tweezers [51] and laser traps [52], have initiated further investigations of various stochastic processes under resetting, including heterogeneous diffusion [26,53] and heterogeneous telegrapher's processes [25]. Therefore, the heterogeneous telegrapher's models under resetting can be useful in description of a random search in a turbulent environment. In this context, run-and-tumble particle motion under resetting [54–57] and different models of Lévy walks under resetting [58] are of interest nowadays.

Next, we study the generalized heterogeneous telegraph processes under stochastic resetting. This means that the particle after a random time interval is reset to its initial position. As a simplest case we consider a Poissonian resetting with the resetting time PDF $p(\tau) = r\,e^{-r\tau}$, where r is the resetting rate [59–62]. Thus, the PDF $P_{r,i}(x,t|x_0)$ of the generalized HTP can be expressed in terms of the PDF $P_i(x,t)$ of corresponding process without resetting via the renewal equation [57,60,63–65]

$$P_{r,i}(x,t|x_0) = e^{-rt}P_i(x,t) + \int_0^t re^{-rt'}P_i(x,t')\,dt'. \tag{56}$$

Here $i = 1, 2$ are for Case 1 and 2, respectively. This means that each reset to the initial position x_0 renews the process at the rate r, while between any two consecutive renewal events, the particle undergoes a generalized heterogeneous telegraph process. The first term on the right-hand side of the equation corresponds to the case when there is no any reset event up to time t, while the second term describes multiple reset events.

The PDFs in case of resetting can be easily calculated in Laplace space. By the Laplace transform of Eq. (56), one finds

$$\hat{P}_{r,i}(x,s|x_0) = \frac{s+r}{s}\hat{P}_i(x,s+r) = \frac{s+r}{s}\frac{1}{\hat{K}_i(s+r)}\hat{p}\left(x,\frac{s+r}{\hat{K}_i(s+r)}\right). \tag{57}$$

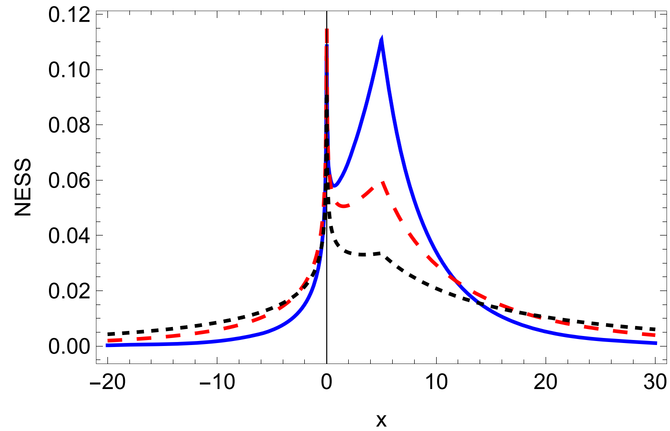

FIG. 3. NESS (62) for $x_0 = 5$, $\tau = 1$, $D_\alpha = 1$, $r = 0.01$, $\alpha = 1/2$ and $\mu = 1/2$ (blue solid line), $\mu = 3/4$ (red dashed line), $\mu = 1$ (black dotted line).

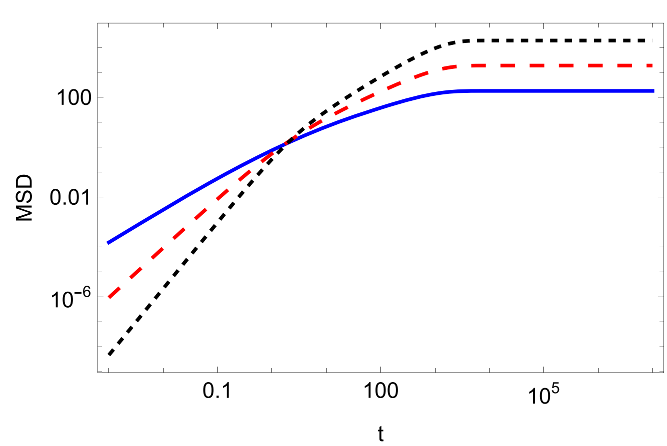

FIG. 4. MSD (64) for $\tau = 1$, $D_\alpha = 1$, $r = 0.001$, $\alpha = 1/2$ and $\mu = 1/2$ (blue solid line), $\mu = 3/4$ (red dashed line), $\mu = 1$ (black dotted line).

In the long time limit, the system approaches a nonequilibrium stationary state (NESS) given by

$$P^{\rm st}_{r,i}(x) = \lim_{t\to\infty} P_{r,i}(x,t|x_0) = \lim_{s\to 0} s\hat{P}_{r,i}(x,s|x_0) = r\hat{P}_i(x,r) = \frac{r}{\hat{K}_i(r)}\hat{p}\left(x, \frac{r}{\hat{K}_i(r)}\right). \quad (58)$$

Respectively, the MSD in case of resetting can be obtained from the corresponding MSD in case of no resetting, i.e.,

$$\langle \hat{x}^2(s)\rangle_{r,i} = \frac{s+r}{s}\langle \hat{x}^2(s+r)\rangle_i = \frac{s+r}{s}\frac{1}{\hat{K}_i(s+r)}\left\langle \hat{x}^2\left(\frac{s+r}{\hat{K}_i(s+r)}\right)\right\rangle, \quad (59)$$

where $\langle \hat{x}^2(s)\rangle$ is given by Eq. (22). In the long time limit it is saturated to

$$\lim_{t\to\infty}\langle x^2(t)\rangle_{r,i} = \frac{r}{\hat{K}_i(r)}\left\langle \hat{x}^2\left(\frac{r}{\hat{K}_i(r)}\right)\right\rangle. \quad (60)$$

A. Case 1

For the telegraph process governed by Eq. (28) under resetting, the PDF reads

$$\hat{P}_{r,i}(x,s|x_0) = \tau^\mu s^{-1}(s+r)^\mu[(s+r)^\mu + \tau^{-\mu}] \times \hat{p}(x, \tau^\mu(s+r)^\mu[(s+r)^\mu+\tau^{-\mu}]). \quad (61)$$

From here the NESS becomes

$$P^{\rm st}_{r,i}(x) = \frac{|x|^{1/\rho-1}}{2\sqrt{D_\alpha/r^\mu}}\sqrt{r^\mu\tau^\mu+1}\exp\left(-\frac{\rho|\mathrm{sgn}(x)\,|x|^{1/\rho} - \mathrm{sgn}(x_0)\,|x_0|^{1/\rho}|\sqrt{r^\mu\tau^\mu+1}}{\sqrt{D_\alpha/r^\mu}}\right). \quad (62)$$

Graphical representation of the NESS (62) is given in Fig. 3.

For the MSD in Laplace space, we have

$$\langle \hat{x}^2(s)\rangle_{r,1} = \frac{\Gamma(1+2\rho)}{\rho^{2\rho}}\left(\frac{D_\alpha}{\tau^\mu}\right)^\rho s^{-1}\frac{(s+r)^{-\rho\mu}}{[(s+r)^\mu+\tau^{-\mu}]^\rho}. \quad (63)$$

The inverse Laplace transform of the expression yields

$$\langle x^2(t)\rangle_{r,1} = \frac{\Gamma(1+2\rho)}{\rho^{2\rho}}(D_\alpha\tau^\mu)^\rho \times \frac{1}{\tau}\int_0^t e^{-rt'}\left(\frac{t'}{\tau}\right)^{2\rho\mu-1}E^\rho_{\mu,2\rho\mu}\left(-\left[\frac{t'}{\tau}\right]^\mu\right)dt'. \quad (64)$$

In the short time limit it behaves as $\langle x^2(t)\rangle_{r,1} \sim t^{4\mu/(2-\alpha)}$ as in the case without resetting, while in the long time limit it saturates,

$$\langle x^2(t)\rangle_{r,1} = \frac{\Gamma(1+2\rho)}{\rho^{2\rho}}\frac{(D_\alpha/r^\mu)^\rho}{(r^\mu\tau^\mu+1)^\rho}. \quad (65)$$

Therefore, there is an interplay between heterogeneity, memory, and stochastic resetting, represented by the parameters α, μ, and r, respectively. In the short time limit the particle does not feel the resetting mechanism and the MSD behaves as in the case without resetting. Thus, in the short time limit the MSD depends only on α and μ. Graphical representation of the MSD (64) for fixed α and r and different μ is given in Fig. 4.

For $\tau = 0$, i.e., for the generalized heterogeneous diffusion process, we have

$$\hat{p}_{r,i}(x,s|x_0) = s^{-1}(s+r)^\mu\hat{p}(x,(s+r)^\mu), \quad (66)$$

and the NESS becomes

$$p^{\rm st}_{r,i}(x) = \frac{|x|^{1/\rho-1}}{2\sqrt{D_\alpha/r^\mu}} \times \exp\left(-\frac{\rho\left|\mathrm{sgn}(x)\,|x|^{1/\rho} - \mathrm{sgn}(x_0)\,|x_0|^{1/\rho}\right|}{\sqrt{D_\alpha/r^\mu}}\right). \quad (67)$$

For the MSD then we have

$$\langle x^2(t)\rangle_{r,1} = \frac{\Gamma(1+2\rho)}{\rho^{2\rho}}\frac{D_\alpha^\rho}{r^{\rho\mu}}\frac{\gamma(\rho\mu,rt)}{\Gamma(\rho\mu)}, \quad (68)$$

where $\gamma(a,z)$ is the lower incomplete gamma function.

B. Case 2

For the telegraph process described by Eq. (9), with resetting, we obtain that the PDF is

$$\hat{P}_{r,2}(x,s|x_0) = \tau s^{-1}(s+r)^\mu[(s+r)+\tau^{-1}] \times \hat{p}(x, \tau(s+r)^\mu[(s+r)+\tau^{-1}]). \quad (69)$$

Then the corresponding NESS observed in the long time limit becomes

$$P^{\rm st}_{r,2}(x) = \frac{|x|^{1/\rho-1}}{2\sqrt{D_\alpha/r^\mu}}\sqrt{r\tau+1}\exp\left(-\frac{\rho\left|\mathrm{sgn}(x)\,|x|^{1/\rho} - \mathrm{sgn}(x_0)\,|x_0|^{1/\rho}\right|\sqrt{r\tau+1}}{\sqrt{D_\alpha/r^\mu}}\right). \quad (70)$$

Graphical representation of the NESS (70) is given in Fig. 5.

The MSD in Laplace space has the form

$$\langle \hat{x}^2(s)\rangle_{r,2} = \frac{\Gamma(1+2\rho)}{\rho^{2\rho}}\left(\frac{D_\alpha}{\tau}\right)^\rho s^{-1}\frac{(s+r)^{-\rho\mu}}{[(s+r)+\tau^{-1}]^\rho}. \quad (71)$$

Thus, we obtain

$$\langle x^2(t)\rangle_{r,2} = \frac{\Gamma(1+2\rho)}{\rho^{2\rho}}(D_\alpha\tau^\mu)^\rho \times \frac{1}{\tau}\int_0^t e^{-rt'}\left(\frac{t'}{\tau}\right)^{(1+\mu)\rho-1}E^\rho_{1,(1+\mu)\rho}\left(-\left[\frac{t'}{\tau}\right]\right)dt'. \quad (72)$$

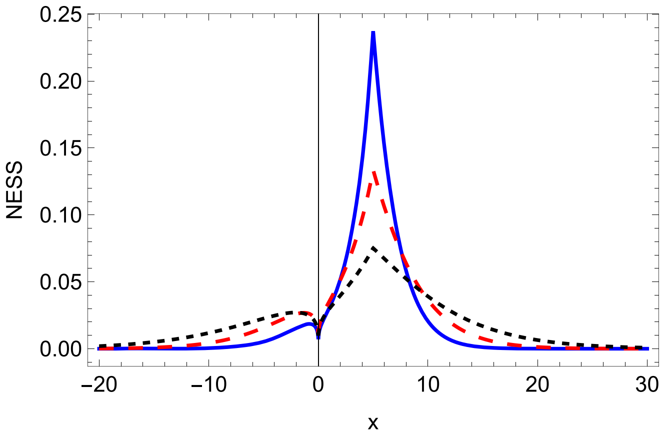

FIG. 5. NESS (70) for $x_0 = 5$, $\tau = 1$, $D_\alpha = 1$, $r = 0.01$, $\alpha = -1/2$ and $\mu = 1/2$ (blue solid line), $\mu = 3/4$ (red dashed line), $\mu = 1$ (black dotted line).

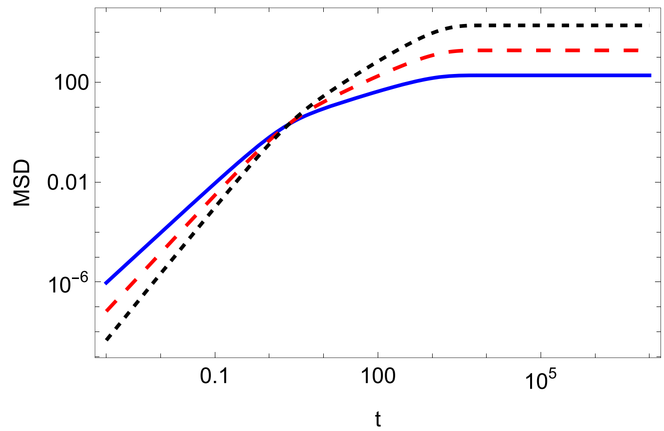

FIG. 6. MSD (72) for $\tau = 1$, $D_\alpha = 1$, $r = 0.001$, $\alpha = 1/2$ and $\mu = 1/2$ (blue solid line), $\mu = 3/4$ (red dashed line), $\mu = 1$ (black dotted line).

In the short time limit, the MSD (72) behaves as $\langle x^2(t)\rangle_{r,2} \sim t^{2(1+\mu)/(2-\alpha)}$, as in the case without resetting, while in the long time limit it saturates due to the resetting. Therefore, there is a complex interplay between heterogeneity (α), memory (μ), and resetting (r). Graphical representation of the MSD (72) is given in Fig. 6. We also note that for $\tau = 0$ we obtain $p_{r,2}(x, t|x_0) = p_{r,1}(x, t|x_0)$ and $\langle x^2(t)\rangle_{r,2} = \langle x^2(t)\rangle_{r,1}$, as expected.

V. SUMMARY

We suggested two different generalizations of the heterogeneous telegraph process by introducing a memory in the corresponding telegrapher's equations for the PDF, and the corresponding solutions are obtained as well. We also analyzed the MSD, and exact results are obtained in terms of the Mittag-Leffler functions. The asymptotic behavior in both the short and long time limits are analyzed and various characteristics of the crossover dynamics are observed. We have shown that there is a competition between the heterogeneity described by the power law in the range-dependent diffusivity, and the memory, described by the fractional derivative with the power law memory kernel. We also have shown that for Case 1, the generalized HTP represents a subordinated HTP, by introducing an operational time. The two generalized telegraph process in the presence resetting are studied, as well. We showed that in the presence of resetting, the particle approaches NESSs in the long time limit. Exact expressions for the stationary PDFs are obtained for both Case 1 and Case 2. Exact results for the MSDs in the presence of resetting are obtained as well. It is shown that from anomalous diffusive behavior in the short time limit, the MSDs saturate for large times, as the result of resetting.

The investigation of random search with memory and space-dependent velocity in the absence and presence of stochastic resetting could be of interest for future research, as a generalization of the previously obtained results for the first-passage time distribution and the mean first-passage time of heterogeneous diffusion processes [66–68], including the case of stochastic resetting to multiple [69] and random positions [70]. Other topics for future research are investigation of ergodic properties [71,72] of generalized HTPs, including the HTPs under noninstantaneous resetting [56,64,73], resetting in an interval [74,75], and partial resetting [76,77].

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