Anomalous conductivity due to relativistic Landau quantization

Gert Brodin [1,*] and Haidar Al-Naseri [2,†]

[1] *Department of Physics, Umeå University, SE–901 87 Umeå, Sweden*

[2] *Stanford PULSE Institute, SLAC National Accelerator Laboratory, Menlo Park, California 94025, USA*

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We use a recently developed a kinetic model derived from the Dirac equation to study electromagnetic wave propagation in superstrong magnetic fields, such as in magnetars, where relativistic Landau quantization is prominent. The leading contribution to the conductivity tensor in such a plasma is calculated. It is found that the electron Hall current has an anomalous contribution, in the quantum relativistic regime, where the effective particle energy has a significant contribution from the diamagnetic and Zeeman energy. As a result, a new quantum resonance frequency appears, and the dispersion relation for the left- and right-hand polarized modes are strongly modified for long and moderate wavelengths. The implications for magnetar physics are discussed.

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I. INTRODUCTION

As is well known, letting the electromagnetic field strength approach the critical field opens for a host of new phenomena. For example, aided by a strong electric field, the Schwinger mechanism allows virtual electron-positron pairs to tunnel out of the Dirac sea and become real [1,2]. It may be noted, however, that field strengths of this magnitude cannot yet be produced in the laboratory. In an astrophysical context, on the other hand, magnetic fields of the order of the critical field, and sometimes larger, are known to exist in magnetars [3–5]. Also in this case, physics induced by the virtual particles can be crucial, as the vacuum polarization induced by the ultrastrong magnetic fields is believed to be the reason for the observed polarization of electromagnetic radiation emitted from magnetars. Specifically, for photons propagating perpendicular to the magnetic field, the vacuum polarization leads to so-called photon splitting, implying that photons in one polarization state can decay into photons with a lower frequency belonging to the opposite polarization state [3,6].

With real particles present and affected by the magnetic field, either at the magnetar surface or in a magnetar accretion disk, the ultrastrong magnetic field will lead to relativistic Landau quantization [4,7–10] of electrons. In a laboratory context, Landau quantization is known to be a key feature in the quantum Hall effect of the two-dimensional electron gas [11], where the transverse resistivity becomes quantized. However, for the case where electrons are subject to relativistic Landau quantization, less is known. While many works have treated certain aspects of relativistic Landau quantization [5,8,9], e.g., computed the energy and effective mass of electrons in different Landau states, rigorous treatments of the current response due to an electric field have not been made. On the other hand, when studying electromagnetic wave propagation in magnetar environments, several authors (e.g., [12–16]) have stressed the significance of the strong field vacuum polarization, but at the same time evaluated the current response of the plasma based on classical theory [12,14,15], or semiclassical models [13,16], not fully accounting for the dynamics of the perturbed eigenstates. There is reasonable justification for this approach since a strong magnetic field mainly influences the perpendicular current response, which tends to be rather effectively suppressed for electrons due to their shorter Larmor radius. Hence, there is room for a classical ion response to dominate the perpendicular current. Moreover, the parallel electron current tends to be classical as a rough approximation, since it is only marginally affected by the magnetic field. While there is a fair bit of merit to this description, nevertheless, we will show that the picture given above is an over simplification.

In the present work, we will use a recently developed kinetic approach [4] in order to evaluate the electron conductivity in a relativistically Landau quantized plasma, of particular relevance for magnetar environments. We find that the quantum relativistic features induce crucial deviations from the classical conductivity. For certain cases, the idea that relativistically Landau quantized electrons behave almost like classical particles, but with an effective mass dependent on the energy state [5,8,9], can be confirmed. However, for other equally common cases, e.g., electromagnetic waves propagating parallel to the magnetic field, the deviations from classical theory are dramatic. Specifically, both the right and left-hand circular polarized modes behave much differently from classical theory. In this case, one of the wave frequencies approaches zero as the wavenumber $k \to 0$, and the other approaches a new type of quantum resonance frequency $\omega_{\rm res}$ given by $\omega_{\rm res} \sim (\hbar\omega_c/mc^2)(\hbar\omega_p/mc^2)\omega_p$, where $\hbar$ is the reduced Planck constant, m is the electron mass, c is the speed of light in vacuum, ω_c is the electron cyclotron frequency, and ω_p is the electron plasma frequency. The present results are a prerequisite for understanding the dynamics of relativistically Landau quantized states, as well as for interpreting spectra emitted from magnetars. [5,7,14,17–22].

*Contact author: gert.brodin@umu.se

†Contact author: hnaseri@stanford.edu

II. BASIC EQUATIONS

In this work, our starting point will be the kinetic theory derived in Ref. [4]:

$$\partial_t W_\pm + \frac{1}{\epsilon}\mathbf{p}\cdot\nabla_r W_\pm + q\mathbf{E}\cdot\nabla_p W_\pm + \frac{q}{\epsilon}\mathbf{p}\times\mathbf{B}\cdot\nabla_p W_\pm = 0, \tag{1}$$

where we use $q = -e$ for the electron charge.

This evolution equation is very similar to the usual relativistic Vlasov equation. However, in order to account for relativistic Landau quantization, the particle energy ϵ is generalized to be a momentum operator with

$$\epsilon = \sqrt{m^2 + p^2 \pm 2\mu_B B_0 - \mu_B^2 B_0^2(\hat{\mathbf{z}}\times\nabla_p)^2} \tag{2}$$

in units where $c = 1$. Here the index $\pm$ on the distribution function (Wigner function [23]) refers to the particle spin state, up or down relative to the external magnetic field $\mathbf{B}_0$, and $\mu_B = e\hbar/2m$ is the Bohr magneton. The kinetic evolution equation is combined with Maxwell's equations, where the current and charge densities are given as

$$\mathbf{j} = \sum_\pm q\int \frac{1}{\epsilon}(\mathbf{p}W_\pm)d^3p \tag{3}$$

and

$$\rho = \sum_\pm q\int W_\pm d^3p. \tag{4}$$

The above kinetic theory is based on a Foldy-Wouthuysen transformation [24,25] of the Dirac Hamiltonian, separating the positive and negative energy states. The model is applicable for an ultrastrong and homogeneous background magnetic field, $\mathbf{B}_0 = B_0\hat{\mathbf{z}}$, allowing for magnetic fields $B_0 \sim B_{cr}$ that can be found in magnetars, where the critical field strength is $B_{cr} = m^2/|q|\hbar$. Perturbations of the electromagnetic field added to $\mathbf{B}_0$ must still be small compared to the critical field, though, as the electron and positron states cannot be separated otherwise. The model is designed to fully include the effect of relativistic Landau quantization, but ignore other quantum effects, except degeneracy, that might be included by simply picking a degenerate initial state. Ignoring dynamical quantum effects, besides those due to Landau quantization, is possible if the relevant dimensionless parameters involving $\hbar$ are small, excluding $\mu_B B_0/m \sim 1$ which is allowed. To be concrete, we thus assume $\hbar k \ll p_{th}$(where p_{th} is characteristic thermal momentum such that particle dispersive effects can be ignored), $\hbar k^2/m\omega \ll 1$ (making the magnetic dipole force due to the spin small), together with $\hbar k \ll m$ and $\hbar\omega \ll m$ such that spin-orbit coupling [26,27] and other quantum relativistic effects, except those where the strong Landau quantization can be dropped. In the above, ω and k represent characteristic frequency and wavenumber scales of the macroscopic variables.

The physics behind the model is that in a superstrong magnetic field, the magnetic dipole energy due to the spin and the energy associated with the orbital angular momentum gives a significant contribution to the particle energy and thereby to the effective gamma factor $\gamma = \epsilon/m$. This is accompanied by a relativistic momentum spread in the energy eigenstates. In the expression (2), the term $\pm 2\mu_B B_0$ is the energy contribution from the Zeeman energy due to the spin, whereas the term $\mu_B^2 B_0^2(\hat{\mathbf{z}}\times\nabla_p)^2$ gives the orbital diamagnetic energy contribution. When the distribution function is in a Landau quantized energy eigenstate, $W_\pm = W_{\pm n}$, we have $\epsilon W_{\pm n} = \sqrt{m^2 + p_z^2 + \mu_B B_0(\pm 1 + n)}\, W_{\pm n}$ (see Eqs. (34) and (35) of Ref. [4] for the expression for the energy eigenstate $W_{\pm n}$). However, this particular expression is of use mostly to evaluate ϵ when acting on the time-independent background, which can be written as a sum of energy eigenstates. For a perturbation of the background, which in general depends on the azimuthal momentum coordinate, using cylindrical coordinates in momentum space, the distribution function cannot be written as a sum over energy eigenstates. To evaluate the operator ϵ in this case, we must use the defining expression, where the root in the energy expression is Taylor expanded to infinite order,

$$\epsilon = m\left(1 + \frac{1}{2}\frac{p^2 \pm 2\mu_B B_0 - \mu_B^2 B_0^2(\hat{\mathbf{z}}\times\nabla_p)^2}{m^2} + \cdots\right). \tag{5}$$

Naturally, the same definition of ϵ applies also when acting on the energy eigenstates, but in this case, the infinite series sums up to a simple energy eigenvalue. In all equations written, note that ϵ is acting on all momentum dependence that stands to the right, e.g., in the last term of (1), ϵ does not act only on $\nabla_p W_\pm$, but also on the momentum in the cross product $\mathbf{p}\times\mathbf{B}$, and similarly for the energy operator ϵ that appears in the current density (3) to be used in Ampere's law.

III. LINEAR THEORY

Next, we use Eq. (1) to study linear waves in a magnetized plasma. We divide the variables according to $W = W_0(\mathbf{p}) + W_1(\mathbf{r},\mathbf{p},t)$ and $\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1(\mathbf{r},t)$. Furthermore, we use a plane wave ansatz $W_1(\mathbf{r},\mathbf{p},t) = \tilde{W}_1(\mathbf{p})e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$. Using the ansatz and linearizing Eq. (1) (noting that the electric field $\mathbf{E}$ also is a small perturbation), we obtain

$$\begin{aligned}\left[-i\omega + i\frac{\mathbf{k}}{\epsilon}\cdot\mathbf{p} + \frac{qB_0}{\epsilon}\frac{\partial}{\partial\varphi_p}\right]\tilde{W}_1\\ = -q\left[\mathbf{E} + \frac{1}{\epsilon}\mathbf{p}\times\mathbf{B}_1\right]\cdot\nabla_p W_0.\end{aligned} \tag{6}$$

Here we have introduced the azimuthal angle φ_p in momentum space, and in what follows we will use the cylindrical momentum variables $p_\perp$, φ_p, and p_z. Due to the energy ϵ being a momentum operator, the standard techniques that work for the linearized Vlasov equation are not directly applicable. However, the equation can be solved using an expansion of $\tilde{W}_1$ according to

$$\tilde{W}_1 = \sum_{n=-\infty}^{\infty}\sum_{r=0}^{\infty} g_{nr}(p_\perp, p_z)e^{in\varphi_p}\left(\frac{k_\perp p_\perp}{m\omega_c}\right)^r. \tag{7}$$

Note that, in principle, we can include terms to an arbitrary order in the summation over n and r. However, in the strongly magnetized regime that we consider (where all other frequencies are small compared to the electron cyclotron frequency), only the low values of n and r will give a significant contribution. Moreover, without loss of generality we let the wave vector be directed in the xy plane, i.e., $\mathbf{k} = k_x\hat{\mathbf{x}} + k_z\hat{\mathbf{z}}$, such that $k_\perp = k_x$. Using the expansion (7), we solve $\tilde{W}_1$ to leading order, including terms up to $n = \pm 1$ and $r = 1$. The need to include terms with $n = \pm 1$ means we cannot neglect the dependence of $\tilde{W}_1$ on φ_p. Combining the solution for $\tilde{W}_1$ with Ampere's law, we obtain the dispersion relation for electromagnetic wave propagation (including also the electrostatic limiting cases) in a magnetized plasma, extending previous classical results [28] to account for relativistic Landau quantization. The dispersion relation can be written as $\det D_{ij} = 0$, where the matrix is written as

$$D_{ij} = \delta_{ij}\left(1 - \frac{k^2}{\omega^2}\right) + \frac{k_i k_j}{\omega^2} + \chi_{ij}, \tag{8}$$

where δ_{ij} comes from the displacement current in Ampere's law, and the terms proportional to the wavenumber come from the curl of the magnetic field. The last term, χ_{ij}, is the plasma susceptibility and is related to the plasma currents as $\chi_{ij} = \sigma_{ij}/i\omega\varepsilon_0$, where σ_{ij} is the conductivity tensor determined from the kinetic evolution equation. With the wave vector in the xz plane, the expression for D_{ij} becomes

$$D_{ij} = \begin{bmatrix} 1 - \frac{k_z^2}{\omega^2} + \chi_{xx} & \chi_{yx} & \frac{k_x k_z}{\omega^2} + \chi_{zx} \\ \chi_{yx}^* & 1 - \frac{k^2}{\omega^2} + \chi_{yy} & \chi_{yz} \\ \frac{k_x k_z}{\omega^2} + \chi_{zx}^* & \chi_{yz}^* & 1 - \frac{k_x^2}{\omega^2} + \chi_{zz} \end{bmatrix}, \tag{9}$$

where

$$\chi_{xy} = \chi_{yx}^* = -\frac{i\omega q 2\pi}{4}\int d^2p \frac{1}{\epsilon_0}\frac{\pm p_\perp}{\omega - kp_z/\epsilon_1 \mp qB_0/\epsilon_1} \times \left[\frac{\partial f_0}{\partial p_\perp} - \frac{k_z}{\omega}\frac{1}{\epsilon_1}\left(p_z\frac{\partial W_0}{\partial p_\perp} - p_\perp\frac{\partial W_0}{\partial p_z}\right)\right] \tag{10}$$

$$\chi_{yz} = \chi_{zy}^* = \frac{i\omega q 2\pi}{4}\int d^2p \frac{1}{\epsilon_0}\frac{\mp p_\perp}{\omega - kp_z/\epsilon_1 \mp qB_0/\epsilon_1} \times \left[\frac{k_z}{\omega}\frac{1}{\epsilon_1}\left(p_z\frac{\partial W_0}{\partial p_\perp} - p_\perp\frac{\partial W_0}{\partial p_z}\right)\right]. \tag{11}$$

Here ϵ_0 and ϵ_1 constitute the energy operator from Eq. (2) acting on different angular momentum dependences. Specifically, when the energy operator acts on a Wigner function with no dependence on φ_p [i.e., the terms with $n = 0$ in the sum of Eq. (7)] we can use the simplification

$$F \equiv (\hat{\mathbf{z}} \times \nabla_p)^2 = \frac{\partial^2}{\partial p_\perp^2} + \frac{1}{p_\perp}\frac{\partial}{\partial p_\perp}. \tag{12}$$

However, when the energy operator acts on a Wigner function with a dependence on φ_p proportional to $\exp(\pm i\varphi_p)$, [i.e., the terms with $n = \pm 1$ in the sum of Eq. (7)] we must use

$$F' \equiv (\hat{\mathbf{z}} \times \nabla_p)^2 = \frac{\partial^2}{\partial p_\perp^2} + \frac{1}{p_\perp}\frac{\partial}{\partial p_\perp} - \frac{1}{p_\perp^2}. \tag{13}$$

As a result, terms with $n = 0$ in the expansion Eq. (7) result in an energy operator of the form

$$\epsilon_0 = \sqrt{m^2 + p^2 \pm 2\mu_B B_0 - \mu_B^2 B_0^2 F}, \tag{14}$$

whereas terms with $n = \pm 1$ result in an energy operator given by

$$\epsilon_1 = \sqrt{m^2 + p^2 \pm 2\mu_B B_0 - \mu_B^2 B_0^2 F'}. \tag{15}$$

As terms with a higher order dependence [for $|n| > 1$ in Eq. (7)] can be omitted in the expansion, only two versions of the energy operator (ϵ_0 and ϵ_1) are needed.

The given components for the electron susceptibility constitute the dominating contributions for electrons in the regime $\omega \ll \omega_c = qB_0/m \sim qB_0/\epsilon_1$. Accordingly, we should replace the denominators in the expressions above according to $\omega - kp_z/\epsilon_1 \mp qB_0/\epsilon_1 \to qB_0/\epsilon_1$. For consistency with other approximations, this should be done whenever applying the expressions. The only reason to show the susceptibilities in Eqs. (10) and (11) without this simplification is to emphasize the similarity of the quantum result with previous well-known classical formulas when the arguments of the Bessel functions are small [28]. The expression for χ_{zz} is not given above, but we note that this term has been computed previously, see Ref. [4]. Contrary to the terms given here, the component for χ_{zz} can essentially be written as for a classical plasma, except that each Landau-quantized energy eigenstate behaves like a separate particle species, which contributes to the plasma frequency with its own effective mass, see Ref. [4] for details.

The remaining susceptibilities for electrons, χ_{xz} and χ_{zx} will not be given here as this contribution can be approximated with zero, since in the regime $\omega \ll \omega_c$ studied here, those components will always be negligible compared to the ion contribution. While the ion contribution has not been written out, a contribution from the ions can always be added to give the total susceptibility. The ions susceptibility will typically be given by the classical textbook results (see, e.g., Ref. [28]), since even for magnetar field strengths $\hbar\omega_{ci} \ll k_B T$, where ω_{ci} is the ion cyclotron frequency, k_B is the Boltzmann constant, and T is the temperature (the condition for neglecting ion Landau quantization) tend to apply.

As studied, e.g., in Refs. [12–16], in addition to plasma currents, vacuum polarization currents can influence wave propagation in the presence of a strong magnetic field. In Ref. [14], the classical linearized plasma susceptibility was computed, in addition to the contribution from vacuum polarization currents, for an arbitrary direction of propagation. The vacuum polarization contribution is proportional to $\alpha(B_0/B_{cr})^2$, as well as another dimensionless factor smaller than unity, where α is the fine structure constant, see, e.g., Eqs. (31)–(36) of Ref. [14] for details. Whether or not contributions of such a magnitude can be important for strong magnetic fields depends on the directions of wave propagation, as well as the wave polarization. An illustrative example of this can be found in Ref. [29], where the dominating modification of the vacuum dispersion relation for the extraordinary mode propagating perpendicular to the magnetic field comes from the vacuum polarization current, whereas the dominating contribution to the ordinary wave mode comes from plasma currents. In general, the vacuum polarization susceptibility computed by Ref. [14] can be added to the plasma susceptibility computed here to provide a a more complete picture.

However, for the example to be studied in the next section (electromagnetic waves propagating parallel to a magnetic field), the vacuum polarization will give a negligible contribution. The reason is that in this field geometry, the contribution from the vacuum polarization vanishes unless there is a nonzero plasma susceptibility (see, e.g., Eq. (60) of Ref. [14]). As a result, the vacuum contribution will only be a small correction of the plasma susceptibility (smaller by a factor of the order of the fine structure constant), and therefore such contributions will not be considered further.

IV. THE LINEAR DISPERSION RELATION

Let us next consider the dispersion relation for electromagnetic waves propagating parallel to the external magnetic field. Although we are interested in the case of ultrastrong magnetic fields, approaching the critical field, in order to obtain simple analytical results, we will make an expansion in the parameter $\mu_B B_0/m \ll 1$. For $\omega/\omega_{ce} \ll 1$, we keep terms to first order in ω/ω_{ce}, but allow for $\omega/\omega_{ci} \sim 1$. Moreover, we keep correction terms to second order in $\mu_B B_0/m$. By considering an electron-ion plasma, and evaluating the ion current in the classical cold limit, we find that the total (including electrons and ions) susceptibility is given by $\chi_{yx} = \chi_{yx,e} + \chi_{yx,i}$ with the electron contribution

$$\chi_{yx,e} = \frac{\omega_{pe}^2}{\omega_{ce}\omega}\left[1 + \frac{3}{2}\left(\frac{\mu_B B_0}{m}\right)^2\right] \tag{16}$$

and the ion contribution

$$\chi_{yx,i} = \frac{\omega_{pi}^2\omega_{ci}}{\omega_{ci}^2 - \omega^2}, \tag{17}$$

such that the total susceptibility is

$$\chi_{yx} = \frac{\omega_{pe}^2}{\omega_{ce}\omega}\left[\frac{\omega^2}{\omega^2 - \omega_{ci}^2} + \frac{3}{2}\left(\frac{\mu_B B_0}{m}\right)^2\right]. \tag{18}$$

Consequently, for reasonably low frequencies, $\omega \ll \omega_{ci}$, the second term $\propto (\mu_B B_0/m)^2$, which is the quantum contribution associated with the electron Landau quantization, will dominate for magnetic fields not too much smaller than the critical field. Calculating the determinant of the dispersion matrix D_{ij} for $k_\perp = 0$, we obtain the dispersion relation

$$\left(1 - \frac{k^2}{\omega^2} - \frac{\omega_{pi}^2}{\omega^2 - \omega_{ci}^2}\right) = \pm\frac{3}{2}\frac{\omega_{pe}^2}{\omega_{ce}\omega}\left(\frac{\mu_B B_0}{m}\right)^2,$$

where the two signs correspond to the left and right hand circularly polarized modes, respectively. Naturally, the dispersion relation can be studied in all generality. However, to illustrate the main new features we limit ourselves to frequencies well below the ion-cyclotron frequency, in which case we obtain

$$\left(1 - \frac{k^2}{\omega^2} + \frac{\omega_{pi}^2}{\omega_{ci}^2}\right) = \pm\frac{3}{2}\frac{\omega_{pe}^2}{\omega_{ce}\omega}\left(\frac{\mu_B B_0}{m}\right)^2. \tag{19}$$

In the short-wavelength limit, the quantum term becomes negligible, and we get

$$\omega^2 = \frac{k^2}{1 + \omega_{pi}^2/\omega_{ci}^2} \tag{20}$$

for both the left and right hand circularly polarized mode. For ω approaching ω_{ci}, naturally we should avoid the low-frequency simplification, and the left and right hand modes will have different frequencies also in the classical limit. The solution to Eq. (19) is

$$\omega = -\frac{3\omega_{pe}^2}{4\omega_{ce}\left(1 + \omega_{pi}^2/\omega_{ci}^2\right)}\left(\frac{\mu_B B_0}{m}\right)^2 \pm \sqrt{\left[\frac{3\omega_{pe}^2}{4\omega_{ce}\left(1 + \frac{\omega_{pi}^2}{\omega_{ci}^2}\right)}\left(\frac{\mu_B B_0}{m}\right)^2\right]^2 + \frac{k^2}{\left(1 + \frac{\omega_{pi}^2}{\omega_{ci}^2}\right)}}. \tag{21}$$

Thus, in the long wavelength limit, we have a new type of quantum resonance frequency ω_q, such that when $k \to 0$, for the left hand circularly polarized mode we get $\omega \to \omega_q$ with

$$\omega_q = -\frac{3\omega_{pe}^2}{2\omega_{ce}\left(1 + \omega_{pi}^2/\omega_{ci}^2\right)}\left(\frac{\mu_B B_0}{m}\right)^2, \tag{22}$$

whereas the right-hand circularly polarized mode fulfills

$$\omega \simeq \frac{k^2}{4\omega_q\left(1 + \omega_{pi}^2/\omega_{ci}^2\right)} \tag{23}$$

for long wavelengths fulfilling $\frac{k^2}{(1+\omega_{pi}^2/\omega_{ci}^2)} \ll \omega_q^2$. We thus see that for a sufficiently strong magnetization of the plasma, when relativistic Landau quantization is significant, both the left and right-hand circularly polarized modes show a distinct quantum behavior for long wavelengths, see Fig. 1 where the left- and right-handed solutions to Eq. (19) are compared with the classical limit. For shorter wavelengths, $\frac{k^2}{(1+\omega_{pi}^2/\omega_{ci}^2)} \gg \omega_q^2$ not covered in Fig. 1, the quantum behavior is suppressed, and we recover the classical limit. The wavelength where the transition from quantum to classical behavior takes place depends on the plasma density. For the high densities at the surface of magnetars, the plasma frequency is of the order $\omega_{pe} \sim 10^{19-20}s^{-1}$. In this case, for magnetars with surface magnetic fields slightly below the critical field, the quantum limits of the dispersion relation shown above apply for wavelengths in the UV region and longer.

V. DISCUSSION AND CONCLUSION

As is known from condensed matter physics, and seen in the quantum Hall effect, the effect of Landau quantization may induce peculiar properties of the conductivity. In condensed matter systems, this may happen already in a nonrelativistic description. In that case, besides Landau quantization, the special quantum features are dependent on the system being a 2D electron gas. In the present theory, the electrons are not limited to two dimensions, but instead, the relativistic aspects of Landau quantization are crucial, accounting for the contribution to the electron's effective mass through the diamagnetic and Zeeman energies. At first glance, the extra quantum contributions to the Hall term of the electron susceptibility [leading to the quantum contribution in the total susceptibility (18), i.e., the second term $\propto (\mu_B B_0/m)^2$] may be

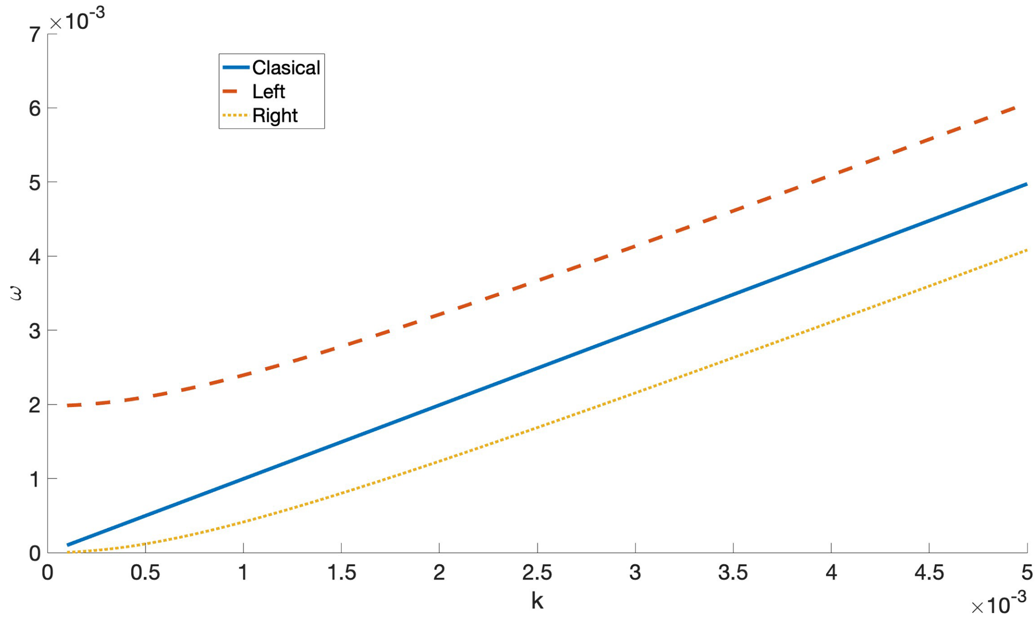

FIG. 1. The frequency ω is plotted as a function of the wave vector k for $\mu_B B_0/m = 0.2$ and $\omega_c/\omega_p = 10$.

expected. Why would the susceptibility (conductivity) be unaffected when the Zeeman energy becomes significant? However, looking more carefully at the result, it appears to challenge a key principle of physics, namely Lorentz invariance.

Normally, in the limit when $k \to 0$ and $\omega \to 0$, the total current density of all species cancels. The reason is as follows: For an electromagnetic field that is effectively static and homogenous, in a reference system moving with the $\mathbf{E} \times \mathbf{B}-$drift, where $\mathbf{v}_E = \mathbf{E} \times \mathbf{B}/B^2$, particles will only feel a static magnetic field. Hence, independent of species, there will only be a gyration and no net drift in this system. This corresponds to all species s having the current density $q_s n_{0s}\mathbf{v}_E$ in the original system. Due to the background neutrality of all species, this implies a vanishing total current density. Fundamentally, a vanishing current density in the limit $k \to 0$ and $\omega \to 0$ is a classical result that follows from Lorentz invariance. In the present case, while our governing equations do not show Lorentz invariance directly (since they are built around a preferred reference system subject to a strong magnetic field B_0), we should still demand our results to be consistent with Lorentz invariance. Clearly, since the model is derived from the Dirac equation and Maxwell's equations, we should not have a theory that is in conflict with the basic principles of special relativity. Thus, the question arises, how can the effective electron drift velocity deviate from the $\mathbf{E} \times \mathbf{B}-$drift in the limit $k \to 0$ and $\omega \to 0$, without contradicting Lorentz invariance?

A first observation is that, contrary to classical theory, it is no longer meaningful to think of individual velocities for different phase space elements, represented by $\mathbf{p}/\varepsilon$. In the limit where this would be accurate, the quantum contribution to χ_{xy} vanish, and the current density for electrons indeed become $q_e n_{0e}\mathbf{v}_E$. A complication of the present theory, where the energy ε is a momentum-operator, is that the effective velocity of the theory depends on the behavior in a region of momentum space, rather than in a single point. Moreover, the effective velocity averaged over all momentum for energy transport ($\mathbf{v}_{en} = \int \frac{1}{\varepsilon}(\mathbf{p}\varepsilon f)d^3p/\int \varepsilon f d^3p$) and for particle transport ($\mathbf{v}_p = \int \frac{1}{\varepsilon}(\mathbf{p}f)d^3p/\int f$) only coincide if the distribution function f is an energy eigenstate, in general $\mathbf{v}_{en} \neq \mathbf{v}_p$. However, even when this ambiguity regarding what constitutes the velocity is noted, one can argue that in a reference system with a static magnetic field, there should be no net current density. If this were true, one would obtain the same cancellation of the total current density as in the classical case.

This argument leaves out a key aspect of the present theory, though. When a Landau quantized eigenstate in a pure magnetic field is exposed to an electric field, for the particle to move at all, the momentum distribution of the particles need to be modified in a way that prevents the particle state from being written as a sum of energy eigenstates (this is not possible, as we break the angular symmetry and get a dependence on φ_p). Breaking the original symmetry contributes to the energy by a positive and quantized amount, since for the perturbed distribution function we must let $B_0^2(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho}) \to B_0^2(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2})$ in the orbital magnetic energy part of the energy operator. This quantized change of energy when modifying the particle state has no classical correspondence and is what determines the momentum-velocity relation when calculating the magnitude of the perturbed distribution function. Importantly, the effective velocity obtained in this step is not the same as the effective velocity that we get when calculating the current density. Here, for the part surviving the momentum integration, the energy operator acts on a function of momentum that is angular symmetric, which leads to a different expression of what can be interpreted as the effective velocity. Ultimately, in the relativistically Landau quantized regime, what constitutes "particle velocity" depends on the details of the definition. As a result, a ratio of different particle energies, which would simply be unity in a classical calculation, is what induces a relativistically quantum-corrected Hall current.

The main conclusion from our analysis is that the anomalous conductivity for relativistically Landau quantized states can modify the electromagnetic wave propagation properties in magnetar environments in a distinct way. However, it is a challenge to relate the properties of the plasma susceptibility tensor to observational magnetar data. Nevertheless, a specific possibility might be to look for the quantum resonance given by Eq. (22) in the observed spectra.

In the present work, we have focused on the simplest aspects of the linearized theory of electromagnetic waves. A more complete study of the linear susceptibility and generalizations to also cover the nonlinear regime are projects for future work. It may then be of interest to include also the physics associated with the magnetic dipole force due to the electron spin [10,26], and the vacuum polarization associated with strong magnetic fields, see, e.g., Refs. [5,13,30,31]

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