Microscopic inclusion statistics in a discrete one-body spectrum

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We present the microscopic formulation of inclusion statistics, a counterpoint to exclusion statistics in which particles tend to coalesce more than ordinary bosons. We derive the microscopic occupation multiplicities of one-body quantum states and show that they factorize into a product of clusters of neighboring occupied states with enhanced statistical weights. Applying this statistics to a one-dimensional gas of particles in a harmonic well leads to a Calogero-like n-body inclusion spectrum with interesting physical properties.

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I. INTRODUCTION

Quantum statistics deviating from Bose-Einstein and Fermi-Dirac statistics have been a part of physics for over 70 years. Parastatistics, originally devised to explain the properties of quarks, followed by anyon statistics [1–3], as a theoretical possibility in two dimensions, non-Abelian statistics in systems with degenerate ground states, and exclusion statistics [4–6] as effective descriptions of some interacting systems, are examples of such generalizations.

In all the above variants of statistics, bosons remain the most "gregarious" type, and the only ones undergoing Bose-Einstein condensation at sufficiently low temperatures. In a recent development [7] we generalized the notion of statistics to *inclusion* statistics, which extends the statistical spectrum beyond bosons and enhances the particles' property to coalesce. The qualitatively new feature of inclusion particles is that they can condense in one space dimension lower than ordinary bosons, opening the possibility for condensation in planar systems. As such, their properties and possible experimental realization are matters of physical interest.

Most types of statistics have a microscopic formulation in terms of individual particles. In a previous work [8] we gave a microscopic formulation of exclusion statistics for an integer statistics parameter in a nondegenerate discrete one-body spectrum and studied its combinatorial properties. In the present work we extend this formulation to inclusion statistics and study its relation to exclusion statistics, via a duality relation generalizing the well-known Bose-Fermi correspondence. The concept of statistical clusters of particles occupying nearby states with nontrivial multiplicities providing the basic building blocks for inclusion statistics emerges in this analysis. Various alternative definitions of inclusion statistics, irrelevant in the thermodynamic limit where the number of one-body quantum states diverges and the energy spectrum effectively becomes continuous, become relevant at the microscopic level.

The paper is organized as follows: in Secs. II and III we review the concepts of exclusion and inclusion statistics in the thermodynamic limit and their duality. In Sec. IV we review the microscopic formulation of exclusion statistics, and in Sec. V we define the microscopic formulation of inclusion statistics for particles on a lineal one-body spectrum (with states ordered on a line) and derive its combinatorial properties. In Sec. VI we present realization of inclusion statistics in a harmonic spectrum and relate it to a Calogero-like model. In Sec. VII we conclude with some speculations and connections with other combinatorial topics. Technical issues such as the formulation of microscopic inclusion statistics on a periodic one-body spectrum (i.e., with one-body quantum states on a circle), the derivation of some mathematical results, and the recovery of the free space limit from a weak potential trap, are presented in the Appendix.

II. g EXCLUSION

It has been known since the nineties [9–12], from the study of quantum systems such as the lowest Landau level anyon model or the Calogero model, that thermodynamical equations can be obtained for particles with generalized statistics described by a statistical parameter g interpolating between the standard $g = 0$ Bose-Einstein and $g = 1$ Fermi-Dirac statistics. The key information for these g statistics is contained in the n-body cluster coefficients $c_{k,n}(g)$; specifically,

$$\ln \mathcal{Z}_g = \sum_{n=1}^{\infty} c_{k,n}(g) z^n,$$

where $\mathcal{Z}_g = \sum_n Z_n z^n$ is the grand partition function for particles with statistical parameter g in k degenerate one-body quantum states (here taken for convenience at zero energy) and $z = e^{\beta\mu}$ is the fugacity. In the thermodynamic limit $k \to \infty$, which is essentially a large-volume limit, the grand

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potential $\ln \mathcal{Z}_g$ scales as k,

$$\ln \mathcal{Z}_g = \sum_{n=1}^{\infty} c_{k,n}(g) z^n \simeq k \sum_{n=1}^{\infty} c_n(g) z^n. \tag{1}$$

This defines effective single-level cluster coefficients $c_n(g)$ and an effective single-level grand partition function y for g statistics,

$$\sum_{n=1}^{\infty} c_n(g) z^n = \ln y. \tag{2}$$

As derived in the context of the lowest Landau level (LLL) anyon model [10], or from combinatorial considerations for exclusion statistics [12], the single-level cluster coefficients are

$$c_n(g) = \frac{1}{n} \prod_{i=1}^{n-1} \frac{i - ng}{i} = (-1)^{n-1} \frac{\binom{gn}{n}}{gn}, \tag{3}$$

and the single-level grand partition function y satisfies the equation

$$y^g - y^{g-1} = z. \tag{4}$$

Details of the derivation of (3) and (4) are given in Refs. [10,12], and Appendix (A 3) gives a first-principles, microscopic derivation of (4).

A natural step is to forget about the specific models that led to these statistics and adopt (1)–(4) as the definition of g statistics in a degenerate spectrum for a general statistical parameter $g \geqslant 0$. For $g \geqslant 1$ one speaks of exclusion statistics, which can also be obtained from an *ad hoc* Hilbert space counting argument *à la* Haldane [4], where each particle excludes g states from being occupied by other particles. This counting is in general ill-defined for fractional g because it can lead to a fractional or even negative number of states and is thus only an effective description of the system in the thermodynamic limit. However, it becomes well defined for integer values of g.

From now on we focus on integer values of g, which allows for a well-defined microscopic formulation of exclusion statistics. In this case, (4) has g generally complex solutions

$$y_1 = 1 + z + \cdots, \quad y_i = e^{i\frac{(2i-1)\pi}{g-1}} z^{\frac{1}{g-1}} + \cdots, \quad i = 2, \ldots, g,$$

of which y_1 is the physical one as defined in (2) and (3), meaning that it is real, equal to one when $z = 0$, and increasing with $z > 0$. For example, for $g = 2$,

$$y_1 = \frac{1}{2}(1 + \sqrt{1 + 4z}) = 1 + z + \cdots,$$

$$y_2 = \frac{1}{2}(1 - \sqrt{1 + 4z}) = -z + \cdots,$$

while for $g = 3$,

$$y_i = \frac{1}{3} - \frac{2}{3} \sin\left(\frac{\arcsin(1 + 27z/2)}{3} - i\frac{2\pi}{3}\right) \tag{5}$$

for $i = 1, 2, 3$ (note that $y_2 = y_3^*$ for $z > 0$). The large-k thermodynamic degenerate grand partition function, which from (1) and (2) rewrites as

$$\mathcal{Z}_g = y_1^k, \tag{6}$$

can be modified to a form that leads to a valid microscopic interpretation for finite k, as explained in detail in the Appendix, namely,

$$\mathcal{Z}_g = \sum_{i=1}^{g} \frac{y_i^{k+g}}{g y_i - g + 1} = \sum_{n=0}^{[(k+g-1)/g]} \frac{[k + (1-g)(n-1)]!}{n![k - (n-1)g - 1]!} z^n. \tag{7}$$

In (7) the summation over n is truncated at $[(k + g - 1)/g]$, as is natural for g exclusion in a spectrum on the line in which $[(k + g - 1)/g]$ particles exclude all states for additional particles. Moreover, the degeneracy factor at order z^n, i.e., the n-body partition function

$$Z_n = \frac{[k + (1-g)(n-1)]!}{n![k - (n-1)g - 1]!} \tag{8}$$

indeed counts the number of ways to put n particles in k slots on a line with at least $g - 1$ unoccupied slots between two occupied slots, a hallmark of g-exclusion statistics for a lineal one-body spectrum. This lineal counting coincides with the original Hilbert space Haldane counting [4,11].

Note that y_i^k for $i = 2, \ldots, g$ scale as $\approx z^{k/(g-1)} \sim e^{-\beta\mu k/(g-1)}$, and thus these terms in (7) make nonperturbative in $1/k$ contributions, while the extra factor with respect to y_1^k in the term $i = 1$ makes perturbative in $1/k$ contributions, all becoming irrelevant in the $k \to \infty$ limit. Therefore, the grand partition function (7) differs from its thermodynamic limit (6) by both perturbative and nonperturbative in $1/k$ terms (the corrections start at order z^2). This is already explicitly visible for $g = 2$ statistics:

$$\mathcal{Z}_2 = \frac{y_1^{k+2} - y_2^{k+2}}{\sqrt{1 + 4z}}, \tag{9}$$

where the term y_2^{k+2} produces nonperturbative corrections, but the additional factor $y_1^2/\sqrt{1 + 4z}$ in y_1^k produces terms of order k^0, down by a factor $1/k$ compared with the thermodynamic result (6). Extending the n-summation in (7) to infinity, including the unphysical values $n > [(k + g - 1)/g]$, leads to (9) with the term y_2^{k+2} absent, still introducing perturbative corrections compared with (6).

We further note that there exists yet another form of g exclusion which also admits a valid microscopic interpretation (see again the Appendix for details), namely,

$$\mathcal{Z}'_g = \sum_{i=1}^{g} y_i^k = \sum_{n=0}^{[k/g]} \frac{k[k + (1-g)n - 1]!}{n!(k - ng)!} z^n, \tag{10}$$

which now corresponds to exclusion on a *periodic* sequence of one-body states (where one-body states are positioned on a circle), involving a periodic version of the lineal exclusion spectrum above (we elaborate later on physical instances of periodic spectra). Indeed, the summation over n in (10) is truncated at $[k/g]$, as is natural for g exclusion in a periodic spectrum in which $[k/g]$ particles exclude all states for additional particles. The degeneracy factor Z'_n at order z^n, i.e., the n-body partition function

$$Z'_n = \frac{k[k + (1-g)n - 1]!}{n!(k - ng)!} \tag{11}$$

counts the number of ways to put n particles in k slots on a circle with at least $g-1$ unoccupied slots between two occupied slots, a hallmark of g-exclusion statistics for a one-body periodic spectrum. The terms $i>1$ in (10) make again nonperturbative in $1/k$ contributions, but the term $i=1$ is identical to the thermodynamic limit (6). Therefore, the expansion in powers of z of (10) is identical to that of (6), but the summation over n in the latter unphysically extends beyond $[k/g]$ all the way to infinity, making a nonperturbative contribution. This is to be contrasted to the lineal case (7) where both nonperturbative and perturbative corrections appear. In this sense, the periodic formulation is closer to the thermodynamic limit than the lineal formulation.

We conclude by pointing out that in the thermodynamic limit, the thermodynamics of g-exclusion particles in an arbitrary discrete one-body spectrum ϵ_i, $i=1,2,\dots,k$, $k\to\infty$, and corresponding density of states $\rho(\epsilon)$, can be inferred from the physical solution y_1 of (4) as

$$\ln \mathcal{Z}_g = \sum_i \ln y_1(z\mathrm{e}^{-\beta\epsilon_i}) \to \int d\epsilon\,\rho(\epsilon)\ln y_1(z\mathrm{e}^{-\beta\epsilon}). \quad (12)$$

The general cluster coefficients, in particular, are obtained by deforming the degenerate cluster coefficients $c_{k,n}(g)=k\,c_n(g)$ appearing in the right-hand side (RHS) of (1) to

$$c_n(g)\sum_i \mathrm{e}^{-n\beta\epsilon_i} \to c_n(g)\int d\epsilon\,\rho(\epsilon)\mathrm{e}^{-n\beta\epsilon}. \quad (13)$$

Note that the sum (or integral) in (13) is simply the one-body partition function in the spectrum ϵ_i with temperature parameter $n\beta$. This would also arise from a path-integral representation [12], where this term is the connected n-body path integral consisting of a single path winding n times in periodic Euclidean time, and statistics enters through the coefficients $c_n(g)$. We stress that relations (12) and (13) hold in general, for any statistics that can be described in terms of an effective single-level grand partition function and the associated cluster coefficients.

III. g-EXCLUSION → $(1-g)$-INCLUSION THERMODYNAMICS

We now turn to inclusion statistics, which we wish to define by equations (1)–(4), as for exclusion statistics, but now with a *negative* exclusion parameter. To do so, we remark [7] that turning $g\to 1-g$ in (3) yields

$$c_n(1-g)=(-1)^{n-1}c_n(g),$$

which implies for the cluster expansion (2)

$$\ln y(z,1-g)=\sum_{n=1}^{\infty}c_n(1-g)z^n = \sum_{n=1}^{\infty}(-1)^{n-1}c_n(g)z^n = -\ln y(-z,g),$$

where now we denote by $y(z,g)$ the physical single-level g-exclusion grand partition function satisfying (4). Therefore,

$$y(z,1-g)=\frac{1}{y(-z,g)}, \quad (14)$$

which provides a duality relation between $g\geqslant 1$ exclusion statistics and $1-g\leqslant 0$ inclusion statistics. This relation can also be directly derived by rewriting (4) as

$$\left(\frac{1}{y(z,g)}\right)^{1-g} - \left(\frac{1}{y(z,g)}\right)^{(1-g)-1} = -z,$$

leading again to (14). For integer $g>0$, (4) with $g\to 1-g$ also has g solutions. The physical inclusion solution $y_1(z,1-g)=1+z+\cdots=1/y_1(-z,g)$ is recovered from the corresponding physical exclusion solution $y_1(z,g)$.

For $g=1\to 1-g=0$, i.e., Fermi → Bose, the duality reduces to

$$y_1(z,0)=\frac{1}{1-z}=\frac{1}{y_1(-z,1)}. \quad (15)$$

For $g=2$ exclusion, and therefore $1-g=-1$ inclusion, the two dual solutions are

$$y_1(z,-1)=\frac{1}{y_1(-z,2)}=\frac{1-\sqrt{1-4z}}{2z},$$
$$y_2(z,-1)=\frac{1}{y_2(-z,2)}=\frac{1+\sqrt{1-4z}}{2z}.$$

We note that z is bounded by $1/4$ and $1\leqslant y_1\leqslant 2$. Similarly, for general $(1-g)$-inclusion the bounds become [7]

$$0\leqslant z\leqslant \frac{(g-1)^{g-1}}{g^g}, \quad 1\leqslant y_1\leqslant \frac{g}{g-1}.$$

The upper bounds for z, and especially for y_1, which occur only for inclusion statistics when $1-g<0$, are at the root of the enhanced condensation properties of inclusion particles and the reduction in the space dimension where condensation occurs [note that in (15), i.e., the Fermi $g=1\ \to$ Bose $1-g=0$ case, $z\leqslant 1$, but y_1 is unrestricted].

To summarize, the duality transformation $g\to 1-g$ amounts to $y\to 1/y$ and $z\to -z$, and thus, in view of (6), to

$$\mathcal{Z}_{1-g}(z)=\frac{1}{\mathcal{Z}_g(-z)}.$$

For a general nondegenerate spectrum, the grand partition function in the thermodynamic limit is the product of effective single-level partition functions, as in (12), and thus again

$$\mathcal{Z}_{1-g}(z)=\prod_i y(ze^{-\beta\epsilon_i},1-g) = \prod_i \frac{1}{y(-ze^{-\beta\epsilon_i},g)} = \frac{1}{\mathcal{Z}_g(-z)}. \quad (16)$$

We stress that the $(1-g)$-inclusion thermodynamic relations (12) and (13) still hold, but now in terms of the physical solution $y(z\mathrm{e}^{-\beta\epsilon},1-g)$ and the cluster coefficients $c_n(1-g)$.

IV. MICROSCOPIC EXCLUSION STATISTICS FOR A NONDEGENERATE 1-BODY SPECTRUM

From here on we focus on a discrete, nondegenerate lineal one-body spectrum $\epsilon_1,\epsilon_2,\dots,\epsilon_k$ where the number of quantum states k can be finite or infinite. Here $i=1,\dots,k$ is a principal quantum number on which the energy depends, and states with neighboring i are considered "close" to each other in the Hilbert space of the particle. The energy ϵ_i can be an arbitrary function of i, not necessarily increasing. The definition of g-exclusion statistics for such a spectrum in terms of occupation numbers amounts to imposing the constraint that individual particles are at least g levels apart (according to their ordering in terms of i). As a consequence, the n-body partition function[1] Z_n of particles with g exclusion has the form

$$Z_n = \sum_{k_1=1}^{k-gn+g} \sum_{k_2=1}^{k_1} \cdots \sum_{k_n=1}^{k_{n-1}} s(k_1+gn-g)\cdots s(k_{n-1}+g)s(k_n), \quad (17)$$

where we defined the "spectral function" $s(i)=\exp(-\beta\epsilon_i)$. In the degenerate case at zero energy, namely, $s(1)=\cdots=s(k)=1$, it is easy to see that Z_n reduces to the lineal counting (8).

From the Z_n in (17) we obtain the g-exclusion grand partition function [8]

$$\mathcal{Z}_g = \sum_{n=0}^{\infty} Z_n z^n = \exp\left(-\sum_{n=1}^{\infty}(-z)^n \sum_{\substack{l_1,l_2,\ldots,l_j \\ g \text{ composition of } n}} c_g(l_1,l_2,\ldots,l_j) \sum_{i=1}^{k-j+1} s^{l_j}(i+j-1)\cdots s^{l_2}(i+1)s^{l_1}(i)\right), \quad (18)$$

where the $c_g(l_1,l_2,\ldots,l_j)$ in the cluster expansion are known combinatorial factors [13], and the summation is over g compositions, which are partitions where the ordering does matter and with up to $g-1$ consecutive zero values for l_i allowed. From (18) one can in turn obtain the single-level grand partition function $y(z,g)$ for g-exclusion statistics in a discrete one-body spectrum (for details on the microscopic g-exclusion statistics, see Ref. [8]).

As advocated in Refs. [8,13], all the above information can be conveniently encapsulated in a $(k+g-1)$-dimensional exclusion matrix with a unit upper-diagonal and $g-1$ empty subdiagonals between it and a nonzero subdiagonal of spectral functions. For example, for $g=3$,

$$H_3 = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ -s(1) & 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & -s(2) & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots & -s(k) & 0 & 0 \end{pmatrix}. \quad (19)$$

Its secular determinant $\det(1-z^{1/g}H_g)$ takes the form

$$\det(1-z^{1/g}H_g) = \sum_{n=0}^{[(k+g-1)/g]} Z_n z^n = \mathcal{Z}_g, \quad (20)$$

where Z_n are the n-body partition functions (17), and thus it is the grand partition function $\mathcal{Z}_g$ of a gas of particles with g exclusion in the one-body lineal spectrum $\epsilon_1,\epsilon_2,\ldots,\epsilon_k$.

We conclude by mentioning that a similar definition for microscopic exclusion statistics is also available for a periodic spectrum, in which one-body states are positioned on a circle and ϵ_k and ϵ_1 are neighbors, reproducing the periodic counting (11) in the degenerate case (see the Appendix for details). Physical situations in which i is a periodic variable (and thus ϵ_i is a periodic function of i with $\epsilon_{k+1}=\epsilon_1$) arise, e.g., in the band structure of solids with i playing the role of Bloch momentum, periodic in the band and discretized at finite volume. In the rest of the paper we focus on the case of a lineal one-body spectrum, relegating the case of a periodic one-body spectrum, in which some additional subtleties arise, to the Appendix.

V. MICROSCOPIC INCLUSION STATISTICS FOR A NONDEGENERATE ONE-BODY SPECTRUM

For inclusion statistics we do not have an *a priori* microscopic definition, such as the "minimal distance g" rule between filled levels in exclusion statistics. We have to devise such a formulation on the basis of some assumptions and give its combinatorial interpretation. The requirements are that it must involve non-negative integer multiplicities of microscopic states, and must also lead to the proper thermodynamic limit of Sec. III for a degenerate spectrum when the number of one-body quantum states k becomes infinite.

Denoting by n_i the occupation number of ϵ_i (i.e., the number of particles occupying the energy level ϵ_i), we recall that, for noninteracting particles, the n-body energy is expressed as the simple sum $E_n=\sum_{i=1}^k n_i\epsilon_i$ with $n=\sum_{i=1}^k n_i$. In the Bose and Fermi cases, where the occupation numbers n_i are independent of each other and the multiplicity of a n-body state with a given $n_1,n_2,\ldots,n_k$ is trivially equal to one, the grand partition function

$$\mathcal{Z} = \sum_{n=0,E_n}^{\infty} z^n e^{-\beta E_n} = \sum_{n_1,\ldots,n_k} (ze^{-\beta\epsilon_1})^{n_1}\cdots(ze^{-\beta\epsilon_k})^{n_k}$$
$$= \sum_{n_1}(ze^{-\beta\epsilon_1})^{n_1}\cdots\sum_{n_k}(ze^{-\beta\epsilon_k})^{n_k}$$

[1] For simplicity, above and in the rest of the paper we keep the same notation for the grand partition functions $\mathcal{Z}_g$ and $\mathcal{Z}_{1-g}$, the ensuing n-body partition functions Z_n, the related exclusion matrices, etc., irrespective of the one-body spectrum being degenerate or nondegenerate.

rewrites in the Bose case, where $n_i = 0, 1, \dots, \infty$, as

$$\mathcal{Z}_0 = (\frac{1}{1-ze^{-\beta\epsilon_1}})\cdots(\frac{1}{1-ze^{-\beta\epsilon_k}}),$$

and in the Fermi case, where $n_i = 0, 1$, as

$$\mathcal{Z}_1 = \big(1+ze^{-\beta\epsilon_1}\big)\cdots\big(1+ze^{-\beta\epsilon_k}\big),$$

so that the standard microscopic Fermi-Bose correspondence

$$\mathcal{Z}_0(z) = \frac{1}{\mathcal{Z}_1(-z)}$$

appears here as the $g = 1 \to 1-g = 0$ particular case of the thermodynamic duality (16).

To derive a microscopic definition of inclusion statistics, we *postulate* that (16) is also valid for the microscopic grand partition functions. This implies, in terms of the inverse of the secular determinant

$$\mathcal{Z}_{1-g}(z) = \frac{1}{\mathcal{Z}_g(-z)} = \frac{1}{\det(1-z^{1/g}H_g)}\bigg|_{z\to -z},$$

which obviously has the appropriate thermodynamic limit. (See Ref. [14] for studies of the inverse of characteristic polynomials of matrices from a combinatorial perspective.) For $g = 2$, i.e., inclusion $1-g = -1$, this yields

$$\mathcal{Z}_{-1} = \sum_{n_1,\dots,n_k=0}^{\infty} (zs(1))^{n_1}\binom{n_1+n_2}{n_1}(zs(2))^{n_2}\binom{n_2+n_3}{n_2}\cdots\binom{n_{k-1}+n_k}{n_{k-1}}(zs(k))^{n_k},$$

while for $g = 3$, i.e., inclusion $1-g = -2$,

$$\mathcal{Z}_{-2} = \sum_{n_1,\dots,n_k=0}^{\infty} (zs(1))^{n_1}\binom{n_1+n_2+n_3}{n_1}(zs(2))^{n_2}\binom{n_2+n_3+n_4}{n_2} \times\cdots\times\binom{n_{k-2}+n_{k-1}+n_k}{n_{k-2}}(zs(k-1))^{n_{k-1}}\binom{n_{k-1}+n_k}{n_{k-1}}(zs(k))^{n_k},$$

etc. This leads to nontrivial inclusion multiplicities $m_{1-g}(n_1,\dots,n_k)$ for states of given occupation numbers $n_1,\dots,n_k$: for $g = 2 \to 1-g = -1$ they are

$$m_{-1}(n_1,\dots,n_k) = \binom{n_1+n_2}{n_1}\binom{n_2+n_3}{n_2}\cdots\binom{n_{k-1}+n_k}{n_{k-1}}.$$

These multiplicities indeed satisfy the criterium to be integer and non-negative, and we can check that, in the degenerate case, they indeed reproduce the inverse of $\mathcal{Z}_2$ in (9) with $z \to -z$, leading to the identity

$$\mathcal{Z}_{-1} = \sum_{n_1,\dots,n_k=0}^{\infty} m_{-1}(n_1,\dots,n_k)z^{n_1+n_2+\cdots+n_k} = \frac{\sqrt{1-4z}}{y_1^{k+2}(-z,2)-y_2^{k+2}(-z,2)}. \tag{21}$$

For $g = 3 \to 1-g = -2$ inclusion the multiplicities are

$$m_{-2}(n_1,\dots,n_k) = \binom{n_1+n_2+n_3}{n_1}\binom{n_2+n_3+n_4}{n_2}\cdots\binom{n_{k-2}+n_{k-1}+n_k}{n_{k-2}}\binom{n_{k-1}+n_k}{n_{k-1}},$$

with $\mathcal{Z}_{-2} = \sum_{n_1,\dots,n_k=0}^{\infty} m_{-2}(n_1,\dots,n_k)z^{n_1+n_2+\cdots+n_k}$ the corresponding degenerate grand partition function obtained by inverting (7) in the case $g = 3$ and trading $z \to -z$.

Generalizing the multiplicities to $(1-g)$ inclusion is straightforward. The $(1-g)$-inclusion grand partition function follows as

$$\mathcal{Z}_{1-g} = \sum_{n_1,\dots,n_k=0}^{\infty} m_{1-g}(n_1,\dots,n_k)(zs(1))^{n_1}(zs(2))^{n_2}\cdots(zs(k))^{n_k}.$$

The resulting multiplicities admit the interpretation of the product of multiplicities of clusters of g adjacent states, with the particles in each cluster considered as distinguishable, divided by the corresponding multiplicities of overlaps between clusters. Defining the shorthand notation

$$[i,j] := \binom{n_i+n_{i+1}+\cdots+n_j}{n_i, n_{i+1},\dots,n_j},$$

the multiplicities for $k \geqslant g$ are

$$m_{1-g}(n_1,\dots,n_k) = \frac{[1,g][2,g+1]\cdots[k+1-g,k]}{[2,g][3,g+1]\cdots[k+1-g,k-1]},$$

while for $k < g$ there is a single cluster, and the multiplicity reduces to

$$[1,k] = \binom{n_1+n_2+\cdots+n_k}{n_1,n_2,\dots,n_k},$$

as for distinguishable particles. Its generating function simplifies to

$$\sum_{n_1,\dots,n_k=0}^{\infty} [1,k]\,(zs(1))^{n_1}\cdots(zs(k))^{n_k} = \frac{1}{1-z\big(s(1)+\cdots+s(k)\big)}$$

The above multiplicities generalize the Bose and Fermi multiplicities, trivially equal to one (or zero in the Fermi case if any $n_i > 1$), by enhancing the weights of neighboring clusters of occupied one-body levels. This enhancement is the counterpart of the dilution of particles in exclusion statistics and a hallmark of inclusion statistics.

Note, finally, that the $(1-g)$-inclusion n-body partition functions follow from $\mathcal{Z}_{1-g}$ as

$$Z_n = \sum_{n_1,\ldots,n_k=0,\; n_1+\cdots+n_k=n}^{\infty} m_{1-g}(n_1,\ldots,n_k)\, s(1)^{n_1} s(2)^{n_2}\cdots s(k)^{n_k}.$$

They are the $g \to 1-g$ counterpart of the g-exclusion n-body partition functions (17) yielding a (1-g) inclusion n-body spectrum for particles on levels ϵ_i, $i = 1, 2, \ldots, k$ with occupancy n_i and overall degeneracy $m_{1-g}(n_1,\ldots,n_k)$.

VI. INCLUSION STATISTICS IN A HARMONIC WELL

As a concrete example let us consider particles with g statistics in a one-dimensional harmonic well, populating the equidistant one-body spectrum $\epsilon_i = (i-1)\omega$, $i = 1, 2, \ldots, \infty$, with spectral function $s(i) = x^{i-1}$ (we denote $x = \mathrm{e}^{-\beta\omega}$). Since there is no upper bound in the energy, this corresponds to a semi-infinite spectrum where $k = \infty$.

A. g exclusion

For $g \geqslant 0$, as is well known [9,10], g-exclusion statistics is realized by the n-body Calogero Hamiltonian [15]

$$H_n = -\frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2} + \sum_{i<j}\frac{g(g-1)}{(x_i-x_j)^2} + \frac{1}{2}\omega^2\sum_{i=1}^{n} x_i^2, \qquad (22)$$

with the n-body spectrum

$$E_n = \omega\left(\sum_{i=1}^{n} l_i + g\frac{n(n-1)}{2}\right), \quad 0 \leqslant l_1 \leqslant l_2 \leqslant \cdots \leqslant l_n \qquad (23)$$

(we eliminated the trivial zero-point energy per particle $\omega/2$ to make the spectrum conform with the convention $\epsilon_i = (i-1)\omega$, $i = 1, 2, \ldots, \infty$). This can be rewritten in terms of the quasi-excitation numbers $l_i' = l_i + g(i-1)$ as

$$E_n = \omega\sum_{i=1}^{n} l_i', \quad 0 \leqslant l_i' \leqslant l_{i+1}' - g, \ldots, \quad 0 \leqslant l_n' \leqslant \infty, \qquad (24)$$

where the l_i' are indeed separated by at least $g-1$ energy quanta, which is nothing but g exclusion. From this n-body spectrum we get the partition function Z_n

$$\begin{aligned} Z_n &= x^{gn(n-1)/2}\sum_{0\leqslant l_1\leqslant l_2\leqslant\cdots\leqslant l_n\leqslant\infty} x^{\sum_{i=1}^{n} l_i} \\ &= \sum_{0\leqslant l_i'\leqslant l_{i+1}'-g,\ldots,0\leqslant l_n'\leqslant\infty} x^{\sum_{i=1}^{n} l_i'} \\ &= x^{gn(n-1)/2}\prod_{j=1}^{n}\frac{1}{1-x^j}. \end{aligned} \qquad (25)$$

We stress here that the n-body partition function obtained above from the n-body Calogero spectrum could as well be derived from the very definition of g-exclusion statistics by means of the Z_n given in (17), in this case for the spectral function $s(i) = x^{i-1}$ and the number of one-body quantum states $k \to \infty$. This reconfirms that the n-body Calogero Hamiltonian (22) is a microscopic dynamical realization of g-exclusion statistics. Likewise, from (18) we obtain the g-exclusion grand partition function and cluster expansion for a one-body harmonic spectrum [8]

$$\mathcal{Z}_g = \sum_{n=0}^{\infty} Z_n z^n = \exp\left(-\sum_{n=1}^{\infty}\frac{(-z)^n}{1-x^n}\sum_{\substack{l_1,l_2,\ldots,l_j\\ g\text{-composition of } n}} c_g(l_1,l_2,\ldots,l_j)\, x^{\sum_{i=1}^{j}(i-1)l_i}\right). \qquad (26)$$

We note that the Hamiltonian (22) is invariant under the mapping $g \to 1-g$. However, the spectrum (23) is not, since it depends linearly on g. The Hermiticity properties of the Hamiltonian (22) restrict the allowed wave functions and impose choosing the greater of g, $1-g$ in (23), leading naturally to exclusion statistics. This is also an early indication that the transition to inclusion statistics will not be trivially obtained by simply trading $g \to 1-g$ in (23).

B. $(1-g)$ inclusion

To reach $(1-g)$-inclusion statistics we use the duality relation (16), inverting the grand partition function $\mathcal{Z}_g$ and turning $z \to -z$, or equivalently multiplying the nth order cluster coefficient by $(-1)^{n-1}$. This amounts to considering the cluster expansion (26) with the minus signs removed, namely,

$$\mathcal{Z}_{1-g} = \exp\left(\sum_{n=1}^{\infty}\frac{z^n}{1-x^n}\sum_{\substack{l_1,l_2,\ldots,l_j\\ g\text{-composition of } n}} c_g(l_1,l_2,\ldots,l_j)\, x^{\sum_{i=1}^{j}(i-1)l_i}\right),$$

which in turn can be expanded in z leading to the $(1-g)$-inclusion n-body partition function[2]

$$Z_n = x^{(1-g)n(n-1)/2} \sum_{(g-1)(i-1)\leqslant l_i \leqslant l_{i+1},\ldots,(g-1)(n-1)\leqslant l_n \leqslant \infty} x^{\sum_{i=1}^n l_i}$$
$$= \sum_{0\leqslant l_i' \leqslant l_{i+1}'-(1-g),\ldots,0\leqslant l_n'\leqslant\infty} x^{\sum_{i=1}^n l_i'}, \quad (27)$$

where $l_i' = l_i + (1-g)(i-1)$ are a new set of quasi-excitation numbers. One can check that the $(1-g)$-inclusion partition functions (27) are indeed identical to those derived from the inclusion occupation multiplicities introduced in Sec. (V). For example, the $g = 2 \to 1-g = -1$ inclusion two-body and three-body partition functions are from (27)

$$Z_2 = \frac{1+x-x^2}{(1-x)(1-x^2)}, \quad Z_3 = \frac{1+2x-x^3-2x^4+x^6}{(1-x)(1-x^2)(1-x^3)}. \quad (28)$$

In the two-body case, the multiplicities are two for adjacent particles ($n_i = n_{i+1} = 1$) and one for all other configurations. This yields the two-body partition function

$$\sum_{k=0}^{\infty} x^{2k}\left(1+2x+x^2+x^3+\cdots\right) = \frac{1}{1-x^2}\left(\frac{1}{1-x}+x\right),$$

which is identical to Z_2 in (28). Similarly, in the three-body case one obtains starting from the configurations with at least one particle populating the one-body ground state, then no particle in the ground state but at least one populating the first-excited state, etc., and plugging the relevant multiplicities

$$\sum_{k=0}^{\infty} x^{3k}(1+3x+4x^2+5x^3+4x^4+5x^5+5x^6+6x^7$$
$$+6x^8+7x^9+7x^{10}+\cdots),$$

which reproduces Z_3 in (28).

Note that the two-body and three-body partition functions (28) become the bosonic ones when one replaces their numerators by one. The polynomials in the numerators, then, account for the additional degeneracies introduced by inclusion statistics. This is entirely general: the n-body partition function for $(1-g)$-inclusion is given by

$$Z_n = \frac{P_{g,n}(x)}{\prod_{j=1}^n (1-x^j)},$$

where $P_{g,n}(x)$ is a polynomial of degree $gn(n-1)/2$ that satisfies $P_{g,n}(0) = P_{g,n}(1) = 1$. The relation $P_{g,n}(0) = 1$ expresses the nondegenerate nature of the ground state, while $P_{g,n}(1) = 1$ arises from the classical thermodynamic limit: for $x \to 1$ ($\omega \to 0$) the n-body partition function becomes independent of statistics, and thus $P_{g,n}(1)$ for inclusion and the corresponding factor $x^{gn(n-1)/2}$ in (25) for exclusion, must become one. The qualitative difference between $P_{g,n}(x)$ and $x^{gn(n-1)/2}$ demonstrates again the nontrivial nature of the transition from g-exclusion to $(1-g)$ inclusion.

The harmonic trap potential serves as a "box" confining the particles. Reducing its strength enlarges the box and amounts to increasing the one-dimensional volume. Therefore, the thermodynamic limit $x \to 1$ ($\beta\omega \to 0$) at constant chemical potential is related to the infinite volume limit, and we can apply the results of Ref. [7] to recover the thermodynamic properties of the system. The one-body spectrum becomes dense in this limit, with a constant density of states $\rho(\epsilon) = 1/\omega$ typical of a free two-dimensional system, and therefore corresponds to the two-dimensional case in Ref. [7] upon identifying the volume (area) with $h^2/(2\pi m\omega)$ (see the Appendix for details). This leads to the result that a gas of particles in a harmonic trap undergoes condensation for *any* inclusion $1-g < 0$ at critical temperature

$$T_c = \frac{\omega N}{k_B \ln \frac{g}{g-1}}, \quad (29)$$

with N being the number of particles and k_B Boltzmann's constant. This is to be contrasted to bosons, which do not condense in a one-dimensional harmonic trap potential (whereas they do condense in a two-dimensional harmonic trap potential).

It should be stressed that the $\omega \to 0$ limit is distinct from the infinite-volume limit in free space, which leads to the density of states $\rho(\epsilon) \sim \epsilon^{-1/2}$. The free infinite-volume limit can nevertheless be recovered from the $\omega \to 0$ limit by scaling z such that the particle density around the origin remains finite and extracting the corresponding part of the (extensive) grand potential. For details on this thermodynamic limit procedure see Ref. [10] and the Appendix.

We conclude by noting that from the Z_n in (27) we can extract the surprisingly simple $(1-g)$-inclusion statistics Calogero-like spectrum

$$E_n = \omega\left(\sum_{i=1}^n l_i + (1-g)\frac{n(n-1)}{2}\right),$$
$$(g-1)(i-1) \leqslant l_i \leqslant l_{i+1}, \quad (g-1)(n-1) \leqslant l_n, \quad (30)$$

or, equivalently, in terms of the quasi-excitations l_i',

$$E_n = \omega \sum_{i=1}^n l_i', \quad 0 \leqslant l_i' \leqslant l_{i+1}' + g - 1, \quad 0 \leqslant l_n'. \quad (31)$$

We stress, however, that the naïve expectation that the inclusion spectrum (31) is obtained from the spectrum (24) by simply substituting $g \to 1-g$ is incorrect. This is due to the inequalities $(g-1)(i-1) \leqslant l_i$ in (30), or equivalently $0 \leqslant l_i'$ in (31): the corresponding l_i' defined by simply taking the exclusion $l_i' = l_i + g(i-1)$ in (24) and turning $g \to 1-g$ satisfy $l_i' \geqslant (1-g)(i-1)$ and thus can dip below zero for $i > 1$. The two spectra differ by the states produced by such negative values of l_i'. In particular, the ground-state energy of (24) with $g \to 1-g$ is negative, while the ground-state energy of (31) is vanishing and non degenerate. Likewise, the naïve spectrum obtained by substituting $g \to 1-g$ in (23),

[2]It rewrites equivalently as

$$Z_n = \sum_{-(g-1)([(n-1)/2-(i-1)]\leqslant l_i \leqslant l_{i+1},\ldots,(g-1)(n-1)/2\leqslant l_n\leqslant\infty} x^{\sum_{i=1}^n l_i}$$
$$= x^{(1-g)n(n-1)/2} \sum_{(g-1)(n-1)/2\leqslant l_i\leqslant l_{i+1}+(g-1),\ldots,(g-1)(n-1)/2\leqslant l_n\leqslant\infty} x^{\sum_{i=1}^n l_i}.$$

namely,

$$E_n = \omega\left((1-g)\frac{n(n-1)}{2} + \sum_{i=1}^{n} l_i\right),$$
$$0 \leqslant l_1 \leqslant l_2 \leqslant \cdots \leqslant l_n,$$

gives rise to the n-body partition function

$$Z_n = x^{(1-g)n(n-1)/2} \sum_{0\leqslant l_1\leqslant l_2\leqslant\cdots\leqslant l_n\leqslant\infty} x^{\sum_{i=1}^n l_i} = \frac{x^{(1-g)n(n-1)/2}}{\prod_{j=1}^n (1-x^j)}.$$

The thermodynamic limit obtained from the above Z_n does not reproduce the thermodynamics of the inclusion model and, in particular, does not exhibit condensation.

VII. CONCLUSIONS

We obtained a microscopic description of inclusion statistics in terms of the inverse of the grand partition function of exclusion statistics, and reexpressed it in terms of nontrivial occupation multiplicities enhancing the weights of neighboring clusters of one-body quantum states, a key property of inclusion statistics. We also presented a duality relation that maps g-exclusion to $(1-g)$-inclusion statistics, the well-known Bose-Fermi correspondence appearing as the $g = 0, 1$ special case.

The expressions of the aforementioned inclusion multiplicities are relatively simple and intuitive, a fact not a priori obvious, as they are obtained by inverting an already quite nontrivial expression for the g-exclusion grand partition function. The same can be said of the quite simple n-body Calogero-like spectrum that arises from the inversion of the Calogero grand partition function and is key to inclusion statistics in a harmonic one-body spectrum.

There are several open questions or directions for further investigation. On the mathematics side, the combinatorial properties of inclusion statistics appear to be quite rich and deserve further study. In addition, in view of the known connection of exclusion statistics and planar lattice paths [13] or forward-moving paths of Dyck, Motzkin, and Lukasiewicz type [16–18], it is worth investigating the potential connection of inclusion statistics with particular types of lattice path or other processes.

The most interesting remaining questions are those related to the physics of inclusion statistics. As already stressed, the experimental consequences of inclusion are striking, in particular due to the propensity of inclusion particles to undergo macroscopic condensation in planar systems. Therefore, the realization of inclusion statistics in a concrete experimental situation is of physical relevance.

Finally, the realization of the Calogero-like n-body spectrum for $(1-g)$-inclusion statistics in (31) in terms of an n-body Calogero-like Hamiltonian remains a fascinating open issue.

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APPENDIX

1. A combinatorial formula for the lineal counting

There is a formula analogous to (8) for the $(1-g)$-inclusion degenerate lineal counting

$$Z_n = \sum_{n_1,\ldots,n_k=0}^{\infty} m_{1-g}(n_1,\ldots,n_k)\, \delta_{n_1+\cdots+n_k,n},$$

valid for $n \leqslant k/(g-1)+1$, namely,

$$Z_n = \frac{[(k+g)(k+g-1) - g(g-1)n](k+g+gn-2)!}{n![k+g+(g-1)n]!},$$

which, for example in the case $g = 2 \to 1-g = -1$, leads to the generating function

$$\sum_{n=0}^{\infty} Z_n z^n = \sqrt{1-4z}/\, y_1^{k+2}(-z,2),$$

i.e., (21), which is the inverse of $\mathcal{Z}_2$ in (9) with $z \to -z$ and only the term y_1 kept. Likewise for $g = 3$ we would obtain the generating function

$$\sum_{n=0}^{\infty} Z_n z^n = [y_1(-z,3) - y_2(-z,3)]$$
$$\times [y_1(-z,3) - y_3(-z,3)]/\, y_1^{k+4}(-z,3),$$

where the y_i are given in (5), again the inverse of (7) for $g = 3$ with $z \to -z$ and only the term y_1 kept. The terms $n > k/(g-1)+1$ make nonperturbative contributions.

2. Periodic spectrum and periodic multiplicities

The circular periodic counting (11) for a periodic degenerate one-body spectrum can also be formulated more generally for a periodic nondegenerate spectrum $\epsilon_1, \epsilon_2, \ldots, \epsilon_k$. Again the key ingredient is the n-body partition function for particles with g-exclusion in the above one-body periodic spectrum

$$Z'_n = \sum_{k_1=1}^{k-gn+\min(g,k_n)} \sum_{k_2=1}^{k_1} \cdots \sum_{k_n=1}^{k_{n-1}} s(k_1+gn-g) \cdots s(k_{n-1}+g)s(k_n). \tag{A1}$$

Clearly (A1) reduces to the $g = 2$ periodic counting (11) when the spectrum is degenerate. Note that (A1) requires $k \geqslant gn$. For $k < g$, even a single particle excludes itself through its periodic image and $Z'_1 = 0$. The corresponding grand partition function is

$$\mathcal{Z}'_g = \sum_{n=0}^{[k/g]} Z'_n z^n.$$

All this information can be encapsulated in the k-dimensional periodic g-exclusion matrix, for $k > g$, which is similar to the lineal one in (19) but with the off-diagonals wrapping around the matrix. For example, for $g = 2$,

$$H_2' = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & -s(k) \\ -s(1) & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & -s(k-1) & 0 \end{pmatrix}.$$

Its secular determinant reproduces $\mathcal{Z}_2'$ up to a residual term,

$$\det(1 - z^{1/2}H_2') = \sum_{n=0}^{[k/2]} Z_n' z^n + \left(-1 - \prod_{i=1}^{k}[-s(i)]\right) z^{k/2},$$

where $\{-1 - \prod_{i=1}^{k}[-s(i)]\} z^{k/2}$ is a spurious "Wilson loop" contribution that can be consistently discarded. Similar results hold for $g > 2$, where the terms $-s(i)$ in the k-dimensional matrix H_g' $(k > g)$ are in the $(g-1)$-lower diagonal wrapping periodically around the matrix. In that case,

$$\det(1 - z^{1/g}H_g') = \sum_{n=0}^{[k/g]} Z_n' z^n + W_g(z),$$

where now the spurious term W_g starts at order $\approx z^{k/[g(g-1)]}$ and goes up to $z^{k/g}$. For $g = 2$, this term can be eliminated by an appropriate modification of the matrix that slightly complicates its form, but its elimination for $g > 2$ becomes more involved.

Microscopic $(1-g)$-inclusion statistics on a periodic one-body spectrum must be defined such that it satisfies appropriate physical and consistency conditions: it must give the correct thermodynamic limit and involve integer, non-negative state multiplicities. Furthermore, the multiplicities must locally agree with these for lineal counting, meaning that the state multiplicity of a cluster of particles on one-body states that span a small subset of the full spectrum must be the same as those for the corresponding cluster on a lineal spectrum, otherwise they could "sense" the topology of the spectrum, introducing a nonlocal element. These conditions largely fix the definition of periodic microscopic inclusion statistics, but still leave some distinct possibilities.

One approach is to postulate as in the lineal case an exact duality with g-exclusion statistics, and define the $(1-g)$-inclusion grand partition function as the inverse of the periodic microscopic g-exclusion partition function with $z \to -z$. Taking the $g = 2$ case as an example, the grand partition function on k degenerate periodic levels is given in (10) as $\mathcal{Z}_2 = y_1^k(z, 2) + y_2^k(z, 2)$. Thus the resulting $1 - g = -1$ inclusion grand partition function obtained by duality is

$$\mathcal{Z}_{-1}' = \frac{1}{y_1^k(-z, 2) + y_2^k(-z, 2)} = \frac{1}{y_1^{-k}(z, -1) + y_2^{-k}(z, -1)},$$

and for general $g \to 1 - g$ and a degenerate spectrum,

$$\frac{1}{\mathcal{Z}_{1-g}'} = \sum_{i=1}^{g} y_i^k(-z, g) = \sum_{i=1}^{g} y_i^{-k}(z, 1-g). \tag{A2}$$

This definition fulfills all the desired conditions.

A different approach to periodic microscopic inclusion would be to define the occupation number multiplicities by periodizing the lineal multiplicities m_{1-g} of Sec. (V) in an obvious way. For the simplest case of $g = 2 \to 1 - g = -1$ one would write

$$m_{-1}'(n_1, \ldots, n_k) = \binom{n_1 + n_2}{n_1}\binom{n_2 + n_3}{n_2} \cdots \binom{n_{k-1} + n_k}{n_{k-1}}\binom{n_k + n_1}{n_k}.$$

Likewise, for $g = 3 \to 1 - g = -2$,

$$m_{-2}'(n_1, \ldots, n_k) = \binom{n_1 + n_2 + n_3}{n_1}\binom{n_2 + n_3 + n_4}{n_2} \cdots \binom{n_{k-1} + n_k + n_1}{n_{k-1}}\binom{n_k + n_1 + n_2}{n_k},$$

and, in general, for $(1-g)$ inclusion,

$$m_{1-g}'(n_1, \ldots, n_k) = \frac{[1, g][2, 1+g] \cdots [k, k+g-1]}{[1, g-1][2, g] \cdots [k, k+g]}.$$

From the above multiplicities one can obtain the grand partition function for a degenerate spectrum. For $g = 2 \to 1 - g = -1$ we obtain

$$\mathcal{Z}_{-1}' = \sum_{n_1, \ldots, n_k = 0}^{\infty} m_{-1}'(n_1, \ldots, n_k) z^{n_1 + \cdots + n_k} = \frac{1}{y_1^k(-z, 2) - y_2^k(-z, 2)} = \frac{1}{y_1^{-k}(z, -1) - y_2^{-k}(z, -1)}. \tag{A3}$$

Comparing with (A2), we see that it differs only in the sign of the term y_2, and thus the two formulas differ only by nonperturbative in $1/k$ terms. This shows that both definitions, via duality of by periodizing the m_{1-g}, lead to the same multiplicities for a number of particles $n < k/g$, which are locally the same as the lineal multiplicities, but start diverging as the particles populate the full one-body spectrum.

We conclude with the remark that one could calculate a periodic inclusion grand partition function for a degenerate spectrum by simply substituting $g \to 1 - g$ in the periodic counting formula (11). Unlike in the g-exclusion case, the corresponding $1 - g$ counting formula gives positive and integer results for all n, including $n > k/g$. This approach is less useful than the previous ones, as it does not obviously generalize to a nondegenerate spectrum, but nevertheless leads to

$$\mathcal{Z}_{1-g}' = \sum_{n=0}^{\infty} \frac{k(k + gn - 1)!}{n!(k + gn - n)!} z^n = y_1^{-k}(-z, g) = y_1^k(z, 1-g),$$

a simple result that, again, differs from (A2) and (A3) only in nonperturbative terms.

3. Proof of the generating function formulas (7) and (10)

Consider the generating function $\mathcal{Z}_{g,k}$ of g-exclusion particles on k degenerate one-body states on a line, and the corresponding one $\mathcal{Z}_{g,k}'$ on a circle (respectively, lineal and periodic counting). For $\mathcal{Z}_{g,k}$, focusing on the state at the end of the chain and putting it to be either empty, leaving an unrestricted system on $k - 1$ levels, or occupied, excluding g

levels and leaving an unrestricted system on $k-g$ levels, we have

$$\mathcal{Z}_{g,k} = \mathcal{Z}_{g,k-1} + z\mathcal{Z}_{g,k-g}. \tag{A4}$$

For $\mathcal{Z}'_{g,k}$, we focus on a set of $g-1$ adjacent states. There can be either no particles in these states, leaving an open (lineal) chain with $k-g$ states, or one particle, excluding $g-1$ states on each side and leaving an open chain with $k-2(g-1)-1$ states. We obtain

$$\mathcal{Z}'_{g,k} = \mathcal{Z}_{g,k-g+1} + (g-1)z\mathcal{Z}_{g,k-2g+1}. \tag{A5}$$

Applying (A4) in (A5) we obtain

$$\begin{aligned}\mathcal{Z}'_{g,k} &= \mathcal{Z}_{g,k-g} + z\mathcal{Z}_{g,k-2g+1} + (g-1)z(\mathcal{Z}_{g,k-2g} + z\mathcal{Z}_{g,k-3g+1})\\ &= \mathcal{Z}_{g,k-g} + (g-1)z\mathcal{Z}_{g,k-2g}\\ &\quad + z[\mathcal{Z}_{g,k-2g+1} + (g-1)z\mathcal{Z}_{g,k-3g+1}]\\ &= \mathcal{Z}'_{g,k-1} + z\mathcal{Z}'_{g,k-g}.\end{aligned}$$

(Note that the same argument works by focusing on g adjacent states, but for no other number. Had we chosen fewer adjacent states, the ends of the broken open chain would be subject to exclusion constraints and would not reproduce $\mathcal{Z}$; had we chosen more states, we could have placed more than one particle in them.)

Thus, both $\mathcal{Z}_{g,k}$ and $\mathcal{Z}'_{g,k}$ satisfy the same linear recursion relation in k. The solution of this recursion equation can be obtained in the usual way by taking an exponential ansatz y^k, which leads to the characteristic equation (4), that is, the exclusion equation. The general solution will be

$$A_1 y_1^k + \cdots + A_g y_g^k,$$

with y_i the g solutions of (4) and A_i arbitrary coefficients.

$\mathcal{Z}_{g,k}$ and $\mathcal{Z}'_{g,k}$ differ only in their initial conditions. We have

$$\mathcal{Z}_{g,k} = 1 + kz, \quad k = 1, 2, \dots, g, \tag{A6}$$

$$\mathcal{Z}'_{g,k} = 1, \quad k = 1, \dots, g-1, \quad \mathcal{Z}'_{g,g} = 1 + gz, \tag{A7}$$

since there can be either no particles (on the periodic spectrum) or one particle (on the lineal spectrum) for $k < g$, and one particle for $k = g$. Writing (4) in factorized form in terms of its roots y_i and taking its logarithm gives

$$\sum_{i=1}^{g} \ln(y - y_i) = \ln(y^g - y^{g-1} - z).$$

Expanding the two sides in $1/y$ and matching powers yields

$$\sum_{i=1}^{g} y_i^k = 1, \quad k = 1, 2, \dots, g-1,$$

$$\sum_{i=1}^{g} y_i^g = 1 + gz.$$

Comparing this with (A7) we see that $A_1 = \cdots = A_g = 1$ for $\mathcal{Z}'_{g,k}$, so

$$\mathcal{Z}'_{g,k} = \sum_{i=1}^{g} y_i^k,$$

i.e., (10). Note that y_i^k for $i > 1$ make purely nonperturbative contributions.

For $\mathcal{Z}_{g,k}$ we need to solve the g linear equations (A6) for the A_i in terms of the y_i. This is tedious, but the result can be obtained analytically. For $g = 2$ we find

$$\mathcal{Z}_{2,k} = \frac{y_1^{k+2} - y_2^{k+2}}{y_1 - y_2} = \frac{y_1^{k+2} - y_2^{k+2}}{\sqrt{1+4z}},$$

for $g = 3$

$$\begin{aligned}\mathcal{Z}_{3,k} &= \frac{y_1^{4+k}}{(y_1-y_2)(y_1-y_3)} + \frac{y_2^{4+k}}{(y_2-y_1)(y_2-y_3)}\\ &\quad + \frac{y_3^{4+k}}{(y_3-y_1)(y_3-y_2)}\\ &= \frac{y_1^{4+k}(y_2-y_3) + y_2^{4+k}(y_3-y_1) + y_3^{4+k}(y_1-y_2)}{\mathrm{i}\sqrt{z(4+27z)}},\end{aligned}$$

and for general g

$$\mathcal{Z}_{g,k} = \sum_{i=1}^{g} y_i^{2(g-1)+k} \prod_{j\neq i, j=1}^{g} \frac{1}{y_i - y_j} = \sum_{i=1}^{g} \frac{y_i^{k+g}}{gy_i - g + 1},$$

i.e., (7). Again the terms $i > 1$ in the sum make nonperturbative in $1/k$ contributions, but the extra factor $y_1^g/(gy_1 - g + 1)$ in the first term also makes perturbative in $1/k$ contributions to the thermodynamic grand partition function y_1^k. In this sense, the periodic spectrum is "closer" to the thermodynamic limit than the lineal one.

4. Thermodynamics in a weak potential trap and in free space

A one-dimensional harmonic trap potential acts as a "box," and taking the strength of the potential to zero should recover the free space result. However, the limit is nontrivial and has to be taken appropriately. In particular, the density of states in the potential will not converge to that of free space. For example, the harmonic-oscillator density of states is constant and equal to $1/\omega$, while the free density of states in a large flat box of length L is $L/\pi\sqrt{2\epsilon}$ (we put the mass of the particle and $\hbar$ to one). Clearly the two have different energy dependence and will never converge in the limit $\omega \to 0$, $L \to \infty$. Thermodynamics "sees" the effect of the potential at large distances.

Assume the system is in a potential $V(x/\lambda)$, with λ a scaling parameter. Taking $\lambda \to \infty$ spreads the potential and corresponds to the infinite-volume limit. In that limit, the potential becomes increasingly smooth, as its derivatives scale like $1/\lambda$. For large enough λ we can cut the system into intervals of size L and take L large enough to contain a macroscopically large number of particles but small enough for the potential inside it to be considered constant. The full grand potential $\ln\mathcal{Z}$ of the system will be the sum of the grand potentials $\ln\mathcal{Z}_n$ of the part of the system in each interval $nL < x < (n+1)L$,

$$\ln\mathcal{Z}(\lambda,\mu) = \sum_n \ln\mathcal{Z}_n(\lambda,\mu),$$

where we displayed the dependence on λ and the chemical potential μ [$z = \exp(\beta\mu)$ being the fugacity]. Since the potential is almost constant in each interval, $\ln\mathcal{Z}_n$ is the grand potential in free space of length L, and the constant potential simply shifts the chemical potential of the system. Calling $\ln \mathcal{Z}_{\text{free}}(L, \mu)$ the free grand potential, we have

$$\ln \mathcal{Z}(\lambda, \mu) \simeq \sum_n \ln \mathcal{Z}_{\text{free}}[L, \mu - V(nL/\lambda)],$$

and in the limit $\lambda \to \infty$, turning the sum to an integral,

$$\frac{1}{\lambda} \ln \mathcal{Z}(\lambda, \mu) = \frac{1}{L} \int dx \ln \mathcal{Z}_{\text{free}}[L, \mu - V(x)]. \tag{A8}$$

The above provides an integral relation between $\mathcal{Z}$ and $\mathcal{Z}_{\text{free}}$. Since $\ln \mathcal{Z}_{\text{free}}$ is extensive it scales as L, and so the left-hand side (LHS) of (A8) is independent of L. Likewise, $\ln \mathcal{Z}(\lambda, \mu)$ will scale as λ, and the LHS is independent of λ in the limit $\lambda \to \infty$. This implies the relation between particle numbers

$$\frac{1}{\lambda} N(\lambda, \mu) = \frac{1}{L} \int dx N_{\text{free}}[L, \mu - V(x)], \tag{A9}$$

and expanding the grand potentials in the fugacity $z = e^{\beta\mu}$ we obtain the relation of the cluster coefficients

$$\frac{1}{\lambda} c_{\lambda,n} = \frac{1}{L} c_{L,n} \int dx e^{-n\beta V(x)}. \tag{A10}$$

This result can also be obtained from the relation (13) derived in Sec. (2). Indeed, in the thermodynamic limit, the single-particle partition function in the LHS of (13) can be well approximated by its semiclassical expression

$$\begin{aligned} \sum_i e^{-n\beta\epsilon_i} &= \int \frac{dxdp}{2\pi} e^{-n\beta H(p,x)} \\ &= \int \frac{dxdp}{2\pi} e^{-n\beta(p^2/2 + V(x))} \\ &= \frac{1}{\sqrt{2\pi n\beta}} \int dx e^{-n\beta V(x)}, \end{aligned}$$

and applying it to a potential $V(x)$, and to a square well of width L, we reproduce (A10). In particular, we obtain for a harmonic potential trap $V(x) = \frac{1}{2}\omega^2 x^2$,

$$c_{\omega,n}(g) = \frac{c_n(g)}{n\beta\omega}, \tag{A11}$$

and for a free system of size L

$$c_{L,n}(g) = L \frac{c_n(g)}{\sqrt{2\pi n\beta}}, \tag{A12}$$

that is, the bosonic results times the statistics factor $c_n(g)$.

The fact that the system of inclusion statistics particles in a one-dimensional harmonic trap manifests condensation, while the free system does not, can be derived from (A9): for $z \to z_{\max} = (g-1)^{g-1}/g^g$, the density of particles for the free system diverges at $x = 0$ but remains finite for other x, and the integral over x is finite, implying a maximal number of particles and condensation.

The g-exclusion harmonic (Calogero) cluster expansion (26) in the $\beta\omega$ small limit, i.e., $x \simeq 1 - \beta\omega$, becomes

$$\ln \mathcal{Z}_g = -\sum_{n=1}^{\infty} (-z)^n \frac{1}{n\beta\omega} \sum_{\substack{l_1, l_2, \ldots, l_j \\ g\text{-composition of } n}} c_g(l_1, l_2, \ldots, l_j),$$

which, thanks to the identity [13]

$$\sum_{\substack{l_1, l_2, \ldots, l_j \\ g\text{-composition of } n}} c_g(l_1, l_2, \ldots, l_j) = \frac{\binom{gn}{n}}{gn},$$

reduces to

$$\ln \mathcal{Z}_g = \sum_{n=1}^{\infty} \frac{1}{n\beta\omega} c_n(g) z^n. \tag{A13}$$

This agrees with the direct calculation from (12) and (13) upon using the constant density of states for the oscillator $\rho(\epsilon) = \omega^{-1}$ and performing the integral in (13).

The above analysis also reproduces the results of Ref. [10] that the free cluster coefficients in volume L can be obtained from the corresponding harmonic cluster coefficients in the thermodynamic limit $\beta\omega \to 0$ by doing the substitution

$$\frac{1}{n\beta\omega} \to \frac{L}{\sqrt{2\pi n\beta}} \tag{A14}$$

arising from comparing (A11) to (A12). With this prescription, the cluster expansion of the harmonic model in the thermodynamic limit maps to that of the free system. Indeed, implementing (A14) in (A13) we obtain

$$\ln \mathcal{Z}_g = L \sum_{n=1}^{\infty} \frac{1}{\sqrt{2\pi n\beta}} c_n(g) z^n. \tag{A15}$$

Rewriting in (A15) $1/\sqrt{2\pi n\beta} = \int_0^\infty \mathrm{e}^{-n\beta k^2/2}\, dk/\pi$ and changing the integration variable to $\epsilon = k^2/2$ we get finally [10]

$$\ln \mathcal{Z}_g = \int_0^\infty \rho(\epsilon) \ln y_1(z\mathrm{e}^{-\beta\epsilon}, z)\, d\epsilon,$$

i.e., (12) where $\rho(\epsilon) = L/(\pi\sqrt{2\epsilon})$ is now the free one-dimensional density of states, as expected.

[1] M. G. G. Laidlaw and C. M. DeWitt, Feynman functional integrals for systems of indistinguishable particles, Phys. Rev. D **3**, 1375 (1971).

[2] J. M. Leinaas and J. Myrheim, On the theory of identical particles, Nuovo Cim. B **37**, 1 (1977).

[3] F. Wilczek, Magnetic flux, angular momentum, and statistics, Phys. Rev. Lett. **48**, 1144 (1982); Quantum mechanics of fractional-spin particles, Nuovo Cim. B **49**, 957 (1982).

[4] F. D. M. Haldane, Fractional statistics in arbitrary dimensions: A generalization of the Pauli principle, Phys. Rev. Lett. **67**, 937 (1991).

[5] D. P. Arovas, R. Schrieffer, and F. Wilczek, Fractional statistics and the quantum Hall effect, Phys. Rev. Lett. **53**, 722 (1984).

[6] C. Nayak and F. Wilczek, Exclusion statistics: Low-temperature properties, fluctuations, duality, and applications, Phys. Rev. Lett. **73**, 2740 (1994).

[7] S. Ouvry and A. P. Polychronakos, Inclusion statistics and particle condensation in two dimensions, Phys. Rev. E **107**, L062102 (2023).

[8] S. Ouvry and A. P. Polychronakos, Exclusion statistics for particles with a discrete spectrum, Nucl. Phys. B **972**, 115573 (2021).

[9] A. P. Polychronakos, Nonrelativistic bosonization and fractional statistics, Nucl. Phys. B **324**, 597 (1989).

[10] A. Dasnières de Veigy and S. Ouvry, Equation of state of an anyon gas in a strong magnetic field, Phys. Rev. Lett. **72**, 600 (1994); One-dimensional statistical mechanics for identical particles: the Calogero and anyon cases, Mod. Phys. Lett. A **10**, 1 (1995); **09**, 271 (1995); S. Ouvry, On the relation between the anyon and the Calogero models, Phys. Lett. B **510**, 335 (2001).

[11] Y. S. Wu, Statistical distribution for generalized ideal gas of fractional-statistics particles, Phys. Rev. Lett. **73**, 922 (1994).

[12] A. P. Polychronakos, Probabilities and path-integral realization of exclusion statistics, Phys. Lett. B **365**, 202 (1996).

[13] S. Ouvry and A. P. Polychronakos, Exclusion statistics and lattice random walks, Nucl. Phys. B **948**, 114731 (2019).

[14] P. L. Giscard and P. Rochet, Algebraic combinatorics on trace monoids: Extending number theory to walks on graphs, SIAM J. Discrete Math. **31**, 1428 (2017).

[15] F. Calogero, Solution of a three-body problem in one dimension, J. Math. Phys. **10**, 2191 (1969); Ground state of a one-dimensional N-body system, **10**, 2197 (1969); Solution of the one-dimensional N-body problems with quadratic and/or inversely quadratic pair potentials, **12**, 419 (1971).

[16] S. Ouvry and A. P. Polychronakos, Hamiltonian and exclusion statistics approach to discrete forward-moving paths, Phys. Rev. E **104**, 014143 (2021).

[17] A. P. Polychronakos, Length and area generating functions for height-restricted Motzkin meanders, Phys. Rev. E **105**, 024102 (2022).

[18] L. Gan, S. Ouvry, and A. P. Polychronakos, Combinatorics of generalized Dyck and Motzkin paths, Phys. Rev. E **106**, 044123 (2022).