Letter

Effective action for dissipative and nonholonomic systems

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We show that the action of a dynamical system can be supplemented by an effective action for its environment to reproduce arbitrary coordinate dependent ohmic dissipation and gyroscopic forces. The action is a generalization of the harmonic bath model and describes a set of massless interacting scalar fields in an auxiliary space coupled to the original system at the boundary. A certain limit of the model implements nonholonomic constraints. In the case of dynamics with nonlinearly realized symmetries the effective action takes the form of a two-dimensional nonlinear σ model. It provides a basis for application of path integral methods to general dissipative and nonholonomic systems.

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Introduction. Dissipation is ubiquitous in nature. The standard way to account for it in the classical theory of dynamical systems is by adding nonconservative forces F_i to the Euler-Lagrange equations of motion,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = F_i, \tag{1}$$

where $L(q, \dot{q})$ is the system Lagrangian and q^i, $\dot{q}^i$ are the generalized coordinates and velocities, respectively. An important type of dissipation is ohmic dissipation when the extra forces are linear in velocities,

$$F_i = -\Gamma_{ij}(q)\,\dot{q}^j \equiv -\frac{\partial F}{\partial \dot{q}^i}. \tag{2}$$

The dissipative coefficients $\Gamma_{ij}(q)$ form a positive-definite symmetric matrix and are in general coordinate dependent. In the last equality we have conventionally written the force as the derivative of the Rayleigh function $F(q, \dot{q}) = (1/2)\Gamma_{ij}(q)\,\dot{q}^i\dot{q}^j$.

Fundamentally, the existence of dissipation is due to the interaction of the system (referred to as *central system* below) with its environment, also called *reservoir* or *bath*. In many applications the microscopic nature of the reservoir is not important and it can be modeled as a collection of infinitely many harmonic oscillators [1]. The action of the harmonic bath coupled to the central system then provides an effective action, from which Eq. (1) can be derived by means of the variational principle. Yet more importantly, the effective action is key for the application of the path integral methods used to study intrinsically quantum phenomena, such as tunneling [2,3], and other aspects of open systems in and out of thermal equilibrium [4–6].

However, as we discuss below, the harmonic bath model fails in the case when dissipation coefficients $\Gamma_{ij}(q)$ have general dependence on the system coordinates. The purpose of this Letter is to provide a reservoir model for this case. As a by-product we also obtain the description of arbitrary gyroscopic forces. Before describing the model, let us discuss two broad classes of situations where the dependence of Γ_{ij} on q is essential.

Dynamics on cosets. The first class are systems whose configuration space represents a group manifold or, more generally, a coset space, and whose dynamics enjoy nonlinearly realized symmetries. Many physically relevant systems can be cast in this form, from dynamics of a rigid body to hydrodynamics [7,8]. They appear in particle physics and condensed matter as a consequence of spontaneous symmetry breaking [9,10]. Development of an effective action for such systems in dissipative environment, besides conceptual interest, is motivated by numerous potential applications, for example to Brownian motion of stiff polymers [11], as well as micro- and nanoparticles of various shapes [12–15].

Following the standard coset construction [9,10,16], we consider a Lie group $\mathbb{G}$ and its subgroup $\mathbb{H}$. The generators of $\mathbb{G}$ are chosen in such a way that the first A of them span the algebra of $\mathbb{H}$; we denote them by H_a, $1 \leqslant a \leqslant A$. The rest of the $\mathbb{G}$ generators are called *broken* and will be denoted by $\hat{G}_i$, $1 \leqslant i \leqslant I$. The coset $\mathbb{G}/\mathbb{H}$ representing the configuration space of the system can be identified with the group elements of the form[1]

$$\hat{g}(q) = e^{q^i \hat{G}_i}. \tag{3}$$

The action of a group element g on the coordinates $q \mapsto \tilde{q}$ is given by the right multiplication,

$$g \cdot \hat{g}(q) = \hat{g}(\tilde{q}) \cdot h, \quad g \in \mathbb{G},\ h \in \mathbb{H}. \tag{4}$$

Next, one constructs the Cartan form,

$$\hat{g}^{-1}d\hat{g} = \Omega^i_j(q)dq^j\,\hat{G}_i + \Omega^a_j(q)dq^j\,H_a \tag{5}$$

and extracts from it the *covariant velocities*

$$D_t q^i = \Omega^i_j(q)\,\dot{q}^j. \tag{6}$$

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[1] We assume summation over repeated indices.

Unlike the ordinary velocities $\dot{q}^i$, the covariant velocities transform linearly under (4). They form a linear representation of the subgroup $\mathbb{H}$.

If the dynamics of the system are to respect the symmetry (4), its Lagrangian and Rayleigh function must be invariants constructed from the covariant velocities.[2] Thus we have

$$F = \frac{1}{2}\gamma_{ij} D_t q^i D_t q^j = \frac{1}{2}\gamma_{ij}\Omega^i_k(q)\Omega^j_l(q)\,\dot{q}^k\dot{q}^l, \tag{7}$$

where γ_{ij} is a constant invariant tensor in the relevant representation of $\mathbb{H}$. In the simplest case when $\mathbb{H}$ is empty ($\mathbb{G}$ is fully broken) γ_{ij} is arbitrary, provided it is symmetric and positive. For a general non-Abelian coset the coefficients of the Cartan form satisfy

$$\frac{\partial\Omega^k_i}{\partial q^j} - \frac{\partial\Omega^k_j}{\partial q^i} \neq 0, \tag{8}$$

so their dependence on coordinates cannot be avoided by any choice of variables, implying the coordinate dependence of the dissipative coefficients $\Gamma_{kl} = \gamma_{ij}\Omega^i_k(q)\Omega^j_l(q)$.

Nonholonomic systems. The second class are systems with constraints on coordinates and velocities,[3]

$$c^\alpha_i(q)\,\dot{q}^i = 0, \quad \alpha = 1,\dots n < I, \tag{9}$$

such that they cannot be integrated into constraints only on coordinates. In other words, Eq. (9) is not equivalent to a set of constraints of the form $\dot{\varphi}^\alpha(q) = 0$. Clearly, this requires

$$\frac{\partial c^\alpha_i}{\partial q^j} - \frac{\partial c^\alpha_j}{\partial q^i} \neq 0. \tag{10}$$

These systems are called nonholonomic and typical examples include rolling of a disk or a ball on a hard surface. Their classical dynamics is well developed and is summarized in excellent textbooks; see, e.g., Refs. [20,21]. Quantization, however, remains an open problem. It was addressed in [22–24] and presents a growing interest due to development of molecular machines [25–27].

Typically, the equations of motion for nonholonomic systems are derived from a modified variational principle restricted to admissible variations δq^i satisfying the constraints $c^\alpha_i\,\delta q^i = 0$. This leads to the appearance of *reaction forces* on the right-hand side (RHS) of the Euler-Lagrange equations (1),

$$F_i = \lambda_\alpha c^\alpha_i(q), \tag{11}$$

where $\lambda_\alpha(t)$ are Lagrange multipliers.[4] Due to the constraints (9), the reaction forces do not produce any work, $F_i\dot{q}^i = 0$, so nonholonomic systems are not truly dissipative. However, they are closely related through the following construction [20,21]. Consider a dissipative system with the Rayleigh function

$$F = \frac{1}{2}\gamma\, c^\alpha_i(q)c^\alpha_j(q)\,\dot{q}^i\dot{q}^j \tag{12}$$

and take the limit $\gamma \to +\infty$. The friction associated with the linear combinations of velocities $c^\alpha_j\dot{q}^j$ becomes very strong and the corresponding combinations quickly die out rendering the constraints (9). On the other hand, the products $\gamma c^\alpha_i\dot{q}^i$ remain finite and become independent variables—the Lagrange multipliers of Eq. (11). Thus the nonholonomic dynamics can be viewed as the limit of infinitely strong viscous friction along the constrained directions.

Reservoir model. Our starting point is the model used in [28] to study environment-induced decoherence. It represents the reservoir as a free massless scalar field $\xi(t,z)$ in one-dimensional space (*bulk*) coupled to the central system at a single point $z = 0$ (*boundary*) and is equivalent to the more common independent-oscillator model [3]. Its straightforward generalization for a central system with several degrees of freedom requires equal number of fields and leads to the following action:

$$S = \int_{z=0} dt\,\big(L(q,\dot{q}) - \beta^j_i q^i\dot{\xi}_j\big) + \int_{z>0} dt\,dz\,\frac{1}{2}\partial_\mu\xi_i\partial^\mu\xi_i, \tag{13}$$

where β^j_i are constant couplings; in the last term we sum over indices $\mu = t, z$ with the Lorentzian metric $\eta^{\mu\nu} = \mathrm{diag}(1,-1)$. Importantly, the coordinate z here is not a physical dimension, but is introduced merely to parametrize the internal dissipative degrees of freedom. By taking variation, one derives the dissipative forces, as well as the equations for the fields,

$$F_i = -\beta^j_i\dot{\xi}_j|, \quad \partial_\mu\partial^\mu\xi_i = 0, \quad \partial_z\xi_i| = -\beta^i_j\dot{q}^j, \tag{14}$$

where the vertical bar means fields evaluated at $z = 0$. The dissipative dynamics is obtained by imposing outgoing boundary conditions on the bulk fields which singles out the solutions of the form $\xi_i(t,z) = \bar{\xi}_i(t-z)$. This implies $\partial_z\xi_i = -\partial_t\xi_i$ and combining the first and third equations in (14) we obtain the forces (2) with $\Gamma_{ij} = \beta^k_i\beta^k_j$. Note that coupling q^i to $\dot{\xi}_i$, rather than the fields themselves, is essential for getting the response local in time.

The above construction fails for general coordinate dependent dissipation. As long as we want to preserve the harmonic nature of the bath, the only option is to generalize its coupling to the central system, $\beta^j_i q^i\dot{\xi}_j \mapsto \beta^j(q)\dot{\xi}_j$, with some arbitrary functions $\beta^i(q)$. Repeating the above derivation we then obtain the dissipative coefficients $\Gamma_{ij} = (\partial\beta^k/\partial q^i)(\partial\beta^k/\partial q^j)$ which, however, do not have the form needed for coset or nonholonomic systems due to the nonintegrability properties (8) and (10).

This failure can be also understood from the symmetry perspective. The system-reservoir coupling in (13) changes by a total time derivative under the shifts of the coordinates $q^i(t) \mapsto q^i(t) + a^i$. This property ensures that the dissipative force is invariant under the coordinate shifts, as it should be for the case of constant Γ_{ij}. In the case of a general non-Abelian coset, however, we do not have at our disposal any functions $\beta^i(q)$ invariant or changing by a constant under the group transformations and hence we cannot construct any

[2]Up to possible Wess-Zumino-Witten terms [17–19].

[3]We only consider constraints linear in velocities.

[4]Note that adding the constraints (9) with Lagrange multipliers into the Lagrangian, instead of the equations of motion, would not reproduce the correct nonholonomic dynamics. Instead, one would obtain a so-called vakonomic system [21].

system-reservoir coupling that would preserve the symmetry of the problem.[5]

To resolve the issue, we apply a duality transformation to the action (13). Performing a change of variables $\tilde{\xi}_i = \beta_i^j \xi_j$ and integrating in a set of vectors $\chi^{\mu i}$, it can be rewritten as

$$S = \int_{z=0} dt\,(L(q,\dot{q}) - q^i \dot{\tilde{\xi}}_i) + \int_{z>0} dt\,dz \left(\chi^{\mu i}\partial_\mu \tilde{\xi}_i - \frac{\Gamma_{ij}}{2}\chi^{\mu i}\chi_\mu^j \right). \qquad (15)$$

We now integrate out $\tilde{\xi}_i$, which gives two equations,

$$\partial_\mu \chi^{\mu i} = 0, \quad \chi^{zi}| = \dot{q}^i. \qquad (16)$$

The first one implies that $\chi^{\mu i}$ are expressed through gradients of scalar functions,

$$\chi^{\mu i} = -\epsilon^{\mu\nu}\partial_\nu \chi^i, \qquad (17)$$

where $\epsilon^{\mu\nu}$ is the two-dimensional Levi-Civita symbol, $\epsilon^{tz} = 1$. The second equation then reduces to $\dot{\chi}^i| = \dot{q}^i$. Using the fact that the fields χ^i are defined up to a constant, we can remove any offset between them and q^i on the boundary and obtain

$$\chi^i| = q^i. \qquad (18)$$

Substituting (17) back into (15) we arrive at the action

$$S = \int dt\, L(q,\dot{q}) + \int_{z>0} dt\,dz\,\frac{1}{2}\Gamma_{ij}\partial_\mu \chi^i \partial^\mu \chi^j, \qquad (19)$$

with the boundary conditions (18). For a single degree of freedom q this action first appeared in [29] and was used in [30] for the derivation of the quantum Langevin equation. More recently, it was extended to describe linear response in dissipative media [31].

So far, we have assumed the dissipative coefficients Γ_{ij} to be constant. However, the action (19) admits a natural generalization. Relation (18) suggests thinking of the fields χ^i as extensions of the original system coordinates into the bulk. Then, to describe coordinate dependent dissipation, we simply need to promote the coefficients in (19) to the functions of χ,

$$\Gamma_{ij} \mapsto \Gamma_{ij}(\chi). \qquad (20)$$

Note that this makes the effective reservoir fields self-interacting. It is a necessary price to pay for modeling coordinate dependent friction.

This is not yet the whole story. We can add to the reservoir action a time-reversal breaking term

$$\int_{z>0} dt\,dz\,\frac{1}{2}\Upsilon_{ij}(\chi)\epsilon^{\mu\nu}\partial_\mu\chi^i\partial_\nu\chi^j, \qquad (21)$$

with antisymmetric coefficients $\Upsilon_{ij}(\chi)$. If Υ_{ij} are constant, this term is a total derivative and reduces to the boundary term $\int dt\, \Upsilon_{ij} q^i \dot{q}^j$ of the Wess-Zumino-Witten type [17–19]. However, for the general field dependent coefficients such reduction is impossible.

Combining all the above ingredients, we write down the action of our reservoir model:

$$S = \int dt\, L(q,\dot{q}) + \int_{z>0} dt\,dz\,\frac{1}{2}(\Gamma_{ij}(\chi)\partial_\mu\chi^i\partial^\mu\chi^j + \Upsilon_{ij}(\chi)\epsilon^{\mu\nu}\partial_\mu\chi^i\partial_\nu\chi^j). \qquad (22)$$

Let us verify that it reproduces the desired equations. Taking its variation and accounting for the relation (18) we obtain in the bulk and on the boundary

$$\partial_\mu(\Gamma_{ij}\partial^\mu\chi^j + \Upsilon_{ij}\epsilon^{\mu\nu}\partial_\nu\chi^j) - \frac{1}{2}\frac{\partial\Gamma_{jk}}{\partial\chi^i}\partial_\mu\chi^j\partial^\mu\chi^k - \frac{1}{2}\frac{\partial\Upsilon_{jk}}{\partial\chi^i}\epsilon^{\mu\nu}\partial_\mu\chi^j\partial_\nu\chi^k = 0, \qquad (23)$$

$$F_i = (\Gamma_{ij}\partial_z\chi^j + \Upsilon_{ij}\dot{\chi}^j)|. \qquad (24)$$

Though the bulk equation (23) looks complicated, it still admits purely outgoing solutions $\chi^i(t,z) = \bar{\chi}^i(t-z)$ with arbitrary functions $\bar{\chi}^i$. The conditions (18) then fix $\bar{\chi}^i(t) = q^i(t)$ and Eq. (24) reduces to

$$F_i = -\Gamma_{ij}(q)\,\dot{q}^j + \Upsilon_{ij}(q)\,\dot{q}^j. \qquad (25)$$

The first term gives the sought-after dissipative forces (2), whereas the second term describes arbitrary gyroscopic forces that arise if the environment breaks time-reversal symmetry, e.g., by magnetization or rotation. The effective reservoir action (22) represents our main result. It covers a much broader class of systems than the original harmonic bath (13).

Nonholonomic limit. To describe a nonholonomic system, we replace $\Gamma_{ij} \mapsto \Gamma_{ij} + \gamma c_i^\alpha c_j^\alpha$ and send γ to infinity. The resulting action can be obtained in a closed form by integrating in a set of auxiliary vectors $\lambda_\alpha^\mu(t,z)$. Omitting for simplicity the time-reversal breaking term, we write

$$S_{\text{nh}} = \int dt\, L(q,\dot{q}) + \int_{z>0} dt\,dz \left(\frac{1}{2}\Gamma_{ij}(\chi)\partial_\mu\chi^i\partial^\mu\chi^j - \lambda_\alpha^\mu c_i^\alpha(\chi)\partial_\mu\chi^i - \frac{1}{2\gamma}\lambda_\alpha^\mu\lambda_{\mu\beta} \right). \qquad (26)$$

In the limit $\gamma \to \infty$ the last term disappears and the fields λ_α^μ become Lagrange multipliers enforcing the constraints $c_i^\alpha(\chi)\partial_\mu\chi^i = 0$. Since χ^i coincide with q^i on the boundary, this implies the nonholonomic constraints (9). The remaining equations also come out right. Varying (26) with respect to χ^i and substituting the outgoing solution into the boundary equation, we obtain the force

$$F_i = -\Gamma_{ij}(q)\,\dot{q}^j + c_i^\alpha(q)\,\lambda_\alpha^z\big|. \qquad (27)$$

The last term gives precisely the reaction forces along the constrained directions (11), with $\lambda_\alpha^z|$ playing the role of the Lagrange multipliers from the standard approach. The first term describes friction along the unconstrained directions. Note that in our approach it cannot be set to zero without making the bulk action degenerate.

Example. The model takes a particularly simple form for motion on cosets:

$$S_{\text{coset}} = \int dt\, L(D_t q) + \int_{z>0} dt\,dz\,\frac{1}{2}\big(\gamma_{ij}\eta^{\mu\nu} + \upsilon_{ij}\epsilon^{\mu\nu}\big)D_\mu\chi^i D_\nu\chi^j, \qquad (28)$$

[5]Integrating by parts the interaction term in (13) and replacing $\dot{q}^i$ with the covariant derivative $D_t q^i$ does not help. We then have ξ_i, instead of $\dot{\xi}_i$ in the coupling, which leads to forces F_i with nonlocal memory of the past motion of the system.

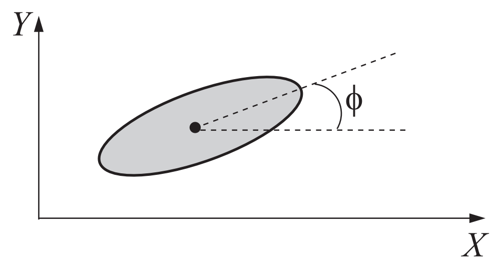

FIG. 1. Oblong particle on a plane.

where $D_\mu \chi^i \equiv \Omega^i_j(\chi)\partial_\mu \chi^j$ are covariant derivatives of the fields χ^i and γ_{ij}, υ_{ij} are constants. One recognizes in the bulk term the action of a two-dimensional nonlinear σ model [32]. It is the most general local action that can be written using the first derivatives of the fields χ^i and invariant under the group $\mathbb{G}$.

Let us illustrate this construction in the case of an oblong particle moving on a two-dimensional plane in a viscous medium [11–13]. Its position is described by the center-of-mass coordinates X, Y and the orientation angle ϕ; see Fig. 1. The friction coefficients are different in the directions along and perpendicular to the particle's main axis. The configuration space coincides with the group of isometries of the Euclidean plane $ISO(2)$ which has two generators of translations P_X, P_Y and a rotation generator J. The commutation relations are

$$[P_X, P_Y] = 0, \quad [P_X, J] = -P_Y, \quad [P_Y, J] = P_X. \tag{29}$$

All generators are broken. We parametrize the group elements as[6].

$$g(X, Y, \phi) = e^{XP_X + YP_Y} e^{\phi J}. \tag{30}$$

From the Cartan form we get the covariant derivatives:

$$D_t X = \dot{X}\, \cos\phi + \dot{Y}\, \sin\phi, \tag{31}$$

$$D_t Y = -\dot{X}\, \sin\phi + \dot{Y}\, \cos\phi, \quad D_t\phi = \dot{\phi}. \tag{32}$$

The Lagrangian coincides with the kinetic energy of the particle and is $ISO(2)$ invariant,

$$L = \frac{m}{2}(\dot{X}^2 + \dot{Y}^2) + \frac{\mathcal{I}}{2}\dot{\phi}^2 = \frac{m}{2}[(D_t X)^2 + (D_t Y)^2] + \frac{\mathcal{I}}{2}(D_t\phi)^2, \tag{33}$$

where $m, \mathcal{I}$ are the particle mass and moment of inertia.

If the viscous medium is homogeneous and isotropic, the effective reservoir action must also enjoy $ISO(2)$ symmetry. To implement it, we introduce the fields $\Xi(t, z)$, $\Psi(t, z)$, and $\Phi(t, z)$, such that at $z = 0$ they coincide with $X(t)$, $Y(t)$, and $\phi(t)$, respectively. We recall that the coordinate z is not a physical dimension. Rather, it parametrizes the internal degrees of freedom of the particle and medium responsible for dissipation. The effective bath action then reads

$$S_{\text{bath}} = \int_{z>0} dt\, dz\, \frac{1}{2}(\gamma_\parallel D_\mu \Xi D^\mu \Xi + \gamma_\perp D_\mu \Psi D^\mu \Psi + \gamma_\phi D_\mu \Phi D^\mu \Phi), \tag{34}$$

where

$$D_\mu \Xi = \cos\Phi\, \partial_\mu \Xi + \sin\Phi\, \partial_\mu \Psi, \tag{35}$$

$$D_\mu \Psi = -\sin\Phi\, \partial_\mu \Xi + \cos\Phi\, \partial_\mu \Psi, \quad D_\mu \Phi = \partial_\mu \Phi. \tag{36}$$

We observe that even in this relatively simple case the bath action is nonlinear if $\gamma_\perp \neq \gamma_\parallel$. In the limit $\gamma_\perp \to +\infty$ we obtain a particle that is constrained to move along its major axis. This is the simplest nonholonomic system known as *Chaplygin sleigh*.

Discussion. We have presented a reservoir model (22) for systems with general coordinate dependent ohmic dissipation and gyroscopic forces. In geometric terms, it can be viewed as a semi-infinite string moving on a curved target space. The coordinate along the string labels the continuum of internal reservoir degrees of freedom. In a certain limit, the model reproduces nonholonomic constraints. If dynamics obey nonlinearly realized symmetries, the reservoir takes the form of a two-dimensional nonlinear σ model.

The model has a vast range of potential applications. It provides a basis for development of path integral methods and quantization[7] in a broad class of dissipative and nonholonomic systems. One possible direction is derivation of classical and quantum Langevin equations for state-dependent diffusion [33] and Brownian motion of extended impurities [11–15], including systematic treatment of the multiplicative and non-Gaussian noise. Another interesting direction is generalization of the model to systems with an infinite number of degrees of freedom. Here promising arenas for applications are dissipative hydrodynamics [34] and open effective field theories [35].

An important question is to what extent the model (22) is universal beyond the classical equations. Does it capture the relevant properties of any ohmic environment? The unique structure of the model for coset dynamics suggests that, at least in this case, it is indeed universal in the sense of effective theory. Namely, we conjecture that the correlators of the q^i variables obtained from the action (28) reproduce the most general long-time behavior of correlators in dissipative systems with given nonlinear symmetries. We leave the exploration of this conjecture for the future. An interesting related work is construction of the effective Schwinger-Keldysh functional for cosets [36].

One of the assumptions of our model is locality in the auxiliary dimension z. This property is reminiscent of holography [37] and it is tempting to speculate that it arises as a consequence of the large number of bath degrees of freedom [38]. Let us note that we have not discussed the metric in the internal (t, z) space-time. At the classical level, it drops out due to the classical Weyl invariance of the action (22). In

[6]This parametrization slightly differs from Eq. (3) and makes the calculations simpler.

[7]Note in this connection that the σ model in 2D is renormalizable.

quantum theory, however, it may become relevant due to the conformal anomaly.

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