Letter

Extreme value statistics of jump processes

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We investigate extreme value statistics (EVS) of general discrete time and continuous space symmetric jump processes. We first show that for unbounded jump processes, the semi-infinite propagator $G_0(x, n)$, defined as the probability for a particle issued from zero to be at position x after n steps whilst staying positive, is the key ingredient needed to derive a variety of joint distributions of extremes and times at which they are reached. Along with exact expressions, we extract universal asymptotic behaviors of such quantities. For bounded, semi-infinite jump processes killed upon first crossing of zero, we introduce the strip probability $\mu_{0,\underline{x}}(n)$, defined as the probability that a particle issued from zero remains positive and reaches its maximum x on its nth step exactly. We show that $\mu_{0,\underline{x}}(n)$ is the essential building block to address EVS of semi-infinite jump processes, and obtain exact expressions and universal asymptotic behaviors of various joint distributions.

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Introduction. In a broad sense, extreme value problems focus on the extrema of a set of random variables $(X_1, \dots, X_n)$. Determining the statistics of such extrema is of high practical interest to understand numerous physical systems driven by rare but extreme events. As an illustration, seismic risk evaluation [1], portfolio management [2,3], or understanding herd behavior [4] are but a few examples of phenomena for which quantifying extreme value statistics (EVS) is key. While EVS of sets of independent random variables have been studied early on [5,6], leading to the renowned Gumbel-Frechet-Weibull universality classes for the distribution of the maximum of n random variables, recent works have also focused on EVS of correlated random variables generated by single-particle trajectories, and more specifically, of continuous stochastic processes. Initiated by Paul Levy's [7,8] derivation of the distribution of the running maximum $M(t)$ of a one dimensional Brownian particle $P(M(t) \leqslant M) = \text{erf}(M/\sqrt{2t})$, and the distribution of the time t_m at which the running maximum is reached (also known as the arc-sine law)

$$P(t_m = u|t) = \frac{1}{\pi\sqrt{u(t-u)}}, \tag{1}$$

a number of important results related to the EVS of one-dimensional Brownian dynamics have followed. In particular, joint distributions of extrema and times at which they are reached have been extensively studied for unbounded Brownian motions and Brownian bridges [9–11], as well as Brownian motions killed upon first passage to zero [12,13].

Jump processes, which are discrete time and continuous space stochastic processes, constitute an alternative model to the continuous description of single particle dynamics. At each discrete time step n, the particle performs a jump of length ℓ drawn from a distribution $p(\ell)$, whose Fourier transform will be denoted $\tilde{p}(k) = \int_{-\infty}^{\infty} e^{ik\ell} p(\ell) d\ell$ [14]. Such processes are involved in various contexts: they constitute paradigmatic models of transport in scattering media [15,16], and of self-propelled particles, living or artificial [17–21].

A striking example is given by the experimentally measured transmission probability of photons through 3D slabs [15,16]. This has been shown to be equivalent to the splitting probability $\pi_{0,\underline{x}}(0)$ that a jump process originated from zero crosses x before zero [22], which is a two-sided EVS observable. In turn, a further characterization of photon states upon exiting slab systems, such as phase distribution, requires more complex two-sided EVS observables, such as the joint distribution $\sigma(x, n_f|0)$ of the maximum x and first passage time n_f through zero for a particle starting from zero, to be derived below.

Most importantly, jump processes are in fact needed to describe experimental or computational data of dynamic processes. Indeed, experimental time series are inherently discrete because sampling times are finite in any measurement protocol. As shown below, continuous stochastic models fail to capture such discretization effects, which can have important consequences that only jump processes can account for—typically the joint distribution σ evaluated with standard methods for continuous processes starting from zero is identically zero and thus useless.

For symmetric jump processes considered hereafter, general EVS results are scarce, and primarily focused on two types of observables. First, the distributions of the time n_m at which the maximum is reached [23,24] and of successive record-breaking times [25] have been shown to be independent of $p(\ell)$, and computed exactly. Second, the asymptotic distribution of the running maximum M_n has been studied in the scaling limit and can be found in Darling [26] (see the SM [27] for details). Note, however, that the specific behavior of M_n stemming from the discrete nature of jump processes has only been characterized at the level of the the expected value of M_n, which has been investigated for processes with $\int \ell p(\ell) d\ell < \infty$. In particular, the leading order large n behavior of $\mathbb{E}(M_n)$ has been shown [28–30] to only depend on the

TABLE I. EVS observables for general unbounded and semi-infinite jump processes. The variables x, n_m, x_f, and n_f respectively denote the maximum, the time at which the maximum is reached, the final position of the process and the first passage time across zero. Our framework allows for the computation of novel exact and asymptotic expressions for all distributions labeled with ☆–explicit expressions are given in Table I of the SM. Entries labeled with √ are already given in the literature.

Unbounded jump processes		Semi-infinite jump processes	
$\mu(x\|n)$ [25,38]	√	$\mu_{\underline{0}}(x\|x_0=0)$	☆
$\rho(n_m\|n)$ [25]	√	$\rho_3(x,n_m\|x_0=0)$	☆
$\rho_1(x,n_m\|n)$	☆	$\rho_4(x,n_m,n_f\|x_0=0)$	☆
$\rho_2(x,n_m,x_f\|n)$	☆	$\sigma(x,n_f\|x_0=0)$	☆

tails of $p(\ell)$, equivalently described by the small k expansion of $\tilde{p}(k)$,

$$\tilde{p}(k) \underset{k\to 0}{=} 1-(a_\mu|k|)^\mu + o(|k|^\mu). \quad (2)$$

Here, the Levy index $\mu \in]0,2]$ describes the large ℓ behavior of $p(\ell)$, and a_μ is the characteristic length scale of the jump process. Importantly, when $\mu < 2$, the jump process is dubbed heavy tailed, and the jump distribution decays algebraically: $p(\ell) \propto \ell^{-(1+\mu)}$.

General outline. In the following, we develop a general framework to systematically analyze EVS of arbitrary symmetric jump processes originating from zero. We show that computing joint distributions of EVS observables reduces to the evaluation of two key quantities: the semi-infinite propagator $G_0(x,n)$, defined as the probability that the particle remains positive and reaches x on its nth step, and the strip probability $\mu_{0,\underline{x}}(n)$, defined as the probability that the particle remains positive and reaches its maximum $x=\underline{x}$ on its nth step exactly. The main result of this Letter is the derivation of an exact expression of $\mu_{0,\underline{x}}(n)$, and the analysis of its large x and n limit for general jump processes. In turn, we obtain exact expressions for a variety of new joint distributions of EVS observables, from which we uncover universal asymptotic behaviors. These joint distributions, summarized in Table I, span both unbounded jump processes with a deterministic number of steps n [Fig. 1(a)], and bounded, semi-infinite jump processes killed upon the first crossing of zero [Fig. 1(b)], for which the discrete nature of the dynamics plays a crucial role. While the main text focuses exclusively on jump processes with continuous $p(\ell)$ originating from zero, our framework is easily extended to nonzero initial conditions, as well as lattice random walks (see the SM).

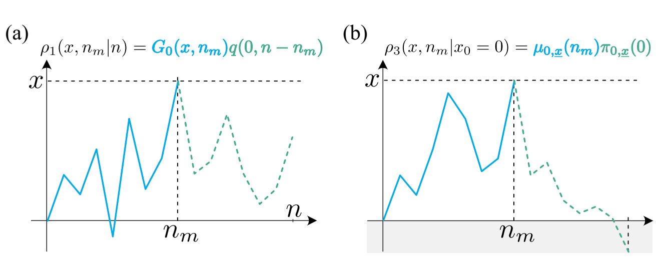

FIG. 1. (a) Sample trajectory contributing to the distribution ρ_1 of the maximum x and time n_m at which it is reached for an unbounded n-step long process. The weight of the solid (resp. dashed) portion is given by $G_0(x,n_m)$ [resp. $q(0,n-n_m)$]. (b) Sample trajectory contributing to the distribution ρ_3 of the maximum x and time at which it is reached n_m for a semi-infinite process. The weight of the solid (resp. dashed) portion is given by $\mu_{0,\underline{x}}(n_m)$ [resp. $\pi_{0,\underline{x}}(0)$].

EVS of unbounded jump processes. In this section, we focus on general n-step long unbounded jump processes issued from zero. By means of introduction, we consider the distribution $\mu(x|n)$ of the running maximum, i.e., the maximum value x reached up to the nth step. To highlight the significant role of the semi-infinite propagator in EVS computations, we first recall a few important known results Eqs. (3)–(5) and (7)]. Defining the survival probability $q(x_0,n)$ that a particle issued from x_0 remains positive during its first n steps, it is easily seen that [25]

$$\mu(x|n) = \frac{d}{dx}q(x,n). \quad (3)$$

In turn, the survival probability is given by $q(x_0,n) = \int_0^\infty G(x,n|x_0)dx$, where the semi-infinite propagator $G(x,n|x_0)$, defined as the probability that the n-step long trajectory issued from x_0 stays positive and is at position x after n steps, is known [25,31], and reads in Laplace and generating function space:

$$\sum_{n=0}^{\infty}\xi^n\left[\int_0^\infty\int_0^\infty e^{-s_1x+s_2x_0}G(x,n|x_0)dxdx_0\right] = \frac{\tilde{G}_0(s_1,\xi)\tilde{G}_0(s_2,\xi)}{s_1+s_2}. \quad (4)$$

Here $\tilde{G}_0(s,\xi) = \sum_{n=0}^\infty \xi^n[\int_0^\infty e^{-sx}G_0(x,n)dx]$ is the generating function of the Laplace transform of the semi-infinite propagator $G_0(x,n) \equiv G(x,n|0)$ of a particle issued from $x_0=0$, and is given in terms of $\tilde{p}(k)$ only by the Pollazcek-Spitzer formula [32,33]

$$\tilde{G}_0(s,\xi) = \exp\left[-\frac{s}{2\pi}\int_{-\infty}^{\infty}\frac{\ln[1-\xi\tilde{p}(k)]}{s^2+k^2}dk\right]. \quad (5)$$

While Eq. (3) is exact, it is clear from Eq. (5) that explicit expressions for the distribution of the running maximum can only be obtained for specific jump distributions. For instance, in the case of the exponential jump process $p(\ell) = 2^{-1}e^{-|\ell|}$, the semi infinite propagator can be found in [34], from which we explicitly derive the generating function of $\mu(x,n)$:

$$\sum_{n=0}^{\infty}\xi^n\mu(x|n) = \frac{(1-\xi-\sqrt{1-\xi})e^{-x\sqrt{1-\xi}}}{\xi-1}. \quad (6)$$

We emphasize that for jump processes for which the semi-infinite propagator cannot be obtained explicitly, Eqs. (3) and (5) still allow for the asymptotic analysis of $\mu(x|n)$, which depends only on the Levy index μ and length scale a_μ. Defining $n_x \equiv (x/a_\mu)^\mu$ as the typical number of steps needed to cover a distance x, we first consider the large n and x scaling limit with $\tau \equiv n/n_x$ fixed. In this limit, jump processes are known to converge to Brownian motion [35] with $D = a_2^2$ when $\mu = 2$, and symmetric α-stable processes [36] when $\mu < 2$. In turn, the limit distribution of the running maximum is given by Darling's result [26] (see the SM for explicit expressions). In the alternative limit regime $1 \ll n \ll n_x$, the behavior of $\mu(x|n)$ for processes with $\mu = 2$ depends on the details of $p(\ell)$. However, for heavy-tailed processes, the distribution of M_n becomes universal, and is readily obtained by extracting the leading order behavior of $G_0(x, n)$ from Eq. (5) (see the SM), yielding

$$\mu(x|n) \underset{1 \ll n \ll n_x}{\sim} \frac{\mu n}{\pi} \sin\left(\frac{\pi\mu}{2}\right)\Gamma(\mu)\left[\frac{a_\mu}{x}\right]^\mu \frac{1}{x}. \tag{7}$$

Importantly, the linear dependence of $\mu(x|n)$ admits a single big jump physical interpretation [37]: the particle has exactly n trials to perform a very large jump bringing it close to x. Of note, the algebraic behavior (7) can also be recovered by analyzing the asymptotic behavior of the maximum distribution of α-stable processes [38]. The semi-infinite propagator is thus an essential tool to derive exact and asymptotic expressions of $\mu(x|n)$. More generally, we claim that it is the necessary and sufficient building block to analyze arbitrary joint space and time EVS distributions, which we illustrate by computing two important quantities.

We first determine the classical joint distribution $\rho_1(x, n_m|n)$ of the maximum x and time n_m at which it is reached, which, so far, has primarily been derived exactly for continuous processes. By splitting the Markovian trajectory at n_m [see Fig. 1(a)], and identifying the probabilistic weights of the first and second independent parts, the joint distribution is given by

$$\rho_1(x, n_m|n) = G_0(x, n_m)q(0, n - n_m). \tag{8}$$

Crucially, considering a trajectory snippet backwards does not change its probability, which allows us to assign the semi-infinite propagator weight to the first part of the trajectory. When $\mu = 2$, the asymptotic behavior of $\rho_1(x, n_m|n)$ is simply given by the corresponding Brownian result obtained in [39]. When $\mu < 2$, no α-stable limit result exists; in turn, we analyze the large x, n_m, and n limit of Eq. (8), and uncover emerging universal behavior of $\rho_1(x, n_m|n)$ which depends only on a_μ and μ:

$$\rho_1(x, n_m|n) \underset{\substack{n_x/n \gg 1 \\ n/n_m \gg 1}}{\sim} \frac{1}{\pi}\sqrt{\frac{n_m}{n - n_m}}\frac{2\mu}{\pi} \sin\left(\frac{\pi\mu}{2}\right)\Gamma(\mu)\left[\frac{a_\mu}{x}\right]^\mu \frac{1}{x}. \tag{9}$$

In fact, our framework permits a more detailed characterization of space and time statistics, as we show by providing the refined multivariate distribution $\rho_2(x, n_m, x_f|n)$ of the maximum x time n_m at which it is reached, and the last position x_f of the particle in terms of $G_0(x, n)$ only:

$$\rho_2(x, n_m, x_f|n) = G_0(x, n_m)G_0(x - x_f, n - n_m). \tag{10}$$

The asymptotic behavior of $\rho_2(x, n_m, x_f|n)$ can be readily obtained for any μ from this general expression as is shown in the SM. Finally, we have shown that studying EVS of unbounded jump processes reduces to the evaluation of a single essential quantity: the semi-infinite propagator $G_0(x, n)$. In the following, we extend these results to the case of bounded, semi-infinite jump processes.

EVS of semi-infinite jump processes. We consider jump processes killed upon crossing zero for the first time, and hereafter choose $x_0 = 0$, although all our results are easily adapted to nonzero initial conditions (see the SM). Note that EVS are properly defined for semi-infinite jump processes starting from zero, in striking contrast to corresponding EVS of continuous processes killed upon first passage to zero, which, by definition, vanish as $x_0 \to 0$. Following the unbounded case, we first compute the distribution $\mu_{\underline{0}}(x|0)$ of the maximum $M_{\underline{0}}$ reached before crossing zero. Recalling that $\pi_{0,\underline{x}}(0)$ is the splitting probability that the walker crosses x strictly before zero, it is clearly seen that the cumulative distribution of $M_{\underline{0}}$ satisfies $\int_0^x \mu_{\underline{0}}(u|0)du = 1 - \pi_{0,\underline{x}}(0)$, yielding

$$\mu_{\underline{0}}(x|0) = -\frac{d}{dx}\pi_{0,\underline{x}}(0), \tag{11}$$

valid for general jump processes. As was recently shown in [22], the splitting probability can only be computed explicitly for a handful of jump distributions; however, in the large x limit, $\pi_{0,\underline{x}}(0)$ takes a universal asymptotic form which we readily exploit to obtain the large x behavior of $\mu_{\underline{0}}(x|0)$:

$$\mu_{\underline{0}}(x|0) \underset{x\to\infty}{\sim} \frac{\mu 2^{\mu-2}}{\sqrt{\pi}}\Gamma\left(\frac{1+\mu}{2}\right)\left[\frac{a_\mu}{x}\right]^{\frac{\mu}{2}}\frac{1}{x}. \tag{12}$$

Of note, the asymptotic decay is much slower than for fixed-length unbounded jump processes (7). Indeed, the survival probability $q(0, n)$ is decaying slowly enough to allow for particles to reach farther maxima before the first crossing of zero.

We now investigate joint space and time distributions. It is clear that being a solely geometrical quantity $\pi_{0,\underline{x}}(0)$ is not sufficient to compute such joint distributions. In fact, in this case of bounded trajectories, $G_0(x, n)$ does not suffice to build EVS distributions. To proceed further, we introduce the strip probability $\mu_{0,\underline{x}}(n)$, defined as the probability that the particle starting from zero stays positive and reaches its maximum x on its nth step exactly (thereby crossing the strip $[0, x]$), and show that $\mu_{0,\underline{x}}(n)$ allows for the systematic derivation of joint distributions. Computing the exact expression of the strip probability requires two auxiliary quantities: (i) the joint distribution $\sigma(x, n_f|0)$ of the maximum x and first passage time n_f through zero, and (ii) the rightward exit time probability (RETP) $F_{0,\underline{x}}(n|x_0)$, defined as the probability that the particle issued from x_0 crosses x before zero on its nth step exactly, which has been studied in [40] (see the SM for a summary of results). First, by partitioning trajectories over the time k at which the maximum is reached, σ is re-expressed in terms of $\mu_{0,\underline{x}}(n)$ and $F_{0,\underline{x}}(n|0)$ only:

$$\sigma(x, n|0) = \sum_{k=1}^{n-1} \mu_{0,\underline{x}}(k)F_{0,\underline{x}}(n - k|0). \tag{13}$$

Next, by now partitioning over the rightmost point reached before crossing zero, we rewrite $F_{\underline{0},x}(n|0) = \int_0^x \sigma(u, n|0)du$ where $F_{\underline{0},x}(n|x_0) = F_{0,\underline{x}}(n|x - x_0)$ by symmetry. Finally, we derive the exact expression of the generating function of the strip probability:

$$\sum_{n=1}^{\infty} \xi^n \mu_{0,\underline{x}}(n) \equiv \tilde{\mu}_{0,\underline{x}}(\xi) = \frac{\frac{d}{dx}\tilde{F}_{\underline{0},x}(\xi|0)}{\tilde{F}_{0,\underline{x}}(\xi|0)}. \tag{14}$$

Computing $\tilde{F}_{0,\underline{x}}(\xi|x_0)$ is thus sufficient to obtain explicit expressions of $\mu_{0,\underline{x}}(n)$. As an illustration, in the specific case of the exponential jump process we obtain

$$\tilde{\mu}_{0,\underline{x}}(\xi) = \frac{\mathrm{sech}(\gamma\sqrt{1-\xi}x)\gamma(1-\xi)\xi}{(2-\xi)\sqrt{1-\xi}\tanh(\gamma\sqrt{1-\xi}x)+2(1-\xi)}, \tag{15}$$

and specific n values of the strip probability are obtained by taking derivatives with respect to ξ: $\mu_{0,\underline{x}}(n) = [n!]^{-1}\tilde{\mu}_{0,\underline{x}}^{(n)}(0)$. For general jump processes for which the RETP cannot be obtained explicitly, we analyze the large x and n behavior of $\mu_{0,\underline{x}}(n)$ and uncover emergent universal behavior.

In the $\mu = 2$ case and in the scaling limit $\tau = n/n_x$ fixed, no overshoot occurs as the particle crosses x for the first time. As a result, the events of crossing x and reaching x on the nth step become statistically equivalent, such that $\mu_{0,\underline{x}}(n) \sim a_2^{-1}F_{0,\underline{x}}(n|0)$, where the proportionality constant is fixed by using the exact exponential distribution result (15). In turn, the asymptotic behavior of the strip probability is given by

$$\mu_{0,\underline{x}}(n) \underset{\tau\ \mathrm{fixed}}{\sim} 2\left[\frac{a_2}{x}\right]^2 \frac{1}{x}\pi^2 \sum_{k=1}^{\infty} k^2(-1)^{k+1}e^{-k^2\pi^2\tau}. \tag{16}$$

For heavy-tailed jump processes, overshoots occur even in the limit $x \to \infty$, such that the identification of the strip probability and the RETP is no longer valid. However, we show in the SM that the strip probability still displays a strikingly simple universal asymptotic behavior:

$$\mu_{0,\underline{x}}(n) \underset{1\ll n\ll n_x}{\sim} \frac{\mu}{\pi}\,\Gamma(\mu)\sin\left(\frac{\pi\mu}{2}\right)\left[\frac{a_\mu}{x}\right]^\mu \frac{1}{x}. \tag{17}$$

Remarkably, $\mu_{0,\underline{x}}(n)$ becomes independent of n, in stark contrast with its unbounded counterpart $G_0(x, n)$. Note also that, surprisingly, $\mu_{0,\underline{x}}(n) \sim p(x)$. We now show that distributions of EVS observables for semi-infinite jump processes can be systematically obtained from the strip probability, and exploit the asymptotic results (16) and (17) to derive explicit universal formulas.

As a first illustration, we determine the joint distribution $\rho_3(x, n_m|0)$ of the maximum and time at which it is reached. Paralleling the unbounded result (8), we decompose the Markovian trajectory into two independent parts around n_m [see Fig. 1(b)], and identify their respective probabilistic weights to obtain

$$\rho_3(x, n_m|0) = \mu_{0,\underline{x}}(n_m)\pi_{0,\underline{x}}(0). \tag{18}$$

Making use of the asymptotic behavior of the strip probability given above, we derive large x and n expressions of ρ_3. For $\mu = 2$, the joint distribution reads

$$\rho_3(x, n_m|0) \underset{\tau\ \mathrm{fixed}}{\sim} 2\left[\frac{a_2}{x}\right]^3 \frac{1}{x}\pi^2 \sum_{k=1}^{\infty} k^2(-1)^{k+1}e^{-k^2\pi^2\tau}, \tag{19}$$

while in the heavy-tailed case one has

$$\rho_3(x, n|0) \underset{1\ll n\ll n_x}{\sim} \frac{2^{\mu-1}\,\mu\,\Gamma\left(\frac{1+\mu}{2}\right)\Gamma(\mu)\,\sin\left(\frac{\pi\mu}{2}\right)}{\pi^{\frac{3}{2}}}\left[\frac{a_\mu}{x}\right]^{\frac{3\mu}{2}}\frac{1}{x}. \tag{20}$$

Importantly, the n independence of the strip probability has drastic effects on ρ_3; indeed, conditioned on the value x of the maximum, the time at which it is reached becomes equiprobable for values of $n \ll n_x$.

As a second illustration, we obtain, thanks to this formalism, the joint distribution $\rho_4(x, n_m, n_f|0)$ of the maximum x, time of maximum n_m, and first passage time n_f across zero, which is given by

$$\rho_4(x, n, n_f|0) = \mu_{0,\underline{x}}(n)F_{0,\underline{x}}(n_f - n). \tag{21}$$

Finally, its asymptotic behavior is readily obtained from that of the strip probability, and we provide universal formulas in the SM, along with the analysis of the joint distribution $\sigma(x, n_f|0)$ of the maximum and first passage time across zero.

Conclusion. We have shown that for general symmetric jump processes, the derivation of joint space and time distributions of EVS observables reduces to the determination of a single key quantity, which only depends on the geometrical constraints imposed on the trajectory. For unbounded jump processes, we identified the sufficient building block to be the semi-infinite propagator $G_0(x, n)$ and made use of its μ-dependent limit behavior to draw a comprehensive picture of large space and time EVS asymptotics. In the case of the semi-infinite jump processes killed upon the first crossing of zero, $G_0(x, n)$ is ill fitted to investigate EVS observables. As a replacement, we introduced the strip probability $\mu_{0,\underline{x}}(n)$, provided exact and asymptotic expressions valid for general symmetric jump distribution $p(\ell)$, and systematically derived joint EVS distributions summarized in Table I. We emphasize that all distributions can be explicitly computed for any n and x values, as soon as $G_0(x, n)$ and $\mu_{0,\underline{x}}(n)$ are known, as is the case for the exponential jump process $p(\ell) = \frac{\gamma}{2}e^{-\gamma|\ell|}$. Importantly, we stress that the asymptotic results depend only on the length scale a_μ and Levy index μ. These parameters can be extracted from empirical time series, as in the case of Levy-like photonic transmission [15] or the well-known run and tumble motion of E. Coli [41]; in turn, our results offer explicit predictions for potential direct experimental comparisons.

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[27] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevE.109.L052101 for additional details on the asymptotic behavior of the running maximum of stable processes; explicit expressions of EVS distributions as well as rightward and leftward exit time distributions; asymptotic behavior of $q(x, n)$, $G_0(x, n)$, and $\mu_{0,\underline{x}}(n)$; nonzero initial conditions and lattice walks, which includes Refs. [42–44].

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