Letter

Logarithmic or algebraic: Roughening of an active Kardar-Parisi-Zhang surface

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The Kardar-Parisi-Zhang (KPZ) equation sets the universality class for growing and roughening of nonequilibrium surfaces without any conservation law and nonlocal effects. We argue here that the KPZ equation can be generalized by including a symmetry-permitted nonlocal nonlinear term of active origin that is of the same order as the one included in the KPZ equation. Including this term, the 2D active KPZ equation is stable in some parameter regimes, in which the interface conformation fluctuations exhibit sublogarithmic or superlogarithmic roughness, with nonuniversal exponents, giving positional generalized quasi-long-ranged order. For other parameter choices, the model is unstable, suggesting a perturbatively inaccessible algebraically rough interface or positional short-ranged order. Our model should serve as a paradigmatic nonlocal growth equation.

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The Kardar-Parisi-Zhang (KPZ) equation [1–3] for growing nonequilibrium surfaces displays a nonequilibrium roughening transition between a smooth phase, whose long wavelength scaling properties are identical to an Edward-Wilkinson (EW) surface [4], to a perturbatively inaccessible rough surface [3,5] when $d > d_c = 2$, its lower critical dimension. Importantly, the local normal velocity of a KPZ surface depends *locally* on surface fluctuations, and hence cannot describe nonequilibrium surface dynamics with nonlocal interactions.

Theoretical studies on nonlocal interactions has a long-standing history in equilibrium systems [6–11]. Examples of their prominent nonequilibrium counterparts include interface dynamics involving nonlocal interactions, e.g., flame front propagation, thin film growth [12], and shading phenomena in surface growth [13]. Kinetic roughening in the presence of nonlocal interactions [14] display generic non-KPZ scaling behavior. Nonlocal effects are often important in biological growth processes; see, e.g., Ref. [15] for a recent study. Furthermore, in many applications, the growth is controlled by fast nonlocal transport not included in the KPZ equation. Prominent examples include diffusion-controlled nonlocal transport [16], dissolution or precipitation processes [17], gas-solid reactions [18], a variety of reaction engineering processes [19], diffusion-limited erosion, that displays nonlocal stabilization of surfaces [20] (see also Ref. [21]), and even geological contexts, e.g., earth surface roughness [22]. Inspired by these past studies, we explore the generic consequences of competition between local contributions and those that depend on the global surface profile, i.e., nonlocal contributions to the local surface velocity, by constructing a purpose-built conceptual model.

In this Letter, we set up and study a generalization of the KPZ equation, where the surface velocity depends, in contrast to the KPZ equation, *nonlocally* on the surface fluctuations. We do this by adding symmetry-permitted nonlocal nonlinear gradient terms that are of the same order as the usual KPZ nonlinear term. These nonlocal, nonlinear terms have the same scaling as the usual local nonlinear term of the KPZ equation. This allows us to study competition and interplay between local and nonlocal nonlinear effects, resulting into stable steady states and roughening transitions distinct from both the usual KPZ equation, or the KPZ equation with truly long-range effects (with either long range nonlinearity or long range noises) [14,23]. To generalize the scope of our study, we also include chiral contributions, which is ubiquitous in soft matter and biologically inspired systems; see, e.g., Refs. [24,25]. The resulting equation in 2D, named active-KPZ or a-KPZ equation, is

$$\frac{\partial h}{\partial t} = \nu\nabla^2 h + \frac{\lambda}{2}(\nabla h)^2 + \lambda_1 Q_{ij}(\mathbf{r})(\nabla_i h\nabla_j h) + \lambda_2 Q_{ij}(\mathbf{r})e_{jm}(\nabla_i h\nabla_m h) + \eta, \quad (1)$$

a *nonlocal* generalization to the usual KPZ equation that is distinct from the one considered in Ref. [14]. Here, the tensor e_{jm} is the 2D totally antisymmetric matrix. Further, $Q_{ij}(\mathbf{r})$ is the longitudinal projection operator that in the Fourier space is $Q_{ij}(\mathbf{k}) = k_i k_j/k^2$, where $\mathbf{k}$ is a Fourier wave vector, and is nonlocal. Physically, $\lambda_1 Q_{ij}(\mathbf{r})(\nabla_i h\nabla_j h) + \lambda_2 Q_{ij}(\mathbf{r})e_{jm}(\nabla_i h\nabla_m h)$ is the contribution to the surface velocity normal to the base plane $v_p = \partial h/\partial t$ that is *nonlocal* in height fluctuations ∇h. Noise η is a zero-mean, Gaussian-distributed white noise with a variance $\langle\eta(\mathbf{x},t)\eta(\mathbf{0},0)\rangle = 2D\delta^d(\mathbf{x})\delta(t)$. We extract the scaling of the stable phases, which exists for a range of the model parameters. In

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particular, we show that the variance $\Delta \equiv \langle h(\mathbf{x},t)^2\rangle \sim [\ln(L/a)]^\mu$ for a surface of lateral size L, where $\mu < (>) 1$ for sub (super) logarithmic roughness and a is a microscopic cutoff. This defines positional generalized quasi-long-ranged order (QLRO), generalizing the well-known QLRO of EW surfaces [3], in which $\Delta \sim \ln(L/a)$, i.e., $\mu = 1$. Further, the time-scale of relaxation $\tau(L) \sim L^2 (\ln L)^{-\kappa}$, $\kappa > 0$, i.e., logarithmically superdiffusive. Both μ and κ are nonuniversal. They vary continuously with λ_1/λ, λ_2/λ.

The form of Eq. (1) can be obtained by first considering the mapping from the KPZ equation to the Burgers equation [26] in terms of the "Burgers velocity $\mathbf{v} = \nabla h$." Now generalizing the Burgers equation nonlinearity $\lambda \nabla v^2$ to $\lambda_1 \nabla_j (v_i v_j) + \lambda_2 e_{jm} \nabla_j (v_i v_m)$, and then writing them in terms of h produces the λ_1 and λ_2 terms in (1); see the Supplemental Material (SM) [27].

The λ_1 and λ_2 terms in (1) can be motivated by considering a nearly flat nonequilibrium surface without any momentum conservation described by a single valued height field $h(\mathbf{x},t)$ in Monge gauge [28,29], with an active conserved density $\rho(\mathbf{x},t)$ living on it. Its hydrodynamic equation, retaining only the lowest order in nonlinearities and spatial gradients, reads

$$\frac{\partial h}{\partial t} = \nu \nabla^2 h + \frac{\lambda}{2} (\nabla h)^2 + v(\rho) + \eta, \tag{2}$$

where $v(\rho)$ is a local density-dependent velocity of the membrane; $v(\rho) = v_0 + g_1 \rho$ to the leading order in ρ; g_1 is a coupling constant of either sign. Further, density ρ follows $\partial_t \rho = -\nabla \cdot \mathbf{J}$, where $\mathbf{J}$ is the current. The specific form of the particle dynamics decides the structure of $\mathbf{J}$. We choose $J_i = -\overline{D}\nabla_i \rho + \nabla_j \sigma_{ij}$, where $\sigma_{ij} \equiv \alpha \nabla_i h \nabla_j h + \beta e_{jm} \nabla_i h \nabla_m h$ is reminiscent of "active stresses" found in active matter theories [30], the β term is a chiral contribution. The quadratic dependence of $\mathbf{J}$ on ∇h implies the active particles (i) respond, unsurprisingly, to the height fluctuations, but not the absolute height; and (ii) ignoring gravity, the particles do not distinguish valleys from the hills (although the surface itself breaks the inversion symmetry). Here, $\overline{D} > 0$ is a diffusivity. We focus on the quasistatic limit of infinitely fast dynamics of ρ, such that $\partial \rho/\partial t \approx 0$, giving $\overline{D}\nabla^2 \rho = \alpha \nabla_i \nabla_j (\nabla_i h \nabla_j h) + \beta e_{jm} \nabla_i \nabla_j (\nabla_i h \nabla_m h)$ neglecting any noise in the ρ dynamics. Now use this to eliminate ρ in (2) to get (1), after absorbing a factor of $\overline{D}$. (We have implicitly assumed α, β to scale with $\overline{D}$, and ignored any advective-type nonlinearity originating from projecting the particle dynamics on the plane of the membrane in the large $\overline{D}$ limit). All of $\lambda, \lambda_1, \lambda_2$ can be individually positive and negative. The chiral term is 2D specific; the other two nonlinear terms with coefficients λ and λ_1 can exist in any dimension d. Thus, the λ_1 and λ_2 terms in (1) are physical, although our active species origin need not be the only possible source of these two terms. See Ref. [31] for a similar mechanism to generate an effective nonlocal dynamics in the noisy Fisher-Kolmogorov equation [32,33] for population dynamics coupled with a fast chemical signal. Further Eq. (1) can be realized microscopically by considering an "active" 2D single-step model for a 2D KPZ surface with point particles living on it. The dynamical update rules of the modified single-step model now depend on the local excess or deficit population of the active particles, instead of being constants as they are in standard single-step models [34–38]. The particle hopping rates to the nearest neighbor sites in turn depends not only on the number inhomogeneities, but also on the height fluctuations (but without distinguishing local valleys from hills). Monte-Carlo simulations of this model, focusing on the limiting case of fast dynamics by the number fluctuations, should bring out the physics described in this Letter. The limit of fast particle dynamics can be implemented by considering time-scale separations in the rates of particle position updates and surface conformation updates.

At one dimension, the λ_2 term vanishes, and the λ_1 term becomes indistinguishable from the λ term. The transformation $x_i' = x_i - (\lambda + 2\lambda_1) c_i t - \lambda_2 e_{ij} c_j t$ and $t' = t$, together with the height function h transforming as $h'(\mathbf{x}', t') = h(\mathbf{x}, t) + \mathbf{c}\cdot\mathbf{x}$, leaves Eq. (1) invariant; see the SM [27]. This generalizes invariance of the usual KPZ equation under a pseudo-Galilean transformation [3].

Similar to the KPZ equation, dimensional analysis via scaling $\mathbf{r} \rightarrow b\mathbf{r}$, $t \rightarrow b^z t$, $h \rightarrow b^\chi h$, where z and χ are the dynamic and roughness exponents, reveals that all of $\lambda, \lambda_1, \lambda_2$ scale similarly, and hence are equally relevant (in the scaling sense). Furthermore, $d = 2$ is the critical dimension of Eq. (1); see the SM [27]. Whether it is the upper or lower critical dimension requires further analysis that follows below. That all of $\lambda, \lambda_1, \lambda_2$ scale the same way is important: it means the nonlocal, nonlinear effects in (1) are as relevant as the short-range, local nonlinear effects in the original KPZ equation [3]. This feature clearly distinguishes the active KPZ equation (1) from generalized KPZ equations with genuine long-range interactions [14]. Indeed, just as the usual KPZ equation [3] is universal in the sense that all short-range growth processes with just one soft mode (height h) and without any conservation laws, inversion symmetry and disorder should be described by it; the active KPZ equation (1) should likewise describe all such nonlocal growth processes having the same scaling properties as the corresponding local growth processes in the KPZ equation, and with just one soft mode (h) but without any conservation laws, inversion symmetry, and disorder, highlighting the universal nature of (1).

We first determine if Eq. (1) has a stable nonequilibrium steady state (NESS), and second, if so, the scaling properties in those NESS. We use renormalization group (RG) framework, well suited to systematically handle the diverging corrections encountered in naïve perturbation theories. The Wilson dynamic RG method for our model closely resembles that for the KPZ equation [1,3,26,39]; see the SM [27] for the one-loop Feynman diagrams. There are no one-loop corrections to $\lambda, \lambda_1, \lambda_2$. However, there are diverging one-loop corrections to ν and D. Dimensional analysis allows us to identify an effective dimensionless coupling constant g and two dimensionless ratios γ_1, γ_2 defined as $g \equiv \frac{\lambda^2 D}{\nu^3}\frac{1}{2\pi}$, $\gamma_1 \equiv \frac{\lambda_1}{\lambda}$, $\gamma_2 \equiv \frac{\lambda_2}{\lambda}$. The RG recursion relations for D, ν at the one-loop order (here l is the "RG time;" $\exp(l)$ is a length scale)

$$\frac{dD}{dl} = D[z - d - 2\chi + g\mathcal{B}(\gamma_1, \gamma_2)], \tag{3}$$

$$\frac{d\nu}{dl} = \nu[z - 2 + g\mathcal{C}(\gamma_1, \gamma_2)], \tag{4}$$

with γ_1, γ_2 being marginal at the one-loop order, stemming from the nonrenormalization of $\lambda, \lambda_1, \lambda_2$ at that

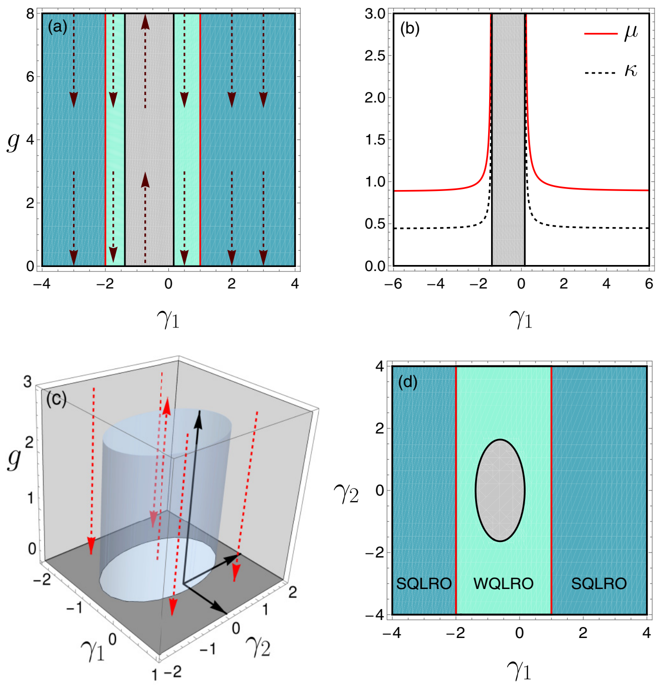

FIG. 1. (a) RG flow diagram in the g-γ_1 plane in the achiral limit ($\gamma_2 = 0$). Arrows indicate RG flows. Flow in the stable (unstable), i.e., toward (away from), $g = 0$ region are marked. (b) Variations of μ and κ as functions of γ_1 in the stable region for the achiral case. (c) RG flow diagram in the space spanned by γ_1-γ_2-g in the full a-KPZ equation. RG flow lines in the stable and unstable regions are shown by the arrows. (d) Phase diagram in the γ_1-γ_2 plane for the a-KPZ equation. The central gray region containing the origin is unstable. Regions with SQLRO and WQLRO are marked (see text).

order. Here, $\mathcal{B}[\gamma_1,\gamma_2] = \frac{3}{8}\gamma_1^2 + \frac{1}{2}\gamma_1 + \frac{1}{8}\gamma_2^2 + \frac{1}{4}$, $\mathcal{C}[\gamma_1,\gamma_2] = \frac{1}{2}\gamma_1^2 + \frac{5}{8}\gamma_1 + \frac{1}{8}\gamma_2^2$; $\mathcal{B} > 0$. Flow Eqs. (3) and (4) yield the flow equation for g:

$$\frac{dg}{dl} = -g^2\mathcal{A}[\gamma_1,\gamma_2], \tag{5}$$

where $\mathcal{A}[\gamma_1,\gamma_2] = \frac{9}{8}\gamma_1^2 + \frac{11}{8}\gamma_1 + \frac{1}{4}\gamma_2^2 - \frac{1}{4}$.

In the achiral case, i.e., $\gamma_2 = 0$, an RG flow diagram in the g-γ_1 plane is shown in Fig. 1(a). The condition $\tilde{\mathcal{A}}(\gamma_1) \equiv \mathcal{A}(\gamma_1, \gamma_2 = 0) = 0$ defines two solid (black) lines $\gamma_1 = \gamma_+, \gamma_-$ parallel to the g axis in the g-γ_1 plane, where $\gamma_+ = 0.161$, $\gamma_- = -1.383$, such that for $\gamma_+ > \gamma_1 > \gamma_-$ (gray region), the RG flow lines flow away parallel to the g axis toward infinity, indicating a perturbatively inaccessible, presumably rough, phase with short-ranged positional order. In this unstable region $g(l)$ diverges as $l \to 1/[|\tilde{\mathcal{A}}(\gamma_1)|]$, reminiscent of the 2D KPZ equation [3], presumably corresponding to algebraically rough phase [40,41]. Outside this region, where $\tilde{\mathcal{A}}(\gamma_1) > 0$, the flow lines flow toward $g = 0$ parallel to the g axis, implying stability, while $g(l) \approx 1/[l\tilde{\mathcal{A}}(\gamma_1)]$ vanishes slowly in the long wavelength limit $l \to \infty$. Although $g^* = 0$ is the only fixed point (FP) in the stable region, the vanishing of $g(l)$ is so slow, being proportional to $1/l$, the parameters D and ν are infinitely renormalized, altering the linear theory scaling in the long wavelength limit. The simplest way to see this is to set $z = 2$, $\chi = 0$ (i.e., their linear theory values) in (3) and (4) with $\gamma_2 = 0$, which gives

$$D(l) = D_0 l^{\tilde{\mathcal{B}}/\tilde{\mathcal{A}}}, \quad \nu(l) = \nu_0 l^{\tilde{\mathcal{C}}/\tilde{\mathcal{A}}}, \tag{6}$$

where $\tilde{\mathcal{B}}(\gamma_1) \equiv \mathcal{B}(\gamma_1, \gamma_2 = 0)$, $\tilde{\mathcal{C}}(\gamma_1) \equiv \mathcal{C}(\gamma_1, \gamma_2 = 0)$, D_0, ν_0 are the small-scale or unrenormalized values of D and ν. Since $\tilde{\mathcal{B}}$ is positive definite, $D(l) \gg D_0$ for $l \to \infty$. On the other hand, $\tilde{\mathcal{C}}$ is positive in stable regions, for which $\nu(l) \gg \nu_0$ for $l \to \infty$, giving the time-scale $\tau(L) \sim L^2[\ln(L/a)]^{-\kappa}$ for relaxation over lateral size L, where $\kappa(\gamma_1) = \tilde{\mathcal{C}}/\tilde{\mathcal{A}}$ is a positive definite but nonuniversal, γ_1-dependent exponent. The logarithmic modulation in $\tau(L)$ implies (i) breakdown of conventional dynamic scaling [42–44], and (ii) *nonuniversally faster* relaxation, being parametrized by γ_1, of fluctuations. Furthermore, by defining RG time $l \simeq \ln(1/aq)$ and using $\nu(q)$, $D(q)$, the variance is

$$\Delta \equiv \langle h^2(\mathbf{x},t)\rangle \sim \int_{1/L}^{1/a} d^2q \frac{D(q)}{\nu(q)q^2} \sim [\ln(L/a)]^{\mu}, \tag{7}$$

where $\mu(\gamma_1) = 1 + (\tilde{\mathcal{B}} - \tilde{\mathcal{C}})/\tilde{\mathcal{A}}$ is also nonuniversal, parametrized by γ_1, and can be more or less than unity, depending upon the sign of $\tilde{\mathcal{B}} - \tilde{\mathcal{C}}$, as mentioned above. Variations of μ and κ as functions of γ_1 are shown in Fig. 1(b). For $\mu(\gamma_1) < 1 (> 1)$, $\Delta(\gamma_1)$ grows with the system size L slower (faster) than positional QLRO, as in the 2D EW equation [4]. We call these stronger (weaker) than QLRO or SQLRO (WQLRO), corresponding to sub (super) logarithmically rough surfaces with positional generalized QLRO, that generalize the well-known QLRO in the 2D EW equation or 2D equilibrium XY model [28]. In particular, the minimum of $\mu = 0.89$. In Fig. 1(a) the blue outer regions (green inner strips) correspond to SQLRO (WQLRO). Solid red lines correspond to positional QLRO. These results are reminiscent of the logarithmic anomalous elasticity in three-dimensional equilibrium smectics [45,46], and a 2D equilibrium elastic sheet having vanishing thermal expansion coupled with Ising spins [47,48]; see also Ref. [44] for similar results.

Including the chiral effects ($\gamma_2 \neq 0$), stability of the RG flow is now determined by $\mathcal{A}(\gamma_1, \gamma_2) > 0$. Flow lines having initial conditions within a narrow elliptical cylinder, containing the origin (0,0,0), and having the axis parallel to the g axis, with its surface given by $\mathcal{A}(\gamma_1, \gamma_2) = 0$ for any g, run away parallel to the g axis, leaving the perturbatively accessible region. Flow lines with initial conditions falling in regions outside of this elliptical cylinder flow toward the γ_1-γ_2 plane with stable states. See Fig. 1(c) depicting the RG flow lines in the space spanned by γ_1-γ_2-g. Outside the elliptical cylinder $g(l) \sim 1/(\mathcal{A}l)$ for large l, similar to its achiral analog. Inside the cylinder, $g(l)$ diverges as $l \to 1/(|\mathcal{A}|)$ from below. Focusing on the γ_1-γ_2 plane, $\mathcal{A}(\gamma_1, \gamma_2) = 0$ sketches out an inner elliptical unstable region, whereas the outer region is stable; see Fig. 1(d). We use the above results to find that in the stable region $\Delta \sim [\ln(L/a)]^{\mu(\gamma_1,\gamma_2)}$, where $\mu = 1 + (\mathcal{B} - \mathcal{C})/\mathcal{A}$ is now parametrized by both γ_1, γ_2. Similar to and quantitatively extending the achiral case, $\mu < 1 (> 1)$ is referred to as SQLRO (WQLRO), giving positional generalised QLRO. The SQLRO and WQLRO regions are demarcated within the stable region in Fig. 1(d).

The equal-time height-difference correlator $C_h(r,0) \equiv \langle[h(\mathbf{x},t)-h(\mathbf{0},t)]^2\rangle \sim \frac{D_0}{\nu_0}[\ln(r/a)]^{\mu}$, for large $r \equiv |\mathbf{x}| \gg a$, indicating logarithmically faster or slower rise with the separation r for large r [42,43], again generalizing the well-known QLRO found in a 2D EW surface.

Our continuously varying scaling exponents are a crucial outcome of the nonrenormalization of λ, λ_1 and λ_2, rendering γ_1,γ_2 marginal, which have been demonstrated at the one-loop order. Unlike the usual KPZ equation, Galilean invariance of the present model ensures nonrenormalization of a combination of $\lambda,\ \lambda_1,\ \lambda_2$, and not each of them individually. Thus, there is no surety that γ_1,γ_2 should remain marginal even at higher-loop orders. We now argue that these possible higher-loop contributions, even though they may exist, actually do not matter. For large l, $g(l)\sim 1/l$ at the one-loop order. At higher-loop orders, the Feynman diagrams will contain higher power of g. Hence, a general scaling solution for $g(l)$ should have the form $g(l)\sim 1/l+\sum_n c_n/l^n$, and $n>1$ is an integer. Thus, the higher-loop corrections to the one-loop solution of $g(l)$ should vanish like $1/l^s$, $s>1$. Therefore, their integrals over l from zero to infinity will be finite, so they will not change the anomalous behavior of D and ν. Similarly, they cannot make any divergent contribution to $\gamma_1(l)$ and $\gamma_2(l)$, even though there can be higher-loop diagrams. Therefore, our one-loop results are, in fact, asymptotically exact. This then implies that the continuous variation of the scaling exponents, making them nonuniversal, is also asymptotically exact in the long wavelength limit. See Refs. [40,42,43,49–53] for similar nonuniversal scaling exponents in other models.

At higher dimensions $d>2$, the chiral term with coupling λ_2 cannot exist. The other two achiral nonlinear terms in Eq. (1) are present at $d>2$. The RG recursion relations for $d>2$ can be obtained from the Feynman diagrams given in the SM [27] with $\gamma_2=0$. Using a $d=2+\epsilon$ expansion as in the KPZ equation [5], we find at the one-loop order or to the lowest order in ϵ,

$$\frac{dg}{dl} = -\epsilon g - \tilde{\mathcal{A}}(\gamma_1)g^2. \tag{8}$$

Parameter γ_1 remains marginal at the lowest order. Therefore, if $\tilde{\mathcal{A}}(\gamma_1)>0$, $g(l)$ flows to zero rapidly, with $g(l)\sim g(0)\exp(-\epsilon\, l)$ in the long wavelength limit; $g^*=0$ is the only FP that is globally stable. This renders the nonlinearities irrelevant in the RG sense. Therefore, scaling in the long wavelength limit is identical to that in the EW equation: $z=2,\ \chi=(2-d)/2$. Furthermore, $d=2$ is then the upper critical dimension. On the other hand, if $\tilde{\mathcal{A}}(\gamma_1)<0$, $g(l)$ has three FPs: $g_c^*=-\epsilon/\tilde{\mathcal{A}}(\gamma_1)$, an unstable FP, parametrized by γ_1 and separating possibly two stable FPs, one being at $g^*=0$ Gaussian FP with EW scaling, and another putative perturbatively inaccessible FP, corresponding presumably to an algebraically rough phase. This gives, with 2D as the lower critical dimension, a roughening transition at $d>2$, very similar to the KPZ equation at $d>2$, but with one caveat. At this unstable FP, using (3) and (4), to $\mathcal{O}(\epsilon)$ $z=2+\epsilon\frac{\tilde{\mathcal{C}}(\gamma_1)}{\tilde{\mathcal{A}}(\gamma_1)},\ \ \chi=-\epsilon\frac{\tilde{\mathcal{C}}(\gamma_1)}{\tilde{\mathcal{A}}(\gamma_1)}$, depend explicitly on γ_1 and deviate from their linear theory (or EW equation) values already at $\mathcal{O}(\epsilon)$. This is in contrast to the KPZ equation at $d>2$, where z and χ at the unstable FP are at least $\mathcal{O}(\epsilon^2)$ [5]. In fact, application of the Cole-Hopf transformation shows that $z=2,\ \chi=0$ at the unstable FP of the KPZ equation at $d>2$ [54].

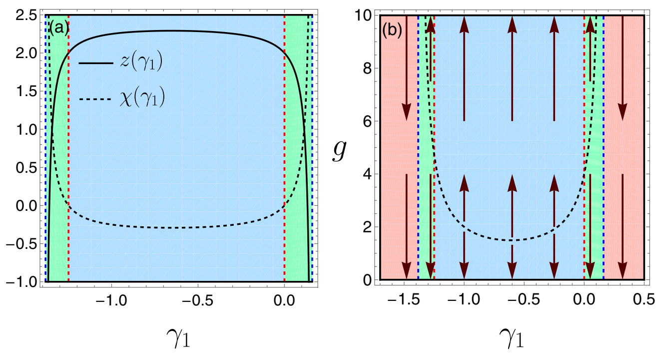

FIG. 2. (a) Variation of z and χ with γ_1 on the fixed line $g_c^*=-1/\tilde{\mathcal{A}}(\gamma_1)$ for $\epsilon=1$. (b) RG flow diagram in the g-γ_1 plane for $d>2$. The black dashed line is the fixed line $g_c^*=-1/\tilde{\mathcal{A}}(\gamma_1)$ for $\epsilon=1$, bounded by lines $\gamma_1=\gamma_-,\gamma_+$ (blue dashed lines). Stable (unstable) flow lines are the arrows pointing toward (away from) $g=0$ (see text).

When $\tilde{\mathcal{A}}(\gamma_1)<0$, the solution of $\tilde{\mathcal{C}}(\gamma_1)=0$ gives the red dashed lines $\gamma_1=\gamma^-,\gamma^\dagger$ where $\gamma^\dagger=0$ and $\gamma^-=-1.25$; see Fig. 2(a) for a variation of z and χ with γ_1 for a fixed ϵ. Green strips correspond to $\tilde{\mathcal{C}}(\gamma_1)>0$ where $\chi>0$ and $z<2$; $\tilde{\mathcal{C}}(\gamma_1)<0$ is the blue region where $\chi<0$ and $z>2$. For a given ϵ, maximum value of z and minimum value of χ are $z_{\max}=2+0.292\epsilon$ and $\chi_{\min}=-0.292\epsilon$ at $\gamma_1=-0.651$, such that the dynamics is slowest and the surface is smoothest at the unstable FP. Since $\chi_{\min}>\chi_{\rm EW}=-\epsilon/2$, an a-KPZ surface at the unstable FP is always rougher than an EW surface.

In the g-γ_1 plane, $g_c^*=-\epsilon/\tilde{\mathcal{A}}(\gamma_1)$ is a fixed line, such that RG flow lines with initial g values above the line flows to perturbatively inaccessible FP, see Fig. 2(b). And for systems with initial g values lying below the line, the RG flow lines run parallel to the g axis toward Gaussian FP, corresponding to the smooth phase belonging to the EW class. This behavior holds within a range $\gamma_->\gamma_1>\gamma_+$. As $\gamma_1\to\gamma_+,\ \gamma_-,\ \tilde{\mathcal{A}}(\gamma_1)$ vanishes and g_c^* diverges. As soon as γ_1 exceeds γ_+ or falls short of γ_-, g_c^* no longer exists with the roughening transition disappearing. RG flow lines starting from any initial condition with $\gamma_1>\gamma_+$ or $\gamma_1<\gamma_-$ (red region) where $\tilde{\mathcal{A}}(\gamma_1)>0$, flow to $g^*=0$ ensuing scaling belonging to the EW class.

In summary, we have proposed and studied an "active KPZ" equation, having a surface velocity v_p depending nonlocally on the surface gradients. Surprisingly, we find stable surfaces with positional generalized QLRO or generalized logarithmic roughness with nonuniversal exponents for wide-ranging choices of the model parameters, unlike the 2D KPZ equation. Physically, this is due to the competition between the nonlocal and local nonlinear terms and the lack of their renormalization. Indeed, this competition between the nonlocal and local nonlinear terms distinguishes our model (1) from that studied in Ref. [14], giving either generalized QLRO with nonuniversal scaling exponents or a novel roughening transition even in 2D controlled by the relative strengths of the local and nonlocal interactions. At $d>2$, sufficiently strong nonlinear nonlocal effects can either entirely suppress the KPZ roughening transitions, resulting into only smooth surfaces, or else give a roughening transition with nonuniversal scaling very different from the well-known roughening transition in the KPZ equation. Heuristically, a nonlocal part in v_p means a local large fluctuation can generate a propulsion not just locally, but over large scales, which when sufficiently strong can suppress local variations in v_p due to the local KPZ-nonlinear term. This in turn has the effect of reducing surface fluctuations. For other parameter choices, a KPZ-like perturbatively inaccessible rough phase is speculated. This may be explored by mode-coupling methods [55]. In that parameter space, the roughening transition survives at $d > 2$, but with significantly different scaling properties, again with nonuniversal exponents. We hope our studies here will provide further impetus to study nonlocal effects on similar nonequilibrium surface dynamics models, e.g., the conserved KPZ [56,57] and the $|\mathbf{q}|$KPZ [58] equations.

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