Letter

Dynamics of packed swarms: Time-displaced correlators of two-dimensional incompressible flocks

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We analytically calculate the scaling exponents of a two-dimensional KPZ-like system: coherently moving incompressible polar active fluids. Using three different renormalization group approximation schemes, we obtain values for the roughness exponent χ and anisotropy exponent ζ that are extremely near the known exact results. This implies our prediction for the previously unknown dynamic exponent z is likely to be quantitatively accurate.

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The Mermin-Wagner theorem [1] states that in equilibrium, two-dimensional systems with finite-range interactions cannot spontaneously break a continuous symmetry at any finite temperature. One of the earliest and most surprising results in the field of active matter was that there is no analog of the Mermin-Wagner theorem in nonequilibrium matter. In particular, it was explicitly demonstrated that active polar units moving on a two-dimensional frictional substrate and with purely short-range interactions can spontaneously break continuous rotation symmetry in two dimensions and form long-range ordered flocks [2–6]. Such flocks have a nonzero mean speed $\langle \mathbf{v} \rangle$ even at finite noise strengths.

Despite the fact that the existence of long-range-ordered two-dimensional flocks has been demonstrated analytically, determining their scaling behavior analytically has proved much more challenging. One class of systems that has proved amenable to analytical treatment is incompressible flocks [7–13].

Incompressibility can arise in many ways. One way is to make the density extremely high. In this limit the effective compressibility of the flockers vanishes, with any departure from the mean density being severely penalized [8,10]. More accurately, as the compressibility goes to 0, the length scale up to which the system is effectively incompressible diverges. A flock formed by a suspension of motile swimmers in an incompressible fluid on a substrate also inherits the incompressibility of the fluid and is therefore incompressible [10]. This is distinct from the model we will consider here due to the presence of an extra conserved quantity: the number of swimmers. However, if the number of active units is not conserved, for instance, due to birth and death [14–16], the dynamics of such a system belongs to the incompressible flock universality class we consider here [10]. Such a system has nonconserved dynamical quantities, like ours; therefore, the only hydrodynamic field is the center of mass (of background fluid plus active particles) velocity field. Furthermore, that velocity must also obey the incompressibility condition $\nabla \cdot \mathbf{v} = 0$. The system also has the same symmetry (rotation invariance) as ours, and breaks that symmetry spontaneously in the same way (by picking out a preferred direction of motion). Therefore, it must be described by the same hydrodynamic equation as ours.

Our understanding of the equal-time behavior of incompressible two-dimensional flocks is quite complete. Although they are nonequilibrium systems, their hydrodynamic equations prove to be equivalent, once terms that are irrelevant in the renormalization group sense are dropped, to those of an equilibrium magnetic system [8,17] with long-ranged interactions. Because of those long-ranged interactions, the Mermin-Wagner theorem does not apply to these magnetic systems. The partition function for this equilibrium system can then be further mapped [8] onto that for a two-dimensional smectic, whose equal-time scaling laws are known exactly via a further mapping [18] onto the $(1+1)$-dimensional KPZ equation [19]. This analysis [8] gives the scaling law for the equal-time fluctuations $\mathbf{u}(\mathbf{r}, t)$ of the local active fluid velocity $\mathbf{v}(\mathbf{r}, t)$ about its mean value $\langle \mathbf{v} \rangle \equiv v_0 \hat{\mathbf{x}}$, where we have defined our coordinate system so that $\hat{\mathbf{x}}$ is along the mean velocity spontaneously chosen by the system. Specifically

$$\langle \mathbf{u}(\mathbf{r}, t) \cdot \mathbf{u}(\mathbf{0}, t) \rangle = |x|^{2\chi} \mathcal{G}_{\mathrm{ET}}\left(\frac{|y|}{|x|^{\zeta}} \right), \tag{1}$$

where ET stands for equal time. The exponents χ and ζ were determined exactly by the aforementioned mappings [8] to be $\chi = -1/2$ and $\zeta = 3/2$.

Unfortunately, since this analysis was based entirely on the partition function for the equivalent equilibrium model, no

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information about the *dynamics* of the system can be obtained by this exact mapping. Most of this missing information can be encoded in a single additional universal exponent, namely, the dynamical exponent z. This can be defined by considering the unequal-time correlation function of the velocity fluctuations, which obeys the scaling law

$$\langle \mathbf{u}(\mathbf{r},t)\cdot\mathbf{u}(\mathbf{0},0)\rangle = |x-v_b t|^{2\chi}\mathcal{G}_{\rm UET}\left(\frac{|y|}{|x-v_b t|^{\zeta}},\frac{|t|}{|x-v_b t|^{z}}\right), \quad (2)$$

and the subscript UET now means unequal time. The boost velocity v_b is a phenomenological parameter of our model.

In this Letter, we obtain the dynamical exponent z for the polar phase of two-dimensional, incompressible flocks using four different dynamic renormalization group (DRG) schemes: two different uncontrolled calculations exactly in two dimensions, and two $d = (d_c - \epsilon)$-expansions, in which we analytically continue our model, which is defined strictly in two dimensions, to higher dimensions. The two ϵ expansions, which we call the hard and soft continuations, have $d_c = 5/2$ and $d_c = 3$, respectively. We get

$$z = 1.67 \pm 0.05, \quad (3)$$

where the error bars correspond to the standard error of the four DRG schemes. We also calculate the roughness exponent χ and the anisotropic exponent ξ in all four schemes.

As mentioned earlier, we already know the static exponents χ and ζ exactly, so our purpose in performing this DRG is only to calculate z. However, our knowledge of the exact values of χ and ζ provides us with a useful check on the quantitative accuracy of our DRG calculations, because we can compare the approximate values of χ and ζ that we get from those schemes with the known exact values. And they prove to be very close; indeed, one of our uncontrolled approximations, and the soft-continuation ϵ expansion, both reproduce the known exact values $\chi = -1/2$ and $\zeta = 3/2$. We therefore believe our prediction (3) is very accurate quantitatively, although this argument is obviously far from rigorous.

In particular, inserting (3) and the exact value for χ into (2), and considering the limit $\mathbf{r} \to \mathbf{0}$, we obtain the temporal part of the velocity correlation:

$$\langle \mathbf{u}(\mathbf{0},t)\cdot\mathbf{u}(\mathbf{0},0)\rangle \sim At^{\frac{2\chi}{z}} = At^{-0.60\pm0.02}, \quad (4)$$

where A is some nonuniversal constant.

We will now present the derivation of the above results. This begins with the hydrodynamic equation of motion (EOM) of incompressible polar active fluids, which can be constructed purely based on symmetry considerations (as has been discussed in Ref. [8]), and is:

$$\partial_t\mathbf{v} + \lambda(\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla\Pi - (\mathbf{v}\cdot\nabla\Pi_1)\mathbf{v} + U(|\mathbf{v}|)\mathbf{v} + \mu_1\nabla^2\mathbf{v} + \mu_2(\mathbf{v}\cdot\nabla)^2\mathbf{v} + \mathbf{f}(\mathbf{r},t), \quad (5)$$

where the pressure Π is a Lagrange multiplier that enforces the incompressibility constraint: $\nabla\cdot\mathbf{v} = 0$, the term involving $U(|\mathbf{v}|)$—a smooth, analytic function of $|\mathbf{v}|$—ensures that there is an ordered phase in which $\mathbf{v}$ has a nonzero mean magnitude v_0. That is, we assume that there is a regime in parameter space of $U(|\mathbf{v}|)$ where $U(|\mathbf{v}|) > 0$ for $|\mathbf{v}| < v_0$, $U(|\mathbf{v}|) = 0$ for $|\mathbf{v}| = v_0$, and $U(|\mathbf{v}|) < 0$ for $|\mathbf{v}| > v_0$. The anisotropic pressure Π_1 is another generic function of $|\mathbf{v}|$. Finally, $\mathbf{f}(\mathbf{r})$ is a zero-mean, Gaussian white noise with the correlation

$$\langle f_i(\mathbf{r},t)f_j(\mathbf{r}',t')\rangle = 2D\delta_{ij}\delta^2(\mathbf{r}-\mathbf{r}')\delta(t-t'), \quad (6)$$

where the indices i, j enumerate the spatial coordinates.

Expanding (5) about an ordered state using $\mathbf{v} = v_0\hat{\mathbf{x}} + \mathbf{u}$ and retaining only relevant terms (i.e., the terms that are important in the limit of large time and length scales), we obtain the EOM governing $\mathbf{u}$

$$\partial_t u_i = -\partial_i\Pi + \mu_\perp\partial_y^2 u_i + \mu\partial_x^2 u_i - \alpha\left(u_x + \frac{u_y^2}{2v_0}\right)\left(\delta_{ix} + \frac{u_y}{v_0}\delta_{iy}\right) + f_i, \quad (7)$$

where $\alpha \equiv -v_0(\frac{dU}{d|\mathbf{v}|})_{|\mathbf{v}|=v_0}$, $\mu_\perp \equiv \mu_1$, and $\mu \equiv \mu_1 + \mu_2 v_0^2$. As discussed in Refs. [8,14], we have performed a Galilean transformation to a reference frame $\mathbf{r}'$, which moves with respect to our original reference frame in the direction of mean flock motion at a speed $v_b = \lambda v_0$; that is, $\mathbf{r}' \equiv \mathbf{r} - \lambda v_0 t\hat{\mathbf{x}}$, or, equivalently, $x' = x - v_b t$. For simplicity, in (7) we have dropped the prime in x. Note that $\mathbf{u}(\mathbf{r},t)$ is also subject to the incompressibility constraint $\nabla\cdot\mathbf{u} = 0$ inherited from $\nabla\cdot\mathbf{v} = 0$. The somewhat nontrivial power counting that leads to (7) is detailed in Ref. [8].

We now perform a DRG analysis to obtain the dynamic exponent for the incompressible flock. Fourier transforming (7) and acting on the projection operator $P_{yi}(\mathbf{q})$ to eliminate the pressure term [20], we obtain

$$-\mathrm{i}\omega u_y = P_{yx}(\mathbf{q})\mathcal{F}_{\tilde{\mathbf{q}}}\left[-\alpha\left(u_x + \frac{u_y^2}{2}\right)\right] + P_{yy}(\mathbf{q})\mathcal{F}_{\tilde{\mathbf{q}}}\left[-\alpha\left(u_x + \frac{u_y^2}{2}\right)u_y + \mu\partial_x^2 u_y + f_y\right], \quad (8)$$

where we have eliminated v_0 by absorbing it into $\mathbf{u}$ (i.e., $\mathbf{u} \to v_0\mathbf{u}$), and the symbol $\mathcal{F}_{\tilde{\mathbf{q}}}$ represents the $\tilde{\mathbf{q}}$-th Fourier component.

Now we implement the standard DRG procedure [21] on the EOM (8). First we decompose the field $\mathbf{u}$ into slow and fast parts $\mathbf{u}^<(\mathbf{q})$ and $\mathbf{u}^>(\mathbf{q})$, which are supported at small $\mathbf{q}$'s and large $\mathbf{q}$'s, respectively. (We will eventually take the large $\mathbf{q}$'s to lie in an infinitesimally thin shell along the outer edge of the Brillouin zone.)

Next we average out $\mathbf{u}^>(\mathbf{q})$ to get the effective EOM for $\mathbf{u}^<(\mathbf{q})$. In this step the various coefficients in (8) are renormalized; we call these changes graphical corrections. Finally we rescale time, lengths, and fields as

$$t \to te^{zd\ell}, \quad x \to xe^{d\ell}, \quad y \to ye^{\zeta d\ell}, \quad (9a)$$

$$u_y \to u_y e^{\chi d\ell}, \quad u_x \to u_x e^{(\chi+1-\zeta)d\ell}, \quad (9b)$$

to restore the Brillouin zone to its original size. Note that the form of the rescaling in u_x is imposed by the incompressibility condition. This whole process is then repeated iteratively, which leads to recursion relations for the three coefficients α, μ, and D in (8). [Note that D is hidden in the correlations of the noise $\mathbf{f}$ (6).]

As usual in DRG calculations [21], we need to make approximations to perform the averaging step. We use four distinct DRG approximation schemes. Two of these are one-loop-order DRG calculations in exactly two dimensions, which are uncontrolled approximations since there is no limit in which they become exact. The third scheme is an $\epsilon_h = 5/2 - d$ expansion to $O(\epsilon_h)$, in which we analytically continue our calculation to $d > 2$ by treating the y direction as $(d-1)$ dimensional, and the fourth scheme is an $\epsilon_s = 3 - d$ expansion to $O(\epsilon_s)$ in which we analytically continue our theory to higher dimensions by treating the x direction to be $(d-1)$ dimensional. We call the former the hard continuation (hence the subscript "h" in ϵ_h), and the latter the soft continuation (hence the subscript "s" in ϵ_s). These last two are controlled approximations, since they formally become exact in the limit $\epsilon_{h,s} \to 0$. Obviously, they are also only approximate in $d = 2$. The first of the two uncontrolled approximations uses a Brillouin zone (BZ) $-\infty < q_y < \infty$, $-\Lambda < q_x < \Lambda$, where Λ is the ultraviolet cutoff, and the second uses the BZ $-\infty < q_x < \infty$, $-\Lambda < q_y < \Lambda$. The ϵ-expansion results are independent of the shape of the Brillouin zone as we explicitly demonstrate in the Supplemental Material [22]. All four approaches yield values of the dynamical exponent that are fairly close to each other. Furthermore, all four obtain values of the already exactly known exponents χ and ζ which prove to be very close to those known exact values; indeed, the first uncontrolled approximation and the soft-continuation ϵ expansion yield the exact values of the χ and ζ.

Crucially, in all four DRG calculations, we use the symmetry properties of the EOM for $\mathbf{u}$ to make two important simplifications. First, the rotation invariance of our hydrodynamic EOM is ensured by choosing the values of χ and ζ to keep all four "α"s appearing in (8) (i.e., the coefficients of u_x, u_y^2, $u_x u_y$, and u_y^3) the same under rescaling, i.e.,

$$\chi = 1 - \zeta. \tag{10}$$

This simplification reduces the total number of recursion relations to three. These are quite generally

$$\frac{d\ln\alpha}{d\ell} = z - 2\zeta + 2 + \eta_\alpha, \tag{11}$$

$$\frac{d\ln\mu}{d\ell} = z - 2 + \eta_\mu, \tag{12}$$

and

$$\frac{d\ln D}{d\ell} = z - \zeta - 2\chi - 1 + \eta_D \tag{13}$$

for the uncontrolled approximations (which are done in $d = 2$), or

$$\frac{d\ln D}{d\ell} = z + (1-d)\zeta - 2\chi - 1 + \eta_D \tag{14}$$

for the hard-continuation ϵ expansion, or

$$\frac{d\ln D}{d\ell} = z - \zeta - 2\chi - (d-1) + \eta_D \tag{15}$$

for the soft-continuation ϵ expansion. In all of these equations (11)–(15), $\eta_{\alpha,u,D}$ denote graphical corrections.

Note that these recursion relations imply that the fixed point values of $\eta_{\alpha,u,D}$ are determined entirely by z, ζ, and χ (and vice versa). Indeed, setting all ℓ derivatives equal to zero in (11), (12), and (13) implies that

$$\eta_\mu = 2 - z, \quad \eta_\alpha = 2\zeta - z - 2, \quad \text{and}$$
$$\eta_D = 2\chi - \zeta - z + 1. \tag{16}$$

Next, the fact that the dynamics of $\mathbf{u}$ obeys detailed balance introduces another simplification. This becomes clear if we formally introduce a friction coefficient in the dynamics for $\mathbf{u}$; $\Gamma\partial_t u_i = -\delta H/\delta u_i + f_i$, which is just the time-dependent Ginzberg-Landau model, or model A [23,24], with $H(\mathbf{M})$ the Hamiltonian for a ordered divergence-free two-dimensional magnet expanded around its minimum at nonzero magnetization, where $\mathbf{M} = (v_0 + u_x)\hat{x} + u_y\hat{y}$. $\Gamma = 1$ in (8) but it does not retain that value under renormalization. This would appear as a renormalization of the coefficient of $-i\omega u_y$; in general, this is an independent quantity. However, here detailed balance implies that the ratio D/Γ cannot change under renormalization, i.e., Γ and D must renormalize in the same way. This implies that the correction to $\ln\Gamma$ is the same as the correction to $\ln D$. If we denote the direct graphical correction to the annealed noise η_D^{dir} and the direct graphical correction to Γ as η_ω, the argument just given implies $\eta_\omega = \eta_D^{\text{dir}}$. To avoid retaining the friction coefficient as another extra parameter, we divide both sides of the renormalized EOM by the coefficient of $-i\omega u_y$ to fix the coefficient of $-i\omega u_y$ at 1. This effectively introduces an additional correction $-\eta_D^{\text{dir}}$ to both α and μ, and $-2\eta_D^{\text{dir}}$ to D. (The factor of 2 arises because D is proportional to the correlation of two noises.) Therefore, the overall graphical corrections to α, μ, and D are given by

$$\eta_\alpha = \eta_\alpha^{\text{dir}} - \eta_D^{\text{dir}}, \quad \eta_\mu = \eta_\mu^{\text{dir}} - \eta_D^{\text{dir}}, \tag{17}$$

$$\eta_D = \eta_D^{\text{dir}} - 2\eta_D^{\text{dir}} = -\eta_D^{\text{dir}}. \tag{18}$$

Using these considerations, and explicitly evaluating the graphical corrections to one-loop order [22], we find the following graphical corrections for the first uncontrolled approximation, i.e., using the BZ $-\infty < q_y < \infty$, $-\Lambda < q_x < \Lambda$ in exactly $d = 2$:

$$\eta_\alpha = -\frac{3g^{U1}}{4}, \quad \eta_\mu = \frac{g^{U1}}{4}, \quad \eta_D = -\frac{g^{U1}}{4}, \tag{19}$$

where

$$g^{U1} = \frac{\alpha^{1/2}D}{4\mu^{3/2}\pi\Lambda}. \tag{20}$$

Using the recursion relations (11), (12), (13), and the definition of g^{U1}, implies the following recursion relation for g^{U1}:

$$\frac{d\ln g^{U1}}{d\ell} = \frac{2 + 2\eta_D + \eta_\alpha - 3\eta_\mu}{2} \approx 1 - g^{U1}, \tag{21}$$

where we have used the identity (10). The first equality is exact, and the second, approximate, equality is valid only to one-loop order, and is obtained from combining (11), (12), (13), (19) with (20). This implies that there is a stable fixed point at $(g^{U1})^* = 1$. While this fixed point is only valid to one-loop order, since g^{U1} does have a nonzero fixed point value at all orders, and since the first equality in (21) is valid to all orders, setting the left-hand side of (21) to 0 leads to an exact identity between the values of $\eta_{\alpha,\mu,D}$ at the fixed point:

$$2 + 2\eta_D + \eta_\alpha - 3\eta_\mu = 0. \tag{22}$$

Using our relations (16) in $d = 2$, we can see that this expression (22) implies (10), which we already required to keep all α's the same.

An exactly analogous calculation for the second uncontrolled approximation, using the BZ $-\infty < q_x < \infty$, $-\Lambda < q_y < \Lambda$, yields the following graphical corrections [22] to one-loop order in exactly $d = 2$:

$$\eta_\alpha = -\frac{7g^{U2}}{8}, \quad \eta_\mu = \frac{5g^{U2}}{8}, \quad \eta_D = -\frac{3g^{U2}}{8}, \tag{23}$$

where

$$g^{U2} = \frac{\zeta\alpha^{1/4}D}{8\sqrt{2}\mu^{5/4}\pi\sqrt{\Lambda}}. \tag{24}$$

This definition of g^{U2} implies the recursion relation

$$\frac{d\ln g^{U2}}{d\ell} = \frac{2\zeta + 4\eta_D + \eta_\alpha - 5\eta_\mu}{4} \approx \frac{4\zeta - 11g^{U2}}{8}, \tag{25}$$

where, again, the first equality is formally valid to all orders, but the second, approximate equality, to only one-loop order. Since $\zeta > 0$, the approximate equality implies a stable fixed point at $(g^{U2})^* = 4\zeta/11$ to one-loop order.

The first, exact, relation implies another exact scaling law at the fixed point:

$$2\zeta + 4\eta_D + \eta_\alpha - 5\eta_\mu = 0. \tag{26}$$

However, this relation again proves to be exactly equivalent to (10), as can be seen by again using our relations (16) in $d = 2$.

Next, we turn to the ϵ-expansion calculations. Analytically continuing our calculation to $d > 2$ by treating the y direction as $(d-1)$ dimensional (hard continuation) and using the BZ $-\infty < q_x < \infty$, $0 < |q_y| < \Lambda$, we obtain the following graphical corrections [22] in the recursion relations (11), (12), (14) to one-loop order:

$$\eta_\alpha = -\frac{7g_h}{8}, \quad \eta_\mu = \frac{5g_h}{8}, \quad \eta_D = -\frac{3g_h}{8}, \tag{27}$$

where

$$g_h = \frac{\zeta\alpha^{1/4}DS_{d-1}\Lambda^{d-5/2}}{8\sqrt{2}(2\pi)^{d-1}\mu^{5/4}}. \tag{28}$$

Combining this definition of g_h with the recursion relations for α, μ, and D implies

$$\frac{d\ln g_h}{d\ell} = \frac{(10-4d)\zeta + 4\eta_D + \eta_\alpha - 5\eta_\mu}{4} \approx \frac{5-2d}{2}\zeta - \frac{11}{8}g_h. \tag{29}$$

This second, approximate equality implies that the critical dimension is $5/2$: for $d < 5/2$, g_h flows to a nonzero stable fixed point $(g_h)^* = \frac{8\epsilon_h}{11}\zeta$ to $O(\epsilon_h)$, where $\epsilon_h = 5/2 - d$; while for $d > 5/2$, g_h is attracted to $(g_h)^* = 0$ (i.e., the Gaussian fixed point). Note once again that the first exact equality implies an exact scaling relation between the anisotropy exponent ζ and the η's for $d < 5/2$:

$$(10-4d)\zeta + 4\eta_D + \eta_\alpha - 5\eta_\mu = 0. \tag{30}$$

Note that this relation reduces to (26) in $d = 2$, as it must.

Finally, we analytically continue our calculation to $d > 2$ by treating the x direction as $(d-1)$ dimensional (soft continuation) and use the BZ $0 < |q_x| < \Lambda$, $-\infty < q_y < \infty$. We obtain the following graphical corrections in the recursion relations (11), (12), and (15) to one-loop order [22]:

$$\eta_\alpha = -\frac{3g_s}{4}, \quad \eta_\mu = \frac{g_s}{4}, \quad \eta_D = -\frac{g_s}{4}, \tag{31}$$

where

$$g_s = \frac{D\alpha^{1/2}S_{d-1}\Lambda^{d-3}}{4\mu^{3/2}(2\pi)^{d-1}}. \tag{32}$$

The closed recursion relation for g_s is

$$\frac{d\ln g_s}{d\ell} = \frac{6 - 2d + 2\eta_D + \eta_\alpha - 3\eta_\mu}{2} \approx 3 - d - g_s. \tag{33}$$

Here the critical dimension is 3. For $d < 3$, the second approximate equality implies a stable fixed point $(g_s)^* = \epsilon_s$ to $O(\epsilon_s)$, where $\epsilon_s = 3 - d$. Note that again the first exact equality implies an exact scaling relation between η's for $d < 3$

$$6 - 2d + 2\eta_D + \eta_\alpha - 3\eta_\mu = 0, \tag{34}$$

which reduces to (22) in $d = 2$.

We now use the trajectory integral-matching formalism [25] to calculate $C(\mathbf{r}, t) = \langle \mathbf{u}(\mathbf{r}, t) \cdot \mathbf{u}(\mathbf{0}, 0)\rangle$, by relating the correlators in the original system to those of the rescaled system, via

$$C(\alpha_0, \mu_{x0}, D_0, \mathbf{r}, t) = e^{2\chi\ell}C\left[\alpha(\ell), \mu(\ell), D(\ell), \frac{|y|}{e^{\zeta\ell}}, \frac{|x - v_b t|}{e^{\ell}}, \frac{|t|}{e^{z\ell}}\right], \tag{35}$$

with α, μ, and D controlling the magnitude of $C(\mathbf{r}, t)$, and the subscript 0 denotes the bare values of the parameters. Note the argument $x - v_b t$ appears in these expressions because we have undone the aforementioned Galilean boost to return to the laboratory coordinate system.

We will illustrate this calculation in detail for the first uncontrolled approximation; for the other approximations the calculation is very similar. We choose ζ and z to fix α, μ, and D [of course, due to the exact relation (10), fixing ζ also fixes χ]. This gives

$$z^{U1} = \frac{7}{4}, \quad \zeta^{U1} = \frac{3}{2}, \quad \chi^{U1} = -\frac{1}{2}. \tag{36}$$

Then setting $\ell = \ln(\Lambda|x - v_b t|)$ casts the right-hand side of (35) in the form (2) where

$$\mathcal{G}_{\text{UET}}\left(\frac{|t|}{|x - v_b t|^z}, \frac{|y|}{|x - v_b t|^\zeta}\right) \equiv \Lambda^{2\chi}C\left[\alpha_0, \mu_0, D_0, \frac{|y|}{(|x - v_b t|\Lambda)^\zeta}, \frac{1}{\Lambda}, \frac{|t|}{(|x - v_b t|\Lambda)^z}\right]. \tag{37}$$

Since the static limit of the correlator $C(\mathbf{r}, t = 0)$ has been obtained in Ref. [8], we focus here on the dynamic limit $C(\mathbf{0}, t)$. In this limit we expect the correlator to be a power law of t only. As a result, the scaling function $\mathcal{G}_{\text{UET}}$ must behave as $(t/|x - v_b t|^z)^{\frac{\chi}{z}}$ to cancel out the prefactor $|x - v_b t|^{2\chi}$ in (2). Thus we obtain (4). We remind the reader that the static exponents ζ^{U1} and χ^{U1} in this approximation, coincidentally, are identical to the exact static exponents obtained in Ref. [8]. We however note that, as these are one-loop calculations, the static exponents have no reason to coincide with their exact values and, indeed, they do not in two of the three other approximations.

Likewise, the second uncontrolled approximation yields

$$z^{U2} = \tfrac{23}{14}, \quad \zeta^{U2} = \tfrac{11}{7}, \quad \chi^{U2} = -\tfrac{4}{7}, \tag{38}$$

the hard continuation

$$z^h = 2 - \tfrac{10}{11}\epsilon_h = \tfrac{17}{11}, \tag{39}$$

$$\zeta^h = 2 - \tfrac{12}{11}\epsilon_h = \tfrac{16}{11}, \tag{40}$$

$$\chi^h = -1 + \tfrac{12}{11}\epsilon_h = -\tfrac{5}{11}, \tag{41}$$

where $\epsilon_h = 5/2 - d$ and in the second equality $d = 2$ is taken, and the soft continuation

$$z^s = 2 - \tfrac{1}{4}\epsilon_s = \tfrac{7}{4}, \tag{42}$$

$$\zeta^s = 2 - \tfrac{1}{2}\epsilon_s = \tfrac{3}{2}, \tag{43}$$

$$\chi^s = -1 + \tfrac{1}{2}\epsilon_s = -\tfrac{1}{2}, \tag{44}$$

where $\epsilon_s = 3 - d$ and in the second equality $d = 2$ is taken.

Taking an average over the four sets of values of the exponents leads to the value $z = 1.67 \pm 0.05$ quoted earlier, $\zeta = 1.51 \pm 0.02$, and $\chi = -0.507 \pm 0.024$. Note that the predictions for ζ and χ are extremely close to their exact values $\zeta = 3/2$ and $\chi = -1/2$, which implies our prediction for z is almost certainly also very accurate. This also implies that, although the linear theory suggested that velocity fluctuations in incompressible flocks are diffusive, nonlinearities lower the dynamical exponent, which makes the dynamics superdiffusive: that is, distance scales like $t^{1/z}$ with $z < 2$, which means distance grows faster with time than the $t^{1/2}$ diffusive law.

In summary, we have calculated the dynamical exponent z characterizing the scaling of velocity fluctuations with time in incompressible two-dimensional flocks. We did so using four different approximation schemes, and demonstrated that the dynamics is significantly modified; specifically, the value of z is substantially changed by nonlinearities, which turn the fluctuations superdiffusive. Interestingly, due to the mapping [8] between this system and the model-A dynamics of a divergence-free magnet [17], this implies that magnetization fluctuations of a divergence-free magnet or a two-dimensional magnet with two-dimensional dipolar interactions are also superdiffusive.

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