Macroscopic finite-difference scheme and modified equations of the general propagation multiple-relaxation-time lattice Boltzmann model

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In this paper we first present the general propagation multiple-relaxation-time lattice Boltzmann (GPMRT-LB) model and obtain the corresponding macroscopic finite-difference (GPMFD) scheme on conservative moments. Then based on the Maxwell iteration method, we conduct the analysis on the truncation errors and modified equations (MEs) of the GPMRT-LB model and GPMFD scheme at both diffusive and acoustic scalings. For the nonlinear anisotropic convection-diffusion equation (NACDE) and Navier-Stokes equations (NSEs), we also derive the first- and second-order MEs of the GPMRT-LB model and GPMFD scheme. In particular, for the one-dimensional convection-diffusion equation (CDE) with the constant velocity and diffusion coefficient, we can develop a fourth-order GPMRT-LB (F-GPMRT-LB) model and the corresponding fourth-order GPMFD (F-GPMFD) scheme at the diffusive scaling. Finally, three benchmark problems, the Gauss hill problem, the CDE with nonlinear convection and diffusion terms, and the Taylor-Green vortex flow in two-dimensional space, are used to test the GPMRT-LB model and GPMFD scheme, and it is found that the numerical results not only are in good agreement with corresponding analytical solutions, but also have a second-order convergence rate in space. Additionally, a numerical study on one-dimensional CDE also demonstrates that the F-GPMRT-LB model and F-GPMFD scheme can achieve a fourth-order accuracy in space, which is consistent with our theoretical analysis.

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I. INTRODUCTION

The kinetic-theory-based lattice Boltzmann (LB) method, as a highly efficient numerical approach at the mesoscopic level, has been widely used to study the fluid flow problems (e.g., the multiphase flows [1,2], fluid flows in porous media [3]) governed by the Navier-Stokes equations (NSEs) [4] for its a second-order accuracy in space [5,6] and advantages in treating complex boundary conditions. On the other hand, the LB method has also been extended to solve some special kinds of partial differential equations (PDEs), including diffusion equations [7–13], convection-diffusion equations (CDEs) [14–23], Burgers' equations [24–26], general real and complex nonlinear convection-diffusion equations [27], and nonlinear anisotropic convection-diffusion equations (NACDEs) [28].

Usually, the LB method suffers from numerical instability when the relaxation parameter is close to 2. To solve the problem, two possible approaches can be adopted. The first one is to introduce the multiple-relaxation-time (MRT) collision operator [29–31] with some adjustable free relaxation parameters [32–35], which is more general and more stable than the single- and two-relaxation-time lattice Boltzmann (SRT-LB [36] and TRT-LB [14]) models. The second one is to use the general propagation LB model within the framework of the time-splitting method where two free parameters are introduced into the propagation step for NSEs [37] or NCDE [38] based on the Lax-Wendroff (LW) scheme [39,40] and fractional propagation (FP) scheme [41]. This model is more stable, and the popular standard LB model, LW and FP schemes can be viewed as its special cases. Considering the advantages of the MRT-LB model and general propagation LB model in the numerical stability, in this work, we will consider the more general propagation MRT-LB (GPMRT-LB) model.

In the framework of LB method, several asymptotic analysis approaches have been used to derive macroscopic PDEs, including the Chapman-Enskog analysis [42], Maxwell iteration [43,44], direct Taylor expansion [30], recurrence equations method [45,46], and equivalent equations method [47–49], although these asymptotic analysis approaches can be adopted to develop higher-order LB models [50–56], while they cannot be applied to clarify the relation between the LB model and the macroscopic PDE-based numerical scheme (the so-called macroscopic numerical scheme). Over recent years, some significant contributions have been made to bridge the gap between the LB model and macroscopic numerical scheme for a specified PDE, and these efforts aim to provide clear and rigorous consistency, accuracy, and derivation of modified equation (ME). Most of the existing works,

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however, are limited to the LB models and the macroscopic finite-difference schemes for diffusion equations and CDEs. On the one hand, Ancona [7] first presented the SRT-LB model with D1Q2 lattice structure for the one-dimensional diffusion equation and found that it is consistent with a macroscopic three-level second-order DuFort-Frankel scheme [57]. Then Suga [10] demonstrated that the SRT-LB model with D1Q3 lattice structure for the one-dimensional diffusion equation is equivalent to a macroscopic four-level fourth-order finite-difference scheme. Lin *et al.* [11] extended this work to consider a more general MRT-LB model and derived a macroscopic four-level sixth-order finite-difference scheme. Silva [12] focused on the TRT-LB model for the diffusion equation with a linear source term and obtained a macroscopic fourth-order finite-difference scheme. However, all of the aforementioned works are limited to only one-dimensional problems. In a recent work, Chen [13] considered the MRT-LB model with the D2Q5 lattice structure for the two-dimensional diffusion equation and obtained a five-level fourth-order finite-difference scheme. On the other hand, Dellacherie [58] analyzed the SRT-LB model for the one-dimensional CDE with D1Q2 lattice structure and illustrated that this LB model is equivalent to a three-level finite-difference scheme called LFCCDF (Leap-Frog difference for the temporal derivative, central difference for the convective term, and DuFort-Frankel approximation for the diffusive term) scheme [59]. Cui *et al.* [34] showed that for the one-dimensional steady CDE, the MRT-LB model can be written as a macroscopic second-order central-difference scheme. Following a similar idea, Wu *et al.* [60] derived a macroscopic finite-difference scheme of the MRT-LB model composed of natural moments and further performed a more general analysis on the discrete effects of some boundary schemes for the CDE. Recently, Chen *et al.* [23] obtained a macroscopic four-level fourth-order finite-difference scheme from the MRT-LB model with D1Q3 lattice structure for the CDE. In addition, Li *et al.* [24] demonstrated that for the one-dimensional Burgers equation, the SRT-LB model with the D1Q2 lattice structure can be expressed as a macroscopic three-level second-order finite-difference scheme. Junk [61] and Inamuro [62] found that the SRT-LB model is equivalent to a macroscopic two-level finite-difference scheme if the relaxation parameter is equal to one, and at the diffusive scaling, it has a second-order convergence rate for the incompressible NSEs [61]. D'Humières and Ginzburg [46] conducted a theoretical analysis on the TRT-LB model with recurrence equations and illustrated that when the magic parameter is fixed as $\Lambda^{eo} = 1/4$, the model can be written as a macroscopic three-level finite-difference scheme with a second-order accuracy in space. Chai *et al.* [30] further presented a more general analysis on the TRT-LB model and also derived the three-level finite-difference schemes for steady and unsteady problems.

It is worth noting that the works mentioned above are limited to some specific problems and/or lattice structures. To obtain the macroscopic finite-difference scheme from a given LB model with the *DdQq* (q discrete velocities in d-dimensional space) lattice structure, Fučík *et al.* [63] developed a general computational tool [64], while the origin of this algorithm remains unclear. In contrast, Bellotti *et al.* [65] conducted a precise algebraic characterization of the LB model and investigated the relationship between the MRT-LB model and macroscopic numerical scheme. They found that the LB model can be exactly expressed as a macroscopic multiple-level finite-difference scheme solely on the conservative variables. Furthermore, they also carried out analysis on the truncation errors and MEs at both diffusive and acoustic scalings [66], which are consistent with the results based on the asymptotic analysis methods [44,49]. However, it should be noted that they considered only the MRT-LB model with a diagonal relaxation matrix and the first-order ME at the diffusive scaling [66]. In this work we will first extend the previous works [65,66] to consider the more general GPMRT-LB models that are developed for NACDE and NSEs, and derive the macroscopic finite-difference (GPMFD) schemes. Then we will conduct detailed analysis on the truncation errors, the first- and second-order MEs of the GPMRT-LB models and GPMFD schemes at both diffusive and acoustic scalings.

The remainder of this paper is organized as follows. In Sec. II we present details on how to derive the GPMFD scheme on conservative moments from the GPMRT-LB model. In Sec. III the truncation errors at both diffusive and acoustic scalings are derived through the Maxwell iteration method, followed by the first- and second-order MEs of the GPMRT-LB model and GPMFD scheme. In Sec. IV we develop a fourth-order GPMRT-LB (F-GPMRT-LB) model and GPMFD (F-GPMFD) scheme at the diffusive scaling for the one-dimensional CDE with the constant velocity and diffusion coefficient. In Sec. V some simulations of the Gauss hill problem, the CDE with nonlinear convection and diffusion terms, the Taylor-Green vortex flow, and one-dimensional CDE are carried out to test the proposed GPMRT-LB model and GPMFD scheme. Finally, conclusions are given in Sec. VI.

II. THE GPMFD SCHEME OF THE GPMRT-LB MODEL

A. Preparation

To begin our analysis, we first discretize the problems in d ($d = 1, 2, 3$) dimensional space without considering the boundary conditions. In the LB method [4], the space is discretized by $\mathcal{L} := \Delta x \mathcal{Z}^d$ with a constant lattice spacing $\Delta x > 0$ in all directions, and the more general rectangular lattice structure [31] is not considered here. The time is uniformly discretized by $\mathcal{T} := \Delta t \mathcal{N}$ with $t_n := n\Delta t$, $n \in \mathcal{N}$, and Δt is the time step. Additionally, we introduce the so-called lattice velocity, defined by $\lambda := \Delta x/\Delta t$. It should be noted that the discretizations of the spatial and temporal domains are completely independent of the scaling between Δx and Δt; this means that it does not have an influence on the derivation of the macroscopic finite-difference schemes of the LB models. However, the scaling has a significant effect on the consistency analysis, i.e., the truncation errors and MEs (see Sec. III for details).

It is known that in the LB method, the evolution process can be split into the collision and propagation steps, and to simplify the following analysis on the derivation of the macroscopic finite-difference scheme, it is necessary to introduce the time and the space operators associated with the discrete velocities.

Definition 1. Let $\mathbf{z} \in \mathcal{Z}^d$ and $z \in \mathcal{Z}$, the space shift operator on the space lattice $\mathcal{L}$ denoted by $T^{\mathbf{z}}_{\Delta x}$ and the time shift operator on the time lattice $\mathcal{T}$ represented by $T^{z}_{\Delta t}$, are defined as follows:

$$\big[T^{\mathbf{z}}_{\Delta x}h\big](\mathbf{x},t) = h(\mathbf{x}+\mathbf{z}\Delta x, t), \mathbf{x} \in \mathcal{L}, t \in \mathcal{T}, \tag{1a}$$

$$\big[T^{z}_{\Delta t}h\big](\mathbf{x},t) = h(\mathbf{x}, t+z\Delta t), \mathbf{x} \in \mathcal{L}, t \in \mathcal{T}, \tag{1b}$$

where $h(\mathbf{x},t)$ is a smooth function: $\mathcal{R}^d \times \mathcal{R} \to \mathcal{R}$. In particular, the space and time shift operators of function $h(\mathbf{x},t)$ in Eq. (1) can also be expressed in the following series forms:

$$\big[T^{\mathbf{z}}_{\Delta x}h\big] = \sum_{k=0}^{+\infty} \frac{\Delta x^k (\mathbf{z}\cdot\nabla)^k}{k!} h(\mathbf{x},t), \tag{2a}$$

$$\big[T^{z}_{\Delta t}h\big](\mathbf{x},t) = \sum_{k=0}^{+\infty} \frac{(z\Delta t)^k \partial_t^k}{k!} h(\mathbf{x},t), \tag{2b}$$

where the gradient operator $\nabla = (\partial_{x_1}, \partial_{x_2}, \dots, \partial_{x_d})$.

B. GPMRT-LB model

In the LB method, some discrete procedures can be used to solve the discrete velocity Boltzmann equation [37,67],

$$\frac{\partial f_i}{\partial t} + \mathbf{c}_i \cdot \nabla f_i = \Omega_i + F_i, \tag{3}$$

where the function f_i represents the particle distribution at position $\mathbf{x}$ and time t, $\{\mathbf{c}_i = c\mathbf{e}_i, i = 1,2,\dots,q\}$ denotes the set of discrete velocities in $DdQq$ lattice structure, Ω_i is the general collision operator, and F_i is the discrete source or force term. Based on [37,38], we can develop a GPMRT-LB model for the NACDE and NSEs.

With the time-splitting method, Eq. (3) can be separated into two steps:

$$\frac{\partial f_i}{\partial t} = \Omega_i + F_i, \tag{4a}$$

$$\frac{\partial f_i}{\partial t} + \mathbf{c}_i \cdot \nabla f_i = 0, \tag{4b}$$

which are the collision and propagation steps, respectively.

Following the approach presented in Ref. [30], the collision step in Eq. (4a) can be reformulated as

$$f_i^\star(\mathbf{x},t) = f_i(\mathbf{x},t) - \mathbf{\Lambda}_{ik} f_k^{ne}(\mathbf{x},t) + \Delta t\left[G_i + F_i + \frac{\Delta t}{2}\overline{D}_i F_i\right](\mathbf{x},t), \quad i = 1,2,\dots,q. \tag{5}$$

Here $f^\star(\mathbf{x},t)$ denotes the postcollision distribution function, $\mathbf{\Lambda} = (\mathbf{\Lambda})_{ik}$ is a $q\times q$ invertible collision matrix and can be defined as $\mathbf{\Lambda} := \mathbf{M}^{-1}\mathbf{S}\mathbf{M}$, where $\mathbf{M}$ and $\mathbf{S}$ are the invertible transform and relaxation matrices, respectively. $f_i^{ne}(\mathbf{x},t) = f_i(\mathbf{x},t) - f_i^{\mathrm{eq}}(\mathbf{x},t)$ represents the nonequilibrium distribution function, and G_i is the auxiliary source distribution function and can be used to remove additional terms. $\overline{D}_i = \partial_t + \gamma\mathbf{c}_i\cdot\nabla$ with $\gamma \in \{0,1\}$ being a parameter to be determined [30], and in this work, we consider $\gamma = 0$ for the NACDE and $\gamma = 1$ for the NSEs.

To discretize the propagation step (4b), we adopt the explicit two-level, three-point scheme [37,38]:

$$f_i(\mathbf{x},t+\Delta t) = p_0 f^\star(\mathbf{x},t) + p_{-1} f_i^\star(\mathbf{x}-\lambda_i\Delta t,t) + p_1 f_i^\star(\mathbf{x}+\lambda_i\Delta t,t), \quad i = 1,2,\dots,q, \tag{6}$$

where the free parameters p_0, p_{-1}, and p_1 should satisfy the following conditions:

$$p_0 = 1-b, \quad p_{-1} = \frac{a+b}{2}, \quad p_1 = \frac{b-a}{2}, \tag{7}$$

with

$$\lambda_i = \lambda\mathbf{e}_i, \quad \lambda = \frac{\Delta x}{\Delta t}, \quad a = \frac{|\mathbf{c}_i|}{|\lambda_i|}, \quad c = a\lambda, \quad (0 < a \leqslant 1). \tag{8}$$

Substituting Eqs. (7) and (8) into Eq. (6) yields

$$f_i(\mathbf{x},t+\Delta t) = f_i^\star(\mathbf{x},t) - \frac{a}{2}[f_i^\star(\mathbf{x}+\lambda_i\Delta t,t) - f_i^\star(\mathbf{x}-\lambda_i\Delta t,t)] + \frac{b}{2}[f_i^\star(\mathbf{x}+\lambda_i\Delta t,t) - 2f^\star(\mathbf{x},t) + f_i^\star(\mathbf{x}-\lambda_i\Delta t,t)], \tag{9}$$

where a and b are considered as two free parameters. Here we would like to point out that Eq. (9) can reduce to the propagation step of the standard LB model [4] when $a = b = 1$, and additionally, based on the stability structure analysis [68], the two parameters a and b should satisfy the following condition:

$$a^2 \leqslant b \leqslant 1. \tag{10}$$

To simplify analysis and for the sake of brevity, the GPMRT-LB model composed of Eqs. (5) and (9) can also be expressed in a matrix form,

$$\mathbf{m}^{\star,n} = (\mathbf{I}_q - \mathbf{S})\mathbf{m}^n + \mathbf{S}\mathbf{m}^{eq,n} + \Delta t\tilde{\mathbf{F}}^n, \tag{11a}$$

$$\mathbf{m}^{n+1}(\mathbf{x}) = \mathbf{M}(p_0\mathbf{M}^{-1}\mathbf{m}^{\star,n}(\mathbf{x}) + p_{-1}\mathbf{M}^{-1}\mathbf{m}^{\star,n}(\mathbf{x}-\lambda_i\Delta t) + p_1\mathbf{M}^{-1}\mathbf{m}^{\star,n}(\mathbf{x}+\lambda_i\Delta t)), \tag{11b}$$

where $\mathbf{I}_q \in R^{q\times q}$ is the identify matrix and

$$\{\mathbf{m}^{\star,n};\mathbf{m}^n;\mathbf{m}^{eq,n};\tilde{\mathbf{F}}^n\} = \mathbf{M}\left\{\mathbf{f}^\star;\mathbf{f};\mathbf{f}^{\rm eq};\mathbf{G}+\mathbf{F}+\frac{\Delta t}{2}\overline{\mathbf{D}}\mathbf{F}\right\}(\mathbf{x},t_n), \tag{12a}$$

$$\mathbf{m}^{\star,n}(\mathbf{x}\pm\lambda_i\Delta t) = \mathbf{M}(f_1(\mathbf{x}\pm\lambda_1\Delta t,t_n),\ldots,f_q(\mathbf{x}\pm\lambda_q\Delta t,t_n))^T, \tag{12b}$$

here $\boldsymbol{\varpi} := (\varpi_1,\varpi_2,\ldots,\varpi_q)^T$ with $\boldsymbol{\varpi}$ representing $\{\mathbf{f}^\star,\mathbf{f},\mathbf{f}^{\rm eq},\mathbf{G},\mathbf{F}\}$ and $\overline{\mathbf{D}} := \mathrm{diag}(\overline{D}_1,\overline{D}_2,\ldots,\overline{D}_q)$.

In the following, we assume that the matrices $\mathbf{M}$ and $\mathbf{S}$ are independent on the space and time [65]. Then according to the space and time shift operators defined by Eqs. (1a) and (1b), the GPMRT-LB model, i.e., Eqs. (11a) and (11b), can be rewritten as

$$\left[T^1_{\Delta t}\mathbf{I}_q-\mathbf{A}\right]\mathbf{m}^n = \mathbf{B}\mathbf{m}^{eq,n}+\Delta t\mathbf{W}\tilde{\mathbf{F}}^n, \tag{13}$$

where

$$\mathbf{A} = \mathbf{W}(\mathbf{I}_q-\mathbf{S}),\,\mathbf{B} = \mathbf{W}\mathbf{S}, \tag{14}$$

with

$$\mathbf{W} = \mathbf{M}\overline{\mathbf{T}}\mathbf{M}^{-1},\quad \overline{\mathbf{T}} = (p_0\mathbf{T}_0+p_{-1}\mathbf{T}_{-1}+p_1\mathbf{T}_1), \tag{15a}$$

$$\mathbf{T}_0 = \mathbf{I}_q,\,\mathbf{T}_{-1} = \mathrm{diag}\left(T^{-\mathbf{e}_1}_{\Delta x},T^{-\mathbf{e}_2}_{\Delta x},\ldots,T^{-\mathbf{e}_q}_{\Delta x}\right),\quad \mathbf{T}_1 = \mathrm{diag}\left(T^{\mathbf{e}_1}_{\Delta x},T^{\mathbf{e}_2}_{\Delta x},\ldots,T^{\mathbf{e}_q}_{\Delta x}\right). \tag{15b}$$

Now we present a remark on the GPMRT-LB model, i.e., Eq. (13).

Remark 1. The term $\overline{D}_iF_i$ in the collision step (5) with $\gamma = 1$ can be discretized by an implicit difference scheme [28],

$$\overline{D}_iF_i = \frac{F_i(\mathbf{x}+\lambda_i\Delta t,t+\Delta t)-F_i(\mathbf{x},t)}{\Delta t},\quad i=1,2,\ldots,q. \tag{16}$$

If we substitute Eq. (16) into the GPMRT-LB model (13), one can obtain

$$\left[T^1_{\Delta t}\mathbf{I}_q-\mathbf{A}\right]\overline{\mathbf{m}}^n = \mathbf{B}\left[\mathbf{m}^{eq,n}-\frac{\Delta t}{2}\mathbf{M}\mathbf{F}^n\right]+\Delta t\mathbf{W}\overline{\mathbf{F}}^n, \tag{17}$$

which can be considered as a modified GPMRT-LB model with $\overline{\mathbf{m}}^n = \mathbf{m}^n-\frac{\Delta t}{2}\mathbf{M}\mathbf{F}^n$ and $\overline{\mathbf{F}}^n = \mathbf{M}(\mathbf{G}^n+\mathbf{F}^n)$.

C. Derivation of the GPMFD scheme

In this part, we will provide some details on how to derive the corresponding GPMFD scheme from the GPMRT-LB model (13). Without loss of generality, we assume that the first N rows in $\mathbf{M}$ correspond to the N conservative moments, and denote $i = 1,2,\ldots,J$ as $i\in\{1\sim J\}$ for brevity. Now we focus on two cases with $N = 1$ and $N > 1$, which are corresponding to the NACDE and NSEs considered in this work.

Proposition 1. For the case of $N = 1$, the GPMRT-LB model (13) can be written as a multiple-level finite-difference scheme on the conservative moment m_1,

$$m_1^{n+1} = -\sum_{k=1}^{q}\gamma_k\mathbf{m}_1^{n+k-q}+\sum_{k=1}^{q}\left[\sum_{l=1}^{k}\gamma_{q+1+l-k}\mathbf{A}^{l-1}(\mathbf{B}\mathbf{m}^{eq,n-k+1}+\Delta t\mathbf{W}\tilde{\mathbf{F}}^{n-k+1})\right]_1 \tag{18}$$

or

$$\det\left(T^1_{\Delta t}\mathbf{I}_q-\mathbf{A}\right)m_1^n = \left[\mathrm{adj}\left(T^1_{\Delta t}\mathbf{I}_q-\mathbf{A}\right)(\mathbf{B}\mathbf{m}^{eq,n}+\Delta t\mathbf{W}\tilde{\mathbf{F}}^n)\right]_1, \tag{19}$$

where $(\gamma_k)_{k=1}^{q+1}$ are the coefficients of the monic characteristic polynomial $p_{\mathbf{A}}(x) = \sum_{k=1}^{q+1}\gamma_kx^{k-1}$ of matrix $\mathbf{A}$, and adj$(\cdot)$ represents the adjugate matrix. The proof is similar to that of Proposition 4 in Ref. [65] and Proposition 2.7 in Ref. [66], and the details are not presented here.

We note that for $N > 1$ conservative moments, it is unclear whether the first N rows of the equation

$$\det\left(T^1_{\Delta t}\mathbf{I}_q-\mathbf{A}\right)\mathbf{m}^n = \left[\mathrm{adj}\left(T^1_{\Delta t}\mathbf{I}_q-\mathbf{A}\right)\left(\mathbf{B}\mathbf{m}^{eq,n}+\Delta t\mathbf{W}\tilde{\mathbf{F}}^n\right)\right] \tag{20}$$

can be considered as the finite-difference schemes of the GPMRT-LB model on the N conservative moments. To this end we first present a corollary of Proposition 1.

Corollary 1. Assuming that the relaxation matrix $\mathbf{S}$ of the GPMRT-LB model with $N\geqslant 1$ conservative moments is a diagonal one with diagonal elements $s_1,s_2,\ldots,s_j,\ldots,s_q$ located in the range (0, 2), then the jth ($j\in\{1\sim N\}$) row of Eq. (20) on the jth conservative moment does not depend on the relaxation parameter s_j. The proof is similar to Proposition 2.12 in Ref. [66], and for brevity, it is not shown here.

Regarding the conclusion of Corollary 1, we give a remark.

Remark 2. It is clear that the jth ($j\in\{1\sim N\}$) row of Eq. (19) can only be independent of s_j, but one cannot show that it is independent on the relaxation parameters associated with other conservative moments. Therefore, for $N > 1$ conservative moments, the first N rows of Eq. (20) cannot be viewed as the finite-difference schemes of the GPMRT-LB model on the N conservative moments. In the following, we will do some treatments on the matrix $\mathbf{A}$ as in the previous work [65] to derive the macroscopic finite-difference scheme of the GPMRT-LB model with $N > 1$ conservative moments, as outlined in Proposition 2. Additionally, we also consider a more general block-lower-triangular relaxation matrix $\mathbf{S}$ in the following Corollary 2.

Proposition 2. For $N \geqslant 1$ conservative moments, the GPMRT-LB model (13) corresponds to a multiple-level finite-difference scheme on the jth conservative moment m_j ($j \in \{1 \sim N\}$),

$$m_j^{n+1} = -\sum_{k=1}^{q+1-N} \gamma_{j,k} m_j^{n+N+k-1-q} + \sum_{k=1}^{q+1-N} \left[\sum_{l=1}^{k} \gamma_{j,q+2-N+l-k} \tilde{\mathbf{A}}_j^{l-1} \overline{\mathbf{A}}_j \mathbf{m}^{n-k+1} \right]_j + \sum_{k=1}^{q+1-N} \left[\sum_{l=1}^{k} \gamma_{j,q+2-N+l-k} \tilde{\mathbf{A}}_j^{l-1} \times (\mathbf{B}\mathbf{m}^{eq|n-k+1} + \Delta t \mathbf{W} \tilde{\mathbf{F}}^{n-k+1}) \right]_j \tag{21}$$

or

$$\det\left(T_{\Delta t}^1 \mathbf{I}_q - \tilde{\mathbf{A}}_j\right) m_j^n = \left[\operatorname{adj}\left(T_{\Delta t}^1 \mathbf{I}_q - \tilde{\mathbf{A}}_j\right) \overline{\mathbf{A}}_j \mathbf{m}^n\right]_j + \left[\operatorname{adj}\left(T_{\Delta t}^1 \mathbf{I}_q - \tilde{\mathbf{A}}_j\right) (\mathbf{B}\mathbf{m}^{eq,n} + \Delta t \mathbf{W} \tilde{\mathbf{F}}^n)\right]_j, \tag{22}$$

where

$$\tilde{\mathbf{A}}_j = \mathbf{A}\mathbf{P}_j, \mathbf{P}_j := \sum_{l=j,N+1,\ldots,q} I_l I_l^T, \tag{23a}$$

$$\overline{\mathbf{A}}_j = \mathbf{A} - \tilde{\mathbf{A}}_j, \tag{23b}$$

and $(\gamma_{j,k})_{k=1}^{q+2-N}$ are the coefficients of the monic characteristic polynomial $p_{\tilde{\mathbf{A}}_j}(x) = \sum_{k=1}^{q+2-N} \gamma_{j,k} x^{k+N-2}$ of matrix $\tilde{\mathbf{A}}_j$. We would like to point out that the proof is similar to that of Proposition 6 in Ref. [65] and Proposition 2.10 in Ref. [66], and the details are not given here. In addition, we would also like to point out that the finite-difference scheme (22) has the following Corollary 2.

Corollary 2. If the relaxation matrix $\mathbf{S}$ of the GPMRT-LB model with $N \geqslant 1$ conservative moments is a block-lower-triangular form with the diagonal elements located in range (0,2) [see Eq. (24)], then for any $j \in \{1 \sim N\}$, the finite-difference scheme (22) on the jth conservative moment is independent on the lower triangular relaxation parameters s_{il} ($l \in \{1 \sim N\}; i \in \{l \sim q\}$),

$$\begin{pmatrix} s_{11} & 0 & \ldots & 0 & \ldots & \ldots & 0 \\ s_{21} & s_{22} & 0 & 0 & \ldots & \ldots & 0 \\ s_{31} & s_{32} & s_{33} & 0 & \ldots & \ldots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ s_{N1} & s_{N2} & s_{N3} & \ldots & s_{NN} & \ldots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & & \\ s_{q1} & s_{q2} & s_{q3} & \ldots & s_{qN} & & \mathbf{S}_r \end{pmatrix}, \tag{24}$$

where $\mathbf{S}_r \in R^{(q-N)\times(q-N)}$ is a block-lower-triangular relaxation matrix.

Proof. Following the idea of proving Proposition 2.12 in Ref. [66], for any fixed $j \in \{1 \sim N\}$, the matrix $\mathbf{B}$ can be decomposed as

$$\mathbf{B} = \mathbf{B}|_{s_{kj}=0,k\geqslant j} + \mathbf{B}_j I_j^T, \tag{25}$$

then with the help of the relations

$$\left[\operatorname{adj}\left(T_{\Delta t}^1 \mathbf{I}_q - \tilde{\mathbf{A}}_j\right)\mathbf{B}_j I_j^T \mathbf{m}^{eq,n}\right]_j = I_j^T \operatorname{adj}\left(T_{\Delta t}^1 \mathbf{I}_q - \tilde{\mathbf{A}}_j\right)\mathbf{B}_j m_j^{eq,n}; \quad m_j^n = m_j^{eq,n}, \tag{26}$$

the finite-difference scheme (22) can be rewritten as

$$\left[\det\left(T_{\Delta t}^1 \mathbf{I}_q - \tilde{\mathbf{A}}_j\right) - I_j^T \operatorname{adj}\left(T_{\Delta t}^1 \mathbf{I}_q - \tilde{\mathbf{A}}_j\right)\mathbf{B}_j\right] m_j^n = \left[\operatorname{adj}\left(T_{\Delta t}^1 \mathbf{I}_q - \tilde{\mathbf{A}}_j\right)\overline{\mathbf{A}}_j \mathbf{m}^n\right]_j + \left[\operatorname{adj}\left(T_{\Delta t}^1 \mathbf{I}_q - \tilde{\mathbf{A}}_j\right)\mathbf{B}\big|_{s_{kj}=0,k\geqslant j} \mathbf{m}^{eq,n}\right]_j. \tag{27}$$

In the following, the proof is divided into four steps.

Step 1: Regarding the term on the left-hand side of Eq. (27), the definition of $\tilde{\mathbf{A}}_j$ (23a) gives

$$[\tilde{\mathbf{A}}_j]_{il} = [\mathbf{A}]_{il}, i \in \{1 \sim q\}; l = \{j\} \cup \{(N+1) \sim q\}, \tag{28a}$$

$$[\tilde{\mathbf{A}}_j]_{il} = 0, i \in \{1 \sim q\}; l \in \{1 \sim N\}/\{j\}, \tag{28b}$$

from which one can find that $\tilde{\mathbf{A}}_j$ and $\mathbf{B}_j$ are independent on s_{il} ($l \in \{1 \sim N\}/\{j\}; i \in \{l \sim q\}$), this also means that the term on the left-hand side of Eq. (27) does not depend on the lth ($l \in \{1 \sim N\}/\{j\}$) column of matrix $\mathbf{S}$.

Step 2: For the terms on the right-hand side of Eq. (27), the definitions of $\overline{\mathbf{A}}_j$ (23b) and $\mathbf{B}|_{s_{kj}=0,k\geqslant j}$ show that $[\overline{\mathbf{A}}_j]_{pr}$ and $[\mathbf{B}|_{s_{kj}=0,k\geqslant j}]_{pr}$ ($p \in \{1 \sim q\}; r \in \{N+1 \sim q\}$), are independent of s_{il} ($l \in \{1 \sim N\}/\{j\}; i \in \{l \sim q\}$). Due to $m_l^n = m_l^{eq,n}$ for $l \in \{1 \sim N\}/\{j\}$, $[\overline{\mathbf{A}}_j]_{il}$ and $[\mathbf{B}|_{s_{kj}=0,k\geqslant j}]_{il}$ can be considered together for $i \in \{1 \sim q\}$ and $l \in \{1 \sim N\}/\{j\}$,

$$[\overline{\mathbf{A}}_j]_{il} + \left[\mathbf{B}|_{s_{kj}=0,k\geqslant j}\right]_{il} = \left[\overline{\mathbf{A}}_j + \mathbf{B}|_{s_{kj}=0,k\geqslant j}\right]_{il} = [\mathbf{A}+\mathbf{B}]_{il} = [\mathbf{W}]_{il} = \left[\mathbf{A}|_{s_{kl}=0,k\geqslant l}\right]_{il} = \left[\overline{\mathbf{A}}_j|_{s_{kl}=0,k\geqslant l}\right]_{il}, \tag{29}$$

thus the terms on the right-hand side of Eq. (27) are independent of the lth ($l \in \{1 \sim N\}/\{j\}$) column of $\mathbf{S}$.

Step 3: Now we consider whether Eq. (27) depends on the jth column of matrix $\mathbf{S}$ or not. After some algebraic manipulations, Eq. (27) can also be reformulated as

$$\left[\det\left(T_{\Delta t}^{1}\mathbf{I}_q-\left(\tilde{\mathbf{A}}_j+\mathbf{B}_j^T I_j\right)\right)\right]m_j^n=\left[\mathrm{adj}\left(T_{\Delta t}^{1}\mathbf{I}_q-\tilde{\mathbf{A}}_j\right)\overline{\mathbf{A}}_j\mathbf{m}^n\right]_j+\left[\mathrm{adj}\left(T_{\Delta t}^{1}\mathbf{I}_q-\tilde{\mathbf{A}}_j\right)\mathbf{B}\big|_{s_{kj}=0,k\geqslant j}\mathbf{m}^{eq,n}\right]_j. \tag{30}$$

From the terms on the the right-hand side of Eq. (30), one can see that the jth column of $\tilde{\mathbf{A}}_j$ depends on only the jth column of matrix $\mathbf{S}$. Therefore, the jth row of $\mathrm{adj}(T_{\Delta t}^{1}\mathbf{I}_q-\tilde{\mathbf{A}}_j)$ does not depend on the jth column of matrix $\mathbf{S}$.

Step 4: From the term on the left-hand side of Eq. (30), for any $i\in\{1\sim q\}$, one can obtain

$$\left[\tilde{\mathbf{A}}_j+\mathbf{B}_j I_j^T\right]_{il}=\begin{cases}[\mathbf{A}]_{il}, & l\in\{(N+1)\sim q\},\\ 0, & l\in\{1\sim N\}/\{j\},\\ [\mathbf{A}+\mathbf{B}]_{il}=[\mathbf{W}]_{il}=\left[\tilde{\mathbf{A}}_j|_{s_{kj}=0,k\geqslant j}\right]_{il}, & l=j,\end{cases} \tag{31}$$

thus the term on the left-hand side of Eq. (30) does depend on the jth column of matrix $\mathbf{S}$. Due to the equivalence of finite-difference scheme (22), Eqs. (27) and (30), one can prove Corollary 2.

We now discuss Corollary 2.

Remark 3. According to the Remark 1 and Proposition 2, if the parameter γ in $\overline{\mathbf{D}}$ is equal to one, the macroscopic finite-difference scheme on the jth ($j\in\{1\sim N\}$) conservative moment of the GPMRT-LB model (17) is given by

$$\det\left(T_{\Delta t}^{1}\mathbf{I}_q-\tilde{\mathbf{A}}_j\right)\overline{\mathbf{m}}_j^n=\left[\mathrm{adj}\left(T_{\Delta t}^{1}\mathbf{I}_q-\tilde{\mathbf{A}}\right)\overline{\mathbf{A}}_j\overline{\mathbf{m}}^n\right]_j+\left[\mathrm{adj}\left(T_{\Delta t}^{1}\mathbf{I}_q-\tilde{\mathbf{A}}_j\right)\mathbf{B}\left(\mathbf{m}^{eq,n}-\frac{\Delta t}{2}\mathbf{M}\mathbf{F}^n\right)\right]_j+\Delta t\left[\mathrm{adj}\left(T_{\Delta t}^{1}\mathbf{I}_q-\tilde{\mathbf{A}}_j\right)\mathbf{W}\overline{\mathbf{F}}^n\right]_j, \tag{32}$$

while it is unclear whether the finite-difference scheme (32) has Corollary 2 or not. We will pay attention to this problem in the following part. It is clear that the last term on the right-hand side of Eq. (32) does not depend on the relaxation matrix $\mathbf{S}$, thus we need to consider only the following scheme:

$$\det\left(T_{\Delta t}^{1}\mathbf{I}_q-\tilde{\mathbf{A}}_j\right)\overline{\mathbf{m}}_j^n=\left[\mathrm{adj}\left(T_{\Delta t}^{1}\mathbf{I}_q-\tilde{\mathbf{A}}_j\right)\overline{\mathbf{A}}_j\overline{\mathbf{m}}^n\right]_j+\left[\mathrm{adj}\left(T_{\Delta t}^{1}\mathbf{I}_q-\tilde{\mathbf{A}}_j\right)\mathbf{B}\left(\mathbf{m}^{eq,n}-\frac{\Delta t}{2}\mathbf{M}\mathbf{F}^n\right)\right]_j. \tag{33}$$

If we write the matrix $\mathbf{B}$ as

$$\mathbf{B}=\mathbf{B}|_{s_{kj}=0,k\geqslant j}+\mathbf{B}_j I_j^T, \tag{34}$$

and according to the relation $\overline{\mathbf{m}}_j=\mathbf{m}^{\mathrm{eq}}-\Delta t/2\mathbf{M}\mathbf{F}$, one can obtain

$$\left[\det\left(T_{\Delta t}^{1}\mathbf{I}_q-\tilde{\mathbf{A}}_j\right)-I_j^T\,\mathrm{adj}\left(T_{\Delta t}^{1}\mathbf{I}_q-\tilde{\mathbf{A}}_j\right)\mathbf{B}_j\right]\overline{\mathbf{m}}_j^n=\left[\mathrm{adj}\left(T_{\Delta t}^{1}\mathbf{I}_q-\tilde{\mathbf{A}}_j\right)\overline{\mathbf{A}}_j\overline{\mathbf{m}}^n\right]_j+\left[\mathrm{adj}\left(T_{\Delta t}^{1}\mathbf{I}_q-\tilde{\mathbf{A}}_j\right)\mathbf{B}|_{s_{kj}=0,k\geqslant j}\left(\mathbf{m}^{eq,n}-\frac{\Delta t}{2}\mathbf{M}\mathbf{F}^n\right)\right]_j, \tag{35}$$

which is similar to Eq. (27), and the detailed proof is not presented here. From above discussion, it can be concluded that the finite-difference scheme (32) also has Corollary 2, like the finite-difference scheme (22).

Remark 4. Based on Corollary 2, the finite-difference scheme (22) on the jth ($j\in\{1\sim N\}$) conservative moment is consistent with the GPMRT-LB model, while Eq. (20) is not. It should also be noted that the lower triangular elements s_{il} ($l\in\{1\sim N\}; i\in\{l\sim q\}$) in the relaxation matrix $\mathbf{S}$ (24) do not affect the forms of difference schemes (21) and (22). To simplify following analysis on the truncation errors and MEs in Sec. III, we assume that the diagonal relaxation parameters corresponding to the conservative moments in the relaxation matrix $\mathbf{S}$ (24) are equal to one, and the nondiagonal relaxation parameters associated with the conservative moments are equal to zero (here we note that the matrix $\mathbf{S}$ must be invertible and this choice is crucial in order to use the Maxwell iteration [44] as we will do in the following), in this case, the finite-difference schemes (21) and (22) can be further simplified; see the following Proposition 3 for details.

Proposition 3. For a given GPMRT-LB model $(\mathbf{M},\mathbf{S})$ with $N\geqslant 1$ conservative moments, let $\hat{\mathbf{S}}=\mathrm{diag}(\mathbf{I}_N,\mathbf{S}_r)$ with $\mathbf{S}_r\in R^{(q-N)\times(q-N)}$ representing the matrix consisting of the $(N+1)$th to qth rows and columns of matrix $\mathbf{S}$, then the finite-difference schemes (21) and (22) of the GPMRT-LB model (13) on the jth ($j\in\{1\sim N\}$) conservative moment can be simplified as

$$m_j^{n+1}=-\sum_{k=N}^{q}\gamma_k m_j^{n+k-q}+\sum_{k=N}^{q}\left[\sum_{l=N}^{k}\gamma_{q+1+l-k}\mathbf{A}^{l-N}\left(\mathbf{B}\mathbf{m}^{eq|n-k+N}+\Delta t\mathbf{W}\tilde{\mathbf{F}}^{n-k+N}\right)\right]_j \tag{36}$$

and

$$\det\left(T_{\Delta t}^{1}\mathbf{I}_q-\mathbf{A}\right)m_j^n=\left[\mathrm{adj}\left(T_{\Delta t}^{1}\mathbf{I}_q-\mathbf{A}\right)\left(\mathbf{B}\mathbf{m}^{eq,n}+\Delta t\mathbf{W}\tilde{\mathbf{F}}^n\right)\right]_j, \tag{37}$$

where $\mathbf{A}=\mathbf{W}(\mathbf{I}_q-\hat{\mathbf{S}})$, $\mathbf{B}=\mathbf{W}\hat{\mathbf{S}}$. In addition, Eq. (37) has the same form as the jth row of Eq. (20).

Proof. Based on the form of relaxation $\hat{\mathbf{S}}$ and the definition of matrix $\mathbf{P}_j$ (23a), one can obtain

$$\tilde{\mathbf{A}}_j=\mathbf{A}\mathbf{P}_j=\mathbf{W}(\mathbf{I}_q-\hat{\mathbf{S}})=\mathbf{A}, \tag{38}$$

which means $\overline{\mathbf{A}}_j = \mathbf{A} - \tilde{\mathbf{A}}_j = \mathbf{0}$, thus the finite-difference schemes (21) and (22) can be simplified by

$$m_j^{n+1} = -\sum_{k=1}^{q+1-N} \gamma_{j,k} m_j^{n+N+k-1-q} + \sum_{k=1}^{q+1-N} \left[\sum_{l=1}^{k} \gamma_{j,q+2-N+l-k} \mathbf{A}^{l-1} (\mathbf{B}\mathbf{m}^{eq|n-k+1} + \Delta t \mathbf{W}\tilde{\mathbf{F}}^{n-k+1}) \right]_j \tag{39}$$

and

$$\det\left(T_{\Delta t}^1 \mathbf{I}_q - \mathbf{A}\right) m_j^n = \left[\operatorname{adj}\left(T_{\Delta t}^1 \mathbf{I}_q - \mathbf{A}\right)(\mathbf{B}\mathbf{m}^{eq,n} + \Delta t \mathbf{W}\tilde{\mathbf{F}}^n)\right]_j. \tag{40}$$

Then one can obtain the finite-difference schemes (36) and (37) through rearranging Eqs. (39) and (40) and with the aid of the following results:

$$\gamma_k = 0, k \in \{1 \sim (N-1)\}; \gamma_k = \gamma_{j,k+1-N}, k \in \{N \sim (q+1)\}, \tag{41}$$

where the relation $p_{\mathbf{A}}(x) = x^{-1} \sum_{k=1}^{q+1} \gamma_k x^k = p_{\tilde{\mathbf{A}}_j}(x) = x^{N-2} \sum_{k=1}^{q+2-N} \gamma_{j,k} x^k$ has been used.

Now we further discuss the issue of the equivalence between two GPMRT-LB models. On one hand, the two GPMRT-LB models $(\mathbf{M}, \mathbf{S})$ and $(\mathbf{M}, \mathbf{S}_1)$ can be considered equivalently if the relaxation parameters associated with nonconservative moments in the relaxation matrices $\mathbf{S}$ and $\mathbf{S}_1$ are identical regardless of whether the relaxation parameters associated with conservative moments are the same or not, and the finite-difference scheme (22) also has this feature (see Corollary 2). On the other hand, the two GPMRT-LB models $(\mathbf{M}, \mathbf{S})$ and $(\mathbf{M}_1, \mathbf{S}_1)$ are also equivalent if the following relations hold:

$$\mathbf{M}_1 = \mathbf{N}\mathbf{M}, \mathbf{S}_1 = \mathbf{N}\mathbf{S}\mathbf{N}^{-1}, \tag{42a}$$

$$[\mathbf{M}_1]_{il} = [\mathbf{M}]_{il}, i \in \{1 \sim N\}; l \in \{1 \sim q\}, \tag{42b}$$

where $\mathbf{N}$ is an invertible block-lower-triangular matrix. This means that the finite-difference scheme of the GPMRT-LB model $(\mathbf{M}_1, \mathbf{S}_1)$ should be the same as that of the GPMRT-LB model $(\mathbf{M}, \mathbf{S})$. In the following, we first show that the first N rows of Eq. (20) corresponding to the two equivalent GPMRT-LB models $(\mathbf{M}, \mathbf{S})$ and $(\mathbf{M}_1, \mathbf{S}_1)$ are identical, and then present another form of the finite-difference scheme from the GPMRT-LB model $(\mathbf{M}_1, \mathbf{S}_1)$, which is identical to schemes (21) and (22).

Theorem 1. The first N rows of Eq. (20) corresponding to the two equivalent GPMRT-LB models $(\mathbf{M}, \mathbf{S})$ and $(\mathbf{M}_1, \mathbf{S}_1)$ satisfying Eq. (42) are totally identical.

Proof. According to the relation (42a), the matrices $\mathbf{A}$ and $\mathbf{A}^1$ satisfy

$$\mathbf{A}^1 = \mathbf{N}\mathbf{A}\mathbf{N}^{-1}, \tag{43}$$

where $\mathbf{A}^1 = \mathbf{W}^1(\mathbf{I} - \mathbf{S}_1)$ with $\mathbf{W}^1 = \mathbf{M}_1\overline{\mathbf{T}}\mathbf{M}_1^{-1}$. Due to the invertible block-lower-triangular matrix $\mathbf{N}$, one can obtain

$$\det\left[T_{\Delta t}^1 \mathbf{I}_q - \mathbf{A}\right] = \det\left[T_{\Delta t}^1 \mathbf{I}_q - \mathbf{A}^1\right]. \tag{44}$$

Based on the relation (42b), we have

$$[\mathbf{A}^1\mathbf{B}^1\mathbf{M}_1]_{il} = [\mathbf{A}\mathbf{B}\mathbf{M}]_{il}; \quad [\mathbf{A}^1\mathbf{W}^1\mathbf{M}_1]_{il} = [\mathbf{A}\mathbf{W}\mathbf{M}]_{il}, \tag{45}$$

where $\mathbf{B}^1 = \mathbf{W}^1\mathbf{S}_1$. Thus, from the algebraic expression of adjugate matrix $\operatorname{adj}(T_{\Delta t}^1 \mathbf{I}_q - \mathbf{A})$,

$$\operatorname{adj}\left(T_{\Delta t}^1 \mathbf{I}_q - \mathbf{A}\right) = T_{\Delta t}^q \sum_{k=1}^{q} \left(\sum_{l=1}^{q-k+1} \gamma_{k+l} \mathbf{A}^{l-1} \right) T_{\Delta t}^{k-q} \mathbf{I}_q, \tag{46}$$

and for any $j \in \{1 \sim N\}$, it is easy to prove

$$\left[\operatorname{adj}\left(T_{\Delta t}^1 \mathbf{I}_q - \mathbf{A}^1\right)\mathbf{B}^1\mathbf{M}_1\mathbf{f}^{eq,n}\right]_j = \left[\operatorname{adj}\left(T_{\Delta t}^1 \mathbf{I}_q - \mathbf{A}\right)\mathbf{B}\mathbf{M}\mathbf{f}^{eq,n}\right]_j, \tag{47a}$$

$$\left[\operatorname{adj}\left(T_{\Delta t}^1 \mathbf{I}_q - \mathbf{A}^1\right)\mathbf{W}^1\mathbf{M}_1\left(\mathbf{F}^n + \mathbf{G}^n + \frac{\Delta t}{2}\mathbf{D}\mathbf{F}^n\right)\right]_j = \left[\operatorname{adj}\left(T_{\Delta t}^1 \mathbf{I}_q - \mathbf{A}\right)\mathbf{W}\mathbf{M}\left(\mathbf{F}^n + \mathbf{G}^n + \frac{\Delta t}{2}\mathbf{D}\mathbf{F}^n\right)\right]_j. \tag{47b}$$

According to the relations (44) and (47), one can prove Theorem 1.

Based on Theorem 1, we can also conclude that the analysis on the truncation errors and MEs shown below will remain identical whether the transform matrix $\mathbf{M}$ is independent of the parameter c [see Eq. (8)] or not. The reason is provided in the following Remark 5.

Remark 5. Considering the following relation between the two GPMRT-LB models $(\mathbf{M}^c, \mathbf{S}^c)$ and $(\mathbf{M}^o, \mathbf{S}^o)$:

$$\mathbf{M}^c = \mathbf{C}_d\mathbf{M}^o, \mathbf{S}^c = \mathbf{C}_d\mathbf{S}^o\mathbf{C}_c^{-1}, \tag{48a}$$

$$[\mathbf{M}^c]_{il} = [\mathbf{C}_d]_{ii}[\mathbf{M}^o]_{il}, i \in \{1 \sim N\}; l \in \{1 \sim q\}, \tag{48b}$$

where $\mathbf{C}_d$ is an invertible diagonal matrix associated with the parameter c. Similar to the proof in Theorem 1, one can obtain

$$\det\left[T^1_{\Delta t}\mathbf{I}_q-\mathbf{A}^c\right]\mathbf{M}^c\mathbf{f}^n=\mathbf{C}_d\det\left[T^1_{\Delta t}\mathbf{I}_q-\mathbf{A}\right]\mathbf{M}^o\mathbf{f}^n \tag{49}$$

and

$$\left[\mathrm{adj}\left(T^1_{\Delta t}\mathbf{I}_q-\mathbf{A}^c\right)\mathbf{B}^c\mathbf{M}^c\mathbf{f}^{eq,n}\right]_j=\mathbf{C}_d\left[\mathrm{adj}\left(T^1_{\Delta t}\mathbf{I}_q-\mathbf{A}\right)\mathbf{B}\mathbf{M}^o\mathbf{f}^{eq,n}\right]_j, \tag{50a}$$

$$\left[\mathrm{adj}\left(T^1_{\Delta t}\mathbf{I}_q-\mathbf{A}^c\right)\mathbf{W}^c\mathbf{M}^c\left(\mathbf{F}^n+\mathbf{G}^n+\frac{\Delta t}{2}\mathbf{D}\mathbf{F}^n\right)\right]_j=\mathbf{C}_d\left[\mathrm{adj}\left(T^1_{\Delta t}\mathbf{I}_q-\mathbf{A}\right)\mathbf{W}\mathbf{M}^o\left(\mathbf{F}^n+\mathbf{G}^n+\frac{\Delta t}{2}\mathbf{D}\mathbf{F}^n\right)\right]_j. \tag{50b}$$

It is obvious that the first N rows of Eq. (20) corresponding respectively to the two GPMRT-LB models ($\mathbf{M}^c,\mathbf{S}^c$) and ($\mathbf{M}^o,\mathbf{S}^o$) satisfying Eq. (48) only differ in the constant matrix $\mathbf{C}_d$, which has no influence on the truncation errors and MEs analysis in Sec. III.

Theorem 2. The finite-difference scheme (22) corresponding to the GPMRT-LB model ($\mathbf{M},\mathbf{S}$) can be rewritten as

$$\det\left(T^1_{\Delta t}\mathbf{I}_q-\tilde{\mathbf{A}}^1_j\right)\mathbf{m}^{n,(1)}_j=\left[\mathrm{adj}\left(T^1_{\Delta t}\mathbf{I}_q-\tilde{\mathbf{A}}^1_j\right)\overline{\mathbf{A}}^1_j\mathbf{m}^{n,(1)}\right]_j+\left[\mathrm{adj}\left(T^1_{\Delta t}\mathbf{I}_q-\tilde{\mathbf{A}}^1_j\right)\mathbf{B}^1\mathbf{m}^{eq,n,(1)}\right]_j+\Delta t\left[\mathrm{adj}\left(T^1_{\Delta t}\mathbf{I}_q-\tilde{\mathbf{A}}^1_j\right)\mathbf{W}\tilde{\mathbf{F}}^{n,(1)}\right]_j, \tag{51}$$

where

$$\mathbf{A}^1=\mathbf{W}^1(\mathbf{I}_q-\mathbf{S}_1),\quad \mathbf{B}^1=\mathbf{W}^1\mathbf{S}_1,\quad \tilde{\mathbf{A}}^1_j=\mathbf{A}^1\mathbf{P}^1_j,\quad \overline{\mathbf{A}}^1_j=\mathbf{A}^1\overline{\mathbf{P}}_j, \tag{52a}$$

$$\mathbf{m}^{n,(1)}=\mathbf{M}_1\mathbf{m}^n,\quad \mathbf{m}^{eq,n,(1)}=\mathbf{M}_1\mathbf{m}^{eq,n},\quad \tilde{\mathbf{F}}^{n,(1)}=\mathbf{M}_1\tilde{\mathbf{F}}^n, \tag{52b}$$

with matrices $\mathbf{M}_1$ and $\mathbf{S}_1$ satisfying Eq. (42) and

$$\mathbf{W}^1:=\mathbf{M}_1\overline{\mathbf{T}}\mathbf{M}_1^{-1},\quad \mathbf{P}^1_j:=\mathbf{N}\mathbf{P}_j\mathbf{N}^{-1},\quad \overline{\mathbf{P}}_j:=\mathbf{I}_q-\tilde{\mathbf{P}}^1_j. \tag{53}$$

The proof is similar to Theorem 1, and the details are not shown here.

We now give some remarks on the conclusions in Theorems 1 and 2.

Remark 6. For two equivalent GPMRT-LB models ($\mathbf{M},\mathbf{S}$) and ($\mathbf{M}_1,\mathbf{S}_1$) in Theorem 1, if we further consider the GPMRT-LB model ($\mathbf{M}_1,\hat{\mathbf{S}}_1$) with

$$\hat{\mathbf{S}}_1=\mathbf{N}\hat{\mathbf{S}}\mathbf{N}^{-1}=\begin{pmatrix}\mathbf{N}_1&\mathbf{0}\\ \mathbf{N}_2&\mathbf{N}_3\end{pmatrix}\begin{pmatrix}\mathbf{I}_N&\mathbf{0}\\ \mathbf{0}&\mathbf{S}_r\end{pmatrix}\begin{pmatrix}\mathbf{N}_1^{-1}&\mathbf{0}\\ -\mathbf{N}_3^{-1}\mathbf{N}_2\mathbf{N}_1^{-1}&\mathbf{N}_3^{-1}\end{pmatrix}=\begin{pmatrix}\mathbf{I}_N&\mathbf{0}\\ \mathbf{N}_2\mathbf{N}_1^{-1}-\mathbf{N}_3\mathbf{S}_r\mathbf{N}_3^{-1}\mathbf{N}_2\mathbf{N}_1^{-1}&\mathbf{N}_3\mathbf{S}_r\mathbf{N}_3^{-1}\end{pmatrix}, \tag{54}$$

where matrix $\mathbf{N}_1\in R^{N\times N}$, $\mathbf{N}_2\in R^{(q-N)\times N}$, and $\mathbf{N}_3\in R^{(q-N)\times(q-N)}$ are the submatrices of the block-lower-triangular matrix $\mathbf{N}$, and the matrix $\hat{\mathbf{S}}$ is defined as that in Proposion 3, one can obtain

$$\hat{\mathbf{S}}=\mathbf{S}\tilde{\mathbf{P}}_j+\overline{\mathbf{P}}_j+(\mathbf{I}_q-\mathbf{S})I_jI_j^T. \tag{55}$$

Substituting Eq. (55) into Eq. (54) yields

$$\hat{\mathbf{S}}_1=\mathbf{S}_1\tilde{\mathbf{P}}^1_j+\overline{\mathbf{P}}^1_j+(\mathbf{I}_q-\mathbf{S}_1)\mathbf{N}I_jI_j^T\mathbf{N}^{-1}, \tag{56}$$

where matrices $\tilde{\mathbf{P}}^1_j$ and $\overline{\mathbf{P}}^1_j$ are those in Eqs. (52) and (53). It is evident that based on Corollary 2, Proposition 3, and Eq. (42), the four GPMRT-LB models ($\mathbf{M}_1,\mathbf{S}_1$), ($\mathbf{M},\mathbf{S}$), ($\mathbf{M},\hat{\mathbf{S}}$), and ($\mathbf{M}_1,\hat{\mathbf{S}}_1$) are equivalent. Moreover, according to Proposition 3 and Theorems 1 and 2, the four equivalent GPMRT-LB models ($\mathbf{M}_1,\mathbf{S}_1$), ($\mathbf{M},\mathbf{S}$), ($\mathbf{M},\hat{\mathbf{S}}$), and ($\mathbf{M}_1,\hat{\mathbf{S}}_1$) satisfy the relations presented in Fig. 1, where the double-line arrow connecting the two boxes indicates that the two GPMRT-LB models have the same finite-difference scheme.

Remark 7. Regarding the finite-difference scheme (22) with $N\geqslant 1$ conservative moments in Proposition 2, the characteristics shown in Corollary 2 and Theorem 2 are consistent with the GPMRT-LB model. Therefore, we refer to it as the macroscopic finite-difference (GPMFD) scheme of the GPMRT-LB model. Here it should be noted that for simplicity, we do not consider the effect of the choice of the initialization schemes for the GPMRT-LB model and GPMFD scheme, and in the numerical simulations presented in Sec. V, we initialize the distribution function f_i with its equilibrium state for the GPMRT-LB model, while for the GPMFD scheme, we adopt some other numerical schemes to obtain the values required for the initialization. Furthermore, since the finite-difference schemes (22) and (37) have the same form (see Proposition 3), we need to consider only the latter in the following discussion, and this will simplify the analysis on truncation errors and MEs.

III. TRUNCATION ERRORS AND MES OF THE GPMRT-LB MODEL AND GPMFD SCHEME

In this section we will conduct some theoretical analysis on the truncation errors and MEs of the GPMRT-LB model (13) and GPMFD scheme (37). It is known that the GPMRT-LB model ($\mathbf{M}^c,\mathbf{S}^c$) can be equivalent to a GPMRT-LB model ($\mathbf{M}^c_N,\mathbf{S}^c_N$) [see Eq. (42)] where the transform matrix $\mathbf{M}^c_N$ is based on the natural moment and $\mathbf{S}^c_N$ is a block-lower-triangular relaxation matrix. Then from the two purple boxes shown in Fig. 1, one can find that the GPMFD schemes corresponding to the equivalent GPMRT-LB models ($\mathbf{M}^c,\mathbf{S}^c$)

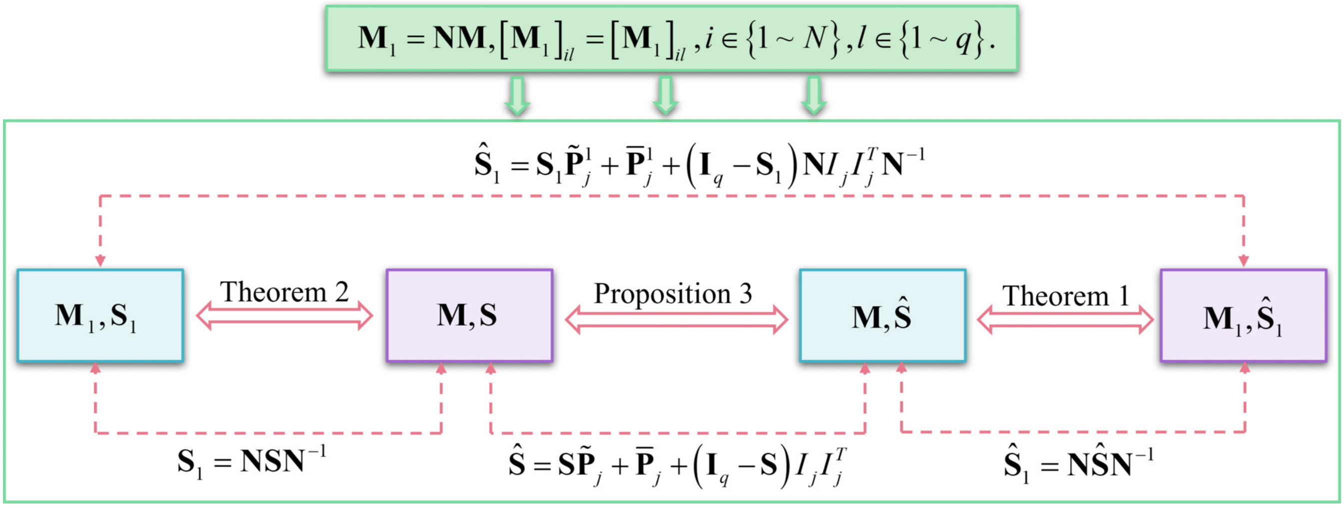

FIG. 1. Relations between the finite-difference schemes of the four equivalent GPMRT-LB models $(\mathbf{M}_1, \mathbf{S}_1)$, $(\mathbf{M}, \mathbf{S})$, $(\mathbf{M}, \hat{\mathbf{S}})$, and $(\mathbf{M}_1, \hat{\mathbf{S}}_1)$.

and $(\mathbf{M}_N^c, \hat{\mathbf{S}}_N^c)$ where the relaxation matrices $\mathbf{S}^c$ and $\hat{\mathbf{S}}_N^c$ [see Eq. (54)] are assumed to be independent on the parameter c, are identical. According to Remark 5 and for convenience of the analysis on the truncation errors and MEs, we need to consider only the GPMRT-LB model $(\mathbf{M}_N, \hat{\mathbf{S}}_N)$ satisfying the relations $\mathbf{M}_N = \mathbf{C}_d^{-1}\mathbf{M}_N^c$ and $\hat{\mathbf{S}}_N = \mathbf{C}_d^{-1}\hat{\mathbf{S}}_N^c\mathbf{C}_d$. Without loss of generality, here we consider that the degree of parameter c does not decrease with the number of rows (columns) in the diagonal matrix $\mathbf{C}_d$. Therefore, for a given GPMRT-LB model $(\mathbf{M}^c, \mathbf{S}^c)$, we focus on the GPMRT-LB model $(\mathbf{M}_N, \hat{\mathbf{S}}_N)$ with the transform matrix $\mathbf{M}_N$ that is independent on the parameter c and its GPMFD scheme (37). In addition, for the inverse of the collision matrix $\overline{\mathbf{\Lambda}} := \mathbf{M}_N\hat{\mathbf{S}}_N\mathbf{M}_N^{-1}$, the following requirements are needed:

$$\sum_{j=1}^{q} \mathbf{e}_{j\alpha}\overline{\mathbf{\Lambda}}_{jk} = S_{\alpha}^{10}\mathbf{i}_k + S_{\alpha\xi_1}^{1}\mathbf{e}_{k\xi_1}, \quad \sum_{j=1}^{q} \mathbf{e}_{j\alpha}\mathbf{e}_{j\beta}\overline{\mathbf{\Lambda}}_{jk} = S_{\alpha\beta}^{20}\mathbf{i}_k + S_{\alpha\beta\xi_1}^{21}\mathbf{e}_{k\xi_1} + S_{\alpha\beta\xi_1\xi_2}^{2}\mathbf{e}_{k\xi_1}\mathbf{e}_{k\xi_2},$$

$$\sum_{j=1}^{q} \mathbf{e}_{j\alpha}\mathbf{e}_{j\beta}\mathbf{e}_{j\gamma}\overline{\mathbf{\Lambda}}_{jk} = S_{\alpha\beta\gamma}^{30}\mathbf{i}_k + S_{\alpha\beta\gamma\xi_1}^{31}\mathbf{e}_{k\xi_1} + S_{\alpha\beta\gamma\xi_1\xi_2}^{32}\mathbf{e}_{k\xi_1}\mathbf{e}_{k\xi_2} + S_{\alpha\beta\gamma\xi_1\xi_2\xi_3}^{33}\mathbf{e}_{k\xi_1}\mathbf{e}_{k\xi_2}\mathbf{e}_{k\xi_3}, \tag{57}$$

$$\sum_{j=1}^{q} \mathbf{e}_{j\alpha}\mathbf{e}_{j\beta}\mathbf{e}_{j\gamma}\mathbf{e}_{j\eta}\overline{\mathbf{\Lambda}}_{jk} = S_{\alpha\beta\gamma\eta}^{40}\mathbf{i}_k + S_{\alpha\beta\gamma\eta\xi_1}^{41}\mathbf{e}_{k\xi_1} + S_{\alpha\beta\gamma\eta\xi_1\xi_2}^{42}\mathbf{e}_{k\xi_1}\mathbf{e}_{k\xi_2} + S_{\alpha\beta\gamma\eta\xi_1\xi_2\xi_3}^{43}\mathbf{e}_{k\xi_1}\mathbf{e}_{k\xi_2}\mathbf{e}_{k\xi_3} + S_{\alpha\beta\gamma\eta\xi_1\xi_2\xi_3\xi_4}^{44}\mathbf{e}_{k\xi_1}\mathbf{e}_{k\xi_2}\mathbf{e}_{k\xi_3}\mathbf{e}_{k\xi_4},$$

where $\mathbf{i}_k$ indicates the kth element of vector $\mathbf{i} = (1, 1, \ldots, 1) \in R^q$, $\mathbf{S}^{li}$ is a $d^l \times d^i$ matrix ($l \in \{1 \sim 2\}; i \in \{0 \sim (l-1)\}$ and $l \in \{3 \sim 4\}; i \in \{0 \sim l\}$), $\mathbf{S}^1$ is an invertible $d \times d$ relaxation matrix associated with the diffusion coefficient matrix of the NACDE, and $\mathbf{S}^2$ is a $d^2 \times d^2$ relaxation matrix associated with the viscosity coefficient of the NSEs. Specifically, we take $\mathbf{S}^{21} = \mathbf{0}$ for the NACDE and $\mathbf{S}^{32} = \mathbf{0}$ for the NSEs.

Due to the equivalence between the GPMRT-LB model (13) and GPMFD scheme (22) discussed in Sec. II and Remark 7, we will focus on the GPMFD scheme (37) and present details to derive its truncation error and ME. First, we decompose the matrix $\mathbf{W}$ in Eq. (15) as

$$\mathbf{W} = p_0\mathbf{I}_q + p_{-1}\mathbf{W}_{-1} + p_1\mathbf{W}_1, \tag{58}$$

where $\mathbf{W}_{-1} = \mathbf{M}_N\mathbf{T}_{-1}\mathbf{M}_N^{-1}$ and $\mathbf{W}_1 = \mathbf{M}_N\mathbf{T}_1\mathbf{M}_N^{-1}$. Due to the space shift operator with a series form (2a), the matrices $\mathbf{W}_{-1}$ and $\mathbf{W}_1$ can be rewritten as

$$\mathbf{W}_{-1} = \left(\sum_{k=0}^{+\infty} \frac{(-\Delta x)^k}{k!}\mathcal{W}_0^k\right) = \exp(-\Delta x\mathcal{W}_{\mathbf{0}}), \tag{59a}$$

$$\mathbf{W}_1 = \left(\sum_{k=0}^{+\infty} \frac{(\Delta x)^k}{k!}\mathcal{W}_0^k\right) = \exp(\Delta x\mathcal{W}_{\mathbf{0}}), \tag{59b}$$

where $\mathcal{W}_0 = \mathbf{M}_N[\mathbf{diag}(\mathbf{e}_1 \cdot \nabla, \mathbf{e}_2 \cdot \nabla, \dots, \mathbf{e}_q \cdot \nabla)]\mathbf{M}_N^{-1}$, and the inverse of matrix $\mathbf{W}$ can be further expressed as

$$\mathbf{W}^{-1} = \sum_{k=0}^{+\infty}[\mathbf{I}_q - p_0\mathbf{I}_q - p_{-1}\exp(-\Delta x\mathcal{W}_\mathbf{0}) - p_1\exp(\Delta x\mathcal{W}_\mathbf{0})]^k, \tag{60}$$

which will be used below.

Based on the Maxwell iteration method [43,44] and the relation between adjugate matrix and determinant,

$$\big(T_{\Delta t}^1\mathbf{I}_q - \mathbf{A}\big)\mathrm{adj}\big(T_{\Delta t}^1\mathbf{I}_q - \mathbf{A}\big) = \mathrm{adj}\big(T_{\Delta t}^1\mathbf{I}_q - \mathbf{A}\big)\big(T_{\Delta t}^1\mathbf{I}_q - \mathbf{A}\big) = \det\big(T_{\Delta t}^1\mathbf{I}_q - \mathbf{A}\big)\mathbf{I}_q, \tag{61}$$

we set $\boldsymbol{\varphi} = \boldsymbol{\varphi}^n$ with $\boldsymbol{\varphi}$ representing $\{\mathbf{m}, \mathbf{m}^{\mathrm{eq}}, \tilde{\mathbf{F}}\}$, and substitute the matrices $\mathbf{M}_N$ and $\hat{\mathbf{S}}_N$ into the GPMFD scheme (37), one can derive [66]

$$\begin{aligned}
\mathbf{0} &= \det\big(T_{\Delta t}^1\mathbf{I}_q - \mathbf{A}\big)\mathbf{m} - \big(\mathrm{adj}\big(T_{\Delta t}^1\mathbf{I}_q - \mathbf{A}\big)\big(\mathbf{B}\mathbf{m}^{eq} + \Delta t\mathbf{W}\tilde{\mathbf{F}}\big)\big)\\
&= \det\big(T_{\Delta t}^1\mathbf{I}_q - \mathbf{A}\big)\big[\mathbf{m} - \big(T_{\Delta t}^1\mathbf{I}_q - \mathbf{W}(\mathbf{I}_q - \hat{\mathbf{S}}_N)\big)^{-1}(\mathbf{W}\hat{\mathbf{S}}_N\mathbf{m}^{eq,n} + \mathbf{W}\Delta t\tilde{\mathbf{F}})\big]\\
&= \det\big(T_{\Delta t}^1\mathbf{I}_q - \mathbf{A}\big)\big[\mathbf{m} - \big(\hat{\mathbf{S}}_N^{-1}\mathbf{W}^{-1}\big(T_{\Delta t}^1\mathbf{I}_q - \mathbf{W}(\mathbf{I}_q - \hat{\mathbf{S}}_N)\big)\big)^{-1}\big(\mathbf{m}^{eq} + \hat{\mathbf{S}}_N^{-1}\Delta t\tilde{\mathbf{F}}\big)\big]\\
&= \det\big(T_{\Delta t}^1\mathbf{I}_q - \mathbf{A}\big)\big[\mathbf{m} - \big(\mathbf{I}_q + \hat{\mathbf{S}}_N^{-1}\big(T_{\Delta t}^1\mathbf{W}^{-1} - \mathbf{I}_q\big)\big)^{-1}\big(\mathbf{m}^{eq} + \hat{\mathbf{S}}_N^{-1}\Delta t\tilde{\mathbf{F}}\big)\big]\\
&:= \det\big(T_{\Delta t}^1\mathbf{I}_q - \mathbf{A}\big)\boldsymbol{\Xi}.
\end{aligned} \tag{62}$$

Here $\mathbf{m} = \mathbf{M}_N\mathbf{f}$, $\mathbf{m}^{\mathrm{eq}} = \mathbf{M}_N\mathbf{f}^{\mathrm{eq}}$, $\tilde{\mathbf{F}} = \mathbf{M}_N(\mathbf{F} + \mathbf{G} + \Delta t/2\overline{\mathbf{D}}\mathbf{F})$, and $\boldsymbol{\Xi}$ is defined as

$$\boldsymbol{\Xi} = \mathbf{m} - \left(\sum_{k=0}^{+\infty}\boldsymbol{\Gamma}^k\right)\big(\mathbf{m}^{eq} + \hat{\mathbf{S}}_N^{-1}\Delta t\tilde{\mathbf{F}}\big), \tag{63}$$

where $\boldsymbol{\Gamma} = -\hat{\mathbf{S}}_N^{-1}(T_{\Delta t}^1\mathbf{W}^{-1} - \mathbf{I}_q)$. With the help of $\boldsymbol{\Gamma}^0 = \mathbf{I}_q$, Eq. (60), and the series form of time operator $T_{\Delta t}^1$ in Eq. (2b), the expression of $\boldsymbol{\Xi}$ in Eq. (63) can be given as

$$\boldsymbol{\Xi} = \mathbf{m} - \big(\mathbf{m}^{\mathrm{eq}} + \Delta t\hat{\mathbf{S}}_N^{-1}\tilde{\mathbf{F}}\big) - \sum_{k=1}^{+\infty}\Delta x^k\boldsymbol{\Xi}^{(k)}, \tag{64}$$

where $\boldsymbol{\Xi}^{(k)} = \boldsymbol{\Gamma}^{(k)}(\mathbf{m}^{eq} + \hat{\mathbf{S}}_N^{-1}\Delta t\tilde{\mathbf{F}})$ ($k \geqslant 1$) is the coefficient before the kth-order term of the series expansion of $\boldsymbol{\Xi}$ with $\boldsymbol{\Gamma}^{(k)}$ denoting that of the series expansion of $\boldsymbol{\Gamma}$.

Based on the fact that $\det(T_{\Delta t}^1\mathbf{I}_q - \mathbf{A}) = \det(\hat{\mathbf{S}}_N) + O(\Delta x)$ [66], we will show that the analysis on the truncation error and ME of scheme (62) is equivalent to the discussion on $\boldsymbol{\Xi}$ (64). The reason is as follows.

According to Eq. (62), we have

$$\begin{aligned}
\mathbf{0} &= \det\big(T_{\Delta t}^1\mathbf{I}_q - \mathbf{A}\big)\boldsymbol{\Xi}\\
&= \det(\hat{\mathbf{S}}_N)\left[\mathbf{m} - \big(\mathbf{m}^{\mathrm{eq}} + \Delta t\hat{\mathbf{S}}_N^{-1}\tilde{\mathbf{F}}\big) - \sum_{k=1}^{+\infty}\Delta x^k\boldsymbol{\Xi}^{(k)}\right] + \left[\big(\mathbf{m} - \big(\mathbf{m}^{\mathrm{eq}} + \Delta t\hat{\mathbf{S}}_N^{-1}\tilde{\mathbf{F}}\big)\big)\times O(\Delta x) - \sum_{k=1}^{+\infty}\Delta x^{k+1}\boldsymbol{\Xi}^{(k)}\right].
\end{aligned} \tag{65}$$

As mentioned previously, to perform the analysis on the truncation errors and the MEs, it is necessary to specify the scaling relationship between Δt and Δx. In particular, it should be noted that at the diffusive scaling ($\Delta t \sim \Delta x^2$), $\boldsymbol{\Xi}^{(k)}$ is at least of order $O(1)$, which arises from the fact that $\mathbf{m}^{\mathrm{eq}}$, $\tilde{\mathbf{F}}$ and $\hat{\mathbf{S}}_N^{-1}$ (54) are at least of order $O(1)$ [the main diagonal element of the block-lower-triangular matrix $\hat{\mathbf{S}}_N^{-1}$ is of order $O(1)$, while the lower triangular element of $\hat{\mathbf{S}}_N^{-1}$ is at least of order $O(1)$]. Due to the conservation law, $m_{i\in\{1\sim N\}} = m_{i\in\{1\sim N\}}^{\mathrm{eq}}$, for any ith ($i \in \{1 \sim N\}$) row of Eq. (65), we have

$$\text{diffusive scaling } (\Delta t \sim \Delta x^2):\ \det(\hat{\mathbf{S}}_N)[\Delta x\boldsymbol{\Xi}^{(1)}]_i = O(\Delta x^2), \tag{66a}$$

$$\text{acoustic scaling } (\Delta t \sim \Delta x):\ \det(\hat{\mathbf{S}}_N)\big[\Delta t\hat{\mathbf{S}}_N^{-1}\tilde{\mathbf{F}} + \Delta x\boldsymbol{\Xi}^{(1)}\big]_i = O(\Delta x^2), \tag{66b}$$

where $\det(\hat{\mathbf{S}}_N)$ is of order $O(1)$; this is because the main diagonal element of the block-lower-triangular matrix $\hat{\mathbf{S}}_N$ is of order $O(1)$. Based on Eq. (66b), one can obtain the third-order ME at the acoustic scaling,

$$\big[\Delta t\hat{\mathbf{S}}_N^{-1}\tilde{\mathbf{F}} + \Delta x\boldsymbol{\Xi}^{(1)} + \Delta x^2\boldsymbol{\Xi}^{(2)}\big]_i = O(\Delta x^3). \tag{67}$$

Then according to Eq. (67), one can further derive fourth- (and higher-) order truncation errors and MEs of the GPMRT-LB model (13) and GPMFD scheme (37) at the acoustic scaling. In addition, it can also be observed that the analysis on the truncation errors and MEs of GPMFD scheme (62) are actually equivalent to the discussion on $\boldsymbol{\Xi}$ (64), which is also true at the diffusive scaling.

A. MEs at the diffusive scaling

Let us begin the analysis at the diffusive scaling, i.e., $\Delta t = \eta\Delta x^2$, $\eta \in \mathcal{R}$. It should be noted that the lattice velocity λ is of order $O(1/\Delta x)$, $\mathbf{m}^{\mathrm{eq}}$, $\tilde{\mathbf{F}}$, and $\hat{\mathbf{S}}_N^{-1}$ are at least of order $O(1)$. We would also like to point out that $\Delta x^{-k_i} m_i$ ($i \in \{1 \sim N\}$) is of order $O(1)$ owing to the fact that the transform matrix $\mathbf{M}$ is of order $O(1)$ and the form of the equilibrium distribution function for some specific problems [see Eq. (91) for NACDE and Eq. (98) for NSEs; $k_i \geqslant 0$ denotes the order of the ith conservative moment m_i in space]. Thus, a higher-order expansion of $\boldsymbol{\Xi}$ (64) beyond $\boldsymbol{\Xi}^{(k)}$ is necessary for the kth-order ME of the GPMRT-LB model (13) and GPMFD scheme (37). Here we expand $\boldsymbol{\Xi}$ (64) up to $\boldsymbol{\Xi}^{(4)}$ and consider the first- to third-order MEs on conservative moment m_i ($i \in \{1 \sim N\}$),

$$[\Delta t\tilde{\mathbf{F}} + \Delta x\hat{\mathbf{S}}_N\boldsymbol{\Xi}^{(1)} + \Delta x^2\hat{\mathbf{S}}_N\boldsymbol{\Xi}^{(2)} + \Delta x^3\hat{\mathbf{S}}_N\boldsymbol{\Xi}^{(3)} + \Delta x^4\hat{\mathbf{S}}_N\boldsymbol{\Xi}^{(4)}]_i = O(\Delta x^5), \tag{68}$$

where the ith row of matrix $\hat{\mathbf{S}}_N$ (54) is identical to I_i^T, which is of order $O(1)$ and has been used.

Moreover, it is also necessary to expand the inverse of matrix $\mathbf{W}$ (60) as

$$\mathbf{W}^{-1} = \mathbf{I}_q + \Delta x a\mathcal{W}_0 + \Delta x^2 a^2\left(1 - \frac{b}{2a^2}\right)\mathcal{W}_0^2 + \Delta x^3 a^3\left(\frac{1}{6a^3} - \frac{b}{a^2} + 1\right)\mathcal{W}_0^3$$
$$+ \Delta x^4 a^4\left(\frac{b^2}{4a^4} + \frac{1}{3a^3} - \frac{3b}{2a^2} + 1 - \frac{b}{24a^4}\right)\mathcal{W}_0 + O(\Delta x^5). \tag{69}$$

By adopting the series form (2b) of the time shift operator $T_{\Delta t}^1$ and Eq. (59), one can obtain

$$-\hat{\mathbf{S}}_N\boldsymbol{\Gamma} = \Delta x A_1 + \Delta x^2(A_{12} + A_2) + \Delta x^3(A_{31} + A_3) + \Delta x^4(A_{41} + A_{42} + A_4) + O(\Delta x^5), \tag{70}$$

where

$$A_1 = a\mathcal{W}_0, \tag{71a}$$

$$A_{21} = \eta\partial_t\mathbf{I}_q, \quad A_2 = \mathcal{W}_0^2\left(a^2 - \frac{b}{2}\right), \tag{71b}$$

$$A_{31} = a\eta\mathcal{W}_0\partial_t, \quad A_3 = \left(\frac{1}{6} - ab + a^3\right)\mathcal{W}_0^3, \tag{71c}$$

$$A_{41} = \eta\left(a^2 - \frac{b}{2}\right)\mathcal{W}_0^2\partial_t, \quad A_{42} = \eta^2\frac{\partial_{tt}\mathbf{I}_q}{2}, \quad A_4 = \left(\frac{b^2}{4} + \frac{a}{3} - \frac{3a^2b}{2} + a^4 - \frac{b}{24}\right)\mathcal{W}_0^4. \tag{71d}$$

In the following, the analysis are based on Eqs. (68) and (71).

1. *First-order ME of the GPMRT-LB model and GPMFD scheme on conservative moment* m_i

For any $i \in \{1 \sim N\}$, to obtain the first-order ME of the GPMRT-LB model and GPMFD scheme on the moment m_i at the diffusive scaling, one needs to consider the following truncation equation of Eq. (68):

$$[\Delta t\tilde{\mathbf{F}} + \Delta x\hat{\mathbf{S}}_N\boldsymbol{\Xi}^{(1)} + \Delta x^2\hat{\mathbf{S}}_N\boldsymbol{\Xi}^{(2)}]_i = O(\Delta x^3). \tag{72}$$

With the aid of Eqs. (64) and (70), Eq. (72) can be written as

$$\left[\Delta t\tilde{\mathbf{F}} - \left[\Delta x A_1 + \Delta x^2 A_{12} + \Delta x^2\left(A_2 - A_1\hat{\mathbf{S}}_N^{-1}A_1\right)\right]\mathbf{m}^{\mathrm{eq}}\right]_i + O(\Delta x^3), \tag{73}$$

and multiplying this equation by $1/\Delta t$ yields

$$\left[\tilde{\mathbf{F}} - \left[\lambda A_1 + \frac{A_{12}}{\eta} + \lambda^2\Delta t\left(A_2 - A_1\hat{\mathbf{S}}_N^{-1}A_1\right)\right]\mathbf{m}^{\mathrm{eq}}\right]_i = O(\Delta x); \tag{74}$$

then substituting Eqs. (71a) and (71b) into Eq. (73), we have

$$(\partial_t\mathbf{m}^{\mathrm{eq}} + c\mathcal{W}_0\mathbf{m}^{\mathrm{eq}})_i = \left[\tilde{\mathbf{F}} + \frac{a^2}{\eta}\mathcal{W}_0\left[\hat{\mathbf{S}}_N^{-1} + \left(\frac{b}{2a^2} - 1\right)\mathbf{I}_q\right]\mathcal{W}_0\mathbf{m}^{\mathrm{eq}}\right]_i + O(\Delta x). \tag{75}$$

2. *Second-order ME of the GPMRT-LB model and GPMFD scheme on conservative moment* m_i

Similar to the discussion in the previous part, we consider the following truncation equation of Eq. (68):

$$[\Delta t\tilde{\mathbf{F}} + \Delta x\hat{\mathbf{S}}_N\boldsymbol{\Xi}^{(1)} + \Delta x^2\hat{\mathbf{S}}_N\boldsymbol{\Xi}^{(2)} + \Delta x^3\hat{\mathbf{S}}_N\boldsymbol{\Xi}^{(3)}]_i = O(\Delta x^4); \tag{76}$$

for any $i \in \{1 \sim N\}$, one can obtain

$$\left[\Delta t\tilde{\mathbf{F}} - \Delta x\Delta t A_1\hat{\mathbf{S}}_N^{-1}\tilde{\mathbf{F}} - \Delta x[A_1 + \Delta x(A_{12} + A_2) + \Delta x^2(A_{31} + A_3)]\mathbf{m}^{\mathrm{eq}}\right.$$
$$\left. + \Delta x^3\left[A_1\hat{\mathbf{S}}_N^{-1}(A_{21} + A_2) + (A_{21} + A_2)\hat{\mathbf{S}}_N^{-1}A_1 - A_1\hat{\mathbf{S}}_N^{-1}A_1\hat{\mathbf{S}}_N^{-1}A_1\right]\mathbf{m}^{\mathrm{eq}}\right]_i = O(\Delta x^4). \tag{77}$$

Substituting Eqs. (71a)–(71c) into Eq. (77) yields

$$[(\partial_t \mathbf{I} + c\mathcal{W}_0)\mathbf{m}^{eq}]_i = \Big[\tilde{\mathbf{F}} + \frac{a^2}{\eta}\mathcal{W}_0\Big(\hat{\mathbf{S}}_N^{-1} + \Big(\frac{b}{2a^2} - 1\Big)\mathbf{I}_q\Big)\mathcal{W}_0\mathbf{m}^{eq} - a\Delta x \partial_t\big(\mathbf{I}_q - \hat{\mathbf{S}}_N^{-1}\big)\mathcal{W}_0\mathbf{m}^{eq}$$
$$+ a\Delta x \mathcal{W}_0\hat{\mathbf{S}}_N^{-1}\Big[\partial_t\mathbf{m}^{eq} - \tilde{\mathbf{F}} - \frac{a^2}{\eta}\mathcal{W}_0\Big(\hat{\mathbf{S}}_N^{-1} + \Big(\frac{b}{2a^2} - 1\Big)\mathbf{I}_q\Big)\mathcal{W}_0\mathbf{m}^{eq}\Big]$$
$$- \frac{a^3\Delta x}{\eta}\Big(\frac{b}{2a^2} - 1\Big)\mathcal{W}_0^2\hat{\mathbf{S}}_N^{-1}\mathbf{W}_0\mathbf{m}^{eq} - \frac{a^3\Delta x}{\eta}\Big(\frac{1}{6a^3} - \frac{b}{a^2} + 1\Big)\mathcal{W}_0^3\mathbf{m}^{eq}\Big]_i + O(\Delta x^2). \tag{78}$$

3. Third-order ME of the GPMRT-LB model and GPMFD scheme on conservative moment m_i

For any $i \in \{1 \sim N\}$, after some manipulations by using Eqs. (64) and (70), we can derive the third-order ME,

$$\big[\Delta t\tilde{\mathbf{F}} - [\Delta x A_1 + \Delta x^2(A_{12} + A_2) + \Delta x^2(A_{31} + A_3) + \Delta x^4(A_{41} + A_{42} + A_4)]\mathbf{m}^{eq}$$
$$- [\Delta x A_1 + \Delta x^2(A_{21} + A_2)]\Delta t\hat{\mathbf{S}}_N^{-1}\tilde{\mathbf{F}} + \Delta x^2\big(\hat{\mathbf{S}}_N^{-1}A_1\big)^2\Delta t\hat{\mathbf{S}}_N^{-1}\tilde{\mathbf{F}}$$
$$+ \hat{\mathbf{S}}_N\big(\hat{\mathbf{S}}_N^{-1}[\Delta x A_1 + \Delta x^2(A_{21} + A_2) + \Delta x^3(A_{31} + A_3)]\big)^2\mathbf{m}^{eq}$$
$$- \hat{\mathbf{S}}_N\big(\hat{\mathbf{S}}_N^{-1}[\Delta x A_1 + \Delta x^2(A_{21} + A_2)]\big)^3\mathbf{m}^{eq} + \Delta x^4\big(\hat{\mathbf{S}}_N^{-1}A_1\big)^4\mathbf{m}^{eq}\big]_i = O(\Delta x^5). \tag{79}$$

Substituting Eqs. (71a)–(71d) into Eq. (79) gives

$$\Big[\tilde{\mathbf{F}} - \Big[c\mathcal{W}_0 + \partial_t\mathbf{I}_q + \frac{a^2}{\eta}\Big(1 - \frac{b}{2a^2}\Big)\mathcal{W}_0^2 + a\Delta x\mathcal{W}_0\partial_t + \frac{a^3\Delta x}{\eta}\Big(1 - \frac{b}{2a^2}\Big)\mathcal{W}_0^2\partial_t\mathbf{I}_q$$
$$+ \frac{\Delta x}{2\lambda}\partial_{tt}\mathbf{I}_q + \frac{a^3\Delta x}{\eta}\Big(\frac{1}{6a^2} - \frac{b}{a^2} - 1\Big)\mathcal{W}_0^3 + \frac{a^4\Delta x^2}{\eta}\Big(\frac{b^2}{4a^2} + \frac{1}{3a^3} - \frac{3b}{2a^2} + 1 - \frac{b}{24a^4}\Big)\mathcal{W}_0^4\Big]\mathbf{m}^{eq}$$
$$- \Big(c\mathcal{W}_0 + \partial_t\mathbf{I}_q + \frac{a^2}{\eta}\Big(1 - \frac{b}{2a^2}\Big)\mathcal{W}_0^2\Big)\frac{\Delta x}{\lambda}\hat{\mathbf{S}}_N^{-1}\tilde{\mathbf{F}} + \Big(\frac{a^2}{\eta}\mathcal{W}_0\hat{\mathbf{S}}_N^{-1}\mathcal{W}_0\Big)\frac{\Delta x}{\lambda}\hat{\mathbf{S}}_N^{-1}\tilde{\mathbf{F}}$$
$$+ \Big[\frac{a^2}{\eta}\mathcal{W}_0\hat{\mathbf{S}}_N^{-1}\mathcal{W}_0 + a\Delta x\mathcal{W}_0\hat{\mathbf{S}}_N^{-1}\partial_t\mathbf{I}_q + \frac{a^3\Delta x}{\eta}\Big(1 - \frac{b}{2a^2}\Big)\mathcal{W}_0\hat{\mathbf{S}}_N^{-1}\mathcal{W}_0^2$$
$$+ a^2\Delta x^2\mathcal{W}_0\hat{\mathbf{S}}_N^{-1}\mathcal{W}_0\partial_t + a\partial_t\mathbf{I}_q\hat{\mathbf{S}}_N^{-1}\mathcal{W}_0 + \frac{\Delta x}{\lambda}\partial_t\hat{\mathbf{S}}_N^{-1}\mathbf{I}_q\partial_t\mathbf{I}_q + a^2\Delta x^2\Big(1 - \frac{b}{2a^2}\Big)\partial_t\mathbf{I}_q\hat{\mathbf{S}}_N^{-1}\mathcal{W}_0^2 + \frac{a^3\Delta x}{\eta}\Big(1 - \frac{b}{2a^2}\Big)\mathcal{W}_0^2\hat{\mathbf{S}}_N^{-1}\mathcal{W}_0$$
$$+ a^2\Delta x^2\Big(1 - \frac{b}{2a^2}\Big)\mathcal{W}_0^2\hat{\mathbf{S}}_N^{-1}\partial_t\mathbf{I}_q + \frac{a^4\Delta x^2}{\eta}\Big(1 - \frac{b}{2a^2}\Big)^2\mathcal{W}_0^2\hat{\mathbf{S}}_N^{-1}\mathcal{W}_0^2 + a^2\Delta x^2\mathcal{W}_0\partial_t\mathbf{I}_q\hat{\mathbf{S}}_N^{-1}\mathcal{W}_0\Big]\mathbf{m}^{eq}$$
$$- \Big[\frac{a^3\Delta x}{\eta}\mathcal{W}_0\hat{\mathbf{S}}_N^{-1}\mathcal{W}_0\hat{\mathbf{S}}_N^{-1}\mathcal{W}_0 + a^2\Delta x^2\mathcal{W}_0\hat{\mathbf{S}}_N^{-1}\mathcal{W}_0\hat{\mathbf{S}}_N^{-1}\partial_t\mathbf{I}_q + a^2\Delta x^2\mathcal{W}_0\partial_t\mathbf{I}_q\hat{\mathbf{S}}_N^{-1}\hat{\mathbf{S}}_N^{-1}\mathcal{W}_0$$
$$+ \frac{a^4\Delta x^2}{\eta}\Big(1 - \frac{b}{2a^2}\Big)\mathcal{W}_0\hat{\mathbf{S}}_N^{-1}\mathcal{W}_0\hat{\mathbf{S}}_N^{-1}\mathcal{W}_0^2 + \frac{a^4\Delta x^2}{\eta}\Big(1 - \frac{b}{2a^2}\Big)\mathcal{W}_0\hat{\mathbf{S}}_N^{-1}\mathcal{W}_0^2\hat{\mathbf{S}}_N^{-1}\mathcal{W}_0$$
$$+ a^2\Delta x^2\partial_t\mathbf{I}_q\hat{\mathbf{S}}_N^{-1}\mathcal{W}_0\hat{\mathbf{S}}_N^{-1}\mathcal{W}_0 + \frac{a^4\Delta x^2}{\eta}\Big(1 - \frac{b}{2a^2}\Big)\mathcal{W}_0^2\hat{\mathbf{S}}_N^{-1}\mathcal{W}_0\hat{\mathbf{S}}_N^{-1}\mathcal{W}_0\Big]\mathbf{m}^{eq}$$
$$+ \frac{a^4\Delta x^2}{\eta}\mathcal{W}_0\hat{\mathbf{S}}_N^{-1}\mathcal{W}_0\hat{\mathbf{S}}_N^{-1}\mathcal{W}_0\hat{\mathbf{S}}_N^{-1}\mathcal{W}_0\mathbf{m}^{eq}\Big]_i = O(\Delta x^3). \tag{80}$$

Now we give a remark on these MEs at the diffusive scaling.

Remark 8. According to the results shown in Secs. III A 1–III A 3, we now consider a specific problem, the d-dimensional NSEs with $d + 1$ conservative moments. In this case the moment m_1 is $O(1)$, while m_i ($i \in \{2 \sim (d + 1)\}$) is of order $O(\Delta x^1)$. Thus, Eqs. (75), (78), and (80) correspond to the first- to third-order MEs of the GPMRT-LB model and GPMFD scheme on conservative moment ρ, but the zeroth- to second-order MEs of the GPMRT-LB model and GPMFD scheme on conservative moment $\rho\mathbf{u}$.

B. MEs at the acoustic scaling

At the acoustic scaling, all the λ, $\mathbf{m}^{\rm eq}$, $\tilde{\mathbf{F}}$, and $\hat{\mathbf{S}}_N^{-1}$ are of order $O(1)$, and only a kth-order expansion of $\boldsymbol{\Xi}$ is needed for the derivation of the kth-order MEs of the GPMRT-LB model (13) and GPMFD scheme (37). We now expand $\boldsymbol{\Xi}$ (64) up to $\boldsymbol{\Xi}^{(2)}$ and consider the first- and second-order MEs on conservative moments m_i for any $i \in \{1 \sim N\}$,

$$[\Delta t\tilde{\mathbf{F}} + \Delta x\hat{\mathbf{S}}_N\boldsymbol{\Xi}^{(1)} + \Delta x^2\hat{\mathbf{S}}_N\boldsymbol{\Xi}^{(2)}]_i = O(\Delta x^3), \tag{81}$$

subsequently, the second-order expansion of $\mathbf{W}^{-1}$ (60) can be given by

$$\mathbf{W}^{-1} = \mathbf{I}_q + \Delta x a\mathcal{W}_0 + \Delta x^2\left(1 - \frac{b}{2a^2}\right)a^2\mathcal{W}_0^2 + O(\Delta x^3). \tag{82}$$

Similar to the analysis at diffusive scaling, one can obtain the expansion of $-\hat{\mathbf{S}}_N\boldsymbol{\Gamma}$ based on Eq. (82),

$$-\hat{\mathbf{S}}_N\boldsymbol{\Gamma} = \Delta x(B_{11} + B_1) + \Delta x^2(B_{21} + B_{22} + B_2) + O(\Delta x^3), \tag{83}$$

where

$$B_{11} = \frac{a\partial_t}{c}\mathbf{I}_q, \quad B_1 = a\mathcal{W}_0, \tag{84a}$$

$$B_{21} = \frac{a^2\partial_{tt}}{2c^2}\mathbf{I}_q, \quad B_{22} = \frac{a^2\mathcal{W}_0\partial_t}{c}, \quad B_2 = a^2\mathcal{W}_0^2\left(1 - \frac{b}{2a^2}\right). \tag{84b}$$

1. First-order ME of the GPMRT-LB model and GPMFD scheme on conservative moment m_i

Considering the first-order truncation equation of Eq. (81), for any $i \in \{1 \sim N\}$, one can obtain

$$[\Delta t\tilde{\mathbf{F}} - \Delta x(B_{11} + B_1)\mathbf{m}^{\rm eq}]_i = O(\Delta x^2), \tag{85}$$

and substituting Eq. (84a) into Eq. (85) gives rise to the first-order ME:

$$[\partial_t\mathbf{m}^{\rm eq} + c\mathcal{W}_0\mathbf{m}^{\rm eq}]_i = [\tilde{\mathbf{F}}]_i + O(\Delta x). \tag{86}$$

2. Second-order ME of the GPMRT-LB model and GPMFD scheme on conservative moment m_i

Considering the second-order equation of Eq. (81), for any $i \in \{1 \sim N\}$, we have

$$\begin{aligned}&\big[\Delta t\tilde{\mathbf{F}} - \Delta x(B_{11} + B_1)\big(\mathbf{m}^{\rm eq} + \hat{\mathbf{S}}_N^{-1}\Delta t\tilde{\mathbf{F}}\big)\\ &\quad + \Delta x^2\big(\hat{\mathbf{S}}_N\big[\hat{\mathbf{S}}_N^{-1}(B_{11} + B_1)\big]^2 - (B_{21} + B_{22} + B_2)\big)\mathbf{m}^{\rm eq}\big]_i\\ &= O(\Delta x^3),\end{aligned} \tag{87}$$

and substituting Eqs. (84a) and (84b) into Eq. (87) yields the following second-order ME:

$$\begin{aligned}&\Bigg[\Delta t\tilde{\mathbf{F}} - \Delta t\big(\partial_t\mathbf{I}_q + c\mathcal{W}_0\big)\mathbf{m}^{\rm eq} - \frac{\Delta t^2}{2}\left(1 - \frac{b}{a^2}\right)c^2\mathcal{W}_0^2\mathbf{m}^{\rm eq}\\ &\quad + \frac{\Delta t^2}{2}\big[\big(2\hat{\mathbf{S}}_N^{-1} - \mathbf{I}_q\big)\partial_{tt} + 2c\big(\hat{\mathbf{S}}_N^{-1}\mathcal{W}_0 + \mathcal{W}_0\hat{\mathbf{S}}_N^{-1} - \mathcal{W}_0\big)\partial_t\\ &\quad + c^2\mathcal{W}_0\big(2\hat{\mathbf{S}}_N^{-1} - \mathbf{I}_q\big)\mathcal{W}_0\big]\mathbf{m}^{\rm eq}\Bigg]_i = O(\Delta x^3).\end{aligned} \tag{88}$$

From above results, one can see that Eqs. (86) and (88) would reduce to the results presented in Ref. [66] when $a = b = 1$ and the term $\tilde{\mathbf{F}}$ is neglected.

C. NACDE: MEs of the GPMRT-LB model and GPMFD scheme

The d-dimensional NACDE with a source term can be expressed as

$$\partial_t\phi + \nabla\cdot\mathbf{B} = \nabla\cdot[\kappa\cdot(\nabla\cdot\mathbf{D})] + R, \tag{89}$$

where ϕ is a scalar variable related to both time t and space $\mathbf{x}$, R denotes the source term. $\mathbf{B} = (B_\alpha)$ is a vector function, $\kappa = (\kappa_{\alpha\beta})$ and $\mathbf{D} = (D_{\alpha\beta})$ are symmetric tensors (matrices), and they can be functions of ϕ, $\mathbf{x}$, and t.

In order to recover the NACDE (89) correctly, some requirements or moment conditions on the equilibrium, auxiliary, and source distribution functions, denoted by $f_i^{\rm eq}$, G_i, and F_i, should be satisfied. For a general $DdQq$ lattice structure, the moment conditions are given by

$$\sum_{i=1}^q f_i = \sum_{i=1}^q f_i^{\rm eq} = \phi, \quad \sum_{i=1}^q F_i = R, \quad \sum_{i=1}^q G_i = 0, \tag{90a}$$

$$\sum_{i=1}^q \mathbf{c}_i f_i^{\rm eq} = \mathbf{B}, \quad \sum_{i=1}^q \mathbf{c}_i F_i = \mathbf{0}, \quad \sum_{i=1}^q \mathbf{c}_i G_i = \mathbf{M}_{1G}, \tag{90b}$$

$$\sum_{i=1}^q \mathbf{c}_i\mathbf{c}_i f_i^{\rm eq} = \chi c_s^2\mathbf{D} + \mathbf{C}, \tag{90c}$$

where c_s is a model parameter related to the lattice velocity λ. The parameter χ is used to adjust the relaxation matrix [see Eq. (94)], $\mathbf{C}$ is an auxiliary moment [30], and $\mathbf{M}_{1G} = (\mathbf{I}_d - (\mathbf{S}^1)^{-1}/2)\partial_t\mathbf{B} + (\mathbf{I}_d - b(\mathbf{S}^1)^{-1}/(2a^2))\nabla\cdot\mathbf{C}$ is the first-order moment of G_i. From Eq. (90) one can determine the expressions of $f_i^{\rm eq}$, G_i, and F_i, while for simplicity, we consider only the following commonly used forms [30]:

$$f_i^{\rm eq} = w_i\left[\phi + \frac{\mathbf{c}_i\cdot\mathbf{B}}{c_s^2} + \frac{\big(\chi c_s^2\mathbf{D} + \mathbf{C} - c_s^2\phi\mathbf{I}_d\big):\big(\mathbf{c}_i\mathbf{c}_i - c_s^2\mathbf{I}_d\big)}{2c_s^4}\right], \tag{91a}$$

$$G_i = w_i\left[\frac{\mathbf{c}_i\cdot\mathbf{M}_{1G}}{c_s^2}\right], F_i = w_iR. \tag{91b}$$

According to the results shown in the Secs. III A 1 and III A 2, it is easy to obtain the first- and second-order MEs of the GPMRT-LB model (13) and GPMFD scheme (37) on the conservative moment $\phi = O(1)$ (see the Appendix A 1 for details) at the diffusive scaling,

$$\partial_t\phi + \partial_\alpha B_\alpha - \Delta t\frac{\partial}{\partial x_\beta}\left[\chi\left(S^1_{\beta\gamma} + \left(\frac{b}{2a^2} - 1\right)\delta_{\beta\gamma}\right)c_s^2\frac{\partial D_{\gamma\theta}}{\partial x_\theta}\right] - R = O(\Delta x), \tag{92a}$$

$$\partial_t\phi + \partial_\alpha B_\alpha - \Delta t\frac{\partial}{\partial x_\beta}\left[\chi\left(S^1_{\beta\gamma} + \left(\frac{b}{2a^2} - 1\right)\delta_{\beta\gamma}\right)c_s^2\frac{\partial D_{\gamma\theta}}{\partial x_\theta}\right] - R = O(\Delta x^2), \tag{92b}$$

and additionally, from Secs. III B 1 and III B 2, one can derive the first- and second-order MEs at the acoustic scaling (see Appendix B 1 for details),

$$\partial_t\phi + \partial_\alpha B_\alpha - R = O(\Delta x), \tag{93a}$$

$$\partial_t\phi + \partial_\alpha B_\alpha - \Delta t\frac{\partial}{\partial x_\beta}\left[\chi\left(S^1_{\beta\gamma} + \left(\frac{b}{2a^2} - 1\right)\delta_{\beta\gamma}\right)c_s^2\frac{\partial D_{\gamma\theta}}{\partial x_\theta}\right] - R = O(\Delta x^2). \tag{93b}$$

It is clear that Eqs. (92b) and (93b) are consistent with the NACDE (89) with

$$\kappa = \chi c_s^2\Delta t\left[\mathbf{S}^1 + \left(\frac{b}{2a^2} - 1\right)\mathbf{I}_d\right]. \tag{94}$$

When the MRT-LB model with the orthogonal moments and D2Q9 lattice structure is considered, it is easy to show that the second-order modified equation (92b) is consistent with the result in Ref. [69].

D. NSEs: MEs of the GPMRT-LB model and GPMFD scheme

We now consider the following d-dimensional NSEs with a force term:

$$\partial_t\rho + \nabla\cdot(\rho\mathbf{u}) = 0, \tag{95a}$$

$$\partial_t(\rho\mathbf{u}) + \nabla\cdot(\rho\mathbf{u}\mathbf{u}) = -\nabla p + \nabla\cdot\sigma + \hat{\mathbf{F}}, \tag{95b}$$

where $p = c_s^2\rho$ is the pressure, $\hat{\mathbf{F}} = (\hat{F}_{x_{\alpha-1}})_{\alpha=2}^{d+1}$ is the force term, and σ is the shear stress defined by

$$\sigma = \mu[\nabla\mathbf{u} + (\nabla\mathbf{u})^T] + \left(\mu_b - \frac{2\mu}{d}\right)(\nabla\cdot\mathbf{u})\mathbf{I}_d = \mu\left[\nabla\mathbf{u} + (\nabla\mathbf{u})^T - \frac{2}{d}(\nabla\cdot\mathbf{u})\mathbf{I}_d\right] + \mu_b(\nabla\cdot\mathbf{u})\mathbf{I}_d, \tag{96}$$

here μ and μ_b are the dynamic and bulk viscosity, respectively.

To recover the macroscopic NSEs (95) from the GPMRT-LB model (13), the equilibrium, auxiliary, and source distribution functions, i.e., f_i^{eq}, F_i, and G_i, should satisfy the following moment conditions:

$$\sum_{i=1}^q f_i = \sum_{i=1}^q f_i^{\text{eq}} = \rho, \quad \sum_{i=1}^q \mathbf{c}_i f_i = \sum_{i=1}^q \mathbf{c}_i f_i^{\text{eq}} = \rho\mathbf{u}, \tag{97a}$$

$$\sum_{i=1}^q \mathbf{c}_i\mathbf{c}_i f_i^{\text{eq}} = \rho\mathbf{u}\mathbf{u} + c_s^2\rho\mathbf{I}, \quad \sum_{i=1}^q \mathbf{c}_i\mathbf{c}_i\mathbf{c}_i f_i^{\text{eq}} = c_s^2\rho\Delta\cdot\mathbf{u}, \tag{97b}$$

$$\sum_{i=1}^q G_i = 0, \quad \sum_{i=1}^q \mathbf{c}_i G_i = \mathbf{0}, \quad \sum_{i=1}^q \mathbf{c}_i\mathbf{c}_i G_i = \mathbf{0}, \tag{97c}$$

$$\sum_{i=1}^q F_i = 0, \quad \sum_{i=1}^q \mathbf{c}_i F_i = \rho\hat{\mathbf{F}}, \quad \sum_{i=1}^q \mathbf{c}_i\mathbf{c}_i F_i = \rho(\hat{\mathbf{F}}\mathbf{u} + (\hat{\mathbf{F}}\mathbf{u})^T); \tag{97d}$$

for the $DdQq$ lattice structure, the explicit expressions of f_i^{eq}, G_i, and F_i can be given by [30]

$$f_i^{\text{eq}} = w_i\rho\left[1 + \frac{\mathbf{c}_i\cdot\mathbf{u}}{c_s^2} + \frac{\mathbf{u}\mathbf{u} : \left(\mathbf{c}_i\mathbf{c}_i - c_s^2\mathbf{I}_d\right)}{2c_s^4}\right], \tag{98a}$$

$$G_i = 0, \quad F_i = w_i\rho\left[\frac{\mathbf{c}_i\cdot\hat{\mathbf{F}}}{c_s^2} + \frac{(\hat{\mathbf{F}}\mathbf{u} + (\hat{\mathbf{F}}\mathbf{u})^T) : \left(\mathbf{c}_i\mathbf{c}_i - c_s^2\mathbf{I}_d\right)}{2c_s^4}\right]. \tag{98b}$$

It should be noted that at the diffusive scaling, the terms $w_i\rho\mathbf{c}_i\cdot\mathbf{u}/c_s^2$ and $w_i\rho[\mathbf{u}\mathbf{u} : (\mathbf{c}_i\mathbf{c}_i - c_s^2\mathbf{I}_d)]/(2c_s^4)$ in the expression of f_i^{eq} are of order $O(\Delta x)$ and $O(\Delta x^2)$, respectively.

For the continuity equation (95a), the derivation process is similar to that of NACDE (see Appendixes A 2 and B 2 for details), and the first- to second-order MEs at the diffusive and acoustic scalings are given by

$$\Delta t \sim \Delta x^2 : \begin{cases} \partial_\alpha(\rho u_\alpha) = O(\Delta x), \\ \partial_t \rho + \partial_\alpha(\rho u_\alpha) = O(\Delta x^2), \end{cases} \tag{99}$$

$$\Delta t \sim \Delta x : \begin{cases} \partial_t \rho + \partial_\alpha(\rho u_\alpha) = O(\Delta x), \\ \partial_t \rho + \partial_\alpha(\rho u_\alpha) + \Delta t\big(\frac{1}{2} - \frac{b}{2a^2}\big)\partial_\beta\partial_\theta\big(\rho u_\beta u_\theta + \rho c_s^2\delta_{\beta\theta}\big) = O(\Delta x^2). \end{cases} \tag{100}$$

We would like to point out that the term $\Delta t[1/2 - b/(2a^2)]\partial_\beta\partial_\theta(\rho u_\beta u_\theta + \rho c_s^2\delta_{\beta\theta})$ in Eq. (100) is of order $O(\Delta t \mathrm{Ma}^2)$ with $\mathrm{Ma} := u/c_s$ being the Mach number, which can be eliminated when $a^2 = b$. However, this term does not appear in Eq. (99) at the diffusive scaling; this is because Ma is of order $O(\Delta x)$, and in this case, it can be rearranged into the truncation error $O(\Delta x^2)$. This also indicates that the LB method is suitable for nearly incompressible flows at both the diffusive and acoustic scalings.

With respect to the momentum equation (95b), the first- and second-order MEs at the diffusive and acoustic scalings are given by (see Appendixes A 2 and B 2 for details)

$$\Delta t \sim \Delta x^2 : \begin{cases} \partial_t(\rho u_\alpha) + \partial_\beta\big(\rho u_\alpha u_\beta + \rho c_s^2\delta_{\alpha\beta}\big) + \Delta t\big(1 - \frac{b}{2a^2}\big)\partial_\beta\big(\rho c_s^2\partial_\theta u_\theta\delta_{\alpha\beta}\big) - \rho\hat{F}_{x_\alpha} \\ \quad -\Delta t\,\partial_\beta\big[S^2_{\alpha\beta\xi_1\xi_2} - \big(1 - \frac{b}{2a^2}\big)\delta_{\xi_1\alpha}\delta_{\xi_2\beta}\big]\big(\rho c_s^2\partial_{\xi_1}u_{\xi_2} + \rho c_s^2\partial_{\xi_2}u_{\xi_1}\big) = O(\Delta x), \\ \partial_t(\rho u_\alpha) + \partial_\beta\big(\rho u_\alpha u_\beta + \rho c_s^2\delta_{\alpha\beta}\big) + \Delta t\big(1 - \frac{b}{2a^2}\big)\partial_\beta\big(\rho c_s^2\partial_\theta u_\theta\delta_{\alpha\beta}\big) - \rho\hat{F}_{x_\alpha} \\ \quad -\Delta t\,\partial_\beta\big[S^2_{\alpha\beta\xi_1\xi_2} - \big(1 - \frac{b}{2a^2}\big)\delta_{\xi_1\alpha}\delta_{\xi_2\beta}\big]\big(\rho c_s^2\partial_{\xi_1}u_{\xi_2} + \rho c_s^2\partial_{\xi_2}u_{\xi_1}\big) = O(\Delta x^2), \end{cases} \tag{101}$$

and

$$\Delta t \sim \Delta x : \begin{cases} \partial_t(\rho u_\alpha) + \partial_\beta\big(\rho u_\alpha u_\beta + \rho c_s^2\delta_{\alpha\beta}\big) - \rho\hat{F}_{x_\alpha} = O(\Delta x), \\ \partial_t(\rho u_\alpha) + \partial_\beta\big(\rho u_\alpha u_\beta + \rho c_s^2\delta_{\alpha\beta}\big) + \frac{\Delta t}{2}\big(1 - \frac{b}{a^2}\big)\partial_\beta\big(\rho c_s^2\partial_\theta u_\theta\delta_{\alpha\beta}\big) - \rho\hat{F}_{x_\alpha} \\ \quad -\Delta t\,\partial_\beta\big[S^2_{\alpha\beta\xi_1\xi_2} - \big(1 - \frac{b}{2a^2}\big)\delta_{\xi_1\alpha}\delta_{\xi_2\beta}\big]\big(\rho c_s^2\partial_{\xi_1}u_{\xi_2} + \rho c_s^2\partial_{\xi_2}u_{\xi_1}\big) = O(\Delta x^2 + \Delta x \mathrm{Ma}^3), \end{cases} \tag{102}$$

where

$$\partial_t\big(\rho u_{\xi_1}u_{\xi_2}\big) = \hat{F}_{x_{\xi_1}}u_{\xi_2} + \hat{F}_{x_{\xi_2}}u_{\xi_1} - c_s^2\big(u_{\xi_1}\partial_{\xi_2}\rho + u_{\xi_2}\partial_{\xi_1}\rho\big) + O(\Delta x + \mathrm{Ma}^3), \tag{103a}$$

$$\partial_\theta\big(\rho c_s^2\Delta_{\alpha\beta\theta\zeta}u_\zeta\big) = \partial_\theta\big(\rho c_s^2 u_\theta\delta_{\alpha\beta}\big) + \partial_\alpha\big(\rho c_s^2 u_\beta\big) + \partial_\beta\big(\rho c_s^2 u_\alpha\big). \tag{103b}$$

$\partial_t(\rho\mathbf{uuu}) = O(\mathrm{Ma}^3)$, $\mathbf{u} = O(\mathrm{Ma})$ and $\nabla\rho = O(\mathrm{Ma}^2)$ are used at the acoustic scaling. From Eqs. (101) and (102), one can see that the viscous terms are different at the diffusive and acoustic scalings. This is because the terms $\Delta t\,\partial_{tt}\hat{m}^{\mathrm{eq}}_{\alpha+1}$ and $\Delta t[\mathcal{W}_0\partial_t\hat{\mathbf{m}}^{\mathrm{eq}}]_{\alpha+1}$ ($\alpha \in \{1 \sim d\}$), where $\hat{\mathbf{m}}^{\mathrm{eq}} := \mathbf{m}^{\mathrm{eq}}/\Delta x^k = O(1)$ with $k = 1$ at the diffusive scaling while $k = 0$ at the acoustic scaling, are of order $O(\Delta x^2)$ at the diffusive scaling while they are of order $O(\Delta x)$ term at the acoustic scaling.

For the term $S^2_{\alpha\beta\xi_1\xi_2}$ shown in Eqs. (101) and (102), we consider the following commonly used form [30]:

$$S^2_{\alpha\beta\xi_1\xi_2} = \begin{cases} \frac{1}{s_{2s}} + \frac{1}{d}\big(\frac{1}{s_{2b}} - \frac{1}{s_{2s}}\big), & \xi_1 = \xi_2 = \alpha = \beta, \\ \frac{1}{d}\big(\frac{1}{s_{2b}} - \frac{1}{s_{2s}}\big), & \alpha = \beta, \xi_1 = \xi_2, \xi_1 \neq \alpha, \\ \frac{1}{s_{2s}}, & \xi_1 = \alpha, \xi_2 = \beta, \alpha \neq \beta, \\ 0, & \text{others}. \end{cases} \tag{104}$$

Then the second-order ME (101) at the diffusive scaling is the same as the momentum euqation (95b) with

$$\upsilon = \left(\frac{1}{s_{2s}} + \frac{b}{2a^2} - 1\right)c_s^2\Delta t, \quad \upsilon_b = \left[\frac{2}{d}\left(\frac{1}{s_{2b}} + \frac{b}{2a^2} - 1\right) + \left(\frac{b}{2a^2} - 1\right)\right]c_s^2\Delta t, \quad \mu = \rho\upsilon, \mu_b = \rho\upsilon_b, \tag{105}$$

while the second-order ME (102) at the acoustic scaling would reduce to the momentum euqation (95b) under the following condition:

$$\upsilon = \left(\frac{1}{s_{2s}} + \frac{b}{2a^2} - 1\right)c_s^2\Delta t, \quad \upsilon_b = \left[\frac{2}{d}\left(\frac{1}{s_{2b}} + \frac{b}{2a^2} - 1\right) + \left(\frac{b}{2a^2} - \frac{1}{2}\right)\right]c_s^2\Delta t, \quad \mu = \rho\upsilon, \mu_b = \rho\upsilon_b. \tag{106}$$

IV. FOURTH-ORDER GPMRT-LB MODEL AND GPMFD SCHEME FOR ONE-DIMENSIONAL CDE

Based on the above results, the GPMFD scheme of a given GPMRT-LB model can be directly derived from Eq. (37), then one can further conduct the accuracy and stability analysis with the help of the traditional tools adopted in the finite-difference method. Similar to our previous work [23], we will present the fourth-order GPMRT-LB model and GPMFD scheme at the diffusive scaling for the one-dimensional CDE,

$$\frac{\partial \phi}{\partial t} + u\frac{\partial \phi}{\partial x} = \kappa \frac{\partial^2 \phi}{\partial x^2}, \tag{107}$$

where ϕ is a scalar function of the position x and time t, u and κ are two constants. For CDE (107), the evolution equation of the GPMRT-LB model can be written as [23,38]

$$f_i^\star(x,t) = f_i(x,t) - (\mathbf{M}^{-1}\mathbf{S}\mathbf{M})_{ik}\big[f_k - f_k^{\rm eq}\big](x,t), \tag{108}$$

$$f_i(x,t+\Delta t) = p_0 f_i^\star(x,t) + p_{-1} f_i^\star(x-\lambda_i\Delta t, t) + p_1 f_i^\star(x+\lambda_i\Delta t,t),\ i=-1,0,1, \tag{109}$$

where the D1Q3 lattice structure, the orthogonal transform matrix $\mathbf{M}$, the diagonal relaxation matrix $\mathbf{S}$ =diag(s_0, s_1, s_2), and the moment conditions required for the equilibrium distribution function are the same as those in Ref. [23]. Here the equilibrium distribution function is given by

$$f_i^{\rm eq} = w_i\phi\left[1 + \frac{c_i u}{c_s^2} + \vartheta\frac{u^2\left(c_i^2 - c_s^2\right)}{2c_s^4}\right], \tag{110}$$

where

$$c_s^2 = (1-w_0)c^2, \quad w_1 = w_{-1} = \frac{1-w_0}{2}; \tag{111}$$

w_i is the weight coefficient, $\vartheta = \zeta\xi$ with $\zeta = 2(1-w_0)/w_0$, and $\xi = (1/s_1 - 1/2)/[1/s_1 + b/(2a^2) - 1]$. We would like to point out that the equilibrium distribution function (110) is different from that in Ref. [23].

With the help of the scheme (18), one can easily obtain the GPMFD scheme on the variable ϕ,

$$\begin{aligned}\phi_j^{n+1} =& \alpha_1\phi_j^n + \alpha_2\phi_{j-1}^n + \alpha_3\phi_{j+1}^n + \beta_1\phi_j^{n-1} + \beta_2\phi_{j-1}^{n-1} \\ &+ \beta_3\phi_{j+1}^{n-1} + \beta_4\phi_{j-2}^{n-1} + \beta_5\phi_{j+2}^{n-1} \\ &+ \gamma_1\phi_j^{n-2} + \gamma_{j-1}^{n-2} + \gamma_3\phi_{j+1}^{n-2} + \gamma_4\phi_{j-2}^{n-2} + \gamma_5\phi_{j+2}^{n-2},\end{aligned} \tag{112}$$

where ϕ_j^n represents $\phi(j\Delta x, t_n)$, $j \in \mathcal{Z}$ and $n \in \mathcal{N}$, the parameters $\alpha_i (i \in \{1 \sim 3\})$, β_k and $\gamma_k (k \in \{1 \sim 5\})$ can be found in Appendix C.

Due to the fact that the accuracy analysis on the above scheme (112) is similar to our previous work [23], here we only present some results and do not show more details. Actually, the second-order ME of the GPMFD scheme (112) can be given by

$$\left[\frac{\partial\phi}{\partial t} + u\frac{\partial\phi}{\partial x}\right]_j^n = \varepsilon\frac{\Delta x^2}{\Delta t}\left[\frac{\partial^2\phi}{\partial x^2}\right]_j^n - \frac{uTR_3}{6s_1}\Delta x^2\left[\frac{\partial^3\phi}{\partial x^3}\right]_j^n + \frac{TR_4(1-w_0)}{24 s_1 s_2}\frac{\Delta x^4}{\Delta t}\left[\frac{\partial^4\phi}{\partial x^4}\right]_j^n + O(\Delta t^2), \tag{113}$$

where $\varepsilon = \kappa\Delta t/\Delta x^2$. To derive a fourth-order GPMRT-LB model (108) and (112), the following conditions need to be satisfied:

$$\varepsilon = a^2\left(\frac{1}{s_1} + \frac{b}{2a^2} - 1\right)(1-w_0), \tag{114a}$$

$$\begin{aligned}TR_3 =& s_1^2 s_2(1-3b) + 12a^2 s_2(s_1-1) \\ &+ 3w_0\big[2a^2\big(s_1 + 2s_2 + s_1^2(s_2-1) - 3s_1 s_2\big) \\ &+ bs_1(s_1(1-s_2)+s_2)\big] = 0,\end{aligned} \tag{114b}$$

$$\begin{aligned}TR_4 =& 6a^4 s_2\big(6s_1 - 4 - 2s_1^3\big) + bs_1^3 s_2(1-3b) \\ &+ 8a^2 s_1^2 s_2(1-s_1)(1-3b) \\ &+ 6a^4 w_0\big[4\big(s_1 + s_2 + s_1^3 + 2s_1^2(s_2-1)\big) \\ &- 10 s_1 s_2 - 2s_1^3 s_2\big] + 3b^2 s_1^3 w_0(2-s_2) \\ &+ 12a^2 b s_1 w_0[2s_1(1-s_1) + s_2((s_1-2)s_1+1)] = 0,\end{aligned} \tag{114c}$$

where $u/c = O(\Delta x)$ has been used, while it has not been take into account in the previous work [23]. In addition, for the special case with $a = b = 1$, the solution of the fourth-order conditions (114) can be derived:

$$\begin{aligned} s_1 &= \frac{12\varepsilon}{6\varepsilon+1}, \\ s_2 &= \frac{2}{6\varepsilon+1}, \\ w_0 &= 1 - 12\varepsilon^2, \end{aligned} \tag{115}$$

which is different from that in Ref. [23].

It should be noted that the F-GPMRT-LB model and F-GPMFD scheme can be obtained once the weight coefficient w_0 and relaxation parameters s_1 and s_2 satisfy the fourth-order conditions (114) for the given parameters ε, a, and b. However, owing to existence of nonlinearity and coupling, it is difficult to derive analytical solution of the fourth-order conditions (114), thus here we only plot the relation between the parameters s_1, s_2, w_0, and ε through selecting four cases of parameters a and b in Fig. 2. Furthermore, based on the solution of Eq. (114) and Corollary 10 in Ref. [65], the F-GPMRT-LB model is stable if and only if the corresponding F-GPMFD scheme is stable in the von Neumann sense, thus we can also consider the numerical stability of the F-GPMRT-LB model and F-GPMFD scheme through judging whether the modulus of the roots of the characteristic polynomial of the amplification matrix $\mathbf{G}$ [see Eq. (D1) in Appendix D] is no larger than the unit. As shown in Fig. 3, one can observe that stability regions of the F-GPMRT-LB model and F-GPMFD scheme can be larger than that of the MRT-LB model through adjusting parameters a and b properly.

V. NUMERICAL RESULTS AND DISCUSSION

In this section we conduct numerical simulations of the Gauss hill problem, the CDE with nonlinear convection and diffusion terms, and the Taylor-Green vortex flow, since they have the analytical solutions, which can also be used to test the convergence rates (CRs) of the GPMRT-LB model (13) and GPMFD scheme (37) for NACDE and NSEs. To measure the difference between the numerical result and analytical solution, we adopt the following root-mean-square error

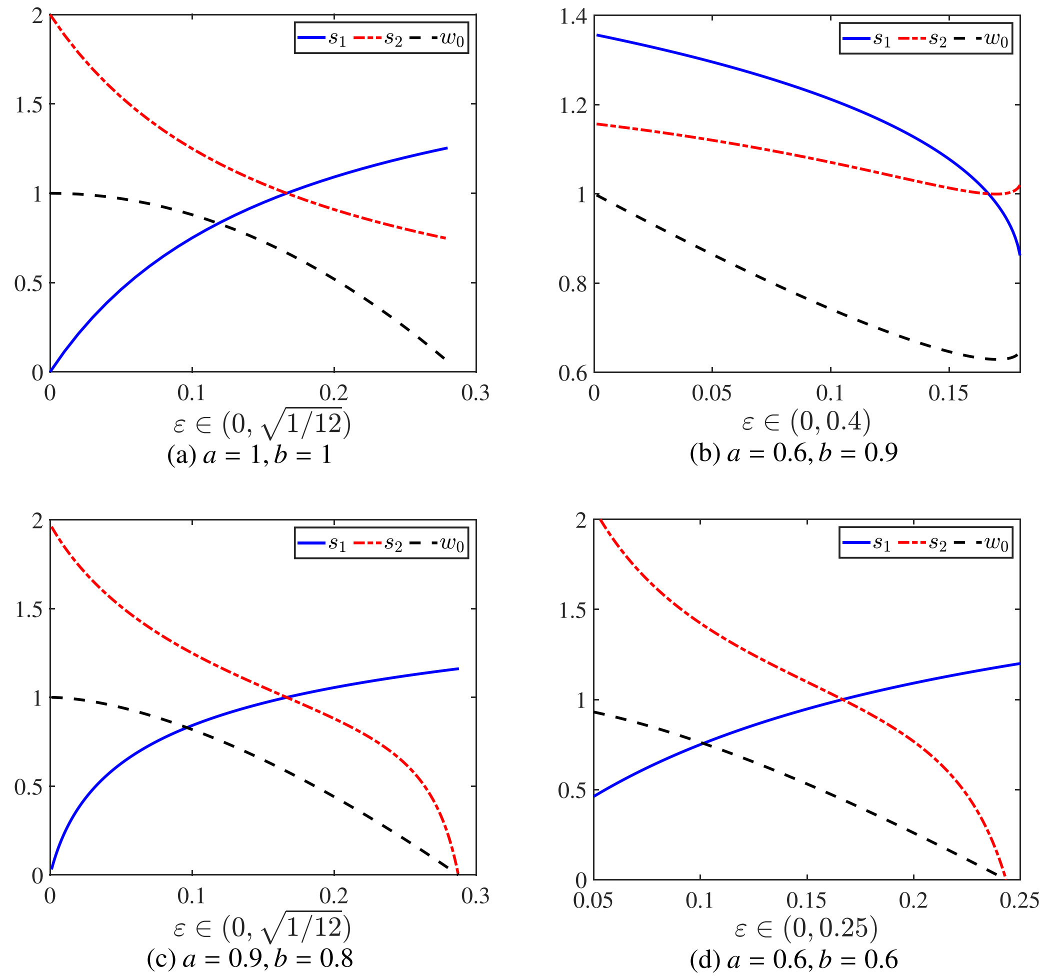

FIG. 2. Weight coefficient w_0 and relaxation parameters s_1 and s_2 as a function of the parameter ε under different values of a and b.

(RMSE) [1]:

$$\mathrm{RMSE} := \sqrt{\frac{\sum_i [\psi(\mathbf{x}_i, t_n) - \psi^\star(\mathbf{x}_i, t_n)]^2}{\prod_{j=1}^{d} N_{x_j}}}, \tag{116}$$

where N_{x_j} is the number of gird points in the j direction, $\mathbf{x}_i$ denotes the grid point, and ψ and $\psi^\star$ are the numerical and analytical solutions, respectively. Based on the definition of RMSE, one can estimate the CR with the following formula:

$$\mathrm{CR} = \frac{\log(\mathrm{RMSE}_{\Delta x}/\mathrm{RMSE}_{\Delta x/2})}{\log 2}. \tag{117}$$

Here we consider the popular D2Q9 lattice structure with $c_s = c/\sqrt{3}$ for two-dimensional problems. For the MRT-LB model, we adopt the orthogonal transform matrix [32],

$$\mathbf{M}^c = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & c & 0 & -c & 0 & c & -c & -c & c \\ 0 & 0 & c & 0 & -c & c & c & -c & -c \\ -4c^2 & -c^2 & -c^2 & -c^2 & -c^2 & 2c^2 & 2c^2 & 2c^2 & 2c^2 \\ 0 & c^2 & -c^2 & c^2 & -c^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c^2 & -c^2 & c^2 & -c^2 \\ 4c^4 & -2c^4 & -2c^4 & -2c^4 & -2c^4 & c^4 & c^4 & c^4 & c^4 \\ 0 & -2c^3 & 0 & 2c^3 & 0 & c^3 & -c^3 & -c^3 & c^3 \\ 0 & 0 & -2c^3 & 0 & 2c^3 & c^3 & c^3 & -c^3 & -c^3 \end{pmatrix}, \tag{118}$$

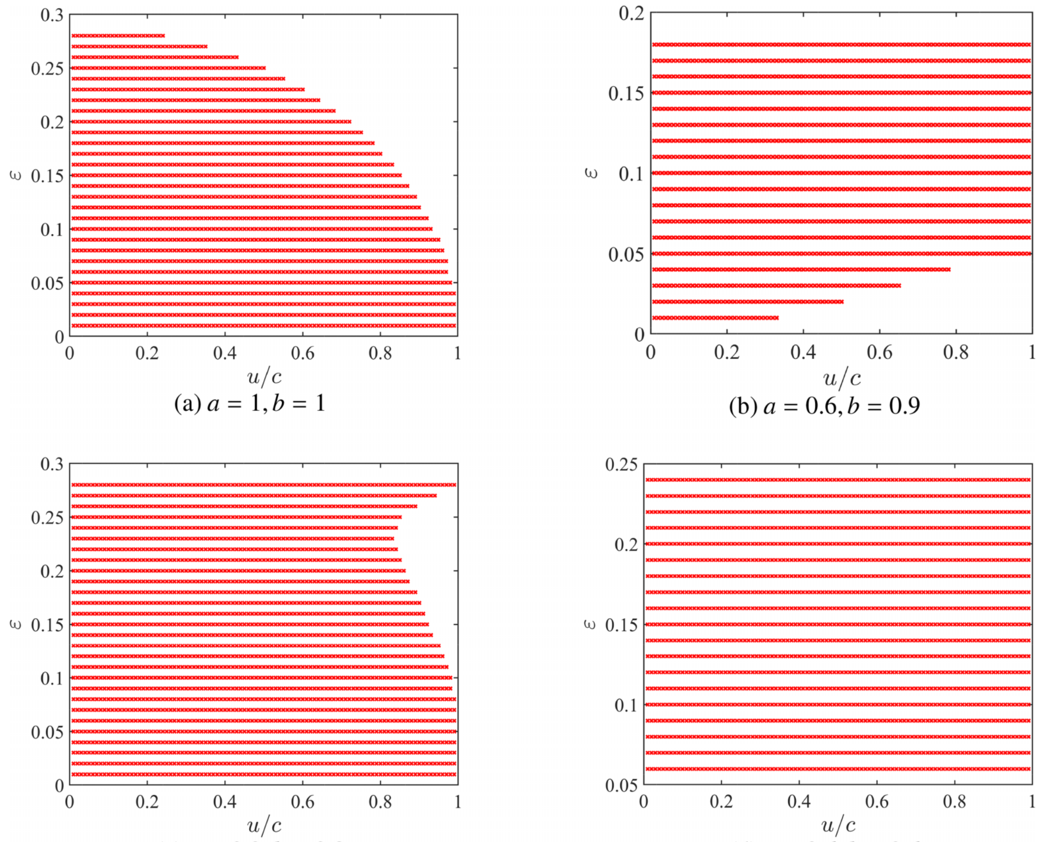

FIG. 3. Stability regions of F-GPMRT-LB model and F-GPMFD scheme for CDE (107) under different values of a and b.

and the relaxation matrix

$$\mathbf{S}^c = \mathrm{diag}(1, (\mathbf{S}^1)^{-1}, 1, 1, 1, 1, 1, 1) \tag{119}$$

is used for the Gauss hill problem and the CDE with nonlinear convection and diffusion terms [see $\mathbf{S}^1$ in Eq. (57)], while

$$\mathbf{S}^c = \mathrm{diag}(1, 1, 1, s_{2s}, s_{2s}, s_{2s}, 1, 1, 1) \tag{120}$$

is applied for the Taylor-Green vortex flow. To satisfy the stability condition [68], the parameter b is located in the range $[a^2, \min\{1, 6\upsilon + a^2\}]$. In addition, we also consider the one-dimensional CDE (107) with the periodic boundary condition to test the F-GPMRT-LB model (108) and F-GPMFD scheme (112), where the parameter ε, weight coefficient w_0, and relaxation parameters s_1 and s_2 are determined by Eq. (114) and the stability region shown in Fig. 3.

Before performing the numerical simulations, we give a remark on the CRs at the acoustic and diffusive scalings.

Remark 9. From the theoretical results in Secs. III C and III D, the GPMRT-LB model and GPMFD scheme have a second-order accuracy at both the acoustic and diffusive scalings for the NACDE and NSEs, and these results are consistent with the asymptotic analysis approaches [30,42–49]. However, it should be noted that when we estimate the CRs of the GPMRT-LB model and GPMFD scheme, for the given physical parameters, e.g., the diffusion or viscosity coefficient, the usually used method is to change the lattice spacing and time step with a fixed $\Delta x/\Delta t$ (the acoustic scaling) or $\Delta x^2/\Delta t$ (the diffusive scaling) while maintaining the other parameters [e.g., the general propagation parameters a and b, the relaxation parameters $\mathbf{S}^1$ (or $\mathbf{S}^2$) related to the diffusion (or viscosity) coefficient, and the weight coefficients, etc.] unchanged, which means that in the LB framework, we cannot estimate the CRs of the GPMRT-LB model and GPMFD scheme at the acoustic scaling in numerical simulations; this also explains why all the works associated with the LB method consider only the CR at the diffusive scaling. In the following simulations, we will consider only the CR at the diffusive scaling.

Example 1. We first consider Gauss hill problem. With the following initial condition:

$$\phi(\mathbf{x}, 0) = \frac{\phi_0}{2\phi\Upsilon_0^2}\exp\left[\left(-\frac{\mathbf{x}^2}{2\Upsilon_0^2}\right)\right], \tag{121}$$

one can obtain the analytical solution of this problem under the constant velocity $\mathbf{u} = (u_x, u_y)^T$ and diffusion coefficient matrix κ,

$$\phi(\mathbf{x}, t) = \frac{\phi_0}{2\pi|\det(\boldsymbol{\Upsilon})|^{1/2}} \times \exp\left\{-\frac{\boldsymbol{\Upsilon}^{-1} : [(\mathbf{x}-\mathbf{u}t)(\mathbf{x}-\mathbf{u}t)]}{2}\right\}, \tag{122}$$

where $\mathbf{x} = (x, y)^T$, $\boldsymbol{\Upsilon} = \Upsilon_0^2\mathbf{I} + 2\kappa t$, and $\boldsymbol{\Upsilon}^{-1}$ $\det(\boldsymbol{\Upsilon})$ represent the inverse matrix and determinant value of $\boldsymbol{\Upsilon}$, respectively. In our simulations, the computational

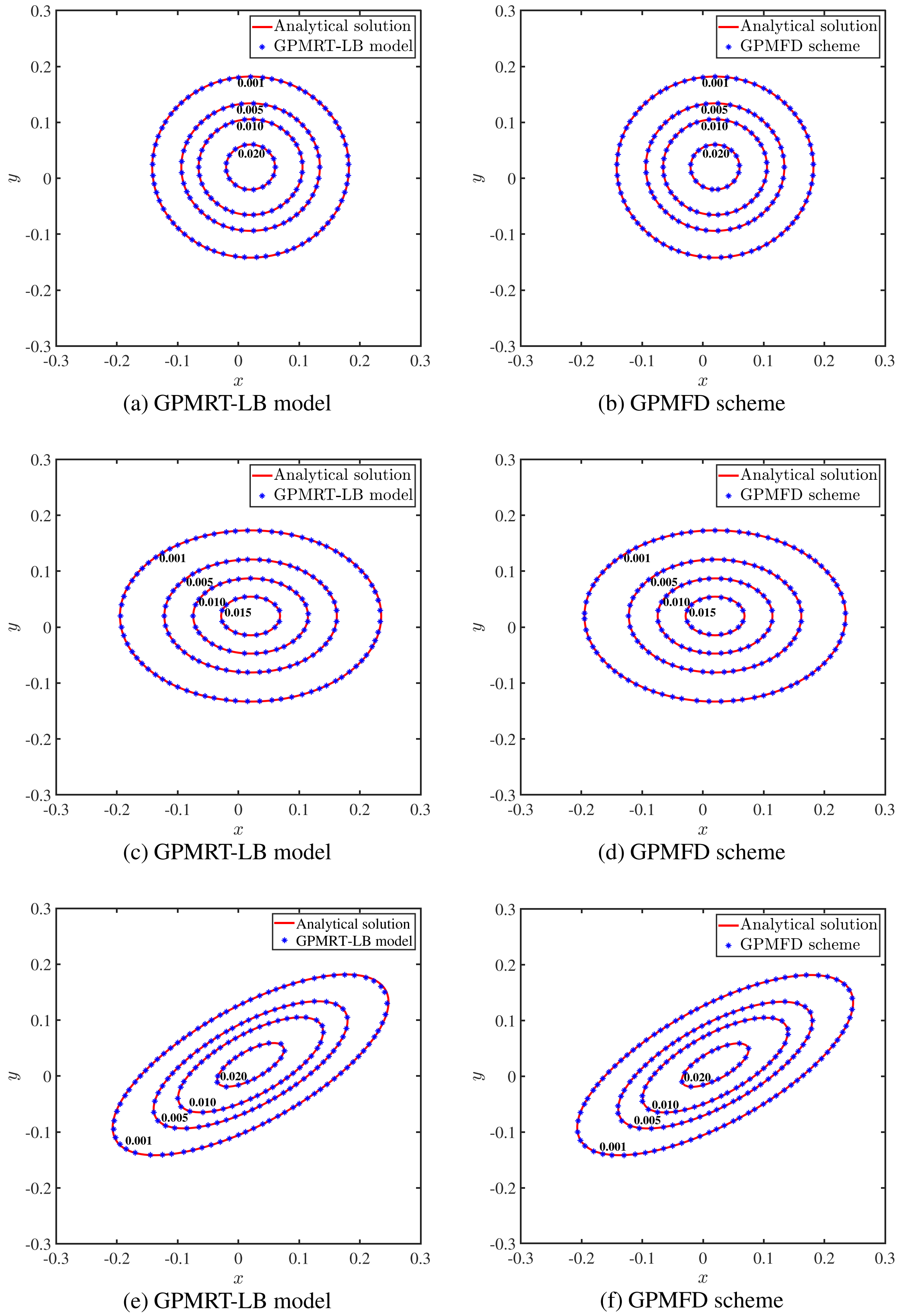

FIG. 4. Contour lines of the scalar variable ϕ at the time $t = 2$ and $\mathbf{u} = (0.01, 0.01)^T$: (a), (b) isotropic diffusion problem, (c), (d) diagonal anisotropic diffusion problem, and (e), (f) full anisotropic diffusion problem.

domain is $[-1, 1] \times [-1, 1]$ and the total concentration is set as $\phi_0 = 2\phi(\Upsilon_0)^2$ with $\Upsilon_0 = 0.01$, which should be small enough when applying the periodic boundary condition.

We first conduct some tests with $u_x = u_y = 0.01$, $\Delta x = \Delta t = 1/200$ and the following three types of diffusion coefficient matrices:

$$\kappa = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \times 10^{-3}, \quad \kappa = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \times 10^{-3},$$
$$\kappa = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \times 10^{-3}, \tag{123}$$

TABLE I. RMSEs and CRs of GPMRT-LB model for five cases of parameters a and b (125) at the diffusive scaling ($t = 2$).

Δx	Δt	(a, b)	$\text{RMSE}_{\Delta x}$	$\text{RMSE}_{\Delta x/2}$	$\text{RMSE}_{\Delta x/4}$	$\text{RMSE}_{\Delta x/8}$	CR
$\frac{1}{80}$	$\frac{1}{50}$	(1,1)	9.5641×10^{-6}	2.3468×10^{-6}	1.0396×10^{-6}	5.8420×10^{-7}	∼2.0108
$\frac{1}{80}$	$\frac{1}{50}$	(0.4,0.2)	2.8713×10^{-5}	6.7951×10^{-6}	2.9940×10^{-6}	1.6794×10^{-6}	∼2.0368
$\frac{1}{80}$	$\frac{1}{50}$	(0.4,0.45)	2.5173×10^{-5}	6.0406×10^{-6}	2.6671×10^{-6}	1.4971×10^{-6}	∼2.0274
$\frac{1}{80}$	$\frac{1}{50}$	(0.6,0.36)	1.4732×10^{-5}	3.5532×10^{-6}	1.5702×10^{-6}	8.8162×10^{-7}	∼2.0240
$\frac{1}{80}$	$\frac{1}{50}$	(0.5,0.5)	1.7176×10^{-5}	4.1824×10^{-6}	1.8512×10^{-6}	1.0400×10^{-6}	∼2.0175

which represent the isotropic diffusion, diagonal anisotropic diffusion, and full anisotropic diffusion problems, and present the results of the LW scheme ($b = a^2$) at the time $t = 2$ in Fig. 4, where $a = 0.5$. As shown in this figure, the numerical results obtained from both the GPMRT-LB model and GPMFD scheme are in good agreement with the analytical solutions.

In addition, we also conduct some simulations at the diffusive scaling under $u_x = u_y = 0.01$ and the full anisotropic diffusion coefficient matrix,

$$\kappa = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \times 10^{-3}, \tag{124}$$

and consider the following five cases with different values of parameters a and b:

$$\text{Case 1: } a = b = 1, \text{ the MRT} - \text{LB model}, \tag{125a}$$

$$\text{Case 2: } a^2 < b < a, \ a = 0.4, \ b = 0.2, \tag{125b}$$

$$\text{Case 3: } a < b < 1, \ a = 0.4, \ b = 0.45, \tag{125c}$$

$$\text{Case 4: } b = a^2, \ a = 0.6, \text{ the LW scheme}, \tag{125d}$$

$$\text{Case 5: } b = a = 0.5, \text{ the FP scheme}. \tag{125e}$$

As seen from Tables I and II, both the GPMRT-LB model and GPMFD scheme can achieve a second-order CR at the diffusive scaling, which is consistent with the theoretical analysis.

Example 2. We would like to point out that in the above example, the anisotropic diffusion of the CDE is considered, while it is only a linear problem. In this example, we will focus on the following more general CDE with the nonlinear convection and diffusion terms [28]:

$$\partial_t \phi + \nabla \cdot (\phi^m \mathbf{u}) = \nabla \cdot [\kappa (\nabla \cdot \mathbf{D}(\phi))] + R, \tag{126}$$

where m is a constant, and κ is the diffusion coefficient. $\mathbf{D}(\phi)$ is a nonlinear diffusion term, which is given by

$$\mathbf{D}(\phi) = \begin{pmatrix} \phi^{n_x} & 0 \\ 0 & \phi^{n_y} \end{pmatrix}, \tag{127}$$

where n_x and n_y are two constants. R is the source term and is defined as

$$\begin{aligned} R = {} & \exp(-At)\{A\cos(2\pi x)\cos(2\pi y) - 4n_x\pi^2\kappa\phi^{n_x-2}[(n_x-1)\exp(-At)\sin^2(2\pi x)\cos^2(2\pi y) + \phi\cos(2\pi x)\cos(2\pi y)] \\ & - 4n_y\pi^2\kappa\phi^{n_y-2}[(n_y-1)\exp(-At)\sin^2(2\pi y)\cos^2(2\pi x) + \phi\cos(2\pi x)\cos(2\pi y)] \\ & + 2\pi m\phi^{m-1}[u_x\sin(2\pi x)\cos(2\pi y) + u_y\cos(2\pi x)\sin(2\pi y)]\}, \end{aligned} \tag{128}$$

here A is a constant. Under the periodic condition and the initial condition,

$$\phi(x, y, 0) = \alpha - \cos(2\pi x)\cos(\pi y), \tag{129}$$

one can derive the analytical solution of Eq. (126),

$$\phi(x, y, t) = \alpha - \exp(-At)\cos(2\pi x)\cos(2\pi y), \tag{130}$$

where α is a constant.

We consider the characteristic velocity $u_x = u_y = 0.1, \alpha = 1.1, A = 1.0, m = 2.0, n_x = 2.0, n_y = 3.0$ with the diffusion coefficient $\kappa = 10^{-4}$ and perform some simulations under different lattice spacings $\Delta x = 1/40, 1/80, 1/160, 1/320$ and the fixed $\Delta x^2/\Delta t = 1/160$. From the results shown in Tables III and IV, one can find that both the GPMRT-LB model and GPMFD scheme for this problem have a second-order

TABLE II. RMSEs and CRs of GPMFD scheme for five cases of parameters a and b (125) at the diffusive scaling ($t = 2$).

Δx	Δt	(a, b)	$\text{RMSE}_{\Delta x}$	$\text{RMSE}_{\Delta x/2}$	$\text{RMSE}_{\Delta x/4}$	$\text{RMSE}_{\Delta x/8}$	CR
$\frac{1}{80}$	$\frac{1}{50}$	(1,1)	6.1121×10^{-6}	1.5280×10^{-6}	6.7934×10^{-7}	3.8222×10^{-7}	∼1.9993
$\frac{1}{80}$	$\frac{1}{50}$	(0.4,0.2)	2.7548×10^{-5}	6.8421×10^{-6}	3.0392×10^{-6}	1.7095×10^{-6}	∼2.0036
$\frac{1}{80}$	$\frac{1}{50}$	(0.4,0.45)	1.0589×10^{-5}	2.6788×10^{-6}	1.1932×10^{-6}	6.7179×10^{-7}	∼1.9041
$\frac{1}{80}$	$\frac{1}{50}$	(0.6,0.36)	7.6951×10^{-6}	1.9097×10^{-6}	8.4799×10^{-7}	4.7691×10^{-7}	∼2.0045
$\frac{1}{80}$	$\frac{1}{50}$	(0.5,0.5)	9.2178×10^{-6}	2.3532×10^{-6}	1.0501×10^{-6}	5.9156×10^{-7}	∼1.9849

TABLE III. RMSEs and CRs of GPMRT-LB model for five cases of parameters a and b at the diffusive scaling ($t = 1$).

Δx	Δt	(a, b)	$\text{RMSE}_{\Delta x}$	$\text{RMSE}_{\Delta x/2}$	$\text{RMSE}_{\Delta x/4}$	$\text{RMSE}_{\Delta x/8}$	CR
$\frac{1}{40}$	$\frac{1}{10}$	(1,1)	2.4101×10^{-3}	4.6829×10^{-4}	1.1172×10^{-4}	2.8655×10^{-5}	~2.1314
$\frac{1}{40}$	$\frac{1}{10}$	(0.4,0.2)	3.9736×10^{-3}	1.0548×10^{-3}	2.6514×10^{-4}	6.5618×10^{-5}	~1.9734
$\frac{1}{40}$	$\frac{1}{10}$	(0.4,0.6)	2.9547×10^{-3}	7.2457×10^{-4}	1.8701×10^{-4}	4.7613×10^{-5}	~1.9852
$\frac{1}{40}$	$\frac{1}{10}$	(0.6,0.36)	4.3929×10^{-3}	1.1640×10^{-3}	2.6713×10^{-4}	6.1312×10^{-5}	~2.0543
$\frac{1}{40}$	$\frac{1}{10}$	(0.4,0.4)	3.8619×10^{-3}	1.1461×10^{-3}	2.6591×10^{-4}	7.4452×10^{-5}	~1.8990

TABLE IV. RMSEs and CRs of GPMFD scheme for five cases of parameters a and b at the diffusive scaling ($t = 1$).

Δx	Δt	(a, b)	$\text{RMSE}_{\Delta x}$	$\text{RMSE}_{\Delta x/2}$	$\text{RMSE}_{\Delta x/4}$	$\text{RMSE}_{\Delta x/8}$	CR
$\frac{1}{40}$	$\frac{1}{10}$	(1,1)	1.1210×10^{-3}	3.0132×10^{-4}	7.8017×10^{-5}	2.1595×10^{-5}	~1.9036
$\frac{1}{40}$	$\frac{1}{10}$	(0.4,0.2)	1.9264×10^{-3}	5.1008×10^{-4}	1.2081×10^{-4}	2.9965×10^{-5}	~2.0021
$\frac{1}{40}$	$\frac{1}{10}$	(0.4,0.6)	2.5461×10^{-3}	6.1005×10^{-4}	1.3577×10^{-4}	3.0083×10^{-5}	~2.1344
$\frac{1}{40}$	$\frac{1}{10}$	(0.6,0.36)	2.1193×10^{-3}	5.0988×10^{-4}	1.3102×10^{-4}	3.2012×10^{-5}	~2.0128
$\frac{1}{40}$	$\frac{1}{10}$	(0.4,0.4)	3.2162×10^{-3}	8.8190×10^{-4}	2.5154×10^{-5}	6.1414×10^{-5}	~1.9035

TABLE V. Errors and CRs of GPMRT-LB model for five cases of parameters a and b at the diffusive scaling ($t = 2$).

Δx	Δt	(a, b)	$\text{Err}^{u_x}_{\Delta x}$	$\text{Err}^{u_x}_{\Delta x/2}$	$\text{Err}^{u_x}_{\Delta x/4}$	$\text{Err}^{u_x}_{\Delta x/8}$	CR^{u_x}
$\frac{1}{16}$	$\frac{1}{400}$	(1,1)	7.5590×10^{-3}	1.8941×10^{-3}	4.7375×10^{-4}	1.1811×10^{-4}	~2.0000
$\frac{1}{16}$	$\frac{1}{400}$	(0.5,0.5)	9.7341×10^{-3}	2.5072×10^{-3}	6.3126×10^{-4}	1.5655×10^{-4}	~1.9861
$\frac{1}{16}$	$\frac{1}{400}$	(0.5,0.25)	1.9696×10^{-3}	4.9446×10^{-4}	1.2378×10^{-4}	3.0824×10^{-5}	~1.9996
$\frac{1}{16}$	$\frac{1}{400}$	(0.5,0.75)	7.0541×10^{-3}	1.7666×10^{-3}	4.4183×10^{-4}	1.1032×10^{-4}	~1.9996
$\frac{1}{16}$	$\frac{1}{400}$	(0.5,0.375)	1.7801×10^{-3}	4.3067×10^{-4}	1.0679×10^{-4}	2.6795×10^{-5}	~2.0180

TABLE VI. Errors and CRs of GPMFD model for five cases of parameters a and b at the diffusive scaling ($t = 2$).

Δx	Δt	(a, b)	$\text{Err}^{u_x}_{\Delta x}$	$\text{Err}^{u_x}_{\Delta x/2}$	$\text{Err}^{u_x}_{\Delta x/4}$	$\text{Err}^{u_x}_{\Delta x/8}$	CR^{u_x}
$\frac{1}{16}$	$\frac{1}{400}$	(1,1)	5.1430×10^{-3}	1.3662×10^{-3}	3.1500×10^{-4}	8.8603×10^{-5}	~1.9530
$\frac{1}{16}$	$\frac{1}{400}$	(0.5,0.5)	6.1180×10^{-3}	1.5801×10^{-3}	4.0917×10^{-4}	9.7914×10^{-5}	~1.9885
$\frac{1}{16}$	$\frac{1}{400}$	(0.5,0.25)	8.8766×10^{-4}	2.2957×10^{-4}	6.1279×10^{-5}	1.5009×10^{-5}	~1.9620
$\frac{1}{16}$	$\frac{1}{400}$	(0.5,0.75)	4.9980×10^{-3}	1.2165×10^{-3}	3.0613×10^{-4}	7.8871×10^{-5}	~1.9953
$\frac{1}{16}$	$\frac{1}{400}$	(0.5,0.375)	9.7244×10^{-4}	2.4097×10^{-4}	6.4010×10^{-5}	1.6419×10^{-5}	~1.9627

TABLE VII. RMSEs and CRs of F-GPMRT-LB model for different value of a and b at the diffusive scaling.

Δx	Δt	(a, b)	$\text{RMSE}_{\Delta x}$	$\text{RMSE}_{\Delta x/2}$	$\text{RMSE}_{\Delta x/4}$	$\text{RMSE}_{\Delta x/8}$	CR
$\frac{1}{10}$	$\frac{1}{50}$	(1,1)	6.1216×10^{-4}	3.7760×10^{-5}	2.3466×10^{-6}	1.4628×10^{-7}	∼4.0103
$\frac{1}{10}$	$\frac{1}{50}$	(0.6,0.9)	1.6485×10^{-3}	1.0545×10^{-4}	6.6188×10^{-6}	4.1442×10^{-7}	∼3.9859
$\frac{1}{10}$	$\frac{1}{50}$	(0.9,0.8)	5.8918×10^{-4}	3.6349×10^{-5}	1.1590×10^{-6}	1.4082×10^{-7}	∼4.0110
$\frac{1}{10}$	$\frac{1}{50}$	(0.6,0.6)	5.4684×10^{-4}	3.3716×10^{-5}	2.0952×10^{-6}	1.3062×10^{-7}	∼4.0105

CR at the diffusive scaling, which is in agreement with the theoretical analysis in Sec. III C.

Example 3. We now consider the Taylor-Green vortex flow, which is unsteady and fully periodic in a domain of size $[0, L] \times [0, L]$, and can obtain the analytical solution of the NSE [1],

$$u_x(x, y, t) = -u_0\sqrt{\frac{k_x}{k_y}}\cos(k_x x)\sin(k_y y)\exp\left(-\frac{t}{t_d}\right), \quad (131a)$$

$$u_y(x, y, t) = u_0\sqrt{\frac{k_x}{k_y}}\sin(k_x x)\cos(k_y y)\exp\left(-\frac{t}{t_d}\right), \quad (131b)$$

$$\rho(x, y, t) = 1 - \frac{u_0^2}{4c_s^2}\left[\frac{k_y}{k_x}\cos(2k_x x) + \frac{k_x}{k_y}\cos(2k_y y)\right]\exp\left(-\frac{2t}{t_d}\right), \quad (131c)$$

where u_0 is the initial characteristic velocity, $k_x = k_y = 2\pi/L$, and the vortex decay time t_d is defined by

$$t_d = \frac{1}{\upsilon(k_x^2 + k_y^2)}. \quad (132)$$

The initial state is determined by $\mathbf{u}(\mathbf{x}, 0)$ and $\rho(\mathbf{x}, 0)$, which are given by Eq. (131). To evaluate the difference between the numerical and analytical solutions, the L^2 norm error is adopted,

$$\text{Err}^{\psi} = \sqrt{\frac{\sum_i(\psi(\mathbf{x}_i, t_n) - \psi^{\star}(\mathbf{x}_i, t_n))^2}{\sum_i(\psi^{\star})^2(\mathbf{x}_i, t_n)}}, \quad (133)$$

where $\psi = \rho, u_x$ or u_y, and the corresponding CR is defined as

$$\text{CR} = \frac{\log(\text{Err}^{\psi}_{\Delta x}/\text{Err}^{\psi}_{\Delta x/2})}{\log 2}; \quad (134)$$

it should be noted that the errors of velocity, Err^{u_x} and Err^{u_y}, are of the same order. In our simulations, $\upsilon = 1/6$, $u_0 = 0.02$ and the characteristic length $L = 2$. We consider the following five cases of parameters a and b [37]:

$$\text{Case 1: } a = 1,\ b = 1,\ \text{the MRT} - \text{LB model}, \quad (135a)$$

$$\text{Case 2: } a = 0.5,\ b = a^2,\ \text{the LW scheme}, \quad (135b)$$

$$\text{Case 3: } a = 0.5,\ b = a,\ \text{the FP scheme}, \quad (135c)$$

$$\text{Case 4: } a = 0.5,\ b = a(2 - a),\ (a < b < 1), \quad (135d)$$

$$\text{Case 5: } a = 0.5,\ b = a^2(2 - a),\ (a^2 < b < a), \quad (135e)$$

and present the results at time $t = 2$ in Tables V and VI, where the lattice spacing $\Delta x = 1/16, 1/32, 1/64, 1/128$ and the time step is determined by the fixed constant $\Delta x^2/\Delta t = 25/16$. We would like to point out that the magnitude of Err^{ρ} is less than 10^{-13}, and thus the CR of the density is not considered here. As shown in Tables V and VI, a second-order CR in space at the diffusive scaling can be observed for the velocity in the x direction, which is consistent with the theoretical analysis in Sec. III D.

Example 4. We further consider the CDE (107) problem with the periodic boundary condition and following initial condition:

$$\phi(x, 0) = \sin(\pi x), \quad -1 \leqslant x \leqslant 1, \quad (136)$$

and obtain the analytical solution of this problem as

$$\phi(x, t) = \sin[\pi(x - ut)]\exp(-\kappa\pi^2 t). \quad (137)$$

In this test the initialization processes for the F-GPMRT-LB model and F-GPMFD scheme are the same as those in our previous work [23]; now we consider the diffusion coefficient $\kappa = 0.08$, velocity $u = 1$, lattice spacing $\Delta x = 1/10, 1/20, 1/40, 1/80$, time step $\Delta t = 1/50, 1/200, 1/800, 1/3200$, and measure the RMSEs between the numerical and analytical solutions at the time $t = 2$. As seen from Tables VII and VIII, both the GPMRT-LB model and GPMFD scheme at the diffusive scaling have a fourth-order CR in space.

TABLE VIII. RMSEs and CRs of F-GPMFD model for different values of a and b at the diffusive scaling.

Δx	Δt	(a, b)	$\text{RMSE}_{\Delta x}$	$\text{RMSE}_{\Delta x/2}$	$\text{RMSE}_{\Delta x/4}$	$\text{RMSE}_{\Delta x/8}$	CR
$\frac{1}{10}$	$\frac{1}{50}$	(1,1)	5.2505×10^{-4}	3.7716×10^{-5}	2.5180×10^{-6}	1.6268×10^{-7}	∼3.8774
$\frac{1}{10}$	$\frac{1}{50}$	(0.6,0.9)	7.2309×10^{-4}	5.1108×10^{-5}	3.3636×10^{-6}	2.1540×10^{-7}	∼3.9043
$\frac{1}{10}$	$\frac{1}{50}$	(0.9,0.8)	2.0699×10^{-4}	1.4997×10^{-5}	1.0076×10^{-6}	6.5284×10^{-8}	∼3.8768
$\frac{1}{10}$	$\frac{1}{50}$	(0.6,0.6)	1.8924×10^{-4}	1.3172×10^{-5}	9.2169×10^{-7}	5.9723×10^{-8}	∼3.8766

VI. CONCLUSIONS

In this paper we first derived the multiple-level GPMFD scheme of the GPMRT-LB model on conservative moments, and then conducted the accuracy analysis for the GPMRT-LB model and GPMFD scheme through the Maxwell iteration method at the diffusive and acoustic scalings. Furthermore, for the NACDE and NSEs, we presented the first- and second-order modified equations of the GPMRT-LB model and GPMFD scheme at both diffusive and acoustic scalings. In particular, based on our previous work [23], we also developed the F-GPMRT-LB model and F-GPMFD scheme at the diffusive scaling for the one-dimensional CDE, which can be more stable than the MRT-LB model and the corresponding macroscopic finite-difference scheme through adjusting parameters a and b properly (see Fig. 3). Finally, some numerical simulations of the Gauss hill problem, the CDE with nonlinear convection and diffusion terms, and Taylor-Green vortex flow were conducted to test the GPMRT-LB model and GPMFD scheme, and the results show that both of them have second-order convergence rates in space. We also performed a numerical test on the F-GPMRT-LB model and F-GPMFD scheme for the one-dimensional CDE, and we found that they are of fourth-order accuracy in space, which is consistent with our theoretical analysis. We would also like to point out that for high-dimensional problems, it is more difficult to develop the high-order GPMRT-LB models and GPMFD schemes.

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APPENDIX A: DERIVATION OF EQS. (92b), (99), AND (101)

For the first-order ME (75), one can obtain

$$[\partial_t \mathbf{m}^{\text{eq}}]_1 = \partial_t \sum_i f_i^{\text{eq}}, \quad [c\mathcal{W}_0\mathbf{m}^{\text{eq}}]_1 = c\partial_\alpha \sum_i \mathbf{e}_{i\alpha} f_i^{\text{eq}}, \quad \tilde{\mathbf{F}}_1 = \sum_{i=1}^{q}\left(F_i + G_i + \frac{\Delta t}{2}\overline{D}_i F_i\right),$$

$$\begin{aligned}
&\left[\Delta t c^2 \mathcal{W}_0\left(\hat{\mathbf{S}}_N^{-1} + \left(\frac{b}{2a^2} - 1\right)\mathbf{I}_q\right)\mathcal{W}_0\mathbf{m}^{\text{eq}}\right]_1\\
&\quad = \Delta t c^2 \partial_\beta \sum_{k=1}^{q}\sum_{j=1}^{q} \mathbf{e}_{j\beta}\left[\overline{\mathbf{\Lambda}}_{jk} + \left(\frac{b}{2a^2} - 1\right)\delta_{jk}\right]\partial_\theta \mathbf{e}_{k\theta} f_k^{\text{eq}}\\
&\quad = \Delta t c^2 \partial_\beta \sum_{k=1}^{q}\left(S_\beta^{10} + S_{\beta\gamma}^{1}\mathbf{e}_{k\gamma}\right)\partial_\theta \mathbf{e}_{k\theta} f_k^{\text{eq}} + \Delta t\left(\frac{b}{2a^2} - \frac{1}{2}\right)\partial_\beta\partial_\theta \sum_{k=1}^{q} \mathbf{c}_{k\beta}\mathbf{c}_{k\theta} f_k^{\text{eq}}.
\end{aligned} \tag{A1}$$

For the second-order ME (78), however, apart from above equalities, we also need the following results:

$$\begin{aligned}
\left[c\mathcal{W}_0\hat{\mathbf{S}}_N^{-1}\tilde{\mathbf{F}}\right]_1 &= c\partial_\alpha \sum_{k=1}^{q} \mathbf{e}_{k\alpha}\left(F_k + G_k + \frac{\Delta t}{2}\overline{D}_k F_k\right) = c\partial_\alpha \sum_{k=1}^{q}\sum_{j=1}^{q} \mathbf{e}_{i\alpha}\overline{\mathbf{\Lambda}}_{jk}\left(F_k + G_k + \frac{\Delta t}{2}\overline{D}_k F_k\right)\\
&= \partial_\alpha \sum_{k=1}^{q}\left(cS_\alpha^{10} + S_{\alpha\beta}^{1}\mathbf{c}_{k\beta}\right)\left(F_k + G_k + \frac{\Delta t}{2}\overline{D}_k F_k\right),\\
\left[c\mathcal{W}_0\hat{\mathbf{S}}_N^{-1}\partial_t \mathbf{m}^{\text{eq}}\right]_1 &= c\partial_\alpha \sum_{k=1}^{q}\sum_{j=1}^{q} \mathbf{e}_{i\alpha}\overline{\mathbf{\Lambda}}_{jk}\partial_t f_k^{\text{eq}} = \partial_\alpha \sum_{k=1}^{q}\left(cS_\alpha^{10} + S_{\alpha\beta}^{1}\mathbf{c}_{k\beta}\right)\partial_t f_k^{\text{eq}},\\
\left[c\partial_t \hat{\mathbf{S}}_N^{-1}\mathcal{W}_0\mathbf{m}^{\text{eq}}\right]_1 &= s^0 \partial_t\partial_\alpha \sum_{k=1}^{q} \mathbf{c}_{k\alpha} f_k^{\text{eq}}, \quad (c\mathcal{W}_0\partial_t \mathbf{m}^{\text{eq}})_1 = \partial_\alpha\partial_t \sum_{k=1}^{q} \mathbf{c}_{k\alpha} f_k^{\text{eq}},
\end{aligned} \tag{A2}$$

and

$$\left[\Delta t^2 c^3 \mathcal{W}_0^3 \mathbf{m}^{\text{eq}}\right]_1 = \Delta t^2 \partial_\alpha\partial_\beta\partial_\gamma \sum_i \mathbf{c}_{i\alpha}\mathbf{c}_{i\beta}\mathbf{c}_{i\gamma} f_i^{\text{eq}},$$

$$\left[\Delta t^2 c^3 \mathcal{W}_0^2 \hat{\mathbf{S}}_N^{-1}\mathcal{W}_0\mathbf{m}^{\text{eq}}\right]_1 = c^3 \Delta t^2 \partial_\alpha\partial_\beta \sum_{k=1}^{q}\left(\sum_{j=1}^{q} \mathbf{e}_{j\alpha}\mathbf{e}_{j\beta}\overline{\mathbf{\Lambda}}_{jk}\right)\partial_\gamma \mathbf{e}_{k\gamma} f_k^{\text{eq}} = \Delta t^2 \partial_\alpha\partial_\beta \sum_{k=1}^{q}\left(c^2 S_{\alpha\beta}^{20} + S_{\alpha\beta\theta\zeta}^{2}\mathbf{c}_{k\theta}\mathbf{c}_{k\zeta}\right)\partial_\gamma \mathbf{c}_{k\gamma} f_k^{\text{eq}},$$

$$\left[\Delta t^2 c^3 \mathcal{W}_0 \hat{\mathbf{S}}_N^{-1} \mathcal{W}_0^2 \mathbf{m}^{\rm eq}\right]_1 = c^3 \Delta t^2 \partial_\alpha \sum_{k=1}^q \left(\sum_{j=1}^q \mathbf{e}_{j\alpha} \overline{\boldsymbol{\Lambda}}_{jk}\right) \partial_\beta \mathbf{e}_{k\beta} \partial_\gamma \mathbf{e}_{k\gamma} f_k^{\rm eq} = \Delta t^2 \partial_\alpha \sum_{k=1}^q \left(c S_\alpha^{10} + S_{\alpha\theta}^1 \mathbf{c}_{k\theta}\right) \partial_\beta \mathbf{c}_{k\beta} \partial_\gamma \mathbf{c}_{k\gamma} f_k^{\rm eq},$$

$$\left[\Delta t^2 c^3 \mathcal{W}_0 \hat{\mathbf{S}}_N^{-1} \mathcal{W}_0 \hat{\mathbf{S}}_N^{-1} \mathcal{W}_0 \mathbf{m}^{\rm eq}\right]_1 = c^3 \Delta t^2 \partial_\alpha \sum_{i=1}^q \left[\sum_{k=1}^q \left(\sum_{j=1}^q \mathbf{e}_{j\alpha} \overline{\boldsymbol{\Lambda}}_{jk}\right) \partial_\beta \mathbf{e}_{k\beta} \overline{\boldsymbol{\Lambda}}_{ki}\right] \partial_\gamma \mathbf{e}_{i\gamma} f_i^{\rm eq}$$

$$= \Delta t^2 \partial_\alpha \sum_{i=1}^q \left[\sum_{k=1}^q \left(c S_\alpha^{10} + S_{\alpha\theta}^1 \mathbf{c}_{k\theta}\right) \partial_\beta \mathbf{c}_{k\beta} \overline{\boldsymbol{\Lambda}}_{ki}\right] \partial_\gamma \mathbf{c}_{i\gamma} f_i^{\rm eq}$$

$$= \Delta t^2 \partial_\alpha \sum_{i=1}^q \left[c S_\alpha^{10} \partial_\beta \left(S_\beta^{10} + S_{\beta\eta}^1 \mathbf{c}_{i\eta}\right) + S_{\alpha\theta}^1 \partial_\beta \left(S_{\theta\beta}^{20} + S_{\theta\beta\eta}^{21} \mathbf{c}_{i\eta} + S_{\theta\beta\zeta\mu}^2 \mathbf{c}_{i\zeta} \mathbf{c}_{i\mu}\right)\right] \partial_\gamma \mathbf{c}_{i\gamma} f_i^{\rm eq}$$

$$= \Delta t^2 \partial_\alpha \sum_{i=1}^q \left[\left(c S_\alpha^{10} \partial_\beta S_\beta^{10} + S_{\alpha\theta}^1 \partial_\beta S_{\theta\beta}^{20}\right) + \left(c S_\alpha^{10} \partial_\beta S_{\beta\eta}^1 + S_{\alpha\theta}^1 \partial_\beta S_{\theta\beta\eta}^{21}\right) \mathbf{c}_{i\eta} + S_{\alpha\theta}^1 \partial_\beta S_{\theta\beta\zeta\mu}^2 \mathbf{c}_{i\zeta} \mathbf{c}_{i\mu}\right] \partial_\gamma \mathbf{c}_{i\gamma} f_i^{\rm eq}. \quad \text{(A3)}$$

1. NACDE: Derivation of Eq. (92b)

Substituting the moment conditions (90) for NACDE into Eq. (A1), one can derive

$$[c\mathcal{W}_0 \mathbf{m}^{\rm eq}]_1 = \partial_\alpha B_\alpha, \quad [\partial_t \mathbf{m}^{\rm eq}]_1 = \partial_t \phi, \quad \tilde{\mathbf{F}}_1 = R,$$

$$\left[\Delta t c^2 \mathcal{W}_0 \left(\hat{\mathbf{S}}_N^{-1} + \left(\frac{b}{2a^2} - 1\right) \mathbf{I}_q\right) \mathcal{W}_0 \mathbf{m}^{\rm eq}\right]_1$$
$$= \Delta t \frac{\partial}{\partial x_\beta} \left\{c S_\beta^{10} \frac{\partial B_\theta}{\partial x_\theta} + \left[S_{\beta\gamma}^1 + \left(\frac{b}{2a^2} - 1\right)\right] \delta_{\beta\gamma} c_s^2 \chi \frac{\partial D_{\gamma\theta}}{\partial x_\theta}\right\} + O(\Delta x^2)$$
$$= \Delta t \chi \frac{\partial}{\partial x_\beta} \left\{\left[S_{\beta\gamma}^1 + \left(\frac{b}{2a^2} - 1\right) \delta_{\beta\gamma}\right] c_s^2 \frac{\partial D_{\gamma\theta}}{\partial x_\theta}\right\} + O(\Delta x); \quad \text{(A4)}$$

then the first-order ME of the GPMRT-LB model and GPMFD scheme for the NACDE can be given by

$$\partial_t \phi + \partial_\alpha B_\alpha - \Delta t \frac{\partial}{\partial x_\beta} \left\{\chi c_s^2 \left[S_{\beta\gamma}^1 - \left(\frac{b}{2a^2} - 1\right) \delta_{\beta\gamma}\right] \frac{\partial D_{\beta\theta}}{\partial x_\theta}\right\} - R = O(\Delta x). \quad \text{(A5)}$$

Similarly, substituting the moment conditions (90) for NACDE (89) into Eqs. (A2) and (A3) yields

$$\left[c\mathcal{W}_0 \hat{\mathbf{S}}_N^{-1} \tilde{\mathbf{F}}\right]_1 = \partial_\alpha \left[c S_\alpha^{10} R + \left(S_{\alpha\beta}^1 - \delta_{\alpha\beta}/2\right) \partial_t B_\beta\right] + O(\Delta x) = c \partial_\alpha S_\alpha^{10} R + O(1),$$

$$\left[c\mathcal{W}_0 \hat{\mathbf{S}}_N^{-1} \partial_t \mathbf{m}^{\rm eq}\right]_1 = c \partial_\alpha S_\alpha^{10} \partial_t \phi + O(1), \left(c \partial_t \hat{\mathbf{S}}_N^{-1} \mathcal{W}_0 \mathbf{m}^{\rm eq}\right)_1 = O(1),$$

$$(c\mathcal{W}_0 \partial_t \mathbf{m}^{\rm eq})_1 = O(1),$$

$$\left[\Delta t^2 c^3 \mathcal{W}_0^3 \mathbf{m}^{\rm eq}\right]_1 = \partial_\alpha \partial_\beta \partial_\gamma \sum_{i=1}^q \mathbf{e}_{i\alpha} \mathbf{e}_{i\beta} \mathbf{e}_{i\gamma} \left\{w_i \left[\phi + \frac{\left[c_s^2 (D_{\eta\xi} - \phi \delta_{\eta\xi}) + C_{\eta\xi}\right] (\mathbf{c}_{i\eta} \mathbf{c}_{i\xi} - c_s^2 \delta_{\eta\xi})}{2c_s^4}\right] + O(\Delta x)\right\} \Delta t^2 c^3 = O(\Delta x^2),$$

$$\left[\Delta t^2 c^3 \mathcal{W}_0^2 \hat{\mathbf{S}}_N^{-1} \mathcal{W}_0 \mathbf{m}^{\rm eq}\right]_1 = \Delta t^2 \partial_\alpha \partial_\beta \left(c^2 S_{\alpha\beta}^{20} \partial_\gamma B_\gamma + S_{\alpha\beta\theta\zeta}^2 \partial_\gamma \Delta_{\theta\zeta\gamma\xi} B_\xi c_s^2\right) = O(\Delta x^2),$$

$$\left[\Delta t^2 c^3 \mathcal{W}_0 \hat{\mathbf{S}}_N^{-1} \mathcal{W}_0^2 \mathbf{m}^{\rm eq}\right]_1 = \Delta t^2 \partial_\alpha \left(c S_\alpha^{10} \partial_\beta \partial_\gamma \left(\chi c_s^2 D_{\beta\gamma} + C_{\beta\gamma}\right) + S_{\alpha\theta}^1 \partial_\beta \partial_\gamma \Delta_{\theta\beta\gamma\xi} B_\xi c_s^2\right)$$
$$= \Delta t^2 \chi \partial_\alpha \left(c S_\alpha^{10} c_s^2 \partial_\beta \partial_\gamma D_{\beta\gamma}\right) + O(\Delta x^2),$$

$$\left[\Delta t^2 c^3 \mathcal{W}_0 \hat{\mathbf{S}}_N^{-1} \mathcal{W}_0 \hat{\mathbf{S}}_N^{-1} \mathcal{W}_0 \mathbf{m}^{\rm eq}\right]_1 = \Delta t^2 \partial_\alpha \left[\left(c S_\alpha^{10} \partial_\beta S_\beta^{10} + S_{\alpha\theta}^1 \partial_\beta S_{\theta\beta}^{20}\right) B_\gamma\right.$$
$$\left. + \left(c S_\alpha^{10} \partial_\beta S_{\beta\eta}^1 + S_{\alpha\theta}^1 \partial_\beta S_{\theta\beta\eta}^{21}\right) \partial_\gamma \left(c_s^2 \chi D_{\eta\gamma} + C_{\eta\gamma}\right) + S_{\alpha\theta}^1 \partial_\beta S_{\theta\beta\zeta\mu}^2 c_s^2 \partial_\gamma \Delta_{\zeta\mu\gamma\delta} B_\delta\right]$$
$$= \Delta t^2 \chi \partial_\alpha \left[c S_\alpha^{10} \partial_\beta S_{\beta\eta}^1 c_s^2 \partial_\gamma D_{\eta\gamma}\right] + O(\Delta x^2); \quad \text{(A6)}$$

thus one can derive the following second-order ME of the GPMRT-LB model and GPMFD scheme for NACDE (89):

$$\partial_t\phi + \partial_\alpha B_\alpha - \Delta t\frac{\partial}{\partial x_\beta}\left\{\chi c_s^2\left[S^1_{\beta\gamma} - \left(\frac{b}{2a^2}-1\right)\delta_{\beta\gamma}\right]\frac{\partial D_{\beta\theta}}{\partial x_\theta}\right\} - R + \underbrace{\Delta t c\left\{\frac{\partial}{\partial x_\beta}\left[S^{10}_\beta\left(\frac{\partial\phi}{\partial t} + \frac{\partial B_\theta}{\partial x_\theta} - \frac{\partial}{\partial x_\theta}\left[\chi c_s^2\Delta t\left[S^1_{\theta\eta} + \left(\frac{b}{2a^2}-1\right)\delta_{\theta\eta}\right]\frac{\partial D_{\eta\gamma}}{\partial x_\gamma}\right] - R\right)\right]\right\}}_{O(\Delta x^2)} = O(\Delta x^2), \tag{A7}$$

where the first-order ME (A5) has been used.

2. NSEs: Derivation of Eqs. (99) and (101)

Substituting the moment conditions (97) for NSEs (95) into Eq. (A1), we have

$$[\partial_t\mathbf{m}^{\rm eq}]_1 = \partial_t\rho,\quad [c\mathcal{W}_0\mathbf{m}^{\rm eq}]_1 = \partial_\alpha(\rho u_\alpha),\quad \tilde{\mathbf{F}}_1 = 0,$$
$$\left[\Delta t c^2\mathcal{W}_0\left(\hat{\mathbf{S}}_N^{-1} + \left(\frac{b}{2a^2}-1\right)\mathbf{I}_q\right)\mathcal{W}_0\mathbf{m}^{\rm eq}\right]_1 = \Delta t\frac{\partial}{\partial x_\beta}\left(cS^{10}_\beta\frac{\partial(\rho u_\theta)}{\partial x_\theta} + \left[S^1_{\beta\gamma} + \left(\frac{b}{2a^2}-1\right)\delta_{\beta\gamma}\right]\frac{\partial\left(\rho u_\gamma u_\theta + \rho c_s^2\delta_{\gamma\theta}\right)}{\partial x_\theta}\right), \tag{A8}$$

and for any $\alpha \in \{1 \sim d\}$,

$$[c\mathcal{W}_0\mathbf{m}^{\rm eq}]_{\alpha+1} = \partial_\beta\left(\rho u_\alpha u_\beta + \rho c_s^2\delta_{\alpha\beta}\right)/c,\quad (\tilde{\mathbf{F}})_{\alpha+1} = O(\Delta x),$$
$$\left[\Delta t c^2\mathcal{W}_0\left(\hat{\mathbf{S}}_N^{-1} + \left(\frac{b}{2a^2}-1\right)\mathbf{I}_q\right)\mathcal{W}_0\mathbf{m}^{\rm eq}\right]_{\alpha+1}$$
$$= \Delta t\partial_\beta\left[cS^{20}_{\alpha\beta}\partial_\gamma(\rho u_\gamma) + S^{21}_{\alpha\beta\xi_1}\partial_\gamma\left(\rho u_{\xi_1}u_\gamma + \rho c_s^2\delta_{\xi_1\gamma}\right) + \frac{1}{c}\left(S^2_{\alpha\beta\xi_1\xi_2} + \left(\frac{b}{2a^2}-1\right)\delta_{\alpha\xi_1}\delta_{\beta\xi_2}\right)\partial_\gamma\left(\rho c_s^2\Delta_{\xi_1\xi_2\gamma\zeta}u_\zeta\right)\right]$$
$$= \Delta t\partial_\beta\left[S^{21}_{\alpha\beta\xi_1}\partial_\gamma\left(\rho c_s^2\delta_{\xi_1\gamma}\right)\right] + O(\Delta x) = O(\Delta x). \tag{A9}$$

From Eqs. (A8) and (A9), the first-order MEs of the GPMRT-LB model and GPMFD scheme on conservative moments $m_1 = \rho$ and $m_{\alpha+1} = (\rho u_\alpha)/c$ can be expressed as

$$[c\mathcal{W}_0\mathbf{m}^{\rm eq}]_1 = \partial_\beta(\rho u_\beta) = O(\Delta x), \tag{A10a}$$
$$[c\mathcal{W}_0\mathbf{m}^{\rm eq}]_{\alpha+1} = \frac{1}{c}\partial_\beta\left(\rho c_s^2\delta_{\alpha\beta}\right) + O(\Delta x) = O(\Delta x), \tag{A10b}$$

and this illustrates that at the diffusive scaling, $\nabla\rho = O(\Delta x^2)$, which will be used below. Subsequently, substituting the moment conditions (97) for NSEs (95) into Eqs. (A2) and (A3) yields

$$\left[\Delta t c\mathcal{W}_0\partial_t\mathbf{m}^{\rm eq}\right]_1 = \Delta t\partial_\alpha\partial_t(\rho u_\alpha) = O(\Delta x^2),$$
$$\left[\Delta t c\mathcal{W}_0\hat{\mathbf{S}}_N^{-1}\partial_t\mathbf{m}^{\rm eq}\right]_1 = \Delta t\left[c\partial_\alpha\left(S^{10}_\alpha\partial_t\rho\right] + \partial_\alpha\left[S^1_{\alpha\beta}\partial_t(\rho u_\beta)\right]\right] = c\partial_\alpha\left(S^{10}_\alpha\partial_t\rho\right] + O(\Delta x^2),$$
$$\left[\Delta t c\partial_t\hat{\mathbf{S}}_N^{-1}\mathcal{W}_0\mathbf{m}^{\rm eq}\right]_1 = \partial_t s_0\partial_\alpha(\rho u_\alpha) = O(\Delta x^2),$$
$$\left[\Delta t c\mathcal{W}_1\hat{\mathbf{S}}_N^{-1}\tilde{\mathbf{F}}\right]_1 = \Delta t\partial_\alpha\left[cS^{10}_\alpha\frac{\Delta t}{2}\partial_\gamma(\rho\hat{\mathbf{F}}_{x_\gamma}) + S^1_{\alpha\beta}(\rho\hat{\mathbf{F}}_{x_\beta}) + O(\Delta x)\right] = O(\Delta x^2),$$
$$\left[\Delta t^2c^3\mathcal{W}_0^3\mathbf{m}^{\rm eq}\right]_1 = \Delta t^2c^3\partial_\alpha\partial_\beta\partial_\gamma\sum_{i=1}^q\mathbf{e}_{i\alpha}\mathbf{e}_{i\beta}\mathbf{e}_{i\gamma}(w_i\rho + O(\Delta x)) = O(\Delta x^2),$$
$$\left[\Delta t^2c^3\mathcal{W}_0^2\hat{\mathbf{S}}_N^{-1}\mathcal{W}_0\mathbf{m}^{\rm eq}\right]_1 = \Delta t^2\partial_\alpha\partial_\beta\left[c^2S^{20}_{\alpha\beta}\partial_\gamma(\rho u_\gamma) + cS^{21}_{\alpha\beta\xi_1}\partial_\gamma\left(\rho u_{\xi_1}u_\gamma + c_s^2\rho\delta_{\xi_1\gamma}\right) + S^2_{\alpha\beta\xi_1\xi_2}\partial_\gamma\left(\rho c_s^2\Delta_{\xi_1\xi_2\gamma\zeta}u_\zeta\right)\right] = O(\Delta x^2),$$
$$\left[\Delta t^2c^3\mathcal{W}_0\hat{\mathbf{S}}_N^{-1}\mathcal{W}_0^2\mathbf{m}^{\rm eq}\right]_1 = \Delta t^2\partial_\alpha\sum_{k=1}^q\left(cS^{10}_\alpha + S^1_{\alpha\xi_1}\mathbf{c}_{k\xi_1}\right)\partial_\beta\mathbf{c}_{k\beta}\partial_\gamma\mathbf{c}_{k\gamma}f_k^{\rm eq}$$
$$= \Delta t^2\partial_\alpha\left[cS^{10}_\alpha\partial_\beta\partial_\gamma\left(\rho u_\beta u_\gamma + c_s^2\rho\delta_{\beta\gamma}\right) + S^1_{\alpha\xi_1}\partial_\beta\partial_\gamma\left(\rho c_s^2\Delta_{\xi_1\beta\gamma\zeta}u_\zeta\right)\right] = O(\Delta x^2),$$

$$\left[\Delta t^2 c^3 \mathcal{W}_0 \hat{\mathbf{S}}_N^{-1} \mathcal{W}_0 \hat{\mathbf{S}}_N^{-1} \mathcal{W}_0 \mathbf{m}^{\rm eq}\right]_1 = \Delta t^2 \partial_\alpha\big[\big(cS_\alpha^{10}\partial_\beta S_\beta^{10} + S^1_{\alpha\xi_1}\partial_\beta S^{20}_{\xi_1\beta}\big)\partial_\gamma(\rho u_\gamma) + \big(cS_\alpha^{10}\partial_\beta S^1_{\beta\zeta_1} + S^1_{\alpha\xi_1}\partial_\beta S^{21}_{\xi_1\beta\zeta_1}\big)\partial_\gamma\big(\rho u_{\zeta_1}u_\gamma + c_s^2\rho\delta_{\zeta_1\gamma}\big) + S^1_{\alpha\xi_1}\partial_\beta S^2_{\xi_1\beta\zeta_1\zeta_2}\partial_\gamma\big(c_s^2\rho\Delta_{\zeta_1\zeta_2\gamma\zeta}u_\zeta\big)\big] = O(\Delta x^2). \tag{A11}$$

Additionally, one can also derive the following second-order ME of the GPMRT-LB model and GPMFD scheme for the continuous equation (95a):

$$\partial_t\rho + \partial_\alpha(\rho u_\alpha) + \underbrace{\Delta t c\left\{\frac{\partial}{\partial x_\beta}\left[S_\beta^{10}\left(\frac{\partial\rho}{\partial t} + \frac{\partial(\rho u_\gamma)}{\partial x_\gamma}\right)\right]\right\}}_{O(\Delta x^2)} = O(\Delta x^2), \tag{A12}$$

where the first-order ME (A10a) has been applied.

For the GPMRT-LB model and the GPMFD scheme for the momentum equation (95b), one can obtain

$$[-\partial_t\mathbf{m}^{\rm eq}]_{\alpha+1} = -\frac{1}{c}\partial_t(\rho u_\alpha),\quad [-c\mathcal{W}_0\mathbf{m}^{\rm eq}]_{\alpha+1} = -\frac{1}{c}\partial_\beta\big(\rho u_\alpha u_\beta + \rho c_s^2\delta_{\alpha\beta}\big),$$

$$[\tilde{\mathbf{F}}]_{\alpha+1} = \hat{F}_{x_\alpha},\quad \left[\Delta t c^2\mathcal{W}_0^2\mathbf{m}^{\rm eq}\right]_{\alpha+1} = -\Delta t\frac{1}{c}\partial_\beta\partial_\gamma\big(\rho c_s^2\Delta_{\alpha\beta\gamma\zeta}u_\zeta\big),$$

$$\left[\Delta t c\mathcal{W}_0\hat{\mathbf{S}}_N^{-1}\partial_t\mathbf{m}^{\rm eq}\right]_{\alpha+1} = \Delta t c\sum_{k=1}^{q}\left[\sum_{j=0}^{q-1}\mathbf{e}_{j\alpha}\mathbf{e}_{j\beta}\partial_\beta\overline{\mathbf{\Lambda}}_{jk}\right]\partial_t f_k^{\rm eq} = \Delta t\partial_\beta\left[cS^{20}_{\alpha\beta}\partial_t\rho + S^{21}_{\alpha\beta\xi_1}\partial_t\big(\rho u_{\xi_1}\big) + \frac{1}{c}S^2_{\alpha\beta\xi_1\xi_2}\partial_t\big(\rho c_s^2\delta_{\xi_1\xi_2}\big)\right],$$

$$\left[\Delta t c^2\mathcal{W}_0\hat{\mathbf{S}}_N^{-1}\mathcal{W}_0\mathbf{m}^{\rm eq}\right]_{\alpha+1} = \Delta t c^2\sum_{k=1}^{q}\left[\sum_{j=1}^{q}\mathbf{e}_{j\alpha}\mathbf{e}_{j\beta}\partial_\beta\overline{\mathbf{\Lambda}}_{jk}\right]\mathbf{e}_{k\gamma}\partial_\gamma f_k^{\rm eq}$$
$$= \Delta t\partial_\beta\left[cS^{20}_{\alpha\beta}\partial_\gamma(\rho u_\gamma) + S^{21}_{\alpha\beta\xi_1}\partial_\gamma\big(\rho u_{\xi_1}u_\gamma + \rho c_s^2\delta_{\xi_1\gamma}\big) + \frac{1}{c}S^2_{\alpha\beta\xi_1\xi_2}\partial_\gamma\big(\rho c_s^2\Delta_{\xi_1\xi_2\gamma\zeta}u_\zeta\big)\right], \tag{A13}$$

and

$$\left[-\Delta t c\mathcal{W}_0\partial_t\mathbf{m}^{\rm eq}\right]_{\alpha+1} = -\Delta t\frac{1}{c}\partial_\beta\partial_t\big(\rho u_\alpha u_\beta + \rho c_s^2\delta_{\alpha\beta}\big) = -\Delta t\frac{1}{c}\partial_\beta\partial_t\big(\rho c_s^2\delta_{\alpha\beta}\big) + O(\Delta x^3),$$

$$\left[-\Delta t c\mathcal{W}_0\hat{\mathbf{S}}_N^{-1}\tilde{\mathbf{F}}\right]_{\alpha+1} = -c\Delta t\sum_{k=1}^{q}\left[\sum_{j=1}^{q}\mathbf{e}_{j\alpha}\mathbf{e}_{j\beta}\partial_\beta\overline{\mathbf{\Lambda}}_{jk}\right]\left(F_k + \frac{\Delta t}{2}D_kF_k\right)$$
$$= -c\Delta t\sum_{k=1}^{q}\partial_\beta\big[S^{20}_{\alpha\beta} + S^{21}_{\alpha\beta\xi_1}\mathbf{e}_{k\xi_1} + S^2_{\alpha\beta\xi_1\xi_2}\mathbf{e}_{k\xi_1}\mathbf{e}_{k\xi_2}\big]\left(F_k + \frac{\Delta t}{2}D_kF_k\right)$$
$$= -\Delta t\partial_\beta\big[S^{21}_{\alpha\beta\xi_1}\big(\rho\hat{F}_{x_\alpha}\big)\big] + O(\Delta x^3),$$

$$\left[\Delta t c\partial_t\hat{\mathbf{S}}_N^{-1}\mathcal{W}_0\mathbf{m}^{\rm eq}\right]_{\alpha+1} = \Delta t c\partial_t\sum_{k=0}^{q-1}\left[\sum_{j=0}^{q-1}\mathbf{e}_{j\alpha}\overline{\mathbf{\Lambda}}_{jk}\right]\mathbf{e}_{k\beta}\partial_\beta f_k^{\rm eq}$$
$$= \Delta t\partial_t S^{10}_\alpha\partial_\beta(\rho u_\beta) + \Delta t\frac{1}{c}\partial_t S^1_{\alpha\xi_1}\partial_\beta\big(\rho u_{\xi_1}u_\beta + \rho c_s^2\delta_{\xi_1\beta}\big) = \Delta t\partial_t\big[S^{10}_\alpha\partial_\beta(\rho u_\beta)\big] + O(\Delta x^3),$$

$$\left[\Delta t^2c^3\mathcal{W}_0^2\hat{\mathbf{S}}_N^{-1}\mathcal{W}_0\mathbf{m}^{\rm eq}\right]_{\alpha+1} = \Delta t^2c^3\sum_{k=1}^{q}\left[\sum_{j=1}^{q}\mathbf{e}_{j\alpha}\mathbf{e}_{j\beta}\partial_\beta\mathbf{e}_{j\gamma}\partial_\gamma\overline{\mathbf{\Lambda}}_{jk}\right]\mathbf{e}_{k\zeta}\partial_{k\zeta}f_k^{\rm eq}$$
$$= \Delta t^2\partial_\beta\partial_\gamma\big[c^2S^{30}_{\alpha\beta\gamma}\partial_\zeta(\rho u_\zeta)\big] + O(\Delta x^3),$$

$$\left[\Delta t^2c^3\mathcal{W}_0\hat{\mathbf{S}}_N^{-1}\mathcal{W}_0^2\mathbf{m}^{\rm eq}\right]_{\alpha+1} = \Delta t^2c^3\sum_{k=1}^{q}\left[\sum_{j=1}^{q}\mathbf{e}_{j\alpha}\mathbf{e}_{j\beta}\partial_\beta\overline{\mathbf{\Lambda}}_{jk}\right]\mathbf{e}_{k\gamma}\mathbf{e}_{k\zeta}\partial_\gamma\partial_\zeta f_k^{\rm eq}$$
$$= \Delta t^2\partial_\beta\big[cS^{20}_{\alpha\beta}\partial_\gamma\partial_\zeta\times O(1) + S^{21}_{\alpha\beta\xi_1}\partial_\gamma\partial_\zeta\big(\rho c_s^2\Delta_{\xi_1\gamma\zeta\chi}u_\chi\big) + \frac{1}{c}S^2_{\alpha\beta\xi_1\xi_2}\partial_\gamma\partial_\zeta$$
$$\times\big[\rho c_s^3\Delta_{\xi_1\xi_2\gamma\zeta} + O(1/\Delta x^2)\big]\big] = \Delta t^2\partial_\beta\big[S^{21}_{\alpha\beta\xi_1}\partial_\gamma\partial_\zeta\big(\rho c_s^2\Delta_{\xi_1\gamma\zeta\chi}u_\chi\big)\big] + O(\Delta x^3),$$

$$\begin{aligned}\left[-\Delta t^2c^3\mathcal{W}_0\hat{\mathbf{S}}_N^{-1}\mathcal{W}_0\hat{\mathbf{S}}_N^{-1}\mathcal{W}_0\mathbf{m}^{\mathrm{eq}}\right]_{\alpha+1} &= -\sum_{i=1}^{q}\left[\sum_{k=1}^{q}\left[\sum_{j=1}^{q}\mathbf{e}_{j\alpha}\mathbf{e}_{j\beta}\partial_\beta\overline{\mathbf{\Lambda}}_{jk}\right]\mathbf{e}_{k\gamma}\partial_\gamma\overline{\mathbf{\Lambda}}_{ki}\right]\mathbf{e}_{i\theta}\partial_\theta f_i^{\mathrm{eq}}\Delta t^2c^3\\ &= -\Delta t^2\partial_\beta\big[S_{\alpha\beta}^{20}\partial_\gamma\big(c^2S_\gamma^{10}\partial_\theta(\rho u_\theta)\big)+S_{\alpha\beta\xi_1}^{21}\partial_\gamma\big(c^2S_{\xi_1\gamma}^{20}\partial_\theta(\rho u_\theta)\\ &\quad+S_{\xi_1\gamma\zeta_1\zeta_2}^{2}\partial_\theta\big(\rho c_s^2\Delta_{\zeta_1\zeta_2\theta\eta}u_\eta\big)\big)+S_{\alpha\beta\xi_1\xi_2}^{2}\partial_\gamma\big(c^2S_{\xi_1\xi_2\gamma}^{30}\partial_\theta(\rho u_\theta)\big)\big]+O(\Delta x^3),\end{aligned} \tag{A14}$$

where $\alpha\in\{1\sim d\}$. Together with Eqs. (A13) and (A14), the second-order ME of GPMRT-LB model and GPMFD scheme on conservative moment $m_{\alpha+1}=(\rho u_\alpha)/c$ can be obtained,

$$\begin{aligned}\left[(\partial_t\mathbf{I}_q+c\mathcal{W}_0)\mathbf{m}^{\mathrm{eq}}\right]_{\alpha+1} &= \left[\tilde{\mathbf{F}}+\Delta tc\mathcal{W}_0\left(\hat{\mathbf{S}}_N^{-1}+\left(\frac{b}{2a^2}-1\right)\mathbf{I}_q\right)c\mathcal{W}_0\mathbf{m}^{\mathrm{eq}}+\Delta tc\mathcal{W}_0\partial_t\hat{\mathbf{S}}_N^{-1}\mathbf{m}^{\mathrm{eq}}\right]_{\alpha+1}\\ &= \Big[\partial_t(\rho u_\alpha)+\partial_\beta\big(\rho u_\alpha u_\beta+\rho c_s^2\delta_{\alpha\beta}\big)-\Delta t\partial_\beta S_{\alpha\beta\xi_1\xi_2}^{2}\partial_t\big(\rho c_s^2\delta_{\xi_1\xi_2}\big)\\ &\quad-\rho\hat{F}_{x_\alpha}-\Delta t\partial_\beta\left[S_{\alpha\beta\xi_1\xi_1}^{2}+\left(\frac{b}{2a^2}-1\right)\delta_{\alpha\xi_1}\delta_{\beta\xi_2}\right]\partial_\gamma\big(\rho c_s^2\Delta_{\xi_1\xi_2\gamma\zeta}u_\zeta\big)\Big]\frac{1}{c}=O(\Delta x^2),\end{aligned} \tag{A15}$$

where the second-order ME of the GPMRT-LB model and GPMFD scheme for the continuous equation (A12) has been used. Actually, at the diffusive scaling, $c=O(1/\Delta x)$, and above ME of the GPMRT-LB model and GPMFD scheme for the momentum equation (95b) is first-order accurate. This also explains why we further conduct the expansion of $\mathbf{\Xi}$ up to $\mathbf{\Xi}^{(4)}$ in Eq. (68). To obtain a second-order ME of the GPMRT-LB model and GPMFD scheme for the momentum equation (95b), it is necessary to compute Eq. (80), and after some manipulations, one can derive

$$\begin{aligned}&\left[-\frac{\Delta t}{2}\partial_{tt}\mathbf{m}^{\mathrm{eq}}\right]_{\alpha+1} = -\frac{1}{c}\frac{\Delta t}{2}\partial_{tt}(\rho u_\alpha),\\ &\left[\Delta t\partial_t\hat{\mathbf{S}}_N^{-1}\partial_t\mathbf{m}^{\mathrm{eq}}\right]_{\alpha+1} = \Delta t\partial_t\sum_{k=0}^{q-1}\left[\sum_{j=0}^{q-1}\mathbf{e}_{j\alpha}\overline{\mathbf{\Lambda}}_{jk}\right]\partial_t f_k^{\mathrm{eq}} = \Delta t\partial_t\sum_{k=0}^{q-1}\left[S_\alpha^{10}+S_{\alpha\xi_1}^{1}\mathbf{e}_{k\xi_1}\right]\partial_t f_k^{\mathrm{eq}}=\Delta t\partial_t\left[S_\alpha^{10}\partial_t\rho\right]+O(\Delta x^3),\\ &\left[\Delta t^2c^2\mathcal{W}_0^2\hat{\mathbf{S}}_N^{-1}\partial_t\mathbf{m}^{\mathrm{eq}}\right]_{\alpha+1} = \Delta t^2c^2\sum_{k=1}^{q}\left[\sum_{j=1}^{q}\mathbf{e}_{j\alpha}\mathbf{e}_{j\beta}\partial_\beta\mathbf{e}_{j\gamma}\partial_\gamma\overline{\mathbf{\Lambda}}_{jk}\right]\partial_t f_k^{\mathrm{eq}} = \Delta t^2c^2\partial_\beta\partial_\gamma S_{\alpha\beta\gamma}^{30}\partial_t\rho+O(\Delta x^3),\\ &\left[\Delta t^2c^2\mathcal{W}_0\hat{\mathbf{S}}_N^{-1}\mathcal{W}_0\partial_t\mathbf{m}^{\mathrm{eq}}\right]_{\alpha+1} = \Delta t^2c^2\sum_{k=1}^{q}\left[\sum_{j=1}^{q}\mathbf{e}_{j\alpha}\mathbf{e}_{j\beta}\partial_\beta\overline{\mathbf{\Lambda}}_{jk}\right]\mathbf{e}_{k\gamma}\partial_\gamma\partial_t f_k^{\mathrm{eq}}\\ &\quad= \Delta t^2\partial_\beta\left[cS_{\alpha\beta}^{20}\partial_\gamma\partial_t\times O(1)+S_{\alpha\beta\xi_1}^{21}\partial_\gamma\partial_t\big(\rho c_s^2\delta_{\xi_1\gamma}\big)+\frac{1}{c}S_{\alpha\beta\xi_1\xi_2}^{2}\partial_\gamma\partial_t\times O(1/\Delta x^2)\right]\\ &\quad= \Delta t^2\partial_\beta\left[S_{\alpha\beta\xi_1}^{21}\partial_\gamma\partial_t\big(\rho c_s^2\delta_{\xi_1\gamma}\big)\right]+O(\Delta x^3),\\ &\left[-\Delta t^2c^2\mathcal{W}_0\hat{\mathbf{S}}_N^{-1}\mathcal{W}_0\hat{\mathbf{S}}_N^{-1}\partial_t\mathbf{m}^{\mathrm{eq}}\right]_{\alpha+1} = \sum_{i=1}^{q}\left[\sum_{k=1}^{q}\left[\sum_{j=1}^{q}\mathbf{e}_{j\alpha}\mathbf{e}_{j\beta}\partial_\beta\overline{\mathbf{\Lambda}}_{jk}\right]\mathbf{e}_{k\gamma}\partial_\gamma\overline{\mathbf{\Lambda}}_{ki}\partial_t f_i^{\mathrm{eq}}\right.\\ &\quad= -\Delta t^2\partial_\beta\big[S_{\alpha\beta}^{20}\partial_\gamma\big(c^2S_\gamma^{10}\partial_t\rho\big)+S_{\alpha\beta\xi_1}^{21}\partial_\gamma\big(c^2S_{\xi_1\gamma}^{20}\partial_t\rho\big)+S_{\alpha\beta\xi_1}^{21}\partial_\gamma\big[S_{\xi_1\gamma\zeta_1\zeta_2}^{2}\partial_t\big(\rho c_s^2\delta_{\zeta_1\zeta_2}\big)\big]\\ &\qquad+S_{\alpha\beta\xi_1\xi_2}^{2}\partial_\gamma\big(c^2S_{\xi_1\xi_2\gamma}^{30}\partial_t\rho\big)\big]+O(\Delta x^3),\end{aligned} \tag{A16}$$

$$\left[\Delta t^2c^3\mathcal{W}_0^3\mathbf{m}^{\mathrm{eq}}\right]_{\alpha+1} = \frac{1}{c}\Delta t^2\partial_\beta\partial_\gamma\partial_\zeta\sum_{i=1}^{q}\mathbf{c}_{i\alpha}\mathbf{c}_{i\beta}\mathbf{c}_{i\gamma}\mathbf{c}_{i\zeta}\left[w_i\rho+\mathbf{c}_{i\eta}\partial_\eta+O(\Delta x^2)\right]=O(\Delta x^3), \tag{A17}$$

$$\left[\Delta t^3c^4\mathcal{W}_0^4\mathbf{m}^{\mathrm{eq}}\right]_{\alpha+1} = \frac{1}{c}\Delta t^3\partial_\beta\partial_\gamma\partial_\zeta\partial_\eta\sum_{i=1}^{q}\mathbf{c}_{i\alpha}\mathbf{c}_{i\beta}\mathbf{c}_{i\gamma}\mathbf{c}_{i\zeta}\mathbf{c}_{i\eta}\left[w_i\rho+O(\Delta x)\right]=O(\Delta x^3), \tag{A18}$$

$$\left[-\Delta t^2c^2\mathcal{W}_0^2\partial_t\mathbf{m}^{\mathrm{eq}}\right]_{\alpha+1} = -\Delta t^2\frac{1}{c}\partial_\beta\partial_\gamma\partial_t\big(\rho c_s^2\Delta_{\alpha\beta\gamma\zeta}u_\zeta\big)=O(\Delta x^3), \tag{A19}$$

$$\left[-\Delta t\partial_t\hat{\mathbf{S}}_N^{-1}\tilde{\mathbf{F}}\right]_{\alpha+1} = -\Delta t\sum_{k=1}^{q}\left[\sum_{j=1}^{q}\mathbf{e}_{j\alpha}\partial_t\overline{\mathbf{\Lambda}}_{jk}\right]\left(F_k+\frac{\Delta t}{2}D_kF_k\right)=O(\Delta x^3), \tag{A20}$$

$$\big[-\Delta t^2c^2\mathcal{W}_0^2\hat{\mathbf{S}}_N^{-1}\tilde{\mathbf{F}}\big]_{\alpha+1} = -\Delta t^2c^2\sum_{k=1}^{q}\partial_\beta\partial_\gamma\left[\sum_{j=1}^{q}\mathbf{e}_{j\alpha}\mathbf{e}_{j\beta}\mathbf{e}_{j\gamma}\overline{\mathbf{\Lambda}}_{jk}\right]\left(F_k+\frac{\Delta t}{2}D_kF_k\right)$$
$$= -c\Delta t^2\sum_{k=1}^{q}\partial_\beta\partial_\gamma\big[S^{30}_{\alpha\beta\gamma}+S^{31}_{\alpha\beta\gamma\xi_1}\mathbf{e}_{k\xi_1}+S^{33}_{\alpha\beta\gamma\xi_1\xi_2\xi_3}\mathbf{e}_{k\xi_1}\mathbf{e}_{k\xi_2}\mathbf{e}_{k\xi_3}\big]\left(F_k+\frac{\Delta t}{2}D_kF_k\right) = O(\Delta x^3), \quad \text{(A21)}$$

$$\big[-\Delta t^2c^2\mathcal{W}_0\hat{\mathbf{S}}_N^{-1}\mathcal{W}_0\hat{\mathbf{S}}_N^{-1}\tilde{\mathbf{F}}\big]_{\alpha+1} = -\Delta t^2c^2\sum_{i=1}^{q}\left[\sum_{k=1}^{q}\partial_\beta\left[\sum_{j=1}^{q}\mathbf{e}_{j\alpha}\mathbf{e}_{j\beta}\overline{\mathbf{\Lambda}}_{jk}\right]\mathbf{e}_{j\gamma}\partial_\gamma\overline{\mathbf{\Lambda}}_{ki}\right]\left(F_i+\frac{\Delta t}{2}D_iF_i\right)$$
$$= -\Delta t^2\partial_\beta\left[S^{20}_{\alpha\beta}\partial_\gamma S^{1}_{\gamma\zeta_1}\big(\rho\hat{F}_{x_{\zeta_1}}\big)+S^{21}_{\alpha\beta\xi_1}\partial_\gamma\left(S^{21}_{\xi_1\gamma\zeta_1}\big(\rho\hat{F}_{x_{\zeta_1}}\big)+\frac{1}{c}S^{2}_{\xi_1\gamma\zeta_1\zeta_2}\times O(1)\right)\right.$$
$$\left.+S^{2}_{\alpha\beta\xi_1\xi_2}\partial_\gamma\left(S^{31}_{\xi_1\xi_2\gamma\zeta_1}\big(\rho\hat{F}_{x_{\zeta_1}}\big)+\frac{1}{c^2}S^{33}_{\xi_1\xi_2\gamma\zeta_1\zeta_2\zeta_3}\times\big(\rho c_s^2\Delta_{\zeta\zeta_2\zeta_3\eta}\hat{F}_{x_\eta}\big)\right)\right]+O(\Delta x^6) = O(\Delta x^4), \quad \text{(A22)}$$

$$\big[\Delta t^2c^2\mathcal{W}_0\partial_t\hat{\mathbf{S}}_N^{-1}\mathcal{W}_0\mathbf{m}^{\text{eq}}\big]_{\alpha+1} = \Delta t^2c^2\sum_{k=1}^{q}\left[\sum_{j=1}^{q}\mathbf{e}_{j\alpha}\mathbf{e}_{j\beta}\partial_\beta\partial_t\overline{\mathbf{\Lambda}}_{jk}\right]\mathbf{e}_{k\gamma}\partial_\gamma f_k^{\text{eq}}$$
$$= \Delta t^2\partial_\beta\partial_t\left[cS^{20}_{\alpha\beta}\partial_\gamma\times O(1)+S^{21}_{\alpha\beta\xi_1}\partial_\gamma\times O(1)+\frac{1}{c}S^{2}_{\alpha\beta\xi_1\xi_2}\partial_\gamma\times O(1/\Delta x^2)\right] = O(\Delta x^3), \quad \text{(A23)}$$

$$\big[\Delta t^2c^2\partial_t\hat{\mathbf{S}}_N^{-1}\mathcal{W}_0^2\mathbf{m}^{\text{eq}}\big]_{\alpha+1} = \Delta t^2c^2\partial_t\sum_{k=1}^{q}\left[\sum_{j=0}^{q-1}\mathbf{e}_{j\alpha}\overline{\mathbf{\Lambda}}^{-1}_{jk}\right]\mathbf{e}_{k\beta}\partial_\beta\mathbf{e}_{k\gamma}\partial_\gamma f_k^{\text{eq}}$$
$$= \Delta t^2\partial_t\sum_{k=1}^{q}\left[S^{10}_{\alpha}\partial_\beta\partial_\gamma\times O(1)+\frac{1}{c}S^{1}_{\alpha\xi_1}\partial_\beta\partial_\gamma\times O(1/\Delta x^2)\right] = O(\Delta x^3), \quad \text{(A24)}$$

$$\big[\Delta t^3c^4\mathcal{W}_0^2\hat{\mathbf{S}}_N^{-1}\mathcal{W}_0^2\mathbf{m}^{\text{eq}}\big]_{\alpha+1} = \Delta t^3c^4\sum_{i=1}^{q}\left[\sum_{j=1}^{q}\mathbf{e}_{j\alpha}\mathbf{e}_{j\beta}\mathbf{e}_{j\gamma}\partial_\beta\partial_\gamma\overline{\mathbf{\Lambda}}_{jk}\right]\partial_\eta\mathbf{e}_{k\eta}\partial_\zeta\mathbf{e}_{k\zeta}f_k^{\text{eq}}$$
$$= \Delta t^3\sum_{i=1}^{q}\left[c^2S^{30}_{\alpha\beta\gamma}\partial_\eta\partial_\zeta\times O(1)+cS^{31}_{\alpha\beta\gamma\xi_1}\partial_\eta\partial_\zeta\times O(1/\Delta x^2)+\frac{1}{c}S^{33}_{\alpha\beta\gamma\xi_1\xi_2\xi_3}\partial_\eta\partial_\zeta\times O(1/\Delta x^4)\right]=O(\Delta x^3), \quad \text{(A25)}$$

$$\big[-\Delta t^3c^4\mathcal{W}_0\hat{\mathbf{S}}_N^{-1}\mathcal{W}_0\hat{\mathbf{S}}_N^{-1}\mathcal{W}_0^2\mathbf{m}^{\text{eq}}\big]_{\alpha+1} = -\Delta t^3c^4\sum_{i=1}^{q}\left[\sum_{k=1}^{q}\left[\sum_{j=1}^{q}\mathbf{e}_{j\alpha}\mathbf{e}_{j\beta}\partial_\beta\overline{\mathbf{\Lambda}}_{jk}\right]\mathbf{e}_{k\gamma}\partial_\gamma\overline{\mathbf{\Lambda}}_{ki}\right]\mathbf{e}_{i\zeta}\partial_\zeta\mathbf{e}_{i\eta}\partial_\eta f_i^{\text{eq}}$$
$$= -\Delta t^3\partial_\beta\left[S^{20}_{\alpha\beta}\partial_\gamma\big(c^2S^{10}_{\gamma}\partial_\zeta\partial_\eta\times O(1)+cS^{1}_{\gamma\zeta_1}\partial_\zeta\partial_\eta\times O(1/\Delta x^2)\big)\right.$$
$$+S^{21}_{\alpha\beta\xi_1}\partial_\gamma\big(c^2S^{20}_{\xi_1\gamma}\partial_\zeta\partial_\eta\times O(1)+cS^{21}_{\xi_1\gamma\zeta_1}\partial_\zeta\partial_\eta\times O(1/\Delta x^2)$$
$$+S^{2}_{\xi_1\gamma\zeta_1\zeta_2}\partial_\zeta\partial_\eta\big[\rho c_s^4\Delta_{\zeta_1\zeta_2\zeta\eta}+O(1/\Delta x^2)\big]\big)+S^{2}_{\alpha\beta\xi_1\xi_2}\partial_\gamma\left(c^2S^{30}_{\xi_1\xi_2\gamma}\partial_\zeta\partial_\eta\times O(1)\right.$$
$$\left.\left.+cS^{31}_{\xi_1\xi_2\gamma\zeta_1}\partial_\zeta\partial_\eta\times O(1/\Delta x^2)+\frac{1}{c}S^{33}_{\xi_1\xi_2\gamma\zeta_1\zeta_2\zeta_3}\partial_\zeta\partial_\eta\times O(1/\Delta x^4)\right)\right] = O(\Delta x^3), \quad \text{(A26)}$$

$$\big[-\Delta t^2c^2\mathcal{W}_0\partial_t\hat{\mathbf{S}}_N^{-1}\hat{\mathbf{S}}_N^{-1}\mathcal{W}_0\mathbf{m}^{\text{eq}}\big]_{\alpha+1} = -\Delta t^2c^2\sum_{i=1}^{q}\left[\sum_{k=0}^{q-1}\left[\sum_{j=1}^{q}\mathbf{e}_{j\alpha}\mathbf{e}_{j\beta}\partial_\beta\partial_t\overline{\mathbf{\Lambda}}_{jk}\right]\overline{\mathbf{\Lambda}}_{ki}\right]\mathbf{e}_{i\gamma}\partial_\gamma f_i^{\text{eq}}$$
$$= -\Delta t^2\partial_\beta\partial_t\left[S^{20}_{\alpha\beta}cs_0\partial_\gamma\times O(1)+S^{21}_{\alpha\beta\xi_1}\big(cS^{10}_{\xi_1}\partial_\gamma\times O(1)+S^{1}_{\xi_1\zeta_1}\partial_\gamma\times O(1)\big)\right.$$
$$\left.+S^{2}_{\alpha\beta\xi_1\xi_2}\left(cS^{20}_{\xi_1\xi_2}\mathbf{c}_{i\gamma}\partial_\gamma+S^{21}_{\xi_1\xi_2\zeta_1}\partial_\gamma\times O(1)+\frac{1}{c}S^{2}_{\xi_1\xi_2\zeta_1\zeta_2}\partial_\gamma\times O(1/\Delta x^2)\right)\right] = O(\Delta x^3), \quad \text{(A27)}$$

$$
\begin{aligned}
\left[-\Delta t^3 c^4 \mathcal{W}_0 \hat{\mathbf{S}}_N^{-1} \mathcal{W}_0^2 \hat{\mathbf{S}}_N^{-1} \mathcal{W}_0 \mathbf{m}^{\mathrm{eq}}\right]_{\alpha+1} &= -\Delta t^3 c^4 \sum_{i=1}^{q} \left[\sum_{k=1}^{q} \left[\sum_{j=0}^{q-1} \mathbf{e}_{j\alpha} \mathbf{e}_{j\beta} \partial_\beta \overline{\boldsymbol{\Lambda}}_{jk} \right] \mathbf{e}_{k\zeta} \partial_\zeta \mathbf{e}_{k\eta} \partial_\eta \overline{\boldsymbol{\Lambda}}_{ki} \right] \mathbf{e}_{i\theta} \partial_\theta f_i^{\mathrm{eq}} \\
&= -\Delta t^3 \partial_\beta \Big[S^{20}_{\alpha\beta} \partial_\zeta \partial_\eta \big(c^3 S^{20}_{\zeta\eta} \partial_\theta \times O(1) + c^2 S^{21}_{\zeta\eta\zeta_1} \partial_\theta \times O(1) + c S^{2}_{\zeta\eta\zeta_1\zeta_2} \partial_\theta \times O(1/\Delta x^2) \big) \\
&\quad + S^{21}_{\alpha\beta\xi_1} \partial_\zeta \partial_\eta \big(c^3 S^{30}_{\xi_1\zeta\eta} \partial_\theta \times O(1) + c^2 S^{31}_{\xi_1\zeta\eta\zeta_1} \partial_\theta \times O(1) \\
&\quad + S^{33}_{\xi_1\zeta\eta\zeta_1\zeta_3} \partial_\theta \big[\rho c_s^4 \Delta_{\zeta_1\zeta_2\zeta_3\theta} + O(1/\Delta x^2) \big] \big) + S^{2}_{\alpha\beta\xi_1\xi_2} \partial_\zeta \partial_\eta \Big(c^3 S^{40}_{\xi_1\xi_2\zeta\eta} \partial_\theta \times O(1) \\
&\quad + c^2 S^{31}_{\xi_1\zeta\eta\zeta_1} \partial_\theta \times O(1) + c S^{42}_{\xi_1\xi_2\zeta\eta\zeta_1\zeta_2} \partial_\theta \times O(1/\Delta x^2) + S^{43}_{\xi_1\xi_2\zeta\eta\zeta_1\zeta_3} \partial_\theta \\
&\quad \times \big(\rho c_s^4 \Delta_{\zeta_1\zeta_2\zeta_3\theta} + O(1/\Delta x^2) \big) + \frac{1}{c} S^{44}_{\xi_1\xi_2\zeta\eta\zeta_1\zeta_3} \partial_\theta \times O(1/\Delta x^4) \Big) \Big] = O(\Delta x^3), \qquad \text{(A28)}
\end{aligned}
$$

$$
\begin{aligned}
\left[-\Delta t^2 c^2 \partial_t \hat{\mathbf{S}}_N^{-1} \mathcal{W}_0 \hat{\mathbf{S}}_N^{-1} \mathcal{W}_0 \mathbf{m}^{\mathrm{eq}}\right]_{\alpha+1} &= -\Delta t^2 c^2 \partial_t \sum_{i=1}^{q} \left[\sum_{k=1}^{q} \left[\sum_{j=1}^{q} \mathbf{e}_{j\alpha} \overline{\boldsymbol{\Lambda}}_{jk} \right] \mathbf{e}_{k\beta} \partial_\beta \overline{\boldsymbol{\Lambda}}_{ki} \right] \mathbf{e}_{i\gamma} \partial_\gamma f_i^{\mathrm{eq}} \\
&= -\Delta t^2 \partial_t \Big[S^{10}_{\alpha} \partial_\beta \big(c S^{10}_{\gamma} \partial_\gamma \times O(1) + S^{1}_{\gamma\zeta_1} \partial_\gamma \times O(1) \big) + S^{1}_{\alpha\xi_1} \partial_\beta \Big(c S^{20}_{\xi_1\beta} \partial_\gamma \times O(1) \\
&\quad + S^{21}_{\xi_1\beta\zeta_1} \partial_\gamma \times O(1) + \frac{1}{c} S^{2}_{\xi_1\beta\zeta_1\zeta_2} \partial_\gamma \times O(1/\Delta x^2) \Big) \Big] = O(\Delta x^3), \qquad \text{(A29)}
\end{aligned}
$$

$$
\begin{aligned}
\left[-\Delta t^3 c^4 \mathcal{W}_0^2 \hat{\mathbf{S}}_N^{-1} \mathcal{W}_0 \hat{\mathbf{S}}_N^{-1} \mathcal{W}_0 \mathbf{m}^{\mathrm{eq}}\right]_{\alpha+1} &= -\Delta t^3 c^4 \sum_{i=1}^{q} \left[\sum_{k=1}^{q} \left[\sum_{j=1}^{q} \mathbf{e}_{j\alpha} \mathbf{e}_{j\beta} \partial_\beta \mathbf{e}_{j\gamma} \partial_\gamma \overline{\boldsymbol{\Lambda}}_{jk} \right] \mathbf{e}_{k\zeta} \partial_\zeta \overline{\boldsymbol{\Lambda}}_{ki} \right] \mathbf{e}_{i\eta} \partial_\eta f_i^{\mathrm{eq}} \\
&= -\Delta t^3 \partial_\gamma \partial_\beta \Big[S^{30}_{\alpha\beta\gamma} \partial_\zeta \big(c^3 S^{10}_{\zeta} \partial_\eta \times O(1) + c^2 S^{31}_{\zeta\zeta_1} \partial_\eta \times O(1) \big) \\
&\quad + S^{31}_{\alpha\beta\gamma\xi_1} \partial_\zeta \big(c^3 S^{20}_{\xi_1\zeta} \partial_\eta \times O(1) + c^2 S^{21}_{\xi_1\zeta\zeta_1} \partial_\eta \times O(1) + c S^{2}_{\xi_1\zeta\zeta_1\zeta_2} \partial_\eta \times O(1/\Delta x^2) \big) \\
&\quad + S^{33}_{\alpha\beta\gamma\xi_1\xi_2\xi_3} \partial_\zeta \Big(c^3 S^{40}_{\xi_1\xi_2\xi_3\zeta} \partial_\eta \times O(1) + c^2 S^{41}_{\xi_1\xi_2\xi_3\zeta\zeta_1} \partial_\eta \times O(1) \\
&\quad + c S^{42}_{\xi_1\xi_2\xi_3\zeta\zeta_1\zeta_2} \partial_\eta \times O(1/\Delta x^2) + S^{43}_{\alpha\beta\gamma\xi_1\xi_2\xi_3\zeta_1\zeta_2\zeta_3} \partial_\eta \times \big[\rho c_s^4 \Delta_{\zeta_1\zeta_2\zeta_3\eta} + O(1/\Delta x^2) \big] \\
&\quad + \frac{1}{c} S^{44}_{\alpha\beta\gamma\xi_1\xi_2\xi_3\zeta_1\zeta_2\zeta_3} \partial_\eta \times O(1/\Delta x^4) \Big) \Big] = O(\Delta x^3), \qquad \text{(A30)}
\end{aligned}
$$

and

$$
\begin{aligned}
&\left[\Delta t^3 c^4 \mathcal{W}_0 \hat{\mathbf{S}}_N^{-1} \mathcal{W}_0 \hat{\mathbf{S}}_N^{-1} \mathcal{W}_0 \hat{\mathbf{S}}_N^{-1} \mathcal{W}_0 \mathbf{m}^{\mathrm{eq}}\right]_{\alpha+1} \\
&= \Delta t^3 c^4 \sum_{l=1}^{q} \left[\sum_{i=1}^{q} \left[\sum_{k=1}^{q} \partial_\beta \left[\sum_{j=1}^{q} \mathbf{e}_{j\alpha} \mathbf{e}_{j\beta} \overline{\boldsymbol{\Lambda}}_{jk} \right] \mathbf{e}_{k\gamma} \partial_\gamma \overline{\boldsymbol{\Lambda}}_{ki} \right] \mathbf{e}_{i\zeta} \partial_\zeta \overline{\boldsymbol{\Lambda}}_{il} \right] \mathbf{e}_{l\eta} \partial_\eta f_l^{\mathrm{eq}} \\
&= \Delta t^3 \partial_\beta \Big[S^{20}_{\alpha\beta} \partial_\gamma S^{10}_{\gamma} \partial_\zeta \big(c^3 S^{10}_{\zeta} \partial_\eta \times O(1) + c^2 S^{1}_{\zeta\chi_1} \partial_\eta \times O(1) \big) \\
&\quad + S^{20}_{\alpha\beta} \partial_\gamma S^{1}_{\gamma\zeta_1} \partial_\zeta \big(c^3 S^{20}_{\zeta_1\zeta} \partial_\eta \times O(1) + c^2 S^{21}_{\zeta_1\zeta\chi_1} \partial_\eta \times O(1) + c S^{2}_{\zeta_1\zeta\chi_1\chi_2} \partial_\eta \times O(1/\Delta x^2) \big) \\
&\quad + S^{21}_{\alpha\beta\xi_1} \partial_\gamma S^{20}_{\xi_1\gamma} \partial_\zeta \big(c^3 S^{10}_{\zeta} \partial_\eta \times O(1) + c^2 S^{1}_{\zeta\chi_1} \partial_\eta \times O(1) \big) + S^{21}_{\alpha\beta\xi_1} \partial_\gamma S^{21}_{\xi_1\gamma\zeta_1} \partial_\zeta \big(c^3 S^{20}_{\zeta_1\zeta} \partial_\eta \times O(1) + c^2 S^{21}_{\zeta_1\zeta\chi_1} \partial_\eta \times O(1) \\
&\quad + c S^{2}_{\zeta_1\zeta\chi_1\chi_2} \partial_\eta \times O(1/\Delta x^2) \big) + S^{21}_{\alpha\beta\xi_1} \partial_\gamma \partial_\zeta S^{2}_{\xi_1\gamma\zeta_1\zeta_2} \partial_\zeta \big(c^3 S^{30}_{\zeta_1\zeta_2\zeta} \partial_\eta \times O(1) + c^2 S^{31}_{\zeta_1\zeta_2\zeta\chi_1} \partial_\eta \times O(1) \\
&\quad + S^{33}_{\zeta_1\zeta_2\zeta\chi_1\chi_2\chi_3} \partial_\eta \big[\rho c_s^4 \Delta_{\chi_1\chi_2\chi_3\eta} + O(1/\Delta x^2) \big] \big) + S^{2}_{\alpha\beta\xi_1\xi_2} \partial_\gamma S^{30}_{\xi_1\xi_2\gamma} \partial_\zeta \big(c^3 S^{10}_{\zeta} \partial_\eta \times O(1) + c^2 S^{1}_{\zeta\chi_1} \partial_\eta \times O(1) \big) \\
&\quad + S^{2}_{\alpha\beta\xi_1\xi_2} \partial_\gamma S^{31}_{\xi_1\xi_2\gamma\zeta_1} \partial_\zeta \big(c^3 S^{20}_{\zeta_1\zeta} \partial_\eta \times O(1) + c^2 S^{21}_{\zeta_1\zeta\chi_1} \partial_\eta \times O(1) + c S^{2}_{\zeta_1\zeta\chi_1\chi_2} \partial_\eta \times O(1/\Delta x^2) \big) \\
&\quad + S^{2}_{\alpha\beta\xi_1\xi_2} \partial_\gamma S^{33}_{\xi_1\xi_2\gamma\zeta_1\zeta_2\zeta_3} \partial_\zeta \Big(c^3 S^{40}_{\zeta_1\zeta_2\zeta_3\zeta} \partial_\eta \times O(1) + c^2 S^{41}_{\zeta_1\zeta_2\zeta_3\zeta\chi_1} \partial_\eta \times O(1) + c S^{42}_{\zeta_1\zeta_2\zeta_3\zeta\chi_1\chi_2} \partial_\eta \times O(1/\Delta x^2) \\
&\quad + S^{43}_{\zeta_1\zeta_2\zeta_3\zeta\chi_1\chi_2\chi_3} \partial_\eta \times \big[\rho c_s^4 \Delta_{\chi_1\chi_2\chi_3\eta} + O(1/\Delta x^2) \big] + \frac{1}{c} S^{44}_{\zeta_1\zeta_2\zeta_3\zeta\chi_1\chi_2\chi_3\chi_4} \partial_\eta \times O(1/\Delta x^4) \Big) \Big] = O(\Delta x^3). \qquad \text{(A31)}
\end{aligned}
$$

With Eqs. (A17)–(A31), the second-order ME of GPMRT-LB model and GPMFD scheme (80) for the momentum equation (95b) becomes

$$\Bigg[\tilde{\mathbf{F}} - \left(c\mathcal{W}_0 + \partial_t \mathbf{I}_q + \Delta t c^2\left(1-\frac{b}{2a^2}\right)\mathcal{W}_0^2 + \Delta t \mathcal{W}_0\partial_t + \frac{\Delta t}{2}\partial_{tt}\mathbf{I}_q\right)\mathbf{m}^{\text{eq}} - \Delta t c\mathcal{W}_0\hat{\mathbf{S}}_N^{-1}\tilde{\mathbf{F}}$$
$$+\Bigg[\Delta t c^2\mathcal{W}_0\hat{\mathbf{S}}_N^{-1}\mathcal{W}_0 + \Delta t c\mathcal{W}_0\hat{\mathbf{S}}_N^{-1}\partial_t\mathbf{I}_q + \Delta t^2 c^3\left(1-\frac{b}{2a^2}\right)\mathcal{W}_0\hat{\mathbf{S}}_N^{-1}\mathcal{W}_0^2$$
$$+\Delta t c\partial_t\mathbf{I}_q\hat{\mathbf{S}}_N^{-1}\mathcal{W}_0 + \Delta t\partial_t\hat{\mathbf{S}}_N^{-1}\mathbf{I}_q\partial_t\mathbf{I}_q + \Delta t^2 c^3\left(1-\frac{b}{2a^2}\right)\mathcal{W}_0^2\hat{\mathbf{S}}_N^{-1}\mathcal{W}_0$$
$$+\Delta t^2 c^2\left(1-\frac{b}{2a^2}\right)\mathcal{W}_0^2\hat{\mathbf{S}}_N^{-1}\partial_t\mathbf{I}_q + \Delta t^2 c^2\mathcal{W}_0\hat{\mathbf{S}}_N^{-1}\mathcal{W}_0\partial_t\mathbf{I}_q\Bigg]\mathbf{m}^{\text{eq}}$$
$$-\left[\Delta t^2 c^3\mathcal{W}_0\hat{\mathbf{S}}_N^{-1}\mathcal{W}_0\hat{\mathbf{S}}_N^{-1}\mathcal{W}_0 + \Delta t^2 c^2\mathcal{W}_0\hat{\mathbf{S}}_N{-1}\mathcal{W}_0\hat{\mathbf{S}}_N^{-1}\partial_t\mathbf{I}_q\right]\mathbf{m}^{\text{eq}}\Bigg]_{\alpha+1} + O(\Delta x^3), \tag{A32}$$

then substituting Eqs. (A13)–(A16) into Eq. (A32), we have

$$\partial_t(\rho u_\alpha) + \partial_\beta(\rho u_\alpha u_\beta + \rho c_s^2\delta_{\alpha\beta}) = \Delta t\partial_\beta\left[\left[S^2_{\alpha\beta\xi_1\xi_2} - \left(1-\frac{b}{2a^2}\right)\delta_{\alpha\xi_1}\delta_{\beta\xi_2}\right]\partial_\gamma(\rho c_s^2\Delta_{\xi_1\xi_2\gamma\zeta}u_\zeta)\right]$$
$$+\Delta t\partial_\beta\left[S^2_{\alpha\beta\xi_1\xi_2}\partial_t(\rho c_s^2\delta_{\xi_1\xi_2})\right] + \rho F_{x_\alpha} = O(\Delta x^2), \tag{A33}$$

where the first-order ME (A15) has been used.

APPENDIX B: DERIVATION OF EQS. (93b), (100), AND (102)

1. NACDE: Derivation of Eq. (93b)

Due to the fact that

$$m_1^{\text{eq}} = \phi, \quad c\mathcal{W}_0 m_1^{\text{eq}} = \sum_{i=1}^q \mathbf{c}_{i\alpha} f_i^{\text{eq}} = B_\alpha,$$
$$\tilde{\mathbf{F}}_1 = \left[\mathbf{M}\left(\mathbf{F} + \mathbf{G} + \frac{\Delta t}{2}\mathbf{D}\mathbf{F}\right)\right]_1 = S + \frac{\Delta t}{2}\sum_{i=1}^q(\partial_t + \mathbf{c}_{i\gamma}\partial_\gamma)F_i = S + \frac{\Delta t}{2}\partial_t S, \tag{B1}$$

one can obtain the first-order ME:

$$\frac{\partial\phi}{\partial t} + \nabla\cdot\mathbf{B} = S + O(\Delta x), \tag{B2}$$

For the second-order ME, we also need to further consider the following equalities:

$$\left[-\frac{\Delta t}{2}\partial_t\tilde{\mathbf{F}}\right]_1 = -\frac{\Delta t}{2}\partial_t S + O(\Delta t^2),$$
$$\left[-\Delta t c\mathcal{W}_0\hat{\mathbf{S}}_N^{-1}\tilde{\mathbf{F}}\right]_1 = -c\Delta t\partial_\beta\sum_{k=1}^q\sum_{j=1}^q[\mathbf{e}_{j\beta}\overline{\mathbf{\Lambda}}_{jk}(F_k+G_k)] + O(\Delta t^2)$$
$$= -\Delta t\partial_\beta\sum_{k=1}^q\left(cS_\beta^{10} + S^1_{\beta\gamma}\mathbf{c}_{k\gamma}\right)(F_k+G_k) + O(\Delta t^2) = -c\Delta t\partial_\beta S_\beta^{10}S - \Delta t\partial_\beta\left(S^1_{\beta\gamma} - \frac{\delta_{\beta\gamma}}{2}\right)\partial_t B_\gamma$$
$$= -\Delta t\partial_\beta\left[S^1_{\beta\gamma} - \left(1-\frac{b}{2a^2}\right)\delta_{\beta\gamma}\right]\partial_\theta C_{\gamma\theta} + O(\Delta x^2),$$
$$\left[c\mathcal{W}_0(\hat{\mathbf{S}}_N^{-1} - \mathbf{I}_q/2)\partial_t\mathbf{m}^{\text{eq}}\right]_1 = \partial_\beta\sum_{k=1}^q\left(cS_\beta^{10} + S^1_{\beta\xi_1}\mathbf{c}_{k\xi_1}\right)\partial_t f_k^{\text{eq}} - \frac{1}{2}\partial_t\partial_\beta B_\beta = \partial_\beta\left[cS_\beta^{10}\partial_t\phi + \left(S^1_{\beta\xi_1} - \frac{1}{2}\delta_{\beta\xi_1}\right)\partial_t B_{\xi_1}\right],$$
$$\left[c^2\mathcal{W}_0\hat{\mathbf{S}}_N^{-1}\mathcal{W}_0\mathbf{m}^{\text{eq}}\right]_1 = \partial_\beta\sum_{k=1}^q\left(cS_\beta^{10} + S^1_{\beta\gamma}\mathbf{c}_{k\gamma}\right)\partial_\theta\mathbf{c}_{k\theta}f_k^{\text{eq}} = \frac{\partial}{\partial x_\beta}\left[cS_\beta^{10}\frac{\partial B_\theta}{\partial x_\theta} + S^1_{\beta\gamma}\frac{\partial(D_{\gamma\theta}c_s^2\chi + \zeta C_{\gamma\theta})}{\partial x_\theta}\right],$$
$$\left[c^2\mathcal{W}_0\mathcal{W}_0\mathbf{m}^{\text{eq}}\right]_1 = \partial_\beta\partial_\theta\sum_{j=1}^q\mathbf{c}_{j\beta}\mathbf{c}_{j\theta}f_j^{\text{eq}} = \frac{\partial^2(\chi c_s^2 D_{\beta\theta} + \zeta C_{\beta\theta})}{\partial x_\beta x_\theta}. \tag{B3}$$

Substituting Eqs. (B1) and (B3) into Eq. (86), one can obtain

$$[(\partial_t \mathbf{I}_q + c\mathcal{W}_0)\mathbf{m}^{\mathrm{eq}}]_1 + \frac{\Delta t}{2}\left(1-\frac{b}{a^2}\right)\left[c\mathcal{W}_0^2\mathbf{m}^{\mathrm{eq}}\right]_1 - \Delta t\left[c\mathcal{W}_0\left(\hat{\mathbf{S}}_N^{-1}-\mathbf{I}_q/2\right)\left(\partial_t + c\mathcal{W}_0\right)\mathbf{m}^{\mathrm{eq}}\right]_1 - \tilde{\mathbf{F}}_1 + \frac{\Delta t}{2}\partial_t\tilde{\mathbf{F}}_1 + \Delta t\left[c\mathcal{W}_0\hat{\mathbf{S}}_N^{-1}\tilde{\mathbf{F}}\right]_1$$
$$= \partial_t\phi + \partial_\alpha B_\alpha - S - \Delta t c_s^2\frac{\partial}{\partial x_\beta}\left(\left[S_{\beta\gamma}^1 + \left(\frac{b}{2a^2}-1\right)\delta_{\beta\gamma}\right]\right)\frac{\partial D_{\gamma\theta}}{\partial x_\theta} + \underbrace{c\Delta t\partial_\beta S_\beta^{10}\left[\partial_t\phi + \frac{\partial B_\theta}{\partial x_\theta} - S\right]}_{O(\Delta x^2)} = O(\Delta x^2), \tag{B4}$$

where the first-order ME (B2) has been used.

2. NSEs: Derivation of Eqs. (100) and (102)

For the first conservative moment, i.e., the density ρ, one can obtain

$$m_1^{\mathrm{eq}} = \rho, \quad \tilde{\mathbf{F}}_1 = \left[\mathbf{M}\left(\mathbf{F} + \frac{\Delta t}{2}\mathbf{DF}\right)\right]_1 = 0 + \frac{\Delta t}{2}\sum_{i=1}^{q}(\partial_t + \mathbf{c}_{i\gamma}\partial_\gamma)F_i = \frac{\Delta t}{2}\partial_\gamma(\rho\hat{F}_\gamma),$$
$$[c\mathcal{W}_0\mathbf{m}^{\mathrm{eq}}]_1 = \sum_{i=1}^{q}\mathbf{c}_{i\alpha}f_i^{\mathrm{eq}} = \rho u_\alpha, \tag{B5}$$

and for the second to $(d+1)$th conservative moments, i.e., the momentum ρu_α, we have

$$m_{\alpha+1}^{\mathrm{eq}} = \frac{\rho u_\alpha}{c}, \quad [c\mathcal{W}_0\mathbf{m}^{\mathrm{eq}}]_{\alpha+1} = \frac{1}{c}\partial_\beta\sum_{i=1}^{q}\mathbf{c}_{i\alpha}\mathbf{c}_{i\beta}f_i^{\mathrm{eq}} = \frac{\left(\rho u_\alpha u_\beta + \rho c_s^2\delta_{\alpha\beta}\right)}{c},$$
$$[\tilde{\mathbf{F}}]_{\alpha+1} = \left[\mathbf{M}\left(\mathbf{F} + \frac{\Delta t}{2}\mathbf{DF}\right)\right]_{\alpha+1} = \frac{\rho\hat{F}_{x_\alpha}}{c} + \frac{\Delta t}{2c}\sum_{i=1}^{q}\mathbf{c}_{i\beta}(\partial_t + \mathbf{c}_{i\gamma}\partial_\gamma)F_i$$
$$= \frac{\rho\hat{F}_{x_\alpha}}{c} + \frac{\Delta t}{2c}\partial_t(\rho\hat{F}_{x_\alpha}) + \frac{\Delta t}{2c}\partial_\gamma(\rho\hat{F}_\beta u_\gamma + \rho\hat{F}_\gamma u_\beta), \tag{B6}$$

then the corrseponding first-order MEs of the GPMRT-LB model and GPMFD scheme for the NSEs (95) are given by

$$\partial_t\rho + \partial_\beta(\rho u_\beta) = O(\Delta x), \tag{B7a}$$
$$\partial_t(\rho u_\alpha) + \partial_\alpha\partial_\beta\left(\rho u_\alpha u_\beta + c_s^2\rho\delta_{\alpha\beta}\right) - \rho\hat{F}_{x_\alpha} = O(\Delta x). \tag{B7b}$$

To obtain the second-order ME, the following equalities are needed:

$$-\Delta t\left[c\mathcal{W}_0\hat{\mathbf{S}}_N^{-1}\tilde{\mathbf{F}}\right]_1 = -\Delta t\partial_\beta\sum_{k=1}^{q}\sum_{j=1}^{q}[\mathbf{c}_{j\beta}\overline{\mathbf{\Lambda}}_{jk}F_k] + O(\Delta t^2) = -\Delta t\partial_\beta\sum_{k=1}^{q}\left(S_\beta^{10} + S_{\beta\gamma}^1\mathbf{c}_{k\xi_1}\right)F_k = -\Delta t\partial_\beta S_{\beta\xi_1}^1\rho\hat{F}_{\xi_1} + O(\Delta x^2),$$

$$\left[-\frac{\Delta t}{2}\partial_t\tilde{\mathbf{F}}\right]_1 = \mathbf{0}, \quad [c\mathcal{W}_0\partial_t\mathbf{m}^{\mathrm{eq}}]_1 = \partial_\beta\partial_t\sum_{i=1}^{q}\mathbf{c}_{i\beta}f_i^{\mathrm{eq}} = \partial_\beta\partial_t(\rho u_\beta),$$

$$\left[c\mathcal{W}_0\hat{\mathbf{S}}_N^{-1}\partial_t\mathbf{m}^{\mathrm{eq}}\right]_1 = \partial_\beta\sum_{k=1}^{q}\sum_{j=1}^{q}\mathbf{c}_{i\beta}\mathbf{\Lambda}_{jk}\partial_t f_k^{\mathrm{eq}} = \partial_\beta\sum_{k=1}^{q}\left(cS_\beta^{10} + S_{\beta\xi_1}^1\mathbf{c}_{k\xi_1}\right)\partial_t f_k^{\mathrm{eq}} = \partial_\beta\left[cS_\beta^{10}\partial_t\rho + S_{\beta\xi_1}^1\partial_t(\rho u_{\xi_1})\right],$$

$$\left[c^2\mathcal{W}_0\hat{\mathbf{S}}_N^{-1}\mathcal{W}_0\mathbf{m}^{\mathrm{eq}}\right]_1 = \partial_\beta\sum_{k=1}^{q}\sum_{j=1}^{q}\mathbf{c}_{j\beta}\mathbf{\Lambda}_{jk}\mathbf{c}_{k\theta}\partial_\theta f_k^{\mathrm{eq}} = \partial_\beta\partial_\theta\sum_{k=1}^{q}\left(cS_\beta^{10} + S_{\beta\xi_1}^1\mathbf{c}_{k\xi_1}\right)\mathbf{c}_{k\theta}\partial_\theta f_k^{\mathrm{eq}}$$
$$= \frac{\partial}{\partial x_\beta}\left[cS_\beta^{10}\frac{\partial}{\partial x_\theta}(\rho u_\theta) + S_{\beta\xi_1}^1\frac{\partial}{\partial x_\theta}\left(\rho u_{\xi_1}u_\theta + \rho c_s^2\delta_{\xi_1\theta}\right)\right],$$

$$[c^2\mathcal{W}_0\mathcal{W}_0\mathbf{m}^{\mathrm{eq}}]_1 = \partial_\beta\partial_\theta\sum_{j=1}^{q}\mathbf{c}_{j\beta}\mathbf{c}_{j\theta}f_j^{\mathrm{eq}} = \frac{\partial^2\left(\rho u_\beta u_\theta + \rho c_s^2\delta_{\beta\theta}\right)}{\partial x_\beta x_\theta}; \tag{B8}$$

combining Eqs. (B5) and (B8) yields

$$\begin{aligned}
&[(\partial_t \mathbf{I}_q + c\mathcal{W}_0)\mathbf{m}^{\mathrm{eq}}]_1 + \frac{\Delta t}{2}\left(1-\frac{b}{a^2}\right)\left[c\mathcal{W}_0^2\mathbf{m}^{\mathrm{eq}}\right]_1 \\
&\quad - \Delta t\left[c\mathcal{W}_0\left(\hat{\mathbf{S}}_N^{-1} - \mathbf{I}_q/2\right)(\partial_t + c\mathcal{W}_0)\mathbf{m}^{\mathrm{eq}}\right]_1 - \tilde{\mathbf{F}}_1 + \frac{\Delta t}{2}\partial_t\tilde{\mathbf{F}}_1 + \Delta t\left[c\mathcal{W}_0\hat{\mathbf{S}}_N^{-1}\tilde{\mathbf{F}}\right]_1 \\
&= \partial_t\rho + \partial_\alpha(\rho u_\alpha) - \Delta t\left(\frac{b}{2a^2} - \frac{1}{2}\right)\frac{\partial^2\left(\rho u_\beta u_\theta + \rho c_s^2\delta_{\beta\theta}\right)}{\partial x_\beta x_\theta} \underbrace{+\partial_\beta c S_\beta^{10}[\partial\rho + \partial_\theta(\rho u_\theta)]}_{O(\Delta x^2)} \\
&\quad \underbrace{-\,\partial_\beta\left(S^1_{\beta\xi_1} - \frac{1}{2}\delta_{\beta\xi_1}\right)\left[\partial_t(\rho u_{\xi_1}) + \partial_\theta\left(\rho u_{\xi_1\theta} + \rho c_s^2\delta_{\xi_1\theta}\right) - \rho\hat{F}_{x_{\xi_1}}\right]}_{O(\Delta x^2)} + O(\Delta x^2) \\
&= \partial_t\rho + \partial_\alpha(\rho u_\alpha) - \Delta t\left(\frac{b}{2a^2} - \frac{1}{2}\right)\frac{\partial^2\left(\rho u_\beta u_\theta + \rho c_s^2\delta_{\beta\theta}\right)}{\partial x_\beta x_\theta} + O(\Delta x^2),
\end{aligned} \tag{B9}$$

where the first-order MEs (B7a) and (B7b) have been used. In addition, for any $\alpha \in \{1 \sim d\}$, we can derive the following equations:

$$\begin{aligned}
&\left[-\frac{\Delta t}{2}\partial_t\tilde{\mathbf{F}}\right]_{\alpha+1} = -\frac{\Delta t}{2c}\partial_t\left(\rho\hat{F}_{x_\alpha}\right) + O(\Delta x^2), \\
&\left[-\Delta t c\mathcal{W}_0\hat{\mathbf{S}}_N^{-1}\tilde{\mathbf{F}}\right]_{\alpha+1} = -\Delta t\frac{1}{c}\sum_{k=1}^{q}\sum_{j=1}^{q}[\mathbf{c}_{j\alpha}\mathbf{c}_{j\beta}\partial_\beta\overline{\mathbf{\Lambda}}_{jk}F_k] + O(\Delta t^2), \\
&\quad = -\Delta t\frac{1}{c}\partial_\beta\sum_{k=1}^{q}\left(S^{20}_{\alpha\beta} + S^{21}_{\alpha\beta\xi_1}\mathbf{c}_{k\xi_1} + S^{22}_{\alpha\beta\xi_1\xi_2}\mathbf{c}_{k\xi_1}\mathbf{c}_{k\xi_2}\right)F_k + O(\Delta t^2) \\
&\quad = -\Delta t\frac{1}{c}\partial_\beta\left[S^{21}_{\alpha\beta\xi_1}\left(\rho\hat{F}_{\xi_1}\right) + S^{22}_{\alpha\beta\xi_1\xi_2}\left(\rho\hat{F}_{\xi_1}u_{\xi_2} + \rho\hat{F}_{\xi_2}u_{\xi_1}\right)\right] + O(\Delta t^2), \\
&[c\mathcal{W}_0\partial_t\mathbf{m}^{\mathrm{eq}}]_{\alpha+1} = \frac{1}{c}\sum_{k=1}^{q}\mathbf{c}_{k\alpha}\mathbf{c}_{k\beta}\partial_\beta\partial_t f_k^{\mathrm{eq}} = \frac{1}{c}\partial_\beta\partial_t\left(\rho c_s^2\delta_{\alpha\beta} + \rho u_\alpha u_\beta\right), \\
&\left[c\mathcal{W}_0\hat{\mathbf{S}}_N^{-1}\partial_t\mathbf{m}^{\mathrm{eq}}\right]_{\alpha+1} = \frac{1}{c}\partial_\beta\sum_{k=0}^{q-1}\left(S^{20}_{\alpha\beta} + S^{21}_{\alpha\beta\xi_1}\mathbf{c}_{k\xi_1} + S^{22}_{\alpha\beta\xi_1\xi_2}\mathbf{c}_{k\xi_1}\mathbf{c}_{k\xi_2}\right)\partial_t f_k^{\mathrm{eq}} \\
&\quad = \frac{1}{c}\partial_\beta\left(S^{20}_{\alpha\beta}\partial_t\rho + S^{21}_{\alpha\beta\xi_1}\partial_t\left(\rho u_{\xi_1}\right) + S^{22}_{\alpha\beta\xi_1\xi_2}\partial_t\left(\rho c_s^2\delta_{\xi_1\xi_2} + \rho u_{\xi_1}u_{\xi_2}\right)\right), \\
&\left[c^2\mathcal{W}_0\hat{\mathbf{S}}_N^{-1}\mathcal{W}_0\mathbf{m}^{\mathrm{eq}}\right]_{\alpha+1} = \frac{1}{c}\partial_\beta\sum_{k=1}^{q}\left(S^{20}_{\alpha\beta} + S^{21}_{\alpha\beta\xi_1}\mathbf{c}_{k\xi_1} + S^{22}_{\alpha\beta\xi_1\xi_2}\mathbf{c}_{k\xi_1}\mathbf{c}_{k\xi_2}\right)\mathbf{c}_{k\theta}\partial_\theta f_k^{\mathrm{eq}} \\
&\quad = \frac{1}{c}\partial_\beta\left[S^{20}_{\alpha\beta}\partial_\theta(\rho u_\theta) + S^{21}_{\alpha\beta\xi_1}\partial_\theta\left(\rho u_\theta u_{\xi_1} + \rho c_s^2\delta_{\xi_1\theta}\right) + S^{22}_{\alpha\beta\xi_1\xi_2}\partial_\theta\left(\rho c_s^2\Delta_{\xi_1\xi_2\theta\zeta}u_\zeta\right)\right], \\
&[c^2\mathcal{W}_0\mathcal{W}_0\mathbf{m}^{\mathrm{eq}}]_{\alpha+1} = \frac{1}{c}\partial_\beta\partial_\theta\sum_{j=1}^{q}\mathbf{c}_{j\alpha}\mathbf{c}_{j\beta}\mathbf{c}_{j\theta}f_j^{\mathrm{eq}} = \frac{1}{c}\partial_\beta\partial_\theta\left(\rho c_s^2\Delta_{\alpha\beta\theta\zeta}u_\zeta\right),
\end{aligned} \tag{B10}$$

considering Eqs. (B6) and (B10) yields

$$\begin{aligned}
&[(\partial_t\mathbf{I}_q + c\mathcal{W}_0)\mathbf{m}^{\mathrm{eq}}]_{\alpha+1} + \frac{\Delta t}{2}\left(1-\frac{b}{a^2}\right)\left[c^2\mathcal{W}_0^2\mathbf{m}^{\mathrm{eq}}\right]_{\alpha+1} - \Delta t\left[c\mathcal{W}_0\left(\hat{\mathbf{S}}_N^{-1} - \mathbf{I}_q/2\right)(\partial_t + c\mathcal{W}_0)\mathbf{m}^{\mathrm{eq}}\right]_{\alpha+1} \\
&\quad - \tilde{\mathbf{F}}_{\alpha+1} + \frac{\Delta t}{2}\partial_t\tilde{\mathbf{F}}_{\alpha+1} + \Delta t\left[c\mathcal{W}_0\hat{\mathbf{S}}_N^{-1}\tilde{\mathbf{F}}\right]_{\alpha+1} = \frac{1}{c}\partial_t(\rho u_\alpha) + \partial_\beta\left(\rho u_\alpha u_\beta + \rho c_s^2\delta_{\alpha\beta}\right) \\
&\quad + \frac{\Delta t}{2c}\left(1-\frac{b}{2a^2}\right)\partial_\beta\partial_\theta\left(\rho c_s^2\Delta_{\alpha\beta\theta\zeta}u_\zeta\right) - \frac{\rho\hat{F}_{x_\alpha}}{c} - \underbrace{\Delta t\frac{1}{c}\partial_\beta S^{20}_{\alpha\beta}\left(\partial_t\rho + \partial_\theta(\rho u_\theta)\right)}_{O(\Delta x^2)}
\end{aligned}$$

$$- \underbrace{\Delta t \frac{1}{c} \partial_\beta S^{21}_{\alpha\beta\xi_1} \big(\partial_t(\rho u_{\xi_1}) + \partial_\theta(\rho u_\theta u_{\xi_1} + \rho c_s^2 \delta_{\xi_1\theta}) - \rho \hat{F}_{x_{\xi_1}}\big)}_{O(\Delta x^2)}$$

$$- \Delta t \frac{1}{c} \partial_\beta \left(S^{22}_{\alpha\beta\xi_1\xi_2} - \frac{1}{2}\delta_{\xi_1\alpha}\delta_{\xi_2\beta}\right)\big(\partial_t\big(\rho c_s^2 \delta_{\xi_1\xi_2} + \rho u_{\xi_1} u_{\xi_2}\big) + \partial_\theta\big(\rho c_s^2 \Delta_{\xi_1\xi_2\theta\zeta} u_\zeta\big) - \rho \hat{F}_{x_{\xi_1}} u_{\xi_2} - \rho \hat{F}_{x_{\xi_2}} u_{\xi_1}\big) = O(\Delta x^2), \tag{B11}$$

where the results of first-order MEs (B7a) and (B7b) have been adopted.

APPENDIX C: THE PARAMETERS OF GPMFD SCHEME (112)

The parameters $\alpha_i (i \in \{1 \sim 3\})$, β_k, $\gamma_k (k \in \{1 \sim 5\})$ of the GPMFD scheme (112) are given by

$$\begin{aligned}
\alpha_1 = {} & \frac{1}{c^2\theta}\big(6a^2c^2 - 4a^2bc^2 - 8a^2c^2s_1 - 2a^2c^2s_2 - bc^2s_1^2 - 2b^2c^2s_1 + 2a^2c^2s_1^2 + b^2c^2s_1^2 \\
& + 3bc^2s_1 - 2a^2bc^2s_1^2 - bc^2s_1s_2 + 6a^2bc^2s_1 + 2a^2c^2s_1s_2 - 2a^2bs_2u^2 + 2a^2bc^2s_2w_0 \\
& + a^2bs_1s_2u^2 + b^2c^2s_1s_2w_0 - 2a^2bc^2s_1s_2w_0\big), \\
\alpha_2 = {} & \frac{1}{2c^2\theta}\big(4a^2bc^2 + 2b^2c^2s_1 - b^2c^2s_1^2 + 2a^2bc^2s_1^2 + 2a^3cs_1u - 6a^2bc^2s_1 + 2a^2bs_2u^2 \\
& - 2a^3cs_1^2u - 2a^2bc^2s_2w_0 - a^2bs_1s_2u^2 - b^2c^2s_1s_2w_0 + abcs_1^2u + 2a^2bc^2s_1s_2w_0\big), \\
\alpha_3 = {} & -\frac{1}{2c^2\theta}\big(b^2c^2s_1^2 - 2b^2c^2s_1 - 4a^2bc^2 - 2a^2bc^2s_1^2 + 2a^3cs_1u + abcs_1^2u - 2a^2bc^2s_1s_2w_0 \\
& + 6a^2bc^2s_1 - 2a^2bs_2u^2 - 2a^3cs_1^2u + 2a^2bc^2s_2w_0 + a^2bs_1s_2u^2 + b^2c^2s_1s_2w_0\big), \\
\beta_1 = {} & -\frac{1}{2c^2\theta}\big(12a^2c^2 + 2a^4c^2 - 16a^2bc^2 - 20a^2c^2s_1 - 8a^2c^2s_2 - 4a^4c^2s_1 - 4bc^2s_1^2 \\
& - 8b^2c^2s_1 + 3b^3c^2s_1 + 2a^4s_2u^2 + 6a^2b^2c^2 + 8a^2c^2s_1^2 + 2a^4c^2s_1^2 + 6b^2c^2s_1^2 \\
& - 3b^3c^2s_1^2 + 6bc^2s_1 - 13a^2bc^2s_1^2 - 12a^2b^2c^2s_1 - 4a^2c^2s_1^2s_2 + 6a^2b^2s_2u^2 \\
& - 2b^2c^2s_1^2s_2 + a^4s_1^2s_2u^2 - 4bc^2s_1s_2 + 6a^2b^2c^2s_1^2 + 29a^2bc^2s_1 + 8a^2bc^2s_2 \\
& + 12a^2c^2s_1s_2 - 4a^2bs_2u^2 + 2bc^2s_1^2s_2 + 4b^2c^2s_1s_2 - 2a^4c^2s_2w_0 - 3a^4s_1s_2u^2 \\
& + 4a^2bc^2s_2w_0 + 6a^2bs_1s_2u^2 + 4a^4c^2s_1s_2w_0 + 2b^2c^2s_1s_2w_0 - 3b^3c^2s_1s_2w_0 \\
& - 6a^2b^2c^2s_2w_0 - 2a^2bs_1^2s_2u^2 - 9a^2b^2s_1s_2u^2 - 2a^4c^2s_1^2s_2w_0 - 2b^2c^2s_1^2s_2w_0 \\
& + 3a^2b^2s_1^2s_2u^2 + 5a^2bc^2s_1^2s_2w_0 + 12a^2b^2c^2s_1s_2w_0 - 6a^2b^2c^2s_1^2s_2w_0, \\
& - 9a^2bc^2s_1s_2w_0 + 4a^2bc^2s_1^2s_2 - 12a^2bc^2s_1s_2 + 3b^3c^2s_1^2s_2w_0\big), \\
\beta_2 = {} & \frac{1}{2c^2\theta}\big(2b^3c^2s_1 - 4b^2c^2s_1 - 8a^2bc^2 + 4a^2b^2c^2 - 4a^2b^2c^2s_1^2s_2w_0 - 4a^2bc^2s_1s_2w_0 \\
& + 3b^2c^2s_1^2 - 2b^3c^2s_1^2 - 6a^2bc^2s_1^2 - 8a^2b^2c^2s_1 + 4a^2b^2s_2u^2 - b^2c^2s_1^2s_2 \\
& + 4a^2b^2c^2s_1^2 + 14a^2bc^2s_1 + 4a^2bc^2s_2 - 2a^2bs_2u^2 + 2b^2c^2s_1s_2 + 2a^3cs_1^2u \\
& + 2a^2bc^2s_2w_0 + 3a^2bs_1s_2u^2 - 2a^3cs_1^2s_2u + b^2c^2s_1s_2w_0 - 2b^3c^2s_1s_2w_0 \\
& - 4a^2b^2c^2s_2w_0 - a^2bs_1^2s_2u^2 - 6a^2b^2s_1s_2u^2 - b^2c^2s_1^2s_2w_0 + 2b^3c^2s_1^2s_2w_0 \\
& - 6a^2bc^2s_1s_2 + 2a^3cs_1s_2u + 2a^2b^2s_1^2s_2u^2 + 2a^2bc^2s_1^2s_2w_0 \\
& + 8a^2b^2c^2s_1s_2w_0 - 2a^3cs_1u + 2a^2bc^2s_1^2s_2 - abcs_1^2u + abcs_1^2s_2u\big), \\
\beta_3 = {} & -\frac{(a^2-b^2)(s_1-1)}{4c^2\theta}[2a^2c^2(1-s_1) + a^2s_2(2-s_1)(u^2 - 2w_0c^2) + bc^2s_1(1-s_2w_0)], \\
\beta_3 = {} & \beta_2, \quad \beta_5 = \beta_4, \quad \gamma_1 = \frac{(s_1-1)(s_2-1)(-a^2+b^2)}{4}(2a^2 + 6b^2 - 8b + 4), \\
\gamma_2 = {} & (s_1-1)(s_2-1)(b-b^2), \quad \gamma_3 = \gamma_2, \gamma_4 = \frac{(s_1-1)(s_2-1)(-a^2+b^2)}{4}, \quad \gamma_5 = \gamma_4.
\end{aligned} \tag{C1}$$

where $\theta = 1/(bs_1 - 2a^2s_1 + 2a^2)$.

APPENDIX D: THE EXPRESSION OF THE AMPLIFICATION MATRIX OF THE F-GPMFD SCHEME (112)

For the F-GPMFD scheme (112), one can obtain the expression of the amplification matrix $\mathbf{G}$,

$$\mathbf{G} = \begin{pmatrix} \alpha_1 + \alpha_2 e^{-i\theta} + \alpha_3 e^{i\theta} & \beta_1 + \beta_2 e^{-i\theta} + \beta_3 e^{i\theta} + \beta_4 e^{-2i\theta} + \beta_5 e^{2i\theta} & \gamma_1 + \gamma_2 e^{-i\theta} + \gamma_3 e^{i\theta} + \gamma_4 e^{-2i\theta} + \gamma_5 e^{2i\theta} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \tag{D1}$$

where $\theta \in [-\pi, \pi]$. However, it is difficult to discuss the von Neumann stability condition of the F-GPMFD scheme (112) from the theoretical perspective, thus we consider the numerical stability for the F-GPMFD scheme (112) in Sec. IV.

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