Insights on phase speed and the critical Reynolds number of falling films

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We revisit the studies of gravity-driven viscous falling films with and without imposed shear stress to provide new perspectives on phase speed and the critical Reynolds number for surface instability. We use the traditional long-wave expansion technique implemented for investigating the linear stability analysis [C. S. Yih, Phys. Fluids **6**, 321 (1963)]. The principal purpose is to create a unified relationship between the leading-order phase speed and the critical Reynolds number that will hold for falling films on impermeable substrates with or without shear stress acting at the liquid film surface. The analytical result demonstrates that the critical Reynolds number for the onset of surface instability is $[5/(2c_0)]\cot\theta$, where c_0 is the leading-order phase speed of the surface mode and θ is the angle of inclination with the horizontal. Clearly, the critical Reynolds number of the surface mode is explicitly dependent on the leading-order phase speed. Furthermore, we reveal that the basic parallel flow with or without imposed shear stress is linearly unstable to infinitesimal disturbances if the modified Reynolds number, $\mathrm{Re}_M = (\mathrm{Re}\, c_0/\cot\theta)$ [Re is the Reynolds number, and $\theta \neq \pi/2$], is greater than its critical value of 5/2, which is independent of the shear stress applied at the film surface. In addition, it is demonstrated that Re_M controls the surface instability in the long-wave regime for both shear-imposed and non-shear-imposed film flows.

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I. INTRODUCTION

The topic of falling liquid film has a long history, dating back to the pioneering experimental research work of Kapitza [1]. Aside from its extensive applications in the technological and chemical industries, it renders an excellent platform for investigating the phenomenon of transition from a primary flow to a secondary instability [2–6]. In this context, Yih [7] and Benjamin [8] initiated the theoretical study many years ago. In particular, Yih formulated the Orr-Sommerfeld boundary value problem for the gravity-driven two-dimensional incompressible viscous liquid film on an inclined plane and analytically derived the critical Reynolds number, $\mathrm{Re}_c = (5/6)\cot\theta$, for surface instability based on the average velocity of a steady parallel undisturbed flow as the characteristic velocity scale. Clearly, the expression of the critical Reynolds number, Re_c, depends only on the inclination angle θ. Moreover, Yih demonstrated that the dimensional value of the leading-order phase speed is three times the average velocity of a steady basic parallel flow. As the linear instability of a primary base flow under an infinitesimal disturbance is ensured by the critical Reynolds number, its precise prediction is a crucial issue for researchers. In particular, the relationship between the leading-order phase speed c_0 and the critical Reynolds number Re_c has not been discussed in the literature. In each study, the expression of c_0 was substituted in the computation of the critical Reynolds number in the first-order long-wave calculation. As a result, $\mathrm{Re}_c/\cot\theta$ is a constant when $\theta \neq \pi/2$. However, the magnitude of this constant alters with the various choices of the characteristic velocity scale used in the different studies of film flows [7–9]. Moreover, the expression, $\mathrm{Re}_c/\cot\theta$, depends on the applied shear stress at the surface of a shear-imposed film flow [10]. Therefore, the expression, $\mathrm{Re}_c/\cot\theta$, is not unique for both shear-imposed and non-shear-imposed film flows. In the present work, our idea is to retrieve a unified relationship between the leading-order phase speed and the critical Reynolds number for falling films with or without imposed shear stress on impermeable substrates because the leading-order phase speed c_0 is not always constant. For example, c_0 varies with the change in applied shear stress at the liquid film surface for a shear-imposed film flow [10–12]. As a result, the critical Reynolds number is not constant, but it alters with the imposed shear stress. Here, we have discovered that the critical Reynolds number not only depends on the inclination angle θ but also depends on the leading-order phase speed c_0. Indeed, we have developed a new expression of the critical Reynolds number for the surface mode, which can be read as $\mathrm{Re}_c = [5/(2c_0)]\cot\theta$ for falling films with or without imposed shear stress. It should be noted that the new expression of the critical Reynolds number recovers the earlier known expression $\mathrm{Re}_c = (5/4)\cot\theta$ [8] once the leading-order phase speed is replaced by its value, $c_0 = 2$. Furthermore, it reproduces the findings of Smith [10], Wei [11], Samanta [12], and Sivapuratharasu *et al.* [13] for a shear-imposed film flow if the expression of the leading-order phase speed is used. In other words, from the study of falling film without shear stress, we can predict the expression of the critical Reynolds number for a shear-imposed film flow with the aid of the current relation $\mathrm{Re}_c = [5/(2c_0)]\cot\theta$ only by using the expression of the leading-order phase speed, c_0, without computing the first-order long-wave solution. In addition, we have found that the modified Reynolds number $\mathrm{Re}_M = \mathrm{Re}\, c_0/\cot\theta$ dominates the linear stability or instability of falling films in the long-wave regime with or without imposed shear stress when $\theta \neq \pi/2$. Based on the definition

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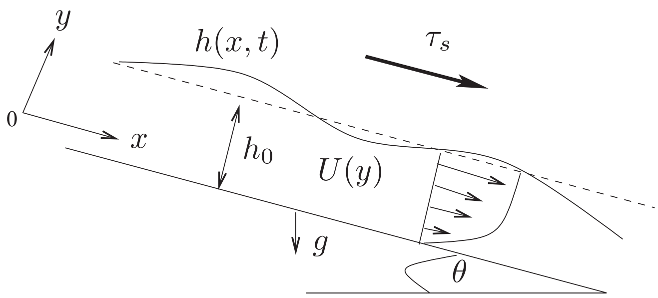

FIG. 1. A schematic diagram of a two-dimensional shear-imposed incompressible disturbed viscous liquid film flowing under the streamwise gravitational force per unit volume. Here, h_0 is the height of the undisturbed liquid film layer and $h(x,t)$ is the height of the disturbed liquid film layer. $U(y)$ represents the streamwise base velocity and τ_s is the dimensional imposed shear stress acting in the streamwise direction.

of the modified Reynolds number Re_M, we can say that the modified critical Reynolds number is now a fixed constant of $5/2$, which is valid for the studies of falling films on an inclined plane with or without imposed shear stress.

II. MATHEMATICAL FORMULATION

Suppose an incompressible shear-imposed viscous film flows down a sloping plane having an angle θ with the horizontal (see Fig. 1). We assume that the current film is isothermal and that all the physical properties of the liquid film, like density ρ, shear viscosity μ, and surface tension σ, are constants. As the liquid film is driven by the streamwise component of the gravitational force, $\rho g \sin\theta$, per unit volume, the flow is governed by the following nondimensional mass conservation, momentum equations, and boundary conditions in the Cartesian coordinate system [10,14]:

$$\partial_x u + \partial_y v = 0, \tag{1}$$

$$\mathrm{Re}(\partial_t u + u\partial_x u + v\partial_y u) = -\partial_x p + (\partial_{xx} u + \partial_{yy} u) + 1, \tag{2}$$

$$\mathrm{Re}(\partial_t v + u\partial_x v + v\partial_y v) = -\partial_y p + (\partial_{xx} v + \partial_{yy} v) - \cot\theta, \tag{3}$$

$$u = v = 0, \quad \text{at } y = 0, \tag{4}$$

$$(\partial_y u + \partial_x v)[1 - (\partial_x h)^2] - 4\partial_x u \partial_x h = \tau\sqrt{1 + (\partial_x h)^2}, \quad \text{at } y = h(x,t), \tag{5}$$

$$P_a - p + \frac{2}{[1 + (\partial_x h)^2]}[\partial_x u(\partial_x h)^2 + \partial_y v - (\partial_y u + \partial_x v)\partial_x h] = \frac{\mathrm{We}\,\partial_{xx} h}{[1 + (\partial_x h)^2]^{3/2}}, \quad \text{at} \quad y = h(x,t), \tag{6}$$

$$\partial_t h + u\partial_x h = v, \quad \text{at} \quad y = h(x,t), \tag{7}$$

where $u(x,y,t)$ and $v(x,y,t)$ are, respectively, the nondimensional x- and y-direction velocity components of the disturbed liquid film, $p(x,y,t)$ is the nondimensional pressure, and t is the time. Here, $\mathrm{Re} = \rho U_0 d/\mu$ is the Reynolds number, $\mathrm{We} = \sigma/(\mu U_0)$ is the Weber number, P_a is the nondimensional ambient pressure, and $\tau = \tau_s h_0/(\mu U_0)$ is the nondimensional imposed shear stress. Clearly, τ will be zero for the non-shear-imposed film flow [7]. To nondimensionalize the governing equations, we have preferred $U_0 = \rho g \sin\theta h_0^2/\mu$ as the characteristic velocity scale, which is obtained by balancing the viscous term $\mu\partial_{yy}u$ and the streamwise gravity term $\rho g \sin\theta$, h_0 is the characteristic length scale, $\mu U_0/h_0$ is the characteristic pressure scale, and h_0/U_0 is the characteristic time scale.

A. Base flow solution

To compute the base flow solution analytically, we assume that the undisturbed flow is steady, unidirectional, and parallel (see Fig. 2) with a constant liquid film layer thickness of $h(x,t) = 1$. In this situation, there is no cross-stream base velocity $[V(y) = 0]$. The base flow assumptions simplify the nondimensional governing equations (1)–(7) in the following forms:

$$-\partial_x p + \partial_{yy} u + 1 = 0, \tag{8}$$

$$\partial_y p + \cot\theta = 0, \tag{9}$$

$$u = 0, \quad \text{at } y = 0, \tag{10}$$

$$\partial_y u = \tau, \quad \text{at } y = 1, \tag{11}$$

$$p = P_a, \quad \text{at } y = 1. \tag{12}$$

Solving Eq. (9) and using the boundary condition (12), we can obtain the following base pressure solution:

$$P(y) = P_a + \cot\theta(1 - y). \tag{13}$$

It should be noted that the base flow pressure is a function of y alone. Hence, the term $\partial_x p$ is deleted from Eq. (8), reducing

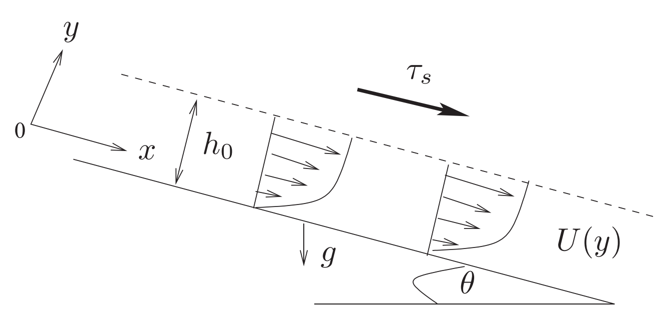

FIG. 2. A schematic diagram of a steady two-dimensional shear-imposed incompressible undisturbed viscous liquid film flowing under the streamwise gravitational force per unit volume.

it to the following form:

$$\partial_{yy}u + 1 = 0. \tag{14}$$

After solving Eq. (14) and using the no-slip boundary condition (10), we can obtain the following expression of the streamwise base velocity:

$$U(y) = (c_k y - y^2/2), \tag{15}$$

where c_k is a nondimensional constant. Clearly, to determine the coefficient c_k, we have not used the tangential stress boundary condition (11) at $y = 1$. Actually, if this boundary condition is used, we will lose the physical significance of the coefficient c_k. Moreover, the base shear stress, $\partial_y U = \tau$ at $y = 1$, is nonzero for the shear-imposed film flow, while $\partial_y U = 0$ at $y = 1$ for the non-shear-imposed film flow. As our purpose is to unify the results of film flows with and without imposed shear stress in one form, we cannot use this boundary condition. Otherwise, we will move on to the study of a particular case [7,8,10]. Hence, the expression (15) of base velocity is valid in both shear-imposed and non-shear-imposed gravity-driven film flows. Now, the value of the streamwise base velocity, $U(y)$, at the liquid film surface, $y = 1$, can be obtained as

$$U(1) = c_k - 1/2, \tag{16}$$

which implies

$$c_k = U(1) + 1/2. \tag{17}$$

Comparing with the expression of the leading-order phase speed c_0 of Smith [10] (given on page 473 in his reference), we can assert that the coefficient c_k coincides with the leading-order phase speed c_0 of the surface mode. This result will be justified further through the long-wave asymptotic expansion [7].

III. PERTURBATION TECHNIQUE

Let us superimpose an infinitesimal perturbation on the basic parallel film flow. Consequently, the flow variables of the perturbation film flow can be decomposed as

$$\left.\begin{aligned} u(x,y,t) &= U(y) + u'(x,y,t),\\ v(x,y,t) &= v'(x,y,t),\\ p(x,y,t) &= P(y) + p'(x,y,t),\\ h(x,t) &= 1 + h'(x,t), \end{aligned}\right\}, \tag{18}$$

where the variables with prime notation denote the perturbation flow variables, and the variables without prime notation denote the base flow variables. Inserting (18) in (1)–(7) and linearizing with respect to the base flow solution, we obtain the following two-dimensional perturbation equations for the gravity-driven viscous liquid film:

$$\partial_x u' + \partial_y v' = 0, \tag{19}$$

$$\mathrm{Re}(\partial_t u' + U\partial_x u' + v' DU) = -\partial_x p' + (\partial_{xx}u' + \partial_{yy}u'), \tag{20}$$

$$\mathrm{Re}(\partial_t v' + U\partial_x v') = -\partial_y p' + (\partial_{xx}v' + \partial_{yy}v'), \tag{21}$$

$$u' = v' = 0, \quad \text{at } y = 0, \tag{22}$$

$$(\partial_y u' + \partial_x v' + h' D^2 U) = 0, \quad \text{at } y = 1, \tag{23}$$

$$-p' - h' DP + 2\partial_y v' - 2DU\,\partial_x h' = \mathrm{We}\,\partial_{xx}h', \quad \text{at } y = 1, \tag{24}$$

$$\partial_t h' + U\partial_x h' = v', \quad \text{at } y = 1, \tag{25}$$

in which $D = d/dy$ is the differential operator. Note that the term DU is still retained in the perturbation normal stress boundary condition (24) because DU is nonzero for a shear-imposed film flow at the surface [$DU = \tau$ at $y = 1$]. In fact, the above two-dimensional perturbation equations are valid in viscous film flows with or without shear stress imposed at the film surface [$DU = 0$ at $y = 1$ in the absence of applied shear stress at the liquid film surface, i.e., when $\tau = 0$]. The perturbation normal component of velocity $v'(x, y, t)$ can be determined from the perturbation mass conservation equation (19) by the following relation:

$$v'(x,y,t) = -\int_0^y \partial_x u' dy. \tag{26}$$

Substituting Eq. (26) in the perturbation evolution (25) for the liquid film surface and using the Leibniz integral rule [15], we get

$$\partial_t h' + U(1)\partial_x h' + \partial_x \int_0^1 u' dy = 0. \tag{27}$$

As the disturbance is infinitesimally small and evolving slowly downstream with respect to space and time, we can assume $\mathcal{O}(\partial_x)$, $\mathcal{O}(\partial_t) \ll \mathcal{O}(\partial_y)$ [9]. In other words, in the leading order, the linearized two-dimensional perturbation equations (19)–(25) are simplified into the following forms:

$$\partial_{yy}u' = 0, \tag{28}$$

$$u' = 0, \quad \text{at } y = 0, \quad \partial_y u' = h', \quad \text{at } y = 1. \tag{29}$$

For our convenience, we have written equations only for the streamwise perturbation velocity. The solution of the perturbation equation (28) with the help of boundary conditions (29) becomes

$$u'(x,y,t) = h' y, \tag{30}$$

which shows that the leading-order streamwise perturbation velocity component $u'(x, y, t)$ is linear in the cross-stream direction y. After inserting the expression of $u'(x, y, t)$ in Eq. (27), the perturbation kinematic equation can be converted into the following form:

$$\partial_t h' + [U(1) + 1/2]\partial_x h' = 0. \tag{31}$$

Equation (31) represents a one-dimensional linear hyperbolic wave equation, where the wave propagates with a constant speed $U(1) + 1/2$, or equivalently, propagates with a speed c_k. Obviously, the above result is consistent with our statement that the leading-order phase speed coincides with c_k.

IV. ORR-SOMMERFELD EQUATION

In an analogous fashion with the work of Yih [7], we introduce the perturbation stream function $\psi'(x, y, t)$ from the perturbation velocity components $u'(x, y, t)$ and $v'(x, y, t)$

with the help of the perturbation mass conservation equation (19) by using the following relations:

$$u' = \partial_y \psi', \quad v' = -\partial_x \psi'. \tag{32}$$

Next, we assume the solution of the two-dimensional perturbation equations (19)–(25) in the normal mode form:

$$\left.\begin{aligned} \psi'(x, y, t) &= \phi(y) \exp[ik(x - ct)] + \text{c.c},\\ h'(x, t) &= \eta \exp[ik(x - ct)] + \text{c.c}, \end{aligned}\right\} \tag{33}$$

where c.c. denotes the complex conjugate. Here, k is the wave number and c is the wave speed. Since we will perform temporal stability analysis, we will assume k is real, but $c = c_r + ic_i$ is complex. Therefore, $c_i > 0$ indicates a condition for linear instability of base flow to infinitesimal disturbances. Otherwise, it will be stable to infinitesimal disturbances if $c_i < 0$. Finally, $c_i = 0$ implies a neutral stability condition for base flow. After that, the normal mode solution (33) is inserted in the perturbation equations (19)–(25) and the perturbation pressure $p'(x, y, t)$ is eliminated from the momentum equations, which yield the following Orr-Sommerfeld equation and associated boundary conditions for the gravity-driven viscous liquid films [16]:

$$(D^2 - k^2)^2\phi = ik\text{Re}[(U - c)(D^2 - k^2) - D^2U]\phi, \tag{34}$$

$$\phi = D\phi = 0, \quad \text{at } y = 0, \tag{35}$$

$$(D^2 + k^2)\phi - \eta = 0, \quad \text{at } y = 1, \tag{36}$$

$$D^3\phi - 3k^2D\phi - ik\text{Re}[(U - c)D\phi - DU\phi] - ik\eta[k^2\text{We} + \cot\theta - 2ikDU] = 0, \quad \text{at } y = 1, \tag{37}$$

$$\phi + (U - c)\eta = 0, \quad \text{at } y = 1, \tag{38}$$

in which $\phi(y)$ is the amplitude of the perturbation stream function $\psi'(x, y, t)$ and η is the amplitude of the liquid film surface deformation $h'(x, t)$. Again, the term DU is kept in Eq. (37) because DU is nonzero for a shear-imposed film flow at the surface [$DU = \tau$ at $y = 1$]. Hence, the above Orr-Sommerfeld equation is valid in viscous film flows with or without shear stress imposed at the liquid film surface [$DU = 0$ at $y = 1$ in the absence of applied shear stress at the liquid film surface].

A. Long-wave solution

The Orr-Sommerfeld equation with boundary conditions (34)–(38) is solved by using the long-wave asymptotic expansion in the limit of $k \to 0$. It is important to mention here that we have used the expression, $U(y) = (c_k y - y^2/2)$, of streamwise base velocity to solve the Orr-Sommerfeld boundary value problem analytically because this expression is valid in both shear-imposed and non-shear-imposed film flows. On the basis of long-wave analysis, we expand the flow variables in the following ways:

$$\left.\begin{aligned} \phi(y) &= \phi_0(y) + k\phi_1(y) + \cdots,\\ c &= c_0 + kc_1 + \cdots,\\ \eta &= \eta_0 + k\eta_1 + \cdots. \end{aligned}\right\} \tag{39}$$

Substituting the above series expansions (39) in Eqs. (34)–(38), we consider the leading-order equations, i.e., equations of $\mathcal{O}(k^0)$:

$$D^4\phi_0 = 0, \tag{40}$$

$$\phi_0 = D\phi_0 = 0, \quad \text{at } y = 0, \tag{41}$$

$$D^2\phi_0 = \eta_0, \;\; D^3\phi_0 = 0, \;\; \phi_0 + (U - c_0)\eta_0 = 0, \quad \text{at } y = 1. \tag{42}$$

The solution of the leading-order equations (40)–(42) yields

$$\phi_0(y) = \eta_0 y^2/2, \;\; c_0 = U(1) + 1/2 = c_k. \tag{43}$$

The analytical expression (43) reveals that the leading-order phase speed, c_0, of the surface mode is exactly equal to c_k. Next, we consider the first-order equations, i.e., equations of $\mathcal{O}(k^1)$:

$$D^4\phi_1 = i\text{Re}[(U - c_0)D^2\phi_0 + \phi_0], \tag{44}$$

$$\phi_1 = D\phi_1 = 0, \quad \text{at } \; y = 0, \tag{45}$$

$$D^2\phi_1 = \eta_1, \quad \text{at } \; y = 1, \tag{46}$$

$$D^3\phi_1 = i\text{Re}[(U - c_0)D\phi_0 - DU\phi_0] + i\eta_0 \cot\theta, \quad \text{at } \; y = 1, \tag{47}$$

$$\phi_1 + (U - c_0)\eta_1 - c_1\eta_0 = 0, \quad \text{at } \; y = 1. \tag{48}$$

Here, we have assumed that the Weber number We is of order $\mathcal{O}(1)$, and thereby, the term involving the Weber number does not appear in the first-order normal stress boundary condition (47). However, this term is essential for computing the cut-off wave number, or equivalently, for figuring out the second boundary of the unstable region when the Reynolds number exceeds the critical value Re_c. After solving the first-order equations (44)–(47), one can obtain the expression of $\phi_1(y)$, which is later inserted in the first-order kinematic boundary condition (48). This leads to the expression of c_1 in the following form:

$$c_1 = i\left[\tfrac{2}{15}c_0\text{Re} - \tfrac{1}{3}\cot\theta\right]. \tag{49}$$

Finally, we use the neutral stability condition, $c_i \approx |kc_1| = 0$, in the limit of $k \to 0$, which evaluates the expression of the

TABLE I. Expression of the critical Reynolds number, $\mathrm{Re}_c = \frac{5}{2c_0}\cot\theta$, for the surface mode corresponding to different types of falling film flow problems. This expression is calculated in the long-wave limit ($k \to 0$).

Falling film models	c_0	$\mathrm{Re}_c = \frac{5}{2c_0}\cot\theta$
Yih [7]	3	$\frac{5}{6}\cot\theta$
Benjamin [8]	2	$\frac{5}{4}\cot\theta$
Ruyer-Quil and Manneville [9]	1	$\frac{5}{2}\cot\theta$
Smith [10]	$(1+\tau)$	$\frac{5}{2(1+\tau)}\cot\theta$

critical Reynolds number, Re_c, for the surface mode

$$\mathrm{Re}_c = \frac{5}{2c_0}\cot\theta. \tag{50}$$

Equation (50) demonstrates that the critical Reynolds number, Re_c, above which the surface instability begins, is explicitly dependent on the leading-order phase speed c_0 and the inclination angle θ. Since the expression of the critical Reynolds number (50) is computed by using the general streamwise base velocity profile $U(y) = (c_k y - y^2/2)$, we can expect that it will recover all the existing results of shear-imposed and non-shear-imposed film flows. To examine that, the critical Reynolds number for the different types of falling film flows is calculated in the long-wave limit based on the current formula (50) and reported in Table I. From the results provided in Table I, we can conclude that gravity-driven viscous liquid film with or without imposed shear stress can be combined in a class of flow problems whose critical Reynolds number is $\mathrm{Re}_c = \frac{5}{2c_0}\cot\theta$. We have further validated our analytical results with the numerical results of Bruin [17] in Table II when the inclination angle is sufficiently small (given on page 269 in his reference). Clearly, the current expression provides an accurate value of the critical Reynolds number for the surface mode when we compare our results with the numerical results of Bruin [17]. If we set $\theta = \pi/2$, the critical Reynolds number Re_c becomes zero, and therefore, the falling film on a vertical substrate will be unstable to an infinitesimal disturbance for any nonzero value of the Reynolds number. Furthermore, the expression $[\mathrm{Re}_c\, c_0/\cot\theta]$ becomes a fixed constant value of $5/2$ for the gravity-driven film flowing down an inclined plane with or without shear stress imposed at the liquid film surface ($\theta \neq \pi/2$). To support our claim, the analytical result is further verified through the numerical solution of the Orr-Sommerfeld boundary value problem (34)–(38) for the shear-imposed film flow [16]. As a result, in the numerical simulations, we have taken $DU = \tau$ at $y = 1$ because the results are computed for the shear-imposed film

TABLE II. Magnitude of the critical Reynolds number, $\mathrm{Re}_c = \frac{5}{2c_0}\cot\theta$, for the surface mode when the inclination angle keeps a small value. Here, $1' = (1/60)^\circ$ and $c_0 = 2$.

Falling film model	$\theta = 1^\circ$	$\theta = 4'$	$\theta = 3'$	$\theta = 1'$	$\theta = 0.5'$
Current result	71.6125	1074.3	1432.39	4297.18	8594.37
Bruin [17]	72	1070	1430	4300	8600

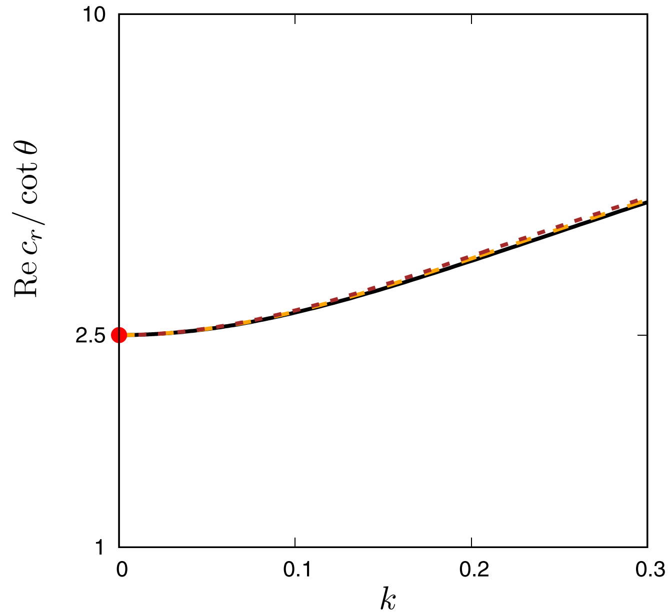

FIG. 3. Variation of the expression $\mathrm{Re}\, c_r/\cot\theta$ on a semilog scale with wave number k when the imposed shear stress τ varies. Solid, dashed, and dotted curves represent the results for $\tau = 0$, $\tau = 0.1$, and $\tau = 0.2$, respectively. The solid point represents the value, $\mathrm{Re}\, c_0/\cot\theta = 5/2$, at $k = 0$.

flow. To compute the numerical results, we have used the continuation software AUTO 07p [18]. In this continuation technique, we have provided a known solution for $k = 0$, i.e., the leading-order long-wave solution of the Orr-Sommerfeld equation is specified as an initial solution. Thereby, we have used $c_r = (1+\tau)$ and $c_i = 0$ at $k = 0$ in the STPNT subroutine of the AUTO software. After that, we continue the parameters up to the desired values. Since this is a continuation technique, the initial solution automatically captures the exact solution for a nonzero k as the parameters continue. In Fig. 3, we have depicted the variation of the expression $[\mathrm{Re}\, c_r/\cot\theta]$ on a semilog scale with wave number k when the imposed shear stress τ changes, where c_r is the phase speed of the surface mode. Here, the numerical results are produced for the glycerin-water mixture with kinematic viscosity $\nu = 5.02 \times 10^{-6}\ \mathrm{m}^2/s$, surface tension $\gamma = 69 \times 10^{-3}$ N/m, density $\rho = 1130\ \mathrm{kg/m}^3$, and inclination angle $\theta = 5.6^\circ$ [19]. Clearly, all the different curves for the expression $\mathrm{Re}\, c_r/\cot\theta$ approach the fixed constant value, $\mathrm{Re}\, c_0/\cot\theta = 5/2$, in the long-wave limit of $k \to 0$, because the phase speed c_r of the surface mode tends to c_0 at $k \to 0$. This fact is completely in favor of the long-wave analytical result given in Eq. (50). Moreover, we can observe that all the curves coalesce in the neighborhood of $k = 0$. Keeping this result in mind, we introduce a new parameter, which can be called the modified Reynolds number, defined as

$$\mathrm{Re}_M = \frac{\mathrm{Re}\, c_0}{\cot\theta}, \tag{51}$$

provided $\theta \neq \pi/2$. The reason behind the choice of this new parameter, Re_M, in the long-wave regime ($k \to 0$) is that $[\mathrm{Re}\, c_r/\cot\theta]$ tends to $[\mathrm{Re}\, c_0/\cot\theta]$ as the wave number k approaches zero. Using this new parameter, Re_M, the

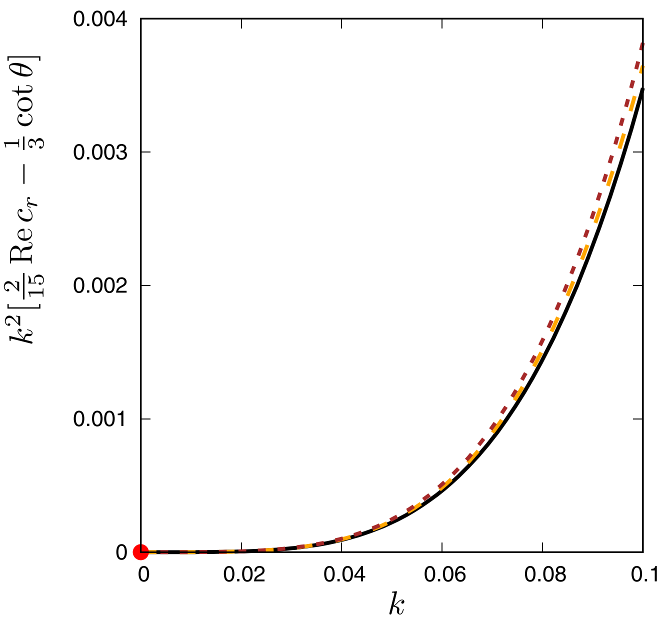

FIG. 4. Variation of the expression $k^2[\frac{2}{15}\mathrm{Re}\,c_r - \frac{1}{3}\cot\theta]$ with wave number k when the imposed shear stress τ varies. Solid, dashed, and dotted curves represent the results for $\tau = 0$, $\tau = 0.1$, and $\tau = 0.2$, respectively. The solid point represents the value, $k^2[\frac{2}{15}\mathrm{Re}_M - \frac{1}{3}]\cot\theta = 0$, at $k = 0$.

first-order expression (49) can be recast into the following form:

$$c_1 = i\left[\tfrac{2}{15}\mathrm{Re}_M - \tfrac{1}{3}\right]\cot\theta. \tag{52}$$

Hence, the basic parallel flow of falling film with or without imposed shear stress under long-wave infinitesimal disturbances will be linearly unstable if $\mathrm{Re}_M > 5/2$ when $\theta \neq \pi/2$. Otherwise, it will be linearly stable to long-wave infinitesimal disturbances if $\mathrm{Re}_M < 5/2$. It is evident that the expression of c_1 is explicitly independent of the imposed shear stress τ even for a shear-imposed film flow [10], but it depends on the new parameter Re_M and the inclination angle θ. As the phase speed c_r tends to c_0 at $k \to 0$, the mathematical expression $k^2[\frac{2}{15}\mathrm{Re}\,c_r - \frac{1}{3}\cot\theta]$ approaches the temporal growth rate

$$kc_i = |k^2c_1| = k^2\left[\tfrac{2}{15}\mathrm{Re}_M - \tfrac{1}{3}\right]\cot\theta,$$

in the limit of $k \to 0$. For this reason, we have computed the expression $k^2[\frac{2}{15}\mathrm{Re}\,c_r - \frac{1}{3}\cot\theta]$ numerically for different values of the imposed shear stress τ. The results are displayed in Fig. 4. As expected, all the different curves for the expression $k^2[\frac{2}{15}\mathrm{Re}\,c_r - \frac{1}{3}\cot\theta]$ merge in the small neighborhood of $k = 0$ when the imposed shear stress τ changes, and they attain a constant value $|k^2c_1| = k^2[\frac{2}{15}\mathrm{Re}_M - \frac{1}{3}]\cot\theta$ at $k \to 0$. Apparently, it seems that the surface instability for the current class of flow problems (gravity-driven viscous liquid films with or without imposed shear stress) is dominated by the new parameter Re_M in the long-wave regime and its critical value is a fixed constant of $5/2$, which is independent of the imposed shear stress even if a constant shear stress is applied at the liquid film surface.

V. CONCLUSIONS

The studies of gravity-driven viscous falling films with or without imposed shear stress are revisited to unify the existing results of the critical Reynolds number for the surface mode in one form. The approach is fully analytical. However, the numerical simulations are also carried out for the shear-imposed film flow. We see that the critical Reynolds number, Re_c, for the surface mode depends not only on the inclination angle, θ, but also on the leading-order phase speed, c_0. Its new analytical expression is $[5/(2c_0)]\cot\theta$, which clearly recovers the previous results for gravity-driven incompressible viscous liquid films with and without imposed shear stress as soon as the expression of leading-order phase speed is substituted. The numerical results of shear-imposed film flow are also presented to support the analytical theory. According to our results, we can predict that gravity-driven falling films with or without imposed shear stress belong to a class of flow problems whose linear stability or instability is triggered by a new parameter $\mathrm{Re}_M = \frac{\mathrm{Re}\,c_0}{\cot\theta}$ in the long-wave regime ($k \to 0$) when $\theta \neq \pi/2$. More specifically, the basic parallel flow of the aforementioned class of flow problems will be susceptible to instability by infinitesimal disturbances if $\mathrm{Re}_M > 5/2$. Indeed, the critical value of Re_M is a fixed constant of $5/2$, and it is independent of the imposed shear stress even if a constant shear stress is imposed at the film surface in the streamwise direction.

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