Macroscopic dynamics of ferromagnetic smectic-*A*

Harald Pleiner [1,*] and Helmut R. Brand [1,2,†]
[1]*Max Planck Institute for Polymer Research, 55021 Mainz, Germany*
[2]*Department of Physics, University of Bayreuth, 95440 Bayreuth, Germany*

(Received 18 March 2024; accepted 3 June 2024; published 26 June 2024)

We derive the macroscopic dynamic equations for ferromagnetic smectic-*A* liquid crystals for which the spontaneous magnetization is parallel to the layer normal of the layering. As additional macroscopic variables when compared to simple fluids, we have the layer displacement u, familiar from smectic liquid crystals, and the magnetization density $\boldsymbol{M}$. We find a number of reversible and dissipative cross-coupling terms to the additional macroscopic variables and discuss possible experiments to detect them. Among other effects, we point out that the velocity of first sound becomes anisotropic due to the influence of the modulus of the magnetization, while the magnitude of the velocity of second sound is modified. As for the static behavior, we find cross-coupling terms between the magnitude of the magnetization, on the one hand, and layer compression as well as osmotic pressure, on the other hand. In addition, we point out that as a dissipative effect, temperature gradients can induce gradients in the magnetization parallel to the layer normal, mediated by layer compressions.

DOI: 10.1103/PhysRevE.109.064612

I. INTRODUCTION

Here we investigate a system combining layering as in a smectic liquid crystal with ferromagnetism. For over three decades, only magnetizable nematic liquid crystals (called ferronematics), without a spontaneous permanent magnetization, have been found [1–7]. These experimental efforts were stimulated by the pioneering paper of Brochard and de Gennes [8] on ferronematics.

This situation completely changed about a decade ago with the synthesis of ferromagnetic nematics [9,10]. Since then, these systems have been investigated by a number of groups, experimentally [11–15] and theoretically [16,17]. More recently, ferromagnetic cholesterics have also been reported and characterized [18].

Regarding smectic systems with magnetic properties, until now only lyotropic systems have been studied [19,20], where a ferrofluid made of magnetic (micellar) particles is put into the layers (compare Ref. [21] for an overview of the macroscopic properties of isotropic magnetic liquids). These lyotropic systems do not show magnetic ordering and so far no ferromagnetic smectic liquid crystals have been reported experimentally.

It seems worthwhile to contrast this situation with the question of ferro *electricity* in liquid crystals. Ferroelectric nematic liquid crystals were prepared and characterized by two groups about five years ago [22,23] and have since been studied experimentally by various groups [24–37]. Also, there has been theoretical work on ferroelectric nematics before their discovery and ever since [38–41]. In 2022, ferroelectric smectic-*A* phases with $C_{\infty v}$ symmetry were also found experimentally [42–44]. In these liquid crystalline phases, the permanent electric polarization is oriented parallel to the layer normal of the smectic layers, which otherwise show in-plane fluidity just like a classical nonpolar smectic-*A* phase [42–44].

Quite recently, a macroscopic description of ferroelectric smectic-A_F phases was elucidated [45]. It was shown that aside from static and dissipative cross-coupling terms, the sound wave spectrum is also significantly altered compared to that of a nonpolar smectic-*A* phase [45]. It should be noted that the macroscopic properties of nonpolar smectic-*A* phases (without being ferroelectric or ferromagnetic) have been studied for decades [46–49].

Based on this situation regarding the multiferroic properties of nematic and smectic liquid crystalline phases, here we study the macroscopic properties of smectic-A_M, a ferromagnetic smectic-*A* phase. We select the simplest possibility, namely, a phase for which the spontaneous magnetization is oriented parallel to the smectic layer normal.

The approach used is that of macroscopic dynamics [50]. This approach has been used to describe a large number of condensed matter systems with spontaneously broken continuous symmetries and macroscopic variables, including nematic liquid crystals [47,50,51], smectic-*A* liquid crystals, crystals [47], magnetic systems [52], and ferroelectric nematics, as well as various types of superfluids, including superfluid ^{4}He [53] and the superfluid phases of ^{3}He in the bulk [54–57] as well as in thin layers [58]. Recently, this approach was applied to ferromagnetic nematics as well as to ferroelectric smectic-A_F [45].

*Contact author: pleiner@mpip-mainz.mpg.de
†Contact author: brand@uni-bayreuth.de

In the smectic-A_M phase, we have as variables, in addition to those from a miscible binary mixture, the layer displacement of the smectic layers, the modulus of the spontaneous magnetization, and the variations in the orientation of the magnetization.

Of particular interest are two main features concerning the macroscopic dynamics: first, the differences between ferromagnetic nematics and ferromagnetic smectics due to the layering, and, second, the differences between ferroelectric smectic-A_F, studied recently, and ferromagnetic smectic-A_M studied here. The differences due to the different behavior of the spontaneous magnetization and the spontaneous macroscopic polarization under parity and time reversal are crucial. These differences in turn lead to different cross-coupling terms to the layer displacement statically and dynamically in the two systems, both of which are multiferroic in the sense that one has a high degree of order with respect to two properties: layering in both cases, and ferromagnetism and ferroelectricity for ferromagnetic smectic-A_M and ferroelectric smectic-A_F, respectively.

The paper is organized as follows. In Sec. II, we discuss in detail the macroscopic variables that are used: conserved quantities, variables associated with spontaneously broken continuous symmetries, and macroscopic variables, which relax on long, but finite and macroscopically relevant timescales. In Sec. III, we present the free-energy functional, thereby defining the thermodynamic conjugates. The macroscopic dynamic equations (Sec. IV) are followed in Secs. V and VI by the reversible and dissipative phenomenological currents. In Sec. VII, we analyze the effects of selected cross-coupling terms which might be suitable for experimental detection in ferromagnetic smectic-A_M and help to identify the phase. A summary and perspective are given in Sec. VIII.

II. VARIABLES

The system we study here is the simplest possibility of a ferromagnetic smectic-A_M liquid crystalline phase. We assume a smectic layer structure (with the layer normal $\hat{p}_i$) of the same nature as a regular smectic-A phase [47], and, in addition, a ferromagnetic structure,

$$M_i = M\hat{m}_i, \tag{1}$$

with a permanent magnetization M along the direction $\hat{m}_i$. In Fig. 1, we have sketched one layer of a ferromagnetic smectic-A_M phase. We consider the case where the two preferred directions are parallel in equilibrium, $\hat{m}_i^0 \parallel \hat{p}_i^0$. In the presence of external fields, they will, in general, no longer be parallel. This feature resembles the analogous situation in ferromagnetic nematics [10,11]. The preferred directions are described by unit vectors $\hat{\boldsymbol{p}}^2 = 1$ and $\hat{\boldsymbol{m}}^2 = 1$.

The set of macroscopic variables, which arises for such a system, comprises, first, the usual conserved quantities of a simple fluid, mass density ρ, density of momentum $\boldsymbol{g}$, and energy density ε, and, second, variables related to the internal structure. The smectic layer structure (the one-dimensional density wave) spontaneously breaks translational symmetry along $\hat{p}_i$, giving rise to a displacement variable $u = u_i\hat{p}_i$ with u_i the displacement vector. Homogeneous translation $u = \text{const}$ does not cost energy, while first-order gradients of

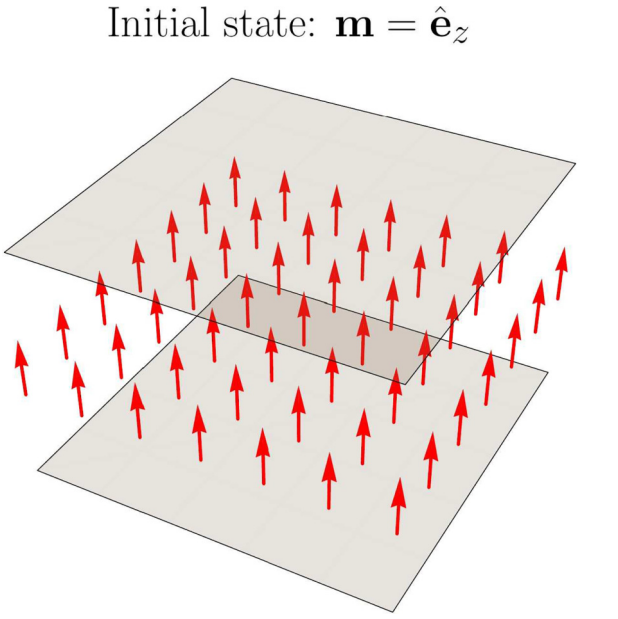

FIG. 1. One layer of the ground state of a ferromagnetic smectic-A_M phase. The arrows indicate the direction of the ferromagnetic magnetization. The normal to the smectic layering is taken to be in the $\hat{z}$ direction. (Adapted from Ref. [59]).

u describe compression or dilation and rotations and second-order gradients account for bent of the layers. Within the perpendicular layers, a smectic-A_M liquid crystal is isotropic and no structural variable occurs.

The existence of the preferred direction $\hat{p}_i$ breaks rotational symmetry and the two rotations δp_i (with $\hat{p}_i\delta\hat{p}_i = 0$) are the appropriate variables. According to the smectic-A structure, δp_i and u are not independent, but related (in linear order) by.

$$\delta\hat{p}_i = -\delta_{ij}^{\perp}\nabla_j u, \tag{2}$$

with the transverse Kronecker symbol $\delta_{ij}^{\perp} = \delta_{ij} - \hat{p}_i\hat{p}_j$. Of course, compression or dilation of the layers, $\hat{p}_i\nabla_i u$, cannot be described by $\delta\hat{p}_i$. Therefore, we will use the gradients of u as the hydrodynamic variables. The smectic and the nematic order parameter will not be taken into account since, generally, they relax very rapidly.

The magnetic structure, given by Eq. (1), gives rise to rotations of the direction, $\delta\hat{m}_i$, and fluctuations of the magnetization, δM (the magnetic order parameter). There is a finite equilibrium magnetization M_0, and $\delta M = M - M_0$ is a macroscopic variable, which relaxes on a long, but finite timescale.

To set up the hydrodynamic equations, we make use of symmetries, such as translational and rotational symmetries, spatial inversion, and time-reversal symmetry. In the present case, we also have to respect the $\hat{p}_i \to -\hat{p}_i$ invariance that is inherited from the smectic-A structure. As a result, $\hat{p}_i$ behaves effectively as an axial vector. This is similar to the director $\hat{n}_i$ in nematic liquid crystals with the $\hat{n}_i \to -\hat{n}_i$ invariance. In particular, the layer displacement $u = \hat{p}_i u_i$, associated with the density wave, is odd in $\hat{p}_i$, odd under space inversion, and even under time reversal. The vectorial variables $\delta\hat{m}_i$, associated with a broken rotational symmetry, are even in $\hat{p}_i$

and even under parity (axial vectors), and odd under time reversal. Finally, the macroscopic variable δM is a true scalar, even under parity and time reversal. To account for mixture effects, we use a concentration variable $\delta\phi$. Depending on the specific material under consideration, it might be interpreted as a mixture effect between magnetic and nonmagnetic materials or as the density of impurities, giving rise to a conserved variable or a nonconserved one, respectively.

Throughout this paper, we will not consider two-fluid effects bringing along a second velocity. Such effects are characteristic for immiscible and/or partially miscible complex fluids that allow for long-living relative motions (e.g., between magnetic and nonmagnetic structures) or due to additional broken symmetries (e.g., gauge symmetry in superfluids [53]). A general systematic macroscopic approach for these systems has been presented in Ref. [60]. Recently, this approach has been applied to a number of systems including nematic liquid crystals with smectic clusters [61].

Assuming local thermodynamic equilibrium, i.e., all fast relaxing quantities being already in equilibrium, the local formulation of the first law of thermodynamics, including the magnetic induction $\boldsymbol{B}$, reads [47,50,62]

$$\begin{aligned} d\varepsilon = {} & T d\sigma + \mu d\rho + \Pi d\phi + v_i dg_i + H_i dB_i \\ & + h_i^m d\hat{m}_i + h^M dM + h_i^u d\nabla_i u, \end{aligned} \tag{3}$$

connecting the macroscopic variables to the entropy density σ. In Eq. (3), the thermodynamic quantities, i.e., chemical potential (μ), temperature (T), relative chemical potential (Π), the molecular field h_i^u associated with the layer displacement u, the magnetic Maxwell field (H_i) and the magnetic molecular fields (h_i^m and h^M), are defined as partial derivatives of the energy density with respect to the appropriate variables [50]. Since the free energy has to be invariant under spatial rotations, time inversion, and $\hat{p}_i \to -\hat{p}_i$ invariance, the conjugate quantities inherit the symmetry properties of the variables.

While we use the assumption of *local thermodynamic equilibrium*, we would like to emphasize that this approach is nevertheless applicable to many nonequilibrium situations. As a typical example, we refer to the Navier-Stokes equations for fluids, which are also used to describe the behavior of hydrodynamic instabilities and the onset of turbulence as well as fully developed turbulence. This situation should be contrasted to approaches assuming the existence of a generalized thermodynamic potential for the complete dynamics, an approach which cannot even describe oscillatory instabilities near the onset of a pattern-forming instability, such as in the case of Rayleigh-Bénard convection.

III. STATIC CONSIDERATIONS

First, we analyze the static properties of smectic-A_M without taking into account the static Maxwell equations in the magnetic domain. Field effects will be discussed later in this section.

The static behavior of the macroscopic system studied here is conveniently described by the energy functional in harmonic approximation; see Refs. [39,47,50]. We obtain, including the kinetic energy density,

$$\begin{aligned} \varepsilon = {} & +\frac{1}{2}c_M(\delta M)^2 + \frac{1}{2}L_{ij}(\nabla_i M)(\nabla_j M) + \frac{1}{2}B(\nabla_z u)^2 + \frac{1}{2}K(\nabla_\perp^2 u)^2 \\ & + \frac{1}{2}c_{\rho\rho}(\delta\rho)^2 + \frac{1}{2}c_{\sigma\sigma}(\delta\sigma)^2 + \frac{1}{2}c_{\phi\phi}(\delta\phi)^2 + c_{\rho\phi}(\delta\rho)(\delta\phi) + c_{\rho\sigma}(\delta\rho)(\delta\sigma) + c_{\sigma\phi}(\delta\sigma)(\delta\phi) \\ & + (\gamma_1\delta\rho + \gamma_2\delta\sigma + \gamma_3\delta\phi + \gamma_4[\nabla_z u])\,\delta M + (\theta_1\delta\rho + \theta_2\delta\sigma + \theta_3\delta\phi)\,\nabla_z u \\ & + \frac{1}{2}F(\hat{\boldsymbol{m}} \times \delta\hat{\boldsymbol{m}} - \hat{\boldsymbol{p}} \times \delta\hat{\boldsymbol{p}})^2 + \frac{1}{2\rho}\boldsymbol{g}^2 + \frac{1}{2}K^m_{ijkl}(\nabla_i\hat{m}_j)(\nabla_k\hat{m}_l), \end{aligned} \tag{4}$$

where ∇_z is a shorthand notation for $\hat{p}_i\nabla_i$. A "δ" denotes deviations from the equilibrium value, in particular $\delta M = M - M_0$, $\delta\hat{m}_i = \hat{m}_i - \hat{m}_i^0$, $\delta\phi = \phi - \phi_0$, $\delta\rho = \rho - \rho_0$, $\delta T = T - T_0$, and $\delta\hat{p}_i = \hat{p}_i - \hat{p}_i^0$, which is given in Eq. (2).

The stiffness of fluctuations of the magnetization is given by c_M. Inhomogeneous deviations of the magnetization are described by $\sim L_{ij} = L_\parallel \hat{p}_i\hat{p}_j + L_\perp\delta_{ij}^\perp$.

Gradients of u describe deformations of the layers, in particular, compression or dilation ($\sim B$) and layer bending ($\sim K$). They are of the standard form familiar from ordinary smectic-A [48]. Bent of the layers is related to splay of $\hat{p}_i$, while bent of $\hat{p}_i$ leads to an energy $\sim\delta_{ij}^\perp(\nabla_z\nabla_i u)(\nabla_z\nabla_j u)$, which is generally neglected. Twist of $\hat{p}_i$ is forbidden in smectic-A structures.

In addition, the energy density of a fluid binary mixture, involving $\delta\sigma$, $\delta\rho$, and $\delta\phi$, is as in the nonmagnetic case. Related to δM are the contributions $\sim\gamma_1$, γ_2, γ_3, and $\sim\gamma_4$. The coupling terms $\sim\delta M$ associated with the contributions $\sim\gamma_1$, γ_2, and γ_3 represent coupling terms between the variations of the modulus of the magnetization, δM, and other scalar variables. These include the variations of density, $\delta\rho$, entropy density $\delta\sigma$, as well as of concentration, $\delta\phi$. A rather specific and quite unusual static cross-coupling term is the contribution $\sim\gamma_4$. It reflects a coupling between variations of the modulus of the magnetization and compression of the layering, $\nabla_z u$. In the present system, δM and $\nabla_z u$ are good scalars, that is, they are invariant under all transformations and can thus lead to a couple of extraordinary effects: a variation of the magnitude of the magnetization leads to a compression of the layering and, vice versa, a layer compression leads to variations of the magnitude of the magnetization. This coupling can arise for ferromagnetic smectics, which possess both variables, and is therefore quite unique.

We also note that as usual, one has static cross-coupling terms between layer compressions, $\nabla_z u$, and variations of density, entropy density, and concentration variations. This

type of static cross-coupling term exists for all layered (smectic) liquid crystalline phases.

The contribution $\sim F$ is zero for a combined rotation of $\hat{m}_i$ and $\hat{p}_i$ reflecting the *spontaneously* broken rotational symmetry for that case. Of course, relative rotations between $\hat{m}_i$ and $\hat{p}_i$ do cost energy, as described by the F term. In harmonic approximation, it reads

$$\tfrac{1}{2}F\delta_{ij}^{\perp}\big[\delta\hat{m}_i\delta\hat{m}_j + (\nabla_i u)(\nabla_j u) + 2\hat{m}_k^0\hat{p}_k^0\delta\hat{m}_i(\nabla_j u)\big]. \tag{5}$$

Relative rotations have been introduced by de Gennes [63] for nematic liquid crystalline elastomers and their effects have also been incorporated in the macroscopic dynamics of uniaxial magnetic gels [64]. In ordinary smectic-A liquid crystals (where no $\delta\hat{m}_i$ exists), this contribution has to be absent, $F = 0$.

The structure of the gradient energy, $\sim K^m_{ijkl}$, is identical with that of Frank's deformation energy for the director field of uniaxial nematic liquid crystals [49] with the director gradients $\nabla_j\hat{n}_i$ replaced by $\nabla_j\hat{m}_i$. It describes splay, bend, and twist deformations of $\hat{m}_i$. Since twist in $\nabla_j\hat{n}_i$ is impossible, the twist in $\nabla_j\hat{m}_i$ also has to be small, in order to keep the quadratic structure of the relative rotations in Eq. (5) valid.

A positive-definite free energy is necessary to guarantee static stability of the system. This is obtained by the conditions $c_M > 0$, $F > 0$, $B > 0$, $L_\perp > 0$, $L_\parallel > 0$, $K_1^m > 0$, $K_2^m > 0$, $K_3^m > 0$, and $\gamma_4^2 < c_M B$, as well as $\gamma_1^2 < c_M c_{\rho\rho}$, $\gamma_2^2 < c_M c_{\sigma\sigma}$, and $\gamma_3^2 < c_M c_{\phi\phi}$, which are related to the magnetic quantities δM, $\nabla_i M$, and $\delta\hat{m}_i$.

Although the energy density given by Eq. (4) is bilinear in deviations from equilibrium, there are intrinsic nonlinear effects since the material parameters can be functions of the (scalar) state variables, such as temperature, density, and magnetization M.

The harmonic approximation is a restriction to sufficiently small deviations from the spatially homogeneous ground state. Big changes, such as a complete reorientation of M_i as in the magnetic Frederiks transition [48], require a fully nonlinear analysis of all the variables that are involved. In particular, a replacement $\nabla_z u \to \nabla_z u - \frac{1}{2}(\nabla_\perp u)^2$ in Eq. (4) takes care of effective layer compressions due to rotations, as explained in Fig. 3(b) of Ref. [65].

We now discuss the influence of an external magnetic field. An external magnetic field constitutes another preferred direction H_i, which we will take as the z direction when necessary. In equilibrium, the magnetization is parallel to the field, $H_i \parallel \hat{m}_i$, due to the static Maxwell coupling energy $-H_i M_i$. For deviations of the equilibrium orientation, the appropriate ferromagnetic energy is

$$\varepsilon_m = \tfrac{1}{2}M_0 H\delta_{ij}^{\perp}\delta\hat{m}_i\delta\hat{m}_j, \tag{6}$$

which is linear in the field. Equation (6) is based on the approximate relation $\hat{\boldsymbol{m}} \approx \{\hat{m}_x, \hat{m}_y, 1 - \frac{1}{2}(\hat{m}_x^2 + \hat{m}_y^2)\}$ for any unit vector that is (slightly) tilted away from the z direction. Up to quadratic order in $\hat{\boldsymbol{m}}_\perp^2$, this expression correctly gives $\hat{\boldsymbol{m}}^2 = 1$

Regarding the smectic layer normal, we assume that in equilibrium the layer normal is parallel to the field. In nonequilibrium, the well-known magnetic anisotropy energy applies,

$$\varepsilon_a = \tfrac{1}{2}\chi_a H^2(\nabla_\perp u)^2, \tag{7}$$

with $\chi_a > 0$ [49].

According to the Gibbs relation, given by Eq. (3), the conjugate quantities to the hydrodynamic and macroscopic variables follow from the free energy as variational or partial derivatives with respect to the appropriate variable, while all the other variables are kept constant. Including the field energies ε_a and ε_m, we get

$$h'^M = \left.\frac{\partial\varepsilon}{\partial\delta M}\right|_{\dots} = c_M\delta M + \gamma_1\delta\rho + \gamma_2\delta\sigma + \gamma_3\delta\phi + \gamma_4\nabla_z u, \tag{8}$$

$$\Phi_i^M = \left.\frac{\partial\varepsilon}{\partial(\nabla_i M)}\right|_{\dots} = (L_\parallel\hat{p}_i\hat{p}_j + L_\perp\delta_{ij}^{\perp})\nabla_j M, \tag{9}$$

$$h_i'^u = \left.\frac{\partial\tilde{\varepsilon}}{\partial\nabla_i u}\right|_{\dots} = \hat{p}_i(B\nabla_z u + \theta_1\delta\rho + \theta_2\delta\sigma + \theta_3\delta\phi + \gamma_4\delta M) + \delta_{ij}^{\perp}(\chi_a H^2\nabla_j u + F[\delta\hat{m}_j + \nabla_j u]), \tag{10}$$

$$\Phi_{ij}^u = \left.\frac{\partial\varepsilon}{\partial\nabla_i\nabla_j u}\right|_{\dots} = \delta_{ij}^{\perp}K\nabla_\perp^2 u, \tag{11}$$

$$h_i'^m = \left.\frac{\partial\varepsilon}{\partial\hat{m}_i}\right|_{\dots} = \delta_{ij}^{\perp}(M_0 H\delta\hat{m}_j + F[\nabla_j u + \delta\hat{m}_j]), \tag{12}$$

$$\Phi_{ij}^m = \left.\frac{\partial\varepsilon}{\partial(\nabla_i\hat{m}_j)}\right|_{\dots} = K_{ijkl}^m\nabla_l\hat{m}_k, \tag{13}$$

$$\delta\mu = \left.\frac{\partial\varepsilon}{\partial\delta\rho}\right|_{\dots} = \gamma_1\delta M + \theta_1\nabla_z u + c_{\rho\rho}\delta\rho + c_{\rho\phi}\delta\phi + c_{\rho\sigma}\delta\sigma, \tag{14}$$

$$\delta T = \left.\frac{\partial\varepsilon}{\partial\delta\sigma}\right|_{\dots} = \gamma_2\delta M + \theta_2\nabla_z u + c_{\sigma\sigma}\delta\sigma + c_{\rho\sigma}\delta\rho + c_{\sigma\phi}\delta\phi, \tag{15}$$

$$\delta\Pi = \left.\frac{\partial\varepsilon}{\partial\delta\phi}\right|_{\dots} = \gamma_3\delta M + \theta_3\nabla_z u + c_{\phi\phi}\delta\phi + c_{\phi\rho}\delta\rho + c_{\phi\sigma}\delta\sigma, \tag{16}$$

$$v_i = \left.\frac{\partial\varepsilon}{\partial g_i}\right|_{\dots} = g_i/\rho, \tag{17}$$

with $\delta_{ij}^{\perp} = \delta_{ij} - \hat{p}_i^0\hat{p}_j^0 = \delta_{ij} - \hat{m}_i^0\hat{m}_j^0$.

For the variables δM, $\delta\hat{m}_i$, and $\nabla_i u$, which enter the energy at different gradient levels, we have split the conjugates $h^M = h'^M - \nabla_i\Phi_i^M$, $h_i^m = h_i'^m - \nabla_j\Phi_{ij}^m$, and $h_i^u = h_i'^u - \nabla_j\Phi_{ij}^u$.

All the static susceptibilities can be a function of the external field. In a ferromagnetic phase, a linear field dependence

$\sim M_0 H$ is possible, but also the conventional $\sim H^2$ dependence independent of M_0 is possible. In addition, M_0 is changed by the external field, $M(H) = M_0 + \chi_\parallel H$ in linear order, with $\chi_\parallel$ the longitudinal magnetic susceptibility. In the general case, $M(H)$ is a function that can be taken from experiment or extended simulations. In the following, we will write M [instead of $M(H)$] for simplicity.

IV. DYNAMICS

The hydrodynamic equations for conserved, broken-symmetry, and slowly relaxing variables are

$$\dot{\rho} + \nabla_i(\rho v_i) = 0, \tag{18}$$

$$\dot{\sigma} + \nabla_i\big(\sigma v_i + j_i^{\sigma \mathrm{R}} + j_i^{\sigma \mathrm{D}}\big) = \frac{2R}{T}, \tag{19}$$

$$\dot{g}_i + \nabla_j\big(g_i v_j + p\,\delta_{ij} - \hat{p}_i h_j^u + \sigma_{ij}^{\mathrm{th}} + \sigma_{ij}^{\mathrm{R}} + \sigma_{ij}^{\mathrm{D}}\big) = 0, \tag{20}$$

$$\dot{\phi} + v_j \nabla_j \phi + J^{\phi \mathrm{R}} + J^{\phi \mathrm{D}} + \nabla_i\big(j_i^{\phi \mathrm{R}} + j_i^{\phi \mathrm{D}}\big) = 0, \tag{21}$$

$$\dot{u} + v_j \nabla_j u - v_z + X^{u \mathrm{R}} + X^{u \mathrm{D}} = 0, \tag{22}$$

$$\dot{M} + v_i \nabla_i M + X^{M \mathrm{R}} + X^{M \mathrm{D}} = 0, \tag{23}$$

$$\dot{\hat{m}}_i + v_j \nabla_j \hat{m}_i + (\hat{\boldsymbol{m}} \times \boldsymbol{\omega})_i + X_i^{m \mathrm{R}} + X_i^{m \mathrm{D}} = 0, \tag{24}$$

where $\omega_i = \frac{1}{2}\epsilon_{ijk}\nabla_j v_k$ is the vorticity. The conserved quantities and the entropy density contain the divergence of a phenomenological current, while quasicurrents (without a divergence) are associated with spontaneously broken continuous symmetry variables or macroscopic variables. In particular, Eq. (23) describes the magnetization order parameter modulus as a slowly relaxing quantity (similar to, e.g., the nematic order parameter modulus [66] or the superfluid degree of order [53,67]). For the concentration variable, we have allowed for both possibilities, conserved or nonconserved. Only for $J^{\phi \mathrm{R}} + J^{\phi \mathrm{D}} = 0$ is $\delta\phi$ conserved.

The superscripts D and R on the currents denote, respectively, the dissipative and reversible parts.

We use the pressure p including the isotropic part of the Maxwell stress,

$$p = \frac{\partial\,(\int \varepsilon dV)}{\partial V} = -\varepsilon + \mu\rho + T\sigma + \boldsymbol{v}\cdot\boldsymbol{g} + B_i H_i, \tag{25}$$

and the off-diagonal terms of the Maxwell and the Ericksen-type stresses,

$$\begin{aligned} 2\sigma_{ij}^{\mathrm{th}} = &-(H_i B_j + B_i H_j) + \Phi_j^M \nabla_i M + \Phi_i^M \nabla_j M \\ &+ \Phi_{ki}^u \nabla_k \nabla_j u + \Phi_{kj}^u \nabla_k \nabla_i u. \end{aligned} \tag{26}$$

The Maxwell stress is of the standard form [68,69] with $B_i = H_i + M_i$ and has been symmetrized with the help of the requirement that the energy density should be invariant under rigid rotations [47,50].

The entropy production $2R/T$ in Eq. (19) is a measure for the energy dissipation of the system. Due to the second law of thermodynamics, the dissipation R must satisfy $\int R\,dV \geqslant 0$: For irreversible processes, R is positive, for reversible ones, it is equal to zero (or a total divergence),

$$\begin{aligned} 2R = &-j_i^{\sigma *}\nabla_i T + J^{\phi *}\Pi - j_i^{\phi *}\nabla_i \Pi - \sigma_{ij}^* A_{ij} \\ &+ X_i^{m *} h_i^m - X^{u *}\nabla_i h_i^u + X^{M *} h^M \geqslant 0, \end{aligned} \tag{27}$$

where the upper sign applies to $* = D$ and the lower one to $* = R$.

The energy conservation law,

$$\dot{\varepsilon} + \nabla_i(\varepsilon + p)\, v_i + \nabla_i\big(j_i^{\varepsilon \mathrm{R}} + j_i^{\varepsilon \mathrm{D}}\big) = 0, \tag{28}$$

is redundant due to the Gibbs relation, given by Eq. (3), and follows from Eqs. (18)–(26) and (27). In particular, $j_i^{\varepsilon \mathrm{D}} = T\, j_i^{\sigma \mathrm{D}} + \Pi\, j_i^{\phi \mathrm{D}} + v_j \sigma_{ij}^{\mathrm{D}} + h_i^u X^{u \mathrm{D}}$.

The phenomenological currents and quasicurrents are the sum of the reversible and the dissipative parts, as can be seen in Eqs. (19)–(23). The various transport contributions in Eqs. (18)–(23) (as well as p and $\sigma_{ij}^{\mathrm{th}}$) are reversible and add up to zero in the entropy production. They are not material dependent, but are given by general symmetry and thermodynamic principles [50], such as transformation behavior under translations (convective terms) or rotations (e.g., $\ldots \times \boldsymbol{\omega}$).

These phenomenological currents and quasicurrents are treated in the following sections within "linear irreversible thermodynamics" (guaranteeing general Onsager relations), i.e., as linear relations between currents and thermodynamic forces. The resulting expressions are nevertheless nonlinear since all material parameters can be functions of the scalar state variables (e.g., σ, ρ, M, ϕ).

V. REVERSIBLE CURRENTS

The reversible phenomenological currents, with superscript R in Eqs. (18)–(24), give rise to zero entropy production, given by Eq. (27). Taking into account the symmetry behavior regarding space inversion and time reversal as well as under $\hat{p}_i \to -\hat{p}_i$ invariance, we obtain

$$j_i^{\sigma R} = -\kappa_{ij}^R \nabla_j T - D_{ij}^{TR} \nabla_j \Pi, \tag{29}$$

$$j_i^{\phi \mathrm{R}} = -D_{ij}^R \nabla_j \Pi - D_{ij}^{TR} \nabla_j T, \tag{30}$$

$$J^{\phi \mathrm{R}} = 0, \tag{31}$$

$$\sigma_{ij}^{\mathrm{R}} = \lambda_{ij}^M h^M - c_{kij}^R h_k^m - \nu_{ijkl}^R A_{kl}, \tag{32}$$

$$X^{u \mathrm{R}} = 0, \tag{33}$$

$$X_i^{m \mathrm{R}} = \gamma_L \epsilon_{ijk} \hat{m}_j h_k^m - c_{ijk}^R A_{jk}, \tag{34}$$

$$X^{M \mathrm{R}} = \lambda_{ij}^M A_{ij}, \tag{35}$$

with $A_{ij} = \frac{1}{2}(\nabla_i v_j + \nabla_j v_i)$. In Eq. (34), we have explicitly written the Larmor-type reversible rotation ($\sim\gamma_L$) of $\hat{m}_i$.

The material tensors with a superscript R have to be odd under time reversal and therefore must contain an odd number of $\hat{m}_i$ factors. In addition they are even in $\hat{p}_i$.

The second rank tensors κ_{ij}^R, D_{ij}^R, and D_{ij}^{TR}, which are the reversible analogues to (the dissipative) heat conduction, diffusion, and thermodiffusion, respectively, have the form

$$\kappa_{ij}^R = \kappa_1^R \epsilon_{ijk} \hat{m}_k^0. \tag{36}$$

They are antisymmetric $\kappa^R_{ij} = -\kappa^R_{ji}$, which ensures $R = 0$, and they fulfill the general Onsager relation $\kappa^R_{ij}(-\hat{m}_i) = -\kappa^R_{ij}(\hat{m}_i)$ based on time-reversal properties $\hat{m}_i \to -\hat{m}_i$ of the magnetization. It should be noted that $j^{\sigma\mathrm{R}}_i$ and $j^{\phi\mathrm{R}}_i$ are divergence free and do not contribute to $\dot{\sigma}$ or $\dot{\phi}$, respectively.

The symmetric tensor $\lambda^M_{ij} = \lambda_\parallel \hat{p}^0_i \hat{p}^0_j + \lambda_\perp \delta^\perp_{ij}$ is of the standard form and connects reversibly magnetization M with flow A_{ij} or stress tensor σ_{ij}. A similar dynamic cross coupling with the magnetic orientation $\hat{m}_i$ is provided by the third rank tensor which is odd under time reversal and is symmetric in the last two indices,

$$c^R_{ijk} = \tfrac{1}{2} c^R \big(\delta^\perp_{ij}\hat{m}^0_k + \delta^\perp_{ik}\hat{m}^0_j\big). \tag{37}$$

We note that Eqs. (36) and (37) are special cases of the appropriate discussions given in Refs. [70] for the ferronematic case.

The reversible analog to the viscosity tensor ν^R_{ijkl} is symmetric under the exchange of indices $i \leftrightarrow j$ and $k \leftrightarrow l$, separately, but is antisymmetric for $\nu^R_{ijkl} \leftrightarrow -\nu^R_{klij}$ and is odd under time reversal, guaranteeing $R = 0$:

$$\begin{aligned}\nu^R_{ijkl} = {} & +\nu^R_1(\epsilon_{ikq}\delta^\perp_{jl} + \epsilon_{ilq}\delta^\perp_{jk} + \epsilon_{jlq}\delta^\perp_{ik} + \epsilon_{jkq}\delta^\perp_{il})\,\hat{m}^0_q \\ & + \nu^R_2\big(\epsilon_{ikq}\hat{m}^0_j\hat{m}^0_l + \epsilon_{ilq}\hat{m}^0_j\hat{m}^0_k \\ & + \epsilon_{jlq}\hat{m}^0_i\hat{m}^0_k + \epsilon_{jkq}\hat{m}^0_i\hat{m}^0_l\big)\,\hat{m}^0_q. \end{aligned} \tag{38}$$

Close inspection of the two terms given in Eq. (38) shows that they are isomorphic to the same type of terms given by Liu [55] for the hydrodynamics of the superfluid A phase of ^{3}He [54]. This common feature between the two systems can be traced back to the fact that both have a unit vector, which is odd under time reversal and is axial in nature: $\hat{l}^0_i$ in the case of ^{3}He-A [54,55] and $\hat{m}^0_i$ in the case of ferromagnetic smectic A_M. Here, $\hat{l}^0_i$ is the preferred direction in (orbit) space in the superfluid A phase of ^{3}He; it characterizes the broken rotational symmetry that exists in ^{3}He-A. In contrast to the director in nematics, $\hat{n}_i$, $\hat{l}^0_i$ is odd under time reversal due to the symmetry properties of the order parameter for the superfluid phases of ^{3}He. From this observation, it emerges that one has reversible coupling terms between different components of the density of momentum for all systems for which such a unit vector exists. These include a number of liquid crystalline phases, including ferronematics [70] und ferromagnetic nematics, as well as various superfluid phases of ^{3}He, including the A_1 phase [56,57] and also systems with an axial dynamic preferred direction [71].

VI. DISSIPATIVE CURRENTS

For the derivation of the dissipative parts of the phenomenological currents, one usually expands the dissipation function R to second order in the thermodynamic forces and then obtains the dissipative currents by taking the variational derivatives with respect to the forces. We find, for the dissipation function of the A_M phase,

$$\begin{aligned} 2R = {} & \kappa_{ij}(\nabla_i T)\big(\nabla_j T\big) + D_{ij}(\nabla_i \Pi)(\nabla_j \Pi) + 2D^{T\Pi}_{ij}(\nabla_i T)(\nabla_j \Pi) + \tilde{\mu}(\delta\Pi)^2 + 2\mu_M(\delta\Pi)h^M + b^D_\parallel (h^M)^2 + b^D_\perp \delta^\perp_{ij} h^m_i h^m_j \\ & + \nu_{ijkl}A_{ij}A_{kl} + 2D^{u\Pi}(\nabla_z \Pi)\big(\nabla_i h^u_i\big) + 2D^{uT}(\nabla_z T)\big(\nabla_i h^u_i\big) + c^D_{ijk}h^m_i A_{jk} + \frac{1}{\gamma_u}\big(\nabla_i h^u_i\big)\big(\nabla_j h^u_j\big). \end{aligned} \tag{39}$$

Here, ν_{ijkl} is the uniaxial viscosity tensor [47,50,72], while by the symmetric second rank tensors κ_{ij}, D_{ij}, and D^T_{ij}, heat conduction, diffusion, and thermodiffusion, respectively, are described. Smectic permeation is given by one coefficient $1/\gamma_u$, while the magnetization relaxation contains a transverse coefficient $b^D_\perp$ (related to $\delta\hat{m}_i$) and a longitudinal one $b^D_\parallel$ (related to δM). Concentration relaxation is governed by $\tilde{\mu}$ and coupled to the relaxation of δM via μ_M. For a conserved concentration variable, $\tilde{\mu} = 0$ and $\mu_M = 0$.

The third rank tensor c^D_{ijk} is specific for smectic-A_M and contains one coefficient,

$$c^D_{ijk} = c^D\big(\epsilon_{imk}\hat{p}^0_m\hat{p}^0_j + \epsilon_{imj}\hat{p}^0_m\hat{p}^0_k\big), \tag{40}$$

describing a dissipative coupling between flow and the orientation of the magnetization.

The range of possible values of the coefficients in Eq. (39) is restricted by the positivity of the entropy production that requires, e.g., $D_\parallel/\gamma_u > (D^{u\Pi})^2$, $\kappa_\parallel/\gamma_u > (D^{uT})^2$, $\nu_3 b^D_\perp > (c^D)^2$, with $b^D_\perp$, $b^D_\parallel$, and γ_u all positive.

To obtain the dissipative parts of the currents and quasicurrents, we take the partial derivatives of R with respect to the appropriate thermodynamic force,

$$j^{\sigma\mathrm{D}}_i = -\left.\frac{\partial R}{\partial(\nabla_i T)}\right|_{\dots} = -\kappa_{ij}\nabla_j T - D^{T\Pi}_{ij}\nabla_j \Pi - D^{uT}\hat{p}_i\nabla_j h^u_j, \tag{41}$$

$$j^{\phi\mathrm{D}}_i = -\left.\frac{\partial R}{\partial(\nabla_j \Pi)}\right|_{\dots} = -D_{ij}\nabla_j \Pi - D^{T\Pi}_{ij}\nabla_j T - D^{u\Pi}\hat{p}_i\nabla_j h^u_j, \tag{42}$$

$$J^{\phi\mathrm{D}} = \left.\frac{\partial R}{\partial \Pi}\right|_{\dots} = \tilde{\mu}\,\delta\Pi + \mu_M h^M, \tag{43}$$

$$\sigma^{\mathrm{D}}_{ij} = -\left.\frac{\partial R}{\partial A_{ij}}\right|_{\dots} = -\nu_{ijkl}A_{kl} - c^D_{kji}h^m_k, \tag{44}$$

$$X^{M\mathrm{D}} = \left.\frac{\partial R}{\partial h^M}\right|_{\dots} = b_{||}h^M + \mu_M\delta\Pi, \tag{45}$$

$$X^{m\mathrm{D}}_i = \left.\frac{\partial R}{\partial h^m_i}\right|_{\dots} = b^D_\perp\delta^\perp_{ij}h^m_j + c^D_{ijk}A_{jk}, \tag{46}$$

$$X^{u\mathrm{D}} = -\frac{\partial R}{\partial\left(\nabla_i h_i^u\right)}\bigg|_{\dots} = -\frac{1}{\gamma_u}\nabla_i h_i^u - D^{u\Pi}\nabla_z \Pi - D^{uT}\nabla_z T. \tag{47}$$

We note that both contributions, $\sim c^R_{ijk}$ (reversible) as well as $\sim c^D_{ijk}$ (dissipative), describe couplings between extensional flow and rotations of the magnetization orientation. It is worthwhile to emphasize that this situation is unknown from other smectic phases. For example, for usual smectic-A phases or for ferroelectric smectic-A_F phases, an effect analogous to the contribution $\sim c^R_{ijk}$ does not exist since one has, in these phases, no unit vector odd under time reversal.

VII. POSSIBLE EXPERIMENTS

A. Static couplings of the magnetization M

In magnetic fluids, the magnetization M generally depends on density and temperature. In the A_M phase, changes in concentration as well as layer compression or dilation also lead to changes in M. In the free energy, given by Eq. (4), this is described by the susceptibilities γ_3 and γ_4, respectively.

We assume an externally fixed layer compression $\nabla_z u \equiv N$ and look for its influence on the magnetization (and concentration). For the incompressible ($\delta\rho = 0$) and isentropic case ($\delta\sigma = 0$), we are left with Eqs. (8) and (16) describing static equilibrium under the condition $N \neq 0$,

$$0 = c_M \delta M + \gamma_3 \delta\phi + \gamma_4 N, \tag{48}$$

$$0 = \gamma_3 \delta M + c_{\phi\phi}\delta\phi + \theta_3 N, \tag{49}$$

with the solution

$$\delta M = (-\gamma_4 c_{\phi\phi} + \gamma_3\theta_3)(\det)^{-1} N, \tag{50}$$

$$\delta\phi = (-c_M\theta_3 + \gamma_3\gamma_4)(\det)^{-1} N, \tag{51}$$

with $\det = c_M c_{\phi\phi} - \gamma_3^2 > 0$. Equations (50) and (51) demonstrate that an externally imposed layer compression leads to changes in the magnitude of the magnetization, as as well as to concentration variations, which are directly proportional to the magnitude of the applied layer compression.

In ferrosmectics (magnetic or magnetizable smectics) that do not have a permanent spontaneous magnetization, the effects $\sim\gamma_3$ and $\sim\gamma_4$ could arise as induced effects in the presence of an external magnetic field. Concerning effects $\sim\gamma_3$, this also applies for ferronematics (magnetic or magnetizable nematics).

We also note that analogous effects $\sim\gamma_3$ and $\sim\gamma_4$ do exist in ferroelectric smectic-A_F, with the electric polarization replacing the magnetization.

B. Reversible dynamic coupling of flow with the magnitude of the magnetization, M.

In simple fluids, ordinary sound is known as the only propagating low-k excitation, $\exp i(k_i r_i - \omega t)$, with $\omega^2 \sim k^2$. In a smectic-A liquid crystal, ferromagnetic or paramagnetic, there is an additional propagating mode, sometimes called second sound. It is due to the spontaneously broken translational symmetry along the normal of the layers. This one-dimensional compression or dilation mode, with susceptibility B, given by Eq. (4), is anisotropic and leads to an anisotropic part in the first sound spectrum. In a ferromagnetic smectic phase A_M, the magnetization δM additionally provides a low-k coupling with $\nabla_i v_i$ and $\nabla_z v_z$ via λ^M_{ij} in Eqs. (32) and (35). As a result, both soundlike excitations are more complicated.

Disregarding dissipation, the solvability condition for the linearized equations of motion, given by Eqs. (19)–(23), leads, after some trivial algebra, to the dispersion relations for first sound,

$$\omega_1^2 = c_{10}^2 k^2 + \frac{B}{\rho_0}\frac{k_z^4}{k^2} + \frac{c_M}{\rho_0}\frac{\left(\lambda_1^M k^2 + \lambda_a^M k_z^2\right)^2}{k^2}, \tag{52}$$

with $\lambda_a^M = \lambda_2^M - \lambda_1^M$ and k_z shorthand for $\hat{p}_i k_i$. For the isotropic first sound velocity (of simple fluids), we get $\rho_0 c_{10}^2 = c_{\rho\rho}^2\rho_0^2 + 2c_{\rho\sigma}^2\rho_0\sigma_0 + c_{\sigma\sigma}^2\sigma_0^2$ in our representation. The second contribution ($\sim B$) shows a uniaxial dependence on the angle ϑ between k_i and $\hat{p}_i$, $k^2\cos^4\vartheta$. The last contribution ($\sim c_M$) reflects the anisotropy of the material tensor, λ^M_{ij}. In general, first sound anisotropy has contributions $\sim k^2\cos^4\vartheta$ and $\sim k^2\cos^2\vartheta$. In writing Eq. (52), we have assumed that $c_{10}^2 k^2$ is the dominant contribution to first sound.

For second sound, we find

$$\omega_{20}^2 = \left(c_B^2 + c_{2\lambda}^2\right)\frac{k_\perp^2 k_z^2}{k^2} \equiv c_2^2 k^2 \sin^2\vartheta\cos^2\vartheta, \tag{53}$$

with $c_B^2 = B/\rho$, related to the smectic compression mode, and $c_{2\lambda}^2 = (c_M/\rho)(\lambda_a^M)^2$, related to the λ^M_{ij} coupling. The perpendicular wave vector is $k_\perp^2 = k^2\sin^2\vartheta$. This mode vanishes for k_i that is either parallel or perpendicular to $\hat{p}_i$.

It is well known that there is no damping for ordinary first sound ($\sim c_{10}^2$) and the smectic compressional wave ($\sim c_B^2$) in lowest order of the wave vector $\omega \sim k$. This is different for the coupling of the magnetization provided by $\sim\lambda^M_{ij}$. The relaxation of δM, given in lowest order in k by $\dot{M} + \lambda^M_{ij}A_{ij} + b_{||}c_M M = 0$, contains the dissipative contribution $\sim b_{||}$ leading to dissipation in the dispersion relations discussed above.

For second sound, we find the implicit relation

$$\omega_2^2 = \left(c_B^2 + \frac{\omega_2}{\omega_2 + i b_{||}c_M}c_{2\lambda}^2\right)\frac{k_\perp^2 k_z^2}{k^2}, \tag{54}$$

where the imaginary part ($\sim b_{||}$) indicates damping. As expected, only the λ^M_{ij} contribution is damped. The influence of the relaxation of δM in the first sound spectrum shows up only in the part ($\sim c_M$) of Eq. (52). The results for second sound discussed above can be taken over accordingly for this part of first sound.

It thus emerges that the presence of a relaxing variable such as the magnitude of the magnetization for ferromagnetic smectic-A_M, or the magnitude of the macroscopic polarization for ferroelectric smectic-A_F, influences all soundlike excitations to lowest order in the wave vector. It is expected that the influence of the relaxation of δM is strongest, and therefore most easily measurable, near a second-order phase transition from smectic-A_M to a phase of higher symmetry. This is based on the fact that in the vicinity of such a phase transition, the lifetime of the order parameter fluctuations is growing.

In closing this section, we note that the effects discussed here exist neither in ferromagnetic nematics nor in usual

smectic-A. On the other hand, they are structurally quite similar to the analogous ones in ferroelectric smectic-A_F [45].

C. Dissipative response to an external temperature gradient

In this section, we investigate the application of a constant temperature gradient across the smectic layers, $\nabla_z T \equiv Q$, which can result in a stationary, space-dependent nonequilibrium layer structure.

We first concentrate on the special case of a pure system without a concentration variable. We take zero velocity $v_i = 0$ (no flow), zero rotation of the direction of the magnetization $\delta\hat{m}_i = 0$, and incompressibility $\delta\rho = 0$. Stationarity $\dot{u} = 0$ in Eq. (22) requires the quasicurrent $X^{u\mathrm{D}}$, given by Eq. (47), to vanish (the reversible part $X^{u\mathrm{R}}$ is zero anyhow), with the result

$$-\frac{1}{\gamma_u}\nabla_i h_i^u = D^{uT} Q. \tag{55}$$

Similarly, using Eqs. (23) and (45) for the magnitude of the magnetization, M, $\dot{M} = 0$ leads to

$$h^M = 0. \tag{56}$$

Relating the molecular fields to the variables, given by Eqs. (8) and (10), we first get an inhomogeneous layer compression,

$$\nabla_z^2 u = \frac{\gamma_u}{\tilde{B}} D^{uT} Q, \tag{57}$$

with $\tilde{B} = B - \gamma_4^2/c_M$, and, second, a gradient in the magnitude of the magnetization,

$$\nabla_z M = -\frac{\gamma_4\gamma_u}{c_M\tilde{B}} D^{uT} Q. \tag{58}$$

Secondly, we have examined, in detail, the presence of a concentration variable $\delta\phi$, that is not conserved. In Eq. (21), $\dot{\phi} = 0$ requires $J^{\phi\mathrm{D}} = 0$. Together with $X^{M\mathrm{D}} = 0$, we have two coupled conditions,

$$0 = \mu\,\delta\Pi + \mu_M h^M, \tag{59}$$

$$0 = b_\parallel h^M + \mu_M \delta\Pi, \tag{60}$$

with the solutions $h^M = 0$ and $\delta\Pi = 0$.

The third stationarity condition in Eq. (47), $X^{u\mathrm{D}} = 0$, can be simplified to Eq. (55) using $\delta\Pi = 0$. Finally, in terms of the variables, the induced quantities read

$$\nabla_z M = \frac{\gamma_u D^{uT}}{\Delta}(c_{\phi\phi}\gamma_4 - \gamma_3\theta_3)\,Q, \tag{61}$$

$$\nabla_z \phi = \frac{\gamma_u D^{uT}}{\Delta}(c_M\theta_3 - \gamma_3\gamma_4)\,Q, \tag{62}$$

$$\nabla_z^2 u = \frac{\gamma_u D^{uT}}{\Delta}(\gamma_3^2 - c_M c_{\phi\phi})\,Q, \tag{63}$$

with $\Delta = B(c_M c_{\phi\phi} - \gamma_3^2) + 2\gamma_3\gamma_4\theta_3 - c_{\phi\phi}\gamma_4^2 - \theta_3^2 c_M$.

This result demonstrates that for an externally applied constant temperature gradient, a dissipative stationary solution without flow, but involving layer compressions and a gradient of the magnitude of the magnetization, can exist for a nonconserved concentration. This observation underscores the significance of the decomposition given in Eq. (1) for the magnetization M_i into the direction of the magnetization $\hat{m}_i$ associated with broken rotational symmetry, and the magnitude of the magnetization, M, a macroscopic variable, which relaxes on a long, but finite timescale.

We have also studied the case of a conserved concentration variable under stationary conditions and without any flow. In this case, we could not find a solution. This indicates that in general, an applied constant temperature gradient will lead to a time-dependent pattern and/or the occurrence of flow.

We close this section by pointing out that to find a stationary solution in an applied constant temperature gradient for such a complex system as ferromagnetic smectic-A_M is rather special. It turns out that for other complex fluids, such as ferroelectric smectic-A_F, recently found experimentally [42–44] and analyzed with respect to their macroscopic properties in Ref. [45], we could not find a stationary solution without flow in an applied constant temperature gradient at all.

VIII. SUMMARY AND PERSPECTIVE

Here we have derived the complete set of macroscopic dynamic equations for ferromagnetic smectic-A_M liquid crystal with the spontaneous average magnetization parallel to the layer normal of the layering. Compared to nonmagnetic smectic-A phases, we find a number of static and reversible as well as irreversible dynamic cross-coupling terms.

As additional macroscopic variables when compared to simple fluids, we have the layer displacement u, familiar from smectic liquid crystals and the magnetization density $\boldsymbol{M}$. We point out that the velocity of first sound becomes anisotropic due to the influence of the modulus of the magnetization, while the magnitude of the velocity of second sound is modified, while preserving the angular dependence familiar from the usual second sound in nonmagnetic smectic-A phases. For the static behavior, we investigate cross-coupling terms between the magnitude of the magnetization, on the one hand, and layer compression as well as osmotic pressure, on the other hand. In addition, we point out that as a stationary dissipative effect without flow, temperature gradients can induce gradients in the magnetization parallel to the layer normal, mediated by layer compressions. Such a stationary solution turns out to be possible as well for an additional nonconserved concentration variable. In general, time-dependent and/or states with a flow are to be expected.

As a perspective, we would like to mention antiferromagnetic smectic-A phases for which the magnetization is pointing "up" and "down" on alternating layers. Apparently, antiferromagnetic liquid crystals have never been reported in the field of liquid crystals, while they are well known in solid-state physics.

We end with a speculative note concerning the synthesis of ferromagnetic smectic-A liquid crystals. So far we have (a) ferromagnetic nematics for which disk-shaped micelles with an easy magnetic axis parallel to the normal of the platelike objects are embedded in a thermotropic low molecular weight nematic [9–11], and (b) lyotropic magnetic smectic-A for

which ferrofluid particles (micelles) have been incorporated into the layers of a lamellar lyotropic smectic with no net magnetic order [19,20]. This leads to the suggestion to put disklike micelles with an easy magnetic axis parallel to the normal of the disks into a lyotropic layered phase with a layer spacing comparable to the "thickness" of the magnetic disks. Clearly, one challenge in this connection is the chemical and sterical compatibility of the two subsystems—a challenge well known from magnetic gels [73,74].

We close this section with a challenge for future work. An intriguing possibility would be to combine ferroelectricity and ferromagnetism for liquid crystalline systems such as nematics or smectics. This would be the first class of triply multiferroic systems at room temperature with fluidity.

ACKNOWLEDGMENTS

It is a pleasure for H.R.B. to thank Darja Lisjak from the Jozef Stefan Institute (Ljubljana) for a stimulating discussion a few years ago about the possibilities to produce ferromagnetic smectics. Partial support of the work of H.R.B. by the Deutsche Forschungsgemeinschaft is gratefully acknowledged.

[1] J. Rault, P. E. Cladis, and J. P. Burger, Phys. Lett. A **32**, 199 (1970).
[2] S. S. Chen and S. H. Chiang, Mol. Cryst. Liq. Cryst. **144**, 359 (1987).
[3] C. F. Hayes, Mol. Cryst. Liq. Cryst. **36**, 245 (1976).
[4] L. Liébert and A. Martinet, J. Phys. Lett. (France) **40**, 363 (1979).
[5] S.-H. Chen and N. M. Amer, Phys. Rev. Lett. **51**, 2298 (1983).
[6] M. Koneracká, V. Závišová, P. Kopčanský, J. Jadżyn, G. Czechowski, and B. Żywuski, J. Magn. Magn. Mater. **157-158**, 589 (1996).
[7] V. Berejnov, J.-C. Bacri, V. Cabuil, R. Perzynski, and Y. L. Raikher, Europhys. Lett. **41**, 507 (1998).
[8] F. Brochard and P. G. de Gennes, J. Phys. France **31**, 691 (1970).
[9] A. Mertelj, D. Lisjak, M. Drofenik, and M. Čopič, Nature (London) **504**, 237 (2013).
[10] A. Mertelj and D. Lisjak, Liq. Cryst. Rev. **5**, 1 (2017).
[11] A. Mertelj, N. Osterman, D. Lisjak, and M. Čopič, Soft Matter **10**, 9065 (2014).
[12] Q. Zhang, P. J. Ackerman, Q. Liu, and I. I. Smalyukh, Phys. Rev. Lett. **115**, 097802 (2015).
[13] A. J. Hess, Q. Liu, and I. I. Smalyukh, Appl. Phys. Lett. **107**, 071906 (2015).
[14] R. Sahoo, M. V. Rasna, D. Lisjak, A. Mertelj, and S. Dahra, Appl. Phys. Lett. **106**, 161905 (2015).
[15] Q. Liu, P. J. Ackerman, T. C. Lubensky, and I. I. Smalyukh, Proc. Natl. Acad. Sci. USA **113**, 10479 (2016).
[16] T. Potisk, D. Svenšek, H. R. Brand, H. Pleiner, D. Lisjak, N. Osterman, and A. Mertelj, Phys. Rev. Lett. **119**, 097802 (2017).
[17] T. Potisk, A. Mertelj, N. Sebastián, N. Osterman, D. Lisjak, H. R. Brand, H. Pleiner, and D. Svenšek, Phys. Rev. E **97**, 012701 (2018).
[18] P. Medle Rupnik, D. Lisjak, M. Čopič, S. Čopar, and A. Mertelj, Sci. Adv. **3**, e1701336 (2017).
[19] P. Fabre, C. Casagrande, M. Veyssie, V. Cabuil, and R. Massart, Phys. Rev. Lett. **64**, 539 (1990).
[20] P. Fabre, R. Ober, M. Veyssie, and V. Cabuil, J. Magn. Magn. Mater. **85**, 77 (1990).
[21] R. E. Rosensweig, *Ferrohydrodynamics* (Cambridge University Press, Cambridge, 1985).
[22] R. J. Mandle, S. J. Cowling, and J. W. Goodby, Chem. Eur. J. **23**, 14554 (2017).
[23] H. Nishikawa, K. Shiroshita, H. Higuchi, Y. Okumura, Y. Hasebo, S. Yamamoto, K. Sago, and H. Kikuchi, Adv. Mater. **29**, 1702354 (2017).
[24] A. Mertelj, L. Cmok, N. Sebastian, R. J. Mandle, R. R. Parker, A. C. Whitwood, J. W. Goodby, and M. Copic, Phys. Rev. X **8**, 041025 (2018).
[25] R. J. Mandle and A. Mertelj, Phys. Chem. Chem. Phys. **21**, 18769 (2019).
[26] X. Chen, E. Korblova, D. Dong, X. Wei, R. Shao, L. Radzihovsky, M. A. Glaser, J. E. Mclennan, D. Bedrov, D. M. Walba, and N. A. Clark, Proc. Natl. Acad. Sci. USA **117**, 14021 (2020).
[27] P. L. M. Connor and R. J. Mandle, Soft Matter **16**, 324 (2020).
[28] N. Sebastian, L. Cmok, R. J. Mandle, M. R. de la Fuente, I. Drevenšek Olenik, M. Copic, and A. Mertelj, Phys. Rev. Lett. **124**, 037801 (2020).
[29] R. J. Mandle, N. Sebastian, J. Martinez-Perdiguero, and A. Mertelj, Nat. Commun. **12**, 4962 (2021).
[30] A. Manabe, M. Bremer, and M. Kraska, Liq. Cryst. **48**, 1079 (2021).
[31] R. J. Mandle, S. J. Cowling, and J. W. Goodby, Liq. Cryst. **48**, 1780 (2021).
[32] N. Sebastian, R. J. Mandle, A. Petelin, A. Eremin, and A. Mertelj, Liq. Cryst. **48**, 2055 (2021).
[33] H. Nishikawa, K. Sano, S. Kunihara, G. Watanabe, A. Nishonyanagi, B. Dahra, and F. Araoka, Commun. Mater. **3**, 89 (2022).
[34] R. J. Mandle, Soft Matter **18**, 5014 (2022).
[35] N. Sebastian, M. Čopič, and A. Mertelj, Phys. Rev. E **106**, 021001 (2022).
[36] E. Zavvou, M. Klasen-Memmer, A. Manabe, M. Bremer, and A. Eremin, Soft Matter **18**, 8804 (2022).
[37] H. Matsukizono, K. Iwamatsu, S. Endo, Y. Okumura, S. Anan, and H. Kikuchi, J. Mater. Chem. C **11**, 6183 (2023).
[38] H. Pleiner and H. R. Brand, Europhys. Lett. **9**, 243 (1989).
[39] H. R. Brand, H. Pleiner, and F. Ziebert, Phys. Rev. E **74**, 021713 (2006).
[40] M. P. Rosseto and J. V. Selinger, Phys. Rev. E **101**, 052707 (2020).
[41] E. I. Kats, Phys. Rev. E **103**, 012704 (2021).
[42] H. Kikuchi, H. Matsukizono, K. Iwamatsu, S. Anam, and Y. Okumura, Adv. Sci. **9**, 2202048 (2022).

[43] Y. Song, M. Deng, Z. Wang, J. Li, H. Lei, Z. Wan, R. Xia, S. Aya, and M. Huang, J. Phys. Chem. Lett. **13**, 9983 (2022).
[44] X. Chen, V. Martinez, P. Nacke, E. Körblova, A. Manabe, M. Klasen-Memmer, G. Freychet, N. Zhernenkov, M. A. Glaser, L. Radzihovsky, J. E. McLennan, D. M. Walba, M. Bremer, F. Giesselmann, and N. A. Clark, Proc. Natl. Acad. Sci. USA **119**, e2210062119 (2022).
[45] H. R. Brand and H. Pleiner, Eur. Phys. J. E **47**, 10 (2024).
[46] P. G. De Gennes, J. Phys. Colloq. **30**, C4-65 (1969).
[47] P. C. Martin, O. Parodi, and P. S. Pershan, Phys. Rev. A **6**, 2401 (1972).
[48] P. G. De Gennes, *The Physics of Liquid Crystals* (Clarendon Press, Oxford, 1975).
[49] P. G. de Gennes and J. Prost, *The Physics of Liquid Crystals* (Clarendon Press, Oxford, 1993).
[50] H. Pleiner and H. R. Brand, Hydrodynamics and electrohydrodynamics of nematic liquid crystals, in *Pattern Formation in Liquid Crystals*, edited by A. Buka and L. Kramer (Springer, New York, 1996), pp. 15.
[51] D. Forster, *Hydrodynamic Fluctuations, Borken Symmertry, and Correlation Functions* (W. A. Benjamin, Reading, MA, 1975).
[52] B. I. Halperin and P. C. Hohenberg, Phys. Rev. **188**, 898 (1969).
[53] I. M. Khalatnikov, *Introduction to the Theory of Superfluidity* (W. A. Benjamin, Reading, MA, 1965).
[54] R. Graham, Phys. Rev. Lett. **33**, 1431 (1974).
[55] M. Liu, Phys. Rev. B **13**, 4174 (1976).
[56] H. Pleiner and R. Graham, J. Phys. C **9**, 4109 (1976).
[57] M. Liu, Phys. Rev. Lett. **43**, 1740 (1979).
[58] H. Pleiner and H. R. Brand, Phys. Rev. B **108**, 054502 (2023).
[59] T. Potisk, Ph.D. dissertation, Macroscopic aspects of ferromagnetic nematic phases, tetrahedral order in ferrogels, and magnetorheological fluids, Universität Bayreuth, 2019 (Fig. 2.10).
[60] H. Pleiner and J. L. Harden, AIP Conf. Proc. **708**, 46 (2004).
[61] H. R. Brand and H. Pleiner, Phys. Rev. E **103**, 012705 (2021).
[62] H. B. Callen, *Thermodynamics and an Introduction to Thermostatics* (Wiley, New York, 1985).
[63] P. G. de Gennes, in *Liquid Crystals of One- and Two-Dimensional Order*, edited by W. Helfrich and G. Heppke (Springer, New York, 1980), p. 231.
[64] S. Bohlius, H. R. Brand, and H. Pleiner, Phys. Rev. E **70**, 061411 (2004).
[65] G. A. Auernhammer, H. R. Brand, and H. Pleiner, Rheol. Acta **39**, 215 (2000).
[66] P. G. De Gennes, Mol. Cryst. Liq. Cryst. **12**, 193 (1971).
[67] L. P. Pitaevskii, Sov. Phys. JETP **35**, 282 (1959).
[68] M. Liu, Phys. Rev. E **50**, 2925 (1994).
[69] L. D. Landau and E. M. Lifshitz, *Electrodynamics of Continuous Media*, 2nd ed (Pergamon Press, Oxford, 1984).
[70] E. Jarkova, H. Pleiner, H.-W. Müller, and H. R. Brand, J. Chem. Phys. **118**, 2422 (2003).
[71] H. R. Brand, H. Pleiner, and D. Svenšek, Eur. Phys. J. E **34**, 128 (2011).
[72] W. P. Mason, *Physical Acoustics and the Properties of Solids* (D. Van Nostrand, New York, 1958).
[73] M. Zrinyi, Trends Polym. Sci. **5**, 280 (1997).
[74] D. Collin, G. K. Auernhammer, O. Gavat, P. Martinoty, and H. R. Brand, Macromol. Rapid Commun. **24**, 737 (2003).