Cascading failures in bipartite networks with directional support links

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We study the cascading failures in a system of two interdependent networks whose internetwork supply links are directional. We will show that, by utilizing generating function formalism, the cascading process can be modeled by a set of recursive relations. Most importantly, the functions involved in these relations are solely dependent upon the choice of the degree distribution of ingoing links. Simulation results in the limit of very large networks based on different choices of degree distributions for outgoing links, e.g., Kronecker delta, Poisson and Pareto, are indeed identical and are in excellent agreement with the theory. However, for Pareto distribution with the shape parameter $1 < \alpha < 2$, the convergence is slow. In general, directional networks can be more vulnerable or less vulnerable than their bidirectional counterparts. For three special settings of interdependent networks, we analytically compare their vulnerability. For practical applications it is important to predict if a system responds to the size of the initial attack continuously or if there is catastrophic collapse of the system if the attack exceeds a specific transition size. We analytically show that systems with lower average degrees are more resilient against this abrupt transition. We also establish an equivalence of this transition with the liquid-gas transition in statistical mechanics. In the last section, we derive the set of recursive relation to describe the cascading process where the initial attack is not restricted to a single network.

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I. INTRODUCTION

Previous studies of cascading failures in a system of interdependent networks [1–9] have focused on the roles played by the settings, such as internetwork and intranetwork degree distribution of nodes or threshold rules which dictate the conditions whether a node is functional or not. However, little attention was spent on the property of the links themselves, e.g., the majority of the models simply considered the case of bidirectional links which simultaneously played the roles of both supply and demand relations.

To mitigate confusion and formalize terminology, we will address all internetwork links as supply links and categorize them as ingoing links and outgoing links in the same manner as [10]. A supply link that starts at a node i and ends at a node j is called the outgoing link of i and the ingoing link of j. Such a supply link represents a supply-demand relation in which node i provides some supply that node j receives to remain functional, i.e., a debtor provides interests and principal to a creditor and the insolvency of such an obligation may cause the bankruptcy of the creditor.

Aforementioned studies that are based on bidirectional internetwork supply links assume a symmetry between ingoing links and outgoing links, but many real-world scenarios of networks are directional and thus asymmetric. Many examples of directional networks has been proposed by [10], i.e., World Wide Web, food webs, and citation networks, but those are limited to a single network. The asset/liability exposures between financial intermediaries [11] are another example of directional network, which can be extended to a network of networks (NON) [9].

Initially, when a loan is made, both an asset exposure and a liability exposure are created, thus making the relation bidirectional. However, the liability exposure can be packaged as derivatives and sold to other financial agents, with subprime mortgage as one of the most famous examples, and this practice will break the symmetry between the asset exposure and the liability exposure. If the financial agents are categorized into debtors and creditors, the interdependence between the network of debtors and the network of creditors can be modeled as directional internetwork links. The insolvency of some debtors propagates along the directed liability relations, and this leads to the cascading failure of both networks. Examples of such a phenomenon occurred during the 2008 financial crisis [12] and more recently when some major banks have just declared or are facing bankruptcy, e.g., Silicon Valley bank, Credit Swiss, First Republic Bank, etc.

A node i in network A may have $k_{\mathrm{out},i}$ supply links by which an important commodity is supplied to nodes in network B. In the aforementioned example of debtor and creditor interdependent networks, a liability exposure is such a commodity and provides the debtors with interests and principals. This number, $k_{\mathrm{out},i}$ is called the out-degree of the node i and is taken from the degree distribution $P_{\mathrm{out},A}(k)$. Some of these supply links may head towards a node j in network B and

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become ingoing links of j. If we count the number of ingoing links of node j in network B coming from different nodes in network A, we define an in-degree $k_{\text{in},j}$ of node j with a degree distribution $P_{\text{in},B}(k)$. We define $P_{\text{out},B}(k)$ and $P_{\text{in},A}(k)$ in the analogous way. In our model we assume that the network is sparse, i.e., that the average in- and out-degrees of both networks are much smaller than the number of nodes in both networks and thus the probability of forming small closed loops is very small. We also assume that, aside from satisfying the given degree distributions, all of the links are connecting to randomly selected nodes.

In [4] all of the links are bidirectional; consequently, the ingoing links in network B also play the role of its outgoing links and, hence, for each node j in network B, $k_{\text{in},j} = k_{\text{out},j}$ and $P_{\text{in},B}(k) = P_{\text{out},B}(k)$. Analogous relations are held for network A. In the directional case, $k_{\text{out},i}$ is not necessarily equal to $k_{\text{in},i}$ and $P_{\text{out},X}(k)$ is not necessarily equal to $P_{\text{in},X}(k)$, where $X \in \{A, B\}$. As a result. the formalism introduced in [4] must be reviewed and revised. In principle, the in-degree and out-degree of the nodes may be correlated or anticorrelated, as well as the ingoing and outgoing links, but for simplicity we assume that ingoing and outgoing links and degrees are totally independent. It is possible to construct an analytically tractable model with both directional and bidirectional links in which ingoing and outgoing links are partially correlated, but we will restrict ourselves to a totally independent case.

The first attempt to study cascading failures between two networks connected by directional dependency links was made in [7], but the survival condition was much simpler, i.e., a node in each network must have had at least one ingoing link coming from a functional node in the other network in order to have been functional. Following [4], we will use a more general functionality condition. We assume that each node i in network X with an in-degree k_i has a predetermined functionality threshold $j_i \leqslant k_i$, such that if the number j of the ingoing links of node i emanating from currently functional nodes is less that j_i, node i fails and becomes nonfunctional. The value of j_i for each node in the network X is a random number taken from the cumulative distribution $r_X(j, k)$, i.e., the probability that $j_i \leqslant j'$ is $r_X(j', k)$. For example, if $r_X(j, k) = 1$ for $j = k$ and $r_X(j, k) = 0$ for $j < k$, then for all i, $j_i = k_i$.

As in [4,7], the cascade of failures starts by an initial attack on one of the networks, e.g., A, after which only a fraction p of nodes in network A remains functional. After the attack, some of the nodes in network B may become nonfunctional because the number of their ingoing links may become less than their functionality threshold. On the second stage of failures even more nodes in network A become nonfunctional because the number of their ingoing links from functional nodes in network B becomes smaller than their functionality threshold, and so on, until at some point no new failures occur in both networks.

Di Muro *et al.* [4] introduced two different generating functions to describe degree distribution and excess degree distribution which are similar to the ones used in heterogeneous k-core percolation [13]:

$$W_X(f) = \sum_{k=0}^{\infty} P_{\text{in},X}(k) \sum_{j=0}^{k} \binom{k}{j} f^j (1-f)^{k-j} r_X(j,k) \tag{1}$$

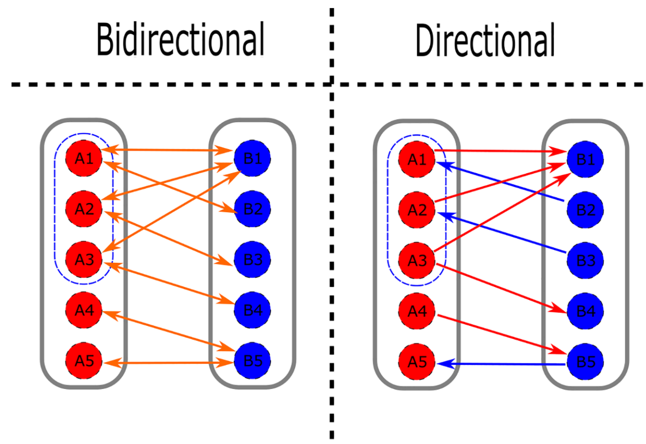

FIG. 1. Toy examples of a bidirectional bipartite network (left) and a directional one (right). All red nodes constitute network A, and all blue nodes constitute network B, as implied by their labels. Both bidirectional and directional examples have the same number of nodes and the same link connections, except that the links are bidirecitonal in the former case and directional in the latter case. Three nodes A_1, A_2, and A_3 are circled with a dashed blue line to emphasize the fact that they all provide ingoing links to node B_1 and thus are in the supply neighborhood of B_1. Both bidirectional and directional networks have the same supply neighborhood for B_1.

and

$$Z_X(f) = \sum_{k=1}^{\infty} \frac{P_{\text{in},X}(k)k}{\langle k \rangle_X} \sum_{j=0}^{k-1} \binom{k-1}{j} \times f^j (1-f)^{k-j-1} r_X(j+1,k), \tag{2}$$

where W_X stands for the degree distribution generating function, Z_X stands for the excess degree distribution generating function, $\langle k \rangle_X$ is the average in-degree of network X, and f is the fraction of functional ingoing links of network X. Similar formulas have been derived in [6] for a variant of the Watts opinion model [14]. By utilizing such a generating function formalism, di Muro *et al.* [4] described the cascading failure in the bidirectional case by a set of recursive relations.

Our goal is to investigate the directional case of the bipartite network and compare it with the bidirectional case [4] illustrated in Fig. 1. In particular, we will study how the results depend on the in- and out-degree distributions. We will explore several distributions which are most commonly used in the theory of complex systems: the delta distribution, in which all nodes have the same degree k, the Poisson degree distribution, which arises when the links are established between random nodes [15], and finally the geometric distribution and the Pareto distributions, which arise in growing economic systems [16]. Both geometric and Pareto distributions emerge in the process known as preferential attachment, i.e., a process when a number of classes (e.g., cities in a country or nodes of a network) receives a new element (a new inhabitant, or link of a node) with a probability proportional to the existing number of elements (total population of a city or a degree of a node). If the total number of classes remains constant, the resulting distribution of the number of elements converges to the geometric distribution $P(k) \sim \theta^k$ [17], but if new classes can be created with certain probability, the resulting distribution converges to the Pareto distribution $P(k) \sim k^{-\alpha-1}$ [18].

In [7], the distribution of the ingoing links in both networks is assumed to be the Poisson distribution because the supply nodes from where the ingoing links originate are chosen totally at random. Here we specify the in-degree distributions in both networks and show that the result is independent from the particular choice of the out-degree distribution in the limit of infinitely large networks, even for the case of scale-free distributions which are Pareto distributions with the exponent of the cumulative distribution $2 > \alpha > 1$. We will also explore the difference in vulnerability of the systems with directional and bidirectional links.

This paper is organized as follows. In Sec. II we will introduce the set of recursion equations based on generating function formalism. In Sec. III we will solve three special settings of directional bipartite networks analytically and compare the results with their bidirectional counterparts. In Sec. IV we will explore the conditions for the existence of a catastrophic cascade of failures, which leads to the almost complete collapse of the system if the size of the initial attack exceeds a certain threshold. In Sec. V we will present simulation results to show that the choice of the degree distribution of the outgoing links does not affect the cascading process, with the exception of the scale-free distributions. In Sec. VI we will generalize initial attack on a single network to attacks on both networks.

II. THE MODEL

A. Recursion relations

In principle, the cascade of failures may develop in many different ways depending on a survival time of a node without sufficient supply. However, the final result will not depend on the details of the process, as is the case in the model of k-core percolation [19], on which the present model is built. Indeed we can define a j-core of the bipartite network as a subset of the nodes in both networks at which each node i has at least j_i ingoing links. Note that, if a node is excluded from a subset, all of its links are also excluded. j-core is thus a topological property of the network which does not depend on the way how it was computed given that all functionality thresholds j_i are set in advance and all of the nodes that fail in the initial attack are excluded in advance. Thus, we study just one of many possible realizations of the cascades of failures in which all nodes without sufficient supply fail simultaneously.

Accordingly, we define the initial state of the cascade as when all nodes in networks A and B are functional and, hence, the fraction of functional nodes in both networks $\mu_{A,0} = \mu_{B,0} = 1$. At the first stage of the cascade, an initial attack on network A takes place after which only fraction $p = \mu_{A,1}$ of nodes in network A remains functional, while $\mu_{B,1} = 1$.

The process of cascading failures in the directional case is illustrated in Fig. 2. We define a directional link to be functional if it is an outgoing link of a functional node. We assume that, starting from stage $n = 2$, each stage of the cascade can be described in two parts, stage n-A and stage n-B. First, in stage n-A, for every node in network B, we check the number of its functional ingoing links, based on the status of the nodes in network A at stage $(n-1)$-B; if a node's threshold condition is not met, then it fails and is no longer called functional together with all of its outgoing links. Next, in stage n-B, we repeat the same process for network A, checking the number of functional ingoing links of every node in network A based on the status of the nodes in network B at stage n-A, and the nodes in network A fail accordingly.

For the directional case, consider node b in network B that is connected to node a in network A via a directional link which serves as an outgoing link of node b and the ingoing link of network A. The functionality status of node b does not directly depend on the status of node a. Therefore, for the directional case, the mathematical expectation of the fraction $f_{B,n}$ of functional outgoing links emanating from the functional nodes in network B at the nth stage of the cascade simply coincides with the fraction $\mu_{B,n}$ of functional nodes of network B at this stage of the cascade. Thus, in the thermodynamic limit, when the number of nodes and links is infinitely large, we can assume that $f_{B,n} = \mu_{B,n}$. Analogously, $f_{A,n} = \mu_{A,n}$. Note that, for the Pareto distribution with $\alpha \leqslant 1$, the first moment of the degree distribution diverges, and hence our recursion formalism is invalid.

Our goal is to relate the fraction of functional nodes in network B to the fraction of functional nodes of network A at the previous stage of the cascade. In Appendix A we show that $f_{B,n} = W_B(f_{A,n-1})$. For the fraction of functional nodes in network A at the nth stage of the cascade, we must combine the effect of the damage in the network B with the totally independent damage due to the initial attack; hence, $f_{A,n} = pW_A(f_{B,n})$. Thus, for the case of directional links: for stage $n > 1$:

$$f_{B,n} = W_B(f_{A,n-1}),$$
$$f_{A,n} = pW_A(f_{B,n})$$

and

$$\mu_{A,n} = f_{A,n},$$
$$\mu_{B,n} = f_{B,n},$$

given that for $n = 1$,

$$f_{A,1} = \mu_{A,1} = p,$$
$$f_{B,1} = \mu_{B,1} = 1.$$

An implication of these equations is that the cascading failure in the directional bipartite network does not depend on the out-degree distributions $P_{\mathrm{out},X}(k)$.

For bidirectional links both ends of the links are ingoing and outgoing and the recursion relations must be modified (see Appendix B), and the meaning of the fraction of the functional links must be changed. We will call $f_{B,n}$ a conditional probability that a bidirectional link ends in a functional node in network B, provided that its other end is a functional node in network A at the nth stage of the cascade. This condition must be included because, at this stage of the cascade, the nodes in network A are assumed to be functional before we start to

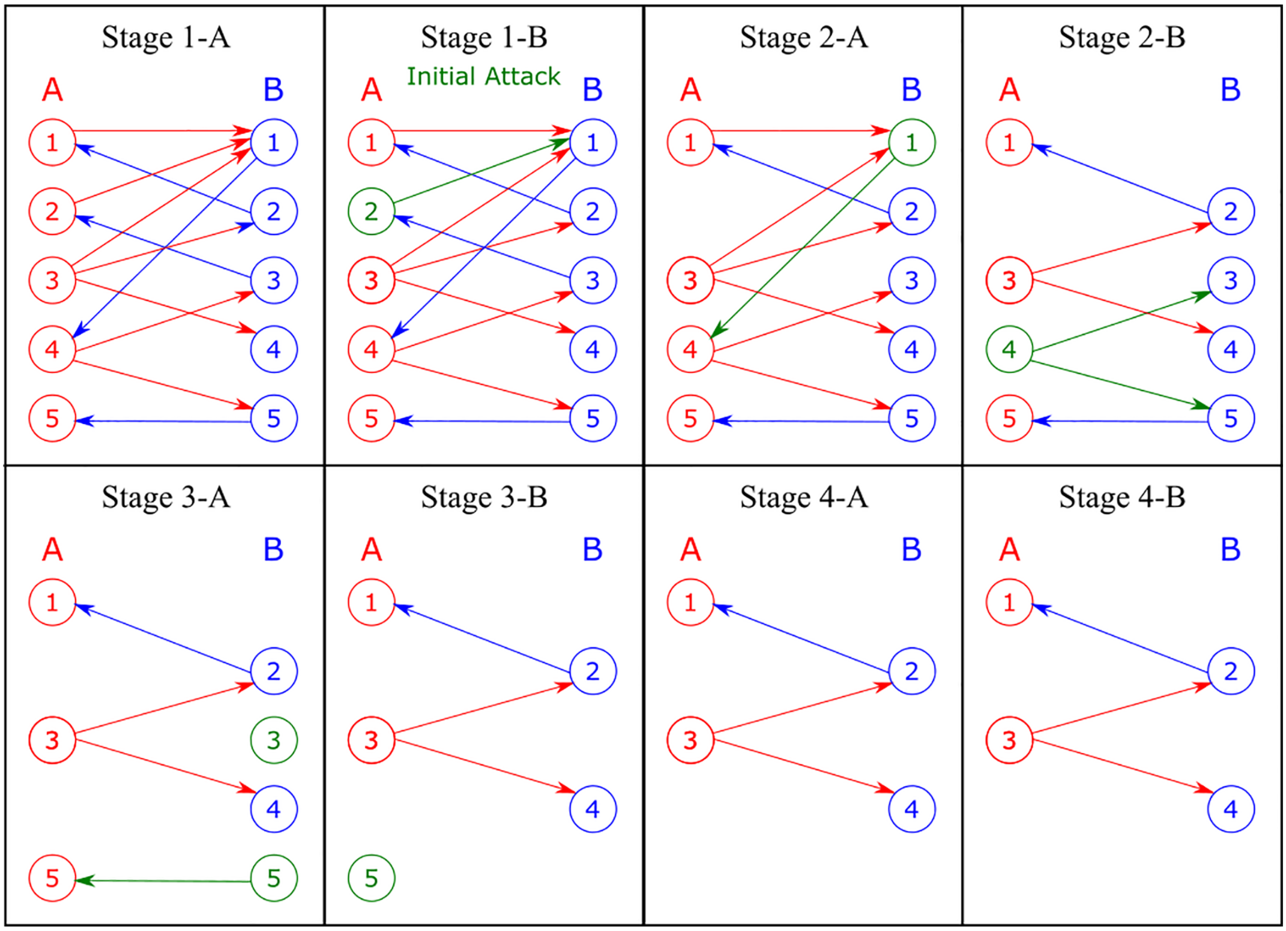

FIG. 2. An example of a cascading process in a bipartite network consisting of two finite networks, A and B. In each stage, starting at stage 2, we first check the functionality of the nodes of network B, taking into account the threshold functionality conditions of all nodes in network B and then those of the nodes of network A. The initial attack happens at stage 1-B and kills node A_2, which is colored green in the panel headed "Stage 1-B." As a consequence, all outgoing links attached to it also become nonfunctional and are colored green. The threshold in this particular example is defined as $r(j,k) = 1$ if $j = k$ or $r(j,k) = 0$ if $j < k$, and thus any node that is pointed to by a single green arrow will fail as part of the cascading process. As a result, node B_1 failed in stage 2-A. In stage 2-B, node A_4, which has a supplied link from the failed node B_1, fails. In stage 3-A, nodes B_3 and B_5, which have supply links from the failed node A_4, fail. In stage 3-B, node A_5, which has a supply link from the failed node B_5, fails, and, since it does not have any outgoing links, no new nodes fail in stage 4-A. Consequently, the cascading process is finished.

compute their new statuses. Analogously, $f_{A,n}$ is a conditional probability that a bidirectional link ends in a functional node in network A provided that its other end is connected to a functional node in network B at the nth stage of the cascade.

Accordingly (see Appendix B), the recursion relations for the bidirectional case become the following:

For stage $n > 1$:

$$f_{B,n} = Z_B(f_{A,n-1}),$$
$$f_{A,n} = pZ_A(f_{B,n})$$

and

$$\mu_{B,n} = W_B(f_{A,n-1}),$$
$$\mu_{A,n} = pW_A(f_{B,n})$$

for $n > 1$ given that

$$f_{A,1} = \mu_{A,1} = p,$$
$$f_{B,1} = \mu_{B,1} = 1.$$

When $n \to \infty$ the successive iterations converge to the limits $\mu_a \equiv \mu_{A,\infty}$ and $\mu_b \equiv \mu_{B,\infty}$, for both directional and bidirectional cases.

B. Gap step threshold function

Before presenting the simulated results, we need to first introduce a new kind of threshold function, which we call the gap step threshold function. Although the choice of threshold function depends on the nature of specific networks, such as social networks, router networks, power grids, etc., many previous studies have used step threshold functions, either explicitly [4] or implicitly [7]. This group of threshold functions is very easy to study, yet it may not be very useful either mathematically or practically. In many cases, we either are more interested in how many links a node has lost or, in the case of scale-free networks, do not want a hub node, i.e., a node with a degree comparable to the number of all links in the network, to be as resilient as a regular node. To focus on

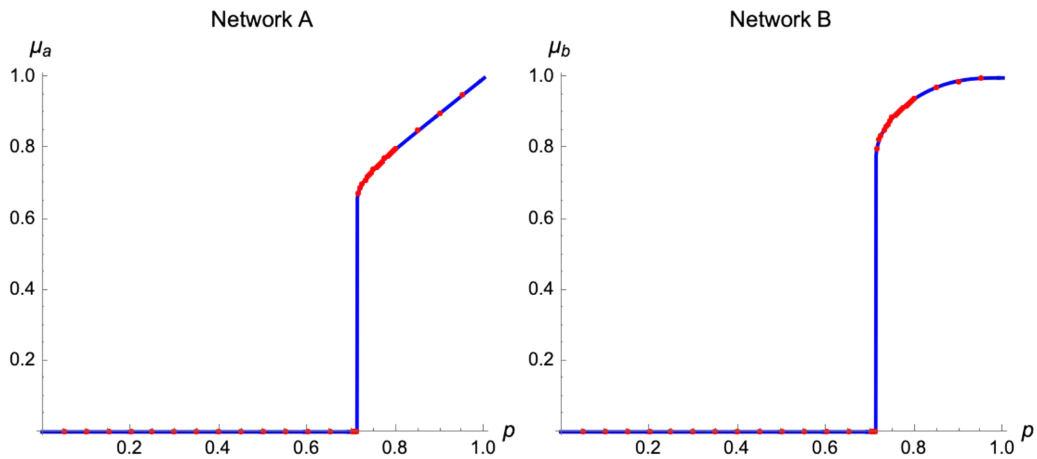

FIG. 3. Fraction of surviving nodes μ when the in-degree distributions follow the delta distribution $P_{\mathrm{in},X}(k) = \delta_5(k)$, and the threshold function is $r_X(j,k) = \mathscr{G}_2$. The blue lines represent theoretical results, and the red dots represent the average of 10 simulations.

the loss of links without losing the simplicity of step functions, we introduce a group of threshold functions, which we call gap step functions:

$$\mathscr{G}_m(j,k) \equiv \begin{cases} 1 & \text{if } k - j \leqslant m \\ 0 & \text{else} \end{cases}. \tag{3}$$

One major benefit of this class of threshold function is that, compared with simple step function, the parameter m can be any non-negative value without causing any contradictions with respect to initial node degrees.

C. Simulated results

We tested the recursion relations for the bipartite network with directed links by computer simulations. All computer simulations have been done for the networks with $N = 10^6$ nodes. The algorithm constructing a bipartite network with given ingoing and outgoing degree distributions is described in Appendix H. For such networks the deviations between theoretical results obtained in the limit $N \to \infty$, and simulations are expected to be approximately $1/\sqrt{N} = 0.1\%$, according to the standard limit theorem, i.e., practically undetectable on the scale of the figures.

Two different simulation results are provided alongside their corresponding theoretical counterparts in Figs. 3 and 4. All x axes represent p, which is the fraction of surviving nodes in network A right after the initial attack, and all y axes represent μ_X, which is the fraction of surviving nodes of network X at the end of the cascading processes. Details of in-degree distributions and threshold functions are provided in the figure caption of each figure, but all simulation settings share the same Poisson out-degree distribution

$$P_{\mathrm{out},X}(k) = \frac{\lambda^k}{k!} e^{-\lambda} \equiv \mathscr{P}_\lambda,$$

with the average degree $\lambda = 5$.

The simulations are in perfect agreement with the theory. As one can see, the behavior is qualitatively similar to the behavior of bipartite graphs with bidirectional links, e.g., they demonstrate an abrupt first-order transition at $p = p_t$ for some point p_t as in many topological models [4,5,7,14].

III. COMPARISON OF THE DIRECTIONAL AND THE BIDIRECTIONAL BIPARTITE NETWORKS

As we see, the recursive relations for the directional and bidirectional cases are very similar, with the only difference being that in the bidirectional case the recursive relations for the cascade of failures use function Z while for the directional case they use only function W. An interesting question is whether the directional network is more vulnerable to the attack than the bidirectional network when all other parameters such as degree distributions are the same. We are able to solve this problem analytically for the case of the delta, Poisson, and geometric in-degree distributions and $r(j,k)$ being a gap step function $\mathscr{G}_m$. We prove that for the delta degree distribution and the gap step function the directional case is more vulnerable than the bidirectional case, for the Poisson degree distribution they are equivalent, and for the geometric degree distribution the bidirectional case is more vulnerable than the directional case.

A. Delta distribution with gap step threshold function

Let the in-degree distribution of internetwork supply links be Kronecker delta

$$P(k) = \delta_{mk} \equiv \delta_m(k) = \begin{cases} 1 & \text{if } k = m \\ 0 & \text{else} \end{cases}. \tag{4}$$

In this case, the degree-generating function is

$$W_X(f) = \sum_{j=0}^{m} \binom{m}{j} f^j (1-f)^{m-j} r_X(j,m),$$

and the excess degree-generating function is

$$Z_X(f) = \sum_{j=0}^{m-1} \binom{m-1}{j} f^j (1-f)^{m-1-j} r_X(j+1,m).$$

Let l denote the maximal degree loss for a node to be functional. Then, for $W_X(f)$, $r_X(j,k) = \mathscr{G}_l(j,k)$ and for $Z_X(f)$, $r_X(j+1,k) = \mathscr{G}_l(j+1,k)$. By substituting the threshold function into the generating functions and changing the summation index j to $j' = j+1$ in the expressions for W and Z,

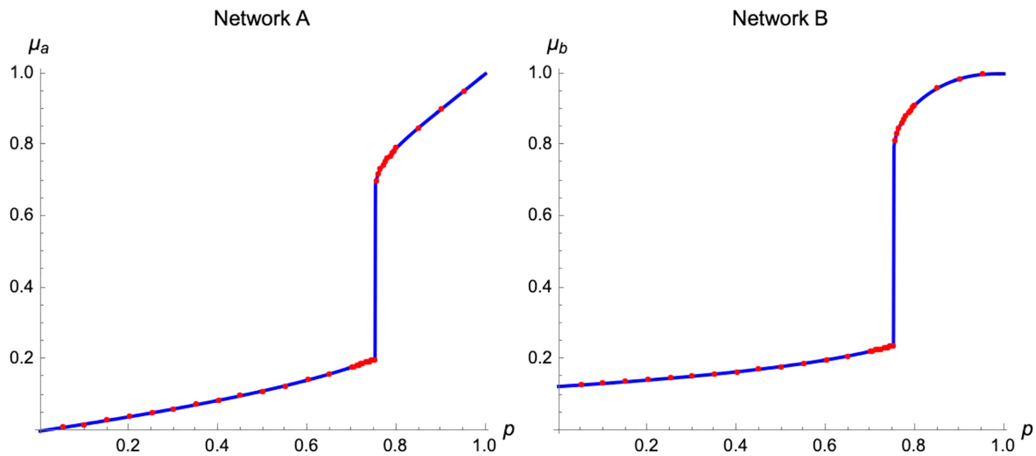

FIG. 4. Fraction of surviving nodes μ when the in-degree distributions follow a Poisson distribution $P_{\text{in},X}(k) = \mathscr{P}_5(k)$, and the threshold function is $r_X(j,k) = \mathscr{G}_2(j,k)$. The blue lines represent theoretical results, and the red dots represent the average of ten simulations. Note that, for $p = 0$, $\mu_B \neq 0$ because the nodes of network B cannot lose more than two links if initially their degrees are $k = 0, 1, 2$. Thus, nodes in network B that have no, one, and two links will always remain functional, as their functionality threshold according to Eq. (3) is $k - 2 \leqslant 0$. The fraction of nodes in network B that survive when network A collapses entirely is expected to be $\mathscr{P}_5(0) + \mathscr{P}_5(1) + \mathscr{P}_5(2) \approx 0.125$, which matches the figure. Note that the data points shown are for the directional case, but as we explain in Sec. III, the same graphs would be produced for the bidirectional counterpart.

we get

$$W_X(f) = \sum_{j=m-l}^{m} \binom{m}{j} f^j (1-f)^{m-j},$$

$$Z_X(f) = \sum_{j=m-l}^{m} \binom{m-1}{j-1} f^{j-1} (1-f)^{m-j}. \tag{5}$$

With details presented in Appendix C, one can show that the difference between $Z_X(f)$ and $W_X(f)$ is

$$(Z_X - W_X)(f) = \binom{m-1}{m-l-1} f^{m-l-1} (1-f)^{l+1} \geqslant 0. \tag{6}$$

In the first stage of the cascading process,

$$f_{A,1} = p,$$
$$f_{B,1} = 1$$

for both directional and bidirectional systems, but for the second stage,

$$f_{B,2} = Z_B(f_{A,1})$$

for the bidirectional network and

$$f_{B,2} = W_B(f_{A,1})$$

for the directional network. Thus, bidirectional $f_{B,2} \geqslant$ directional $f_{B,2}$. Combine this result with the facts that both W and Z are nondecreasing functions and with that

$$f_{A,2} = pZ_A(f_{B,2})$$

for the bidirectional network and

$$f_{A,2} = pW_A(f_{B,2})$$

for the directional network, we can arrive at the conclusion that bidirectional $f_{A,2} \geqslant$ directional $f_{A,2}$.

By extending this process recursively, bidirectional $f_{X,n} \geqslant$ directional $f_{X,n}$ for any stage $n > 1$ and bidirectional $\mu_{X,n} \geqslant$ directional $\mu_{X,n}$ for any stage $n > 1$. As a result, directional systems with the delta in-degree distribution are always more vulnerable than bidirectional ones. However, for other degree distributions, for example, for the geometric distribution $P(k) = \theta^k(1-\theta)$, the opposite is true. We will show that, for the gap step threshold with gap l equal to any value and any geometric distribution, $W(f) \geqslant Z(f)$.

B. Poisson distribution with gap step threshold function

A particular phenomenon arises when we are dealing with a specific kind of Erdős-Rényi graphs, in which case the recursive relations for the directional system is exactly the same as for the bidirectional one.

Let the internetwork supply links follow a Poisson distribution, $P_{\text{in},X}(k) = \mathscr{P}_\lambda$, and let the threshold function $r_X(j,k) = \mathscr{G}_l$, which has a very nice property:

$$r_X(j,k) = r_X(j+1, k+1).$$

With details presented in Appendix D, the following result can be deduced:

$$Z_X(f) = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} e^{-\lambda} \sum_{j=0}^{k} \binom{k}{j} f^j (1-f)^{k-j} r_X(j,k)$$
$$= W_X(f). \tag{7}$$

Utilizing the facts presented in the previous section, the cascading failures of this particular setting are identical for both directional or bidirectional cases.

Another simplification can be made in this setting. With details presented in Appendix E, it can be shown that the function W and, consequently, also Z simplify to be the following:

$$W(f) = \frac{\Gamma(l+1, (1-f)\lambda)}{\Gamma(l+1)}, \tag{8}$$

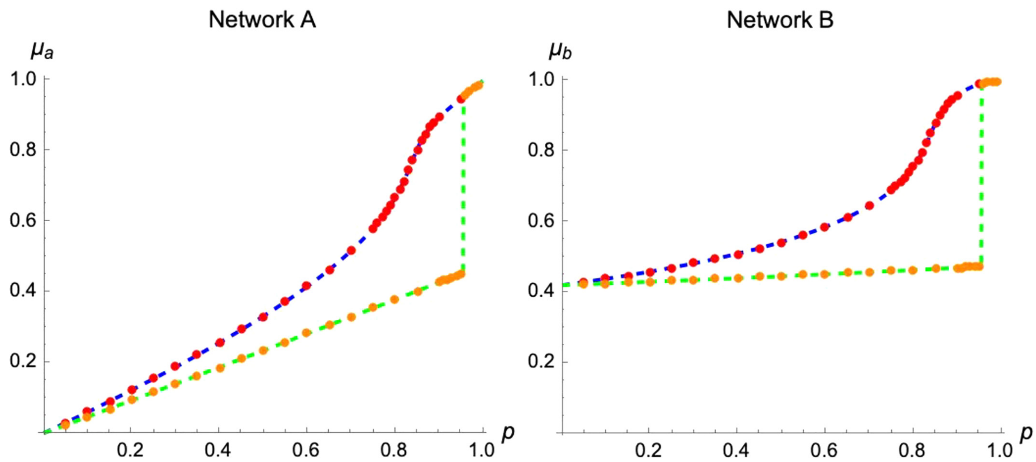

FIG. 5. Fraction of surviving nodes μ_X when the in-degree distributions follow the geometric distribution $P_{\text{in},X}(k) = \theta^k(1-\theta)$, where $\theta = \frac{5}{6}$ so that the mean degree of each node will be 5, and the threshold functions are $r_X(j,k) = \mathscr{G}_2$. The blue and green dashed lines represent theoretical results, and the red and orange dots represent the average of 10 simulations, respectively, for the directional and bidirectional cases. The figures show that the bidirectional case is more vulnerable than its directional counterpart, as the former has fewer surviving nodes than the latter. Note that the directional case happens to not have a discontinuous jump in fraction of survived nodes for the given parameters.

where $\Gamma(x,y)$ is the upper incomplete gamma function. Having the ability to express W in terms of already known, albeit transcendental, functions instead of an infinite sum has many possible theoretical and mathematical benefits. Properties such as derivatives of W are much easier to take in this form.

C. Geometric distribution with gap step threshold function

Let the internetwork supply links follows a geometric distribution: $P_{\text{in},X}(k) = \theta^k(1-\theta)$, where θ is a number between 0 and 1, and let the threshold function $r_X(j,k) = \mathscr{G}_l$. With details presented in Appendix F, the following result can be deduced:

$$W(f) - Z(f) = \sum_{k=0}^{\infty}(k+1)\theta^{k+1}(1-\theta) \times \binom{k}{k-m} f^{k-m}(1-f)^{m+1}, \tag{9}$$

or, more simply,

$$W(f) - Z(f) = -\theta(1-\theta)\frac{\partial W}{\partial \theta}. \tag{10}$$

Upon inspection of Eq. (9), it is clear that each factor in our sum is positive or 0, as both θ and f range between 0 and 1. We therefore find that $W(f) \geqslant Z(f)$. By utilizing the facts presented in the previous sections, for this case bidirectional systems are always more vulnerable than directional ones.

Just as with the Poisson distribution, another simplification can be made in this setting. With details presented in Appendix G, it can be shown that the functions W and Z simplify to be the following:

$$W(f) = 1 - \left(\frac{\theta(1-f)}{1-\theta f}\right)^{m+1}, \tag{11}$$

$$Z(f) = 1 - \left(\frac{\theta(1-f)}{1-\theta f}\right)^{m+1}\left[1 + \frac{(m+1)(1-\theta)}{1-f\theta}\right]. \tag{12}$$

Figures 4–6 display results that support these mathematical findings. For the Poisson distribution, our results showed complete agreement between the directional case and bidirectional case, and therefore their corresponding data points completely overlap. We recycle Fig. 4 to display how these overlapping results look.

IV. CONTINUOUS VERSUS DISCONTINUOUS TRANSITION

The discontinuous transition from an almost intact system to a completely collapsed one at a transitional value of p as the strength of the initial attack, $1-p$, increases is a hallmark of the models of interdependent networks. It is known, however, that the existence of a discontinuous transition is not always the case [4]. The recurrent equations for both directional and bidirectional cases in the limit $n \to \infty$ must converge to a stable fixed point satisfying the following equation:

$$f_{A,\infty} = pW_A[W_B(f_{A,\infty})], \tag{13}$$

for the directional case and the following equation:

$$f_{A,\infty} = pZ_A[Z_B(f_{A,\infty})], \tag{14}$$

for the bidirectional case. These are equations of the type $f = pF(f)$, where f plays the role of $f_{A,\infty}$ and $F(f)$ is a nonlinear function. These equations can be graphically solved by finding the crossing of a straight line representing the left-hand side and a curve representing the right-hand side (Fig. 7). Note that $F(f)$ monotonically increases and $F(1) = 1$.

Usually, $F(f)$ has an inflection point corresponding to the maximum of the first derivative. Accordingly, for large p, $y = pF(f)$ crosses $y = f$ near $f = 1$. At intermediate values of p, it may cross $y = f$ three times, but at a certain value of p, the larger root to which the iterations converge may disappear. At this value of p, the graph of $pF(f)$ becomes tangential to the straight line $y = f$. We now have $f = pF(f)$ and its derivative with respect to f, $1 = pF'(f)$. The system of these two equations will determine the transition point $f = f_t$, $p = p_t$.

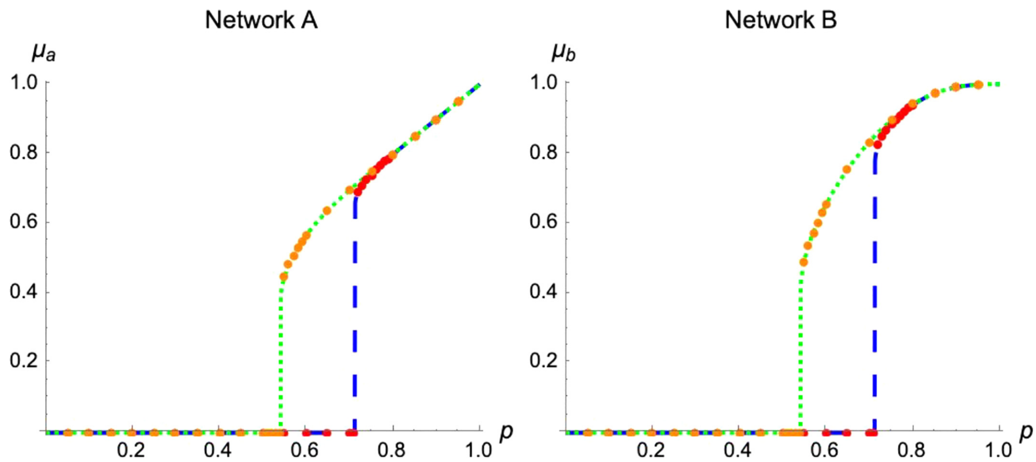

FIG. 6. Fraction of surviving nodes μ_X when the in-degree distributions follow the delta distribution $P_{\text{in},X}(k) = \delta_5(k)$ and the threshold function is $r_X(j,k) = \mathscr{G}_2$. The blue and green dashed lines represent theoretical results and the red and orange dots represent the average of 10 simulations, respectively, for the directional and bidirectional cases. The figures show that the directional case is more vulnerable than its bidirectional counterpart, as the former has fewer surviving nodes than the latter.

If the inflection point f_i of the function $F(f)$ is less than f_t, then the maximum of the derivative of $pF(f)$ is at $f_i < f_t$. Exactly at $f = f_t$, $pF'(f_t) = 1$. Thus, in the entire region $f_i \leqslant f < f_t$, $pF'(f) > 1$, and hence in this region $pF(f) < f$, so any smaller root $f_2 < f_t$ of the equation $pf(f) = f$ must be below f_i, $f_2 < f_i < f_t$. Therefore, we observe a discontinuous jump in the solution from f_t to f_2 by infinitesimal decrease in p, which is a characteristic of the discontinuous first order phase transition [20,21].

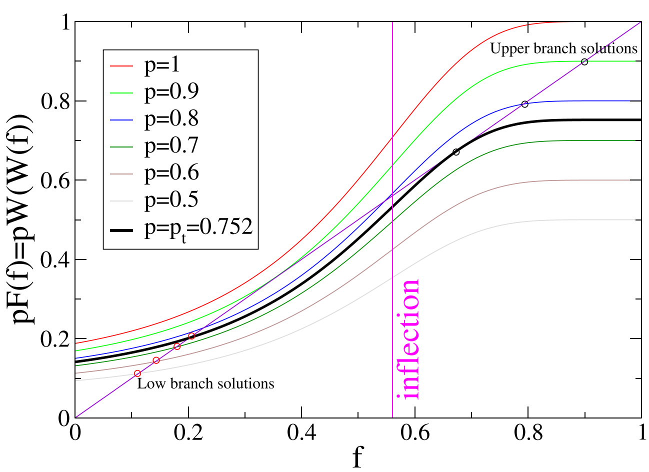

FIG. 7. Graphical solution of the recursive Eq. (13) for the case when networks A and B are identical and the degree distributions in both networks are $P_{X,\text{in}}(k) = \mathscr{P}_5(k)$ and the threshold functions are $r_X(j,k) = \mathscr{G}_2(j,k)$. Different curves correspond to different values of p. All of the curves have the same inflection point f_i. For a given value of p, when its corresponding curve intersects the straight line $f = f$, at possibly more than one place, a solution to the equation is found. We call solutions $f > f_i$ upper branch solutions and solutions $f < f_i$ lower branch solutions. There may be one, two, or three mathematical solutions. For $p > p_t$, the curves representing the right-hand side of Eq. (13) cross the straight line at values of $f > f_i$, giving the upper branch solutions of the equations (black circles). At $p = p_t = 0.752$, the straight line becomes tangential to the curve. At $p < p_t$, the upper branch solution disappears, and only a low branch solution emerges at $f < f_i$.

Let us assume that we have a family of functions $F(f,\lambda)$, characterized by a parameter λ, for example, the average degree. For a certain λ the function $F(f,\lambda)$ may have an inflection point at the transition point. Then $pF'(f) < 1$ for $f < f_t$, which means that by decreasing p below p_t, the crossing point will not jump but rather will continuously decrease. Thus, the condition of the disappearance of the discontinuity gives the third equation $F''(f_t,\lambda) = 0$, from which the critical value λ_c can be found.

The entire phase diagram in the parameter space f, p, λ (Fig. 8) resembles the phase diagram of a liquid [20–22], characterized by its density ρ, pressure, P, and temperature, T, near its critical point ρ_c, P_c, T_c, where the line of the first order transition between liquid and gas disappears. In order to find the critical point from the equation of state $P = P(\rho, T)$ one needs to solve the system of three equations:

$$\begin{aligned} P &= P(\rho, T), \\ \frac{\partial P(\rho,T)}{\partial \rho} &= 0, \\ \frac{\partial^2 P(\rho,T)}{\partial \rho^2} &= 0. \end{aligned} \tag{15}$$

Similarly, for the bipartite networks the equations are

$$\begin{aligned} f &= pF(f,\lambda), \\ p\frac{\partial F(f,\lambda)}{\partial f} &= 1, \\ \frac{\partial^2 F(f,\lambda)}{\partial f^2} &= 0. \end{aligned} \tag{16}$$

Here f plays the role of the order parameter, density, p the role of pressure, and λ the role of temperature. See Fig. 8 for a graphical comparison of these two phase diagrams, where we demonstrate this phenomenon for the Poisson in-degree distribution and the gap step function described above. In

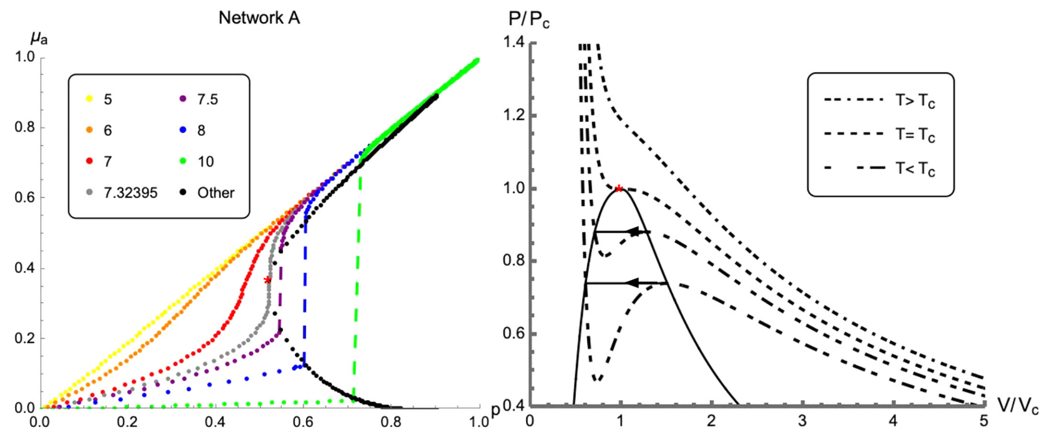

FIG. 8. Figure on the left shows the surviving fraction of nodes as the initial attack on network A varies when the degree distributions of both networks follow the Poisson distribution $P_{\text{in},X}(k) = \mathscr{P}_\lambda$ and the threshold function is $r_X(j,k) = \mathscr{G}_4$. The dotted lines represent theoretical results, and the dashed lines follow the discontinuous jumps. The black dots plot the μ_a values before and after the discontinuous jump for many values of λ. The red star indicates the critical point. For all sets of data, the parameter λ varies as presented in the legend. When λ increases until about 7.32395, the graph reaches the breaking point of its continuity, such as when λ exceeds about 7.32395 the continuity in the graph ceases, as seen by the curve created by the black dots. The critical value of continuity is found using derivatives of the W function, as referenced at the end of Sec. IV. Figure on the right plots pressure of a substance as a function of the volume near the vapor-liquid phase transition [20]. Both pressure and volume have been divided by their critical values such that the break in continuity of the graph takes place at the point (1,1). The horizontal straight lines show the collapse of the vapor sample at constant temperature when the increasing pressure reaches the stability limit (vapor-liquid spinodal) and the supersaturated vapor spontaneously condenses into liquid, which occupies a much smaller volume. The red star indicates the critical point. Note that if the diagram for the network stability is rotated 90° clockwise it becomes equivalent to the phase diagram of liquid-vapor phase transitions.

this case, the recurrent equations for the directional and bidirectional cases are identical and in the limit $n \to \infty$ must converge to a stable fixed point satisfying the equation

$$f_{A,\infty} = pW_A[W_B(f_{A,\infty})]. \tag{17}$$

Figure 9 shows the dependence of the critical parameters of the model as functions of the survival threshold m. One can see that as m increases the discontinuous phase transition emerges for larger values of average degree $\lambda > \lambda_c$, while at this point the networks become more and more resilient as can be seen by the decrease of p_c. Interestingly, λ_c is approximately proportional to $1.35m$ for large m, while p_c and f_c appear to be inversely proportional to $\sqrt{m}$. The rationale for this behavior is given at the end of Sec. VI.

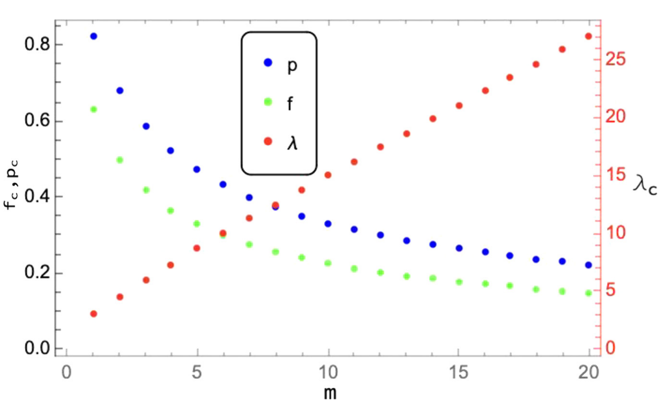

FIG. 9. Critical point parameters f_c, p_c, and λ_c as functions of the survival threshold m when the degree distributions all follow the Poisson distribution $P_{\text{in},X}(k) = \mathscr{P}_\lambda$ and the threshold function is $r_X(j,k) = \mathscr{G}_m$. Note that f_c and p_c follow the vertical axis on the left and that λ_c follows the axis on the right. All values change with respect to the horizontal axis. The critical values are found using derivatives of the W function, as referenced at the end of Sec. IV. The graph shows that f_c and p_c decrease and λ_c increases as m increases.

V. EFFECTS OF OUT-DEGREE DISTRIBUTION

A major point of interest in our theoretical model is the fact that the out-degree distribution does not appear anywhere in the formula. This means that, if our theory is correct, the cascading process is solely determined by the choice of the in-degree distribution. In other words, if the in-degree distribution is fixed, the same cascading process will occur even if the out-degree distributions are different. This is a strong result and should be studied carefully.

The analysis can be categorized into two scenarios, depending on whether the out-degree distribution does or does not have nodes that function as hubs and provide outgoing supply links to a great portion of nodes in the other network. In the case of the Pareto degree distribution, $P_{\text{out},x}(k) \sim k^{-\alpha-1}$ with $\alpha \leqslant 2$, the second moment of the distribution diverges, and, hence, hubs start to emerge. In random realizations of the cascading process, the death of a hub will produce much larger damage than the death of a node with a small degree. For non-scale-free networks, the statistical error of the fraction of survived nodes at the end of the cascade is expected to decrease according to the central limit theorem as $1/\sqrt{N}$, where N is the number of nodes. Thus, for non-scale-free networks of the size of a million nodes, the deviation of the theoretical and simulated result is expected to be about 0.1%. In a scale-free network, with the diverging second moment but the converging first moment ($1 < \alpha \leqslant 2$), according to our preliminary numerical results the error decreases as $N^{1/\alpha-1}$.

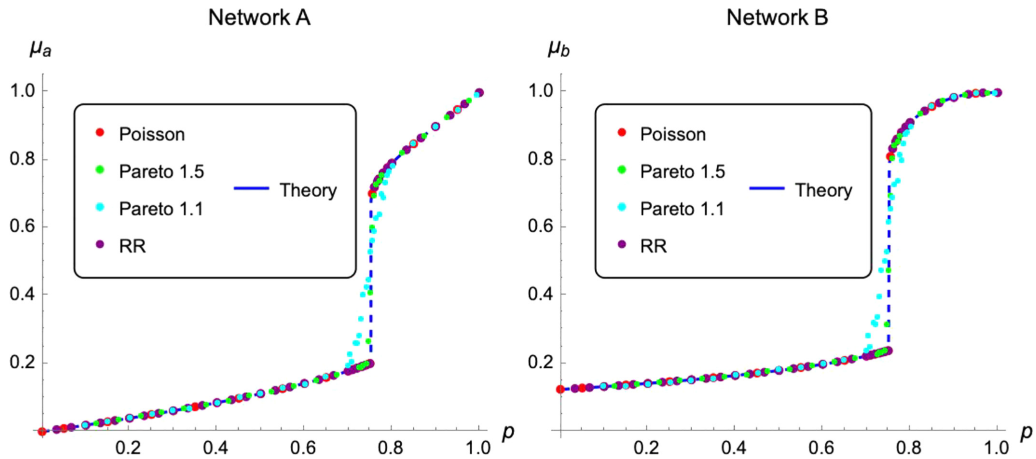

FIG. 10. Fraction of surviving nodes when the in-degree distributions follow a Poisson distribution $P_{\text{in},X}(k) = \mathscr{P}_5$ and the threshold function is $r_X(j,k) = \mathscr{G}_2$. The out-degrees correspond to four different distributions, as shown in the legend. The simulation results from delta (RR) and Poisson out-degrees are averages of 10 samples at each p value, while those from Pareto distributions are the average of 100 samples at each p values. As one can see, all simulations apart from the one that is sampled from the Pareto distribution with a power of 1.1 agree with the theoretical values.

The heuristic justification of this formula is that for $\alpha = 2$ the exponent $1/\alpha - 1 = -1/2$ and we recover the Gaussian behavior of the error $\sim 1/\sqrt{N}$ for $\alpha > 2$, while for $\alpha = 1$, $1/\alpha - 1 = 0$, and indeed for $\alpha \leqslant 1$, we find that the error does not decrease with N at all.

As an example, we provide the simulations with the same in-degree distribution $P_{\text{in},X}(k) = \mathscr{P}_5$ and the same threshold function $r_X(j,k) = \mathscr{G}_2$, but with different out degree distributions: Poisson, delta, and the scale-free with $\alpha = 1.5$ and $\alpha = 1.1$. We see that the difference in the fraction of survived nodes at the end of the cascade, $\mu_{X,\infty}$ between the simulations with Poisson and delta out-degree distribution (Fig. 10) are invisible on the scale of the graph, while for scale-free out-degree distribution with $N = 10^6$ (Fig. 11) the error is quite visible and constitutes about 10% of the predicted result. The error is especially big near the point of the abrupt transition $p = 0.77$. The above referenced figures all share the same in-degree distribution.

The discrepancy between the theory and simulations for the Pareto out-degree distribution with $\alpha = 1.1$ near the transition point is because for p close to p_t some cascades stop before the network complete collapse while others proceed to the complete destruction of the network. Accordingly, when the results are averaged over 100 realizations, we average the results from these two cases, the fraction of which changes with the distance from the transition points. Figure 11 shows the results for all 100 realizations as clouds of points. One can see that the clouds for $\alpha = 1.5$ and for $\alpha = 1.1$

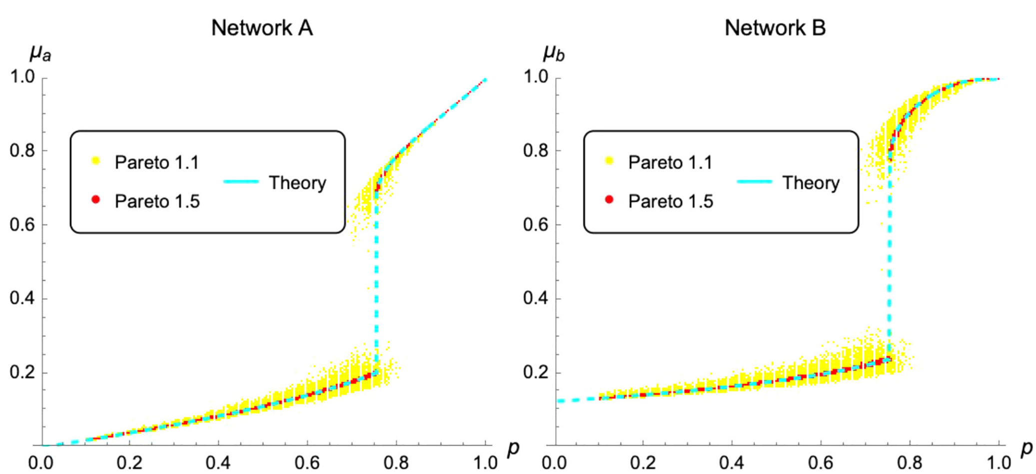

FIG. 11. Fraction of surviving nodes when the in-degree distributions follow a Poisson distribution $P_{\text{in},X}(k) = \mathscr{P}_5$, and the threshold function is $r_X(j,k) = \mathscr{G}_2$. The out-degrees follow two different Pareto distributions with shape parameters $\alpha = 1.1$ and $\alpha = 1.5$, as shown in the legend. The simulation results are scatter plots with 100 samples at each of the p values. As one can see, the scatter plot of Pareto 1.5 agrees with the theoretical values with a very narrow spread, while that of Pareto 1.1 has a large spread. This fact does not mean the theory has failed at this point, as the scatter points of Pareto 1.1 still follow the trend as predicted by the theory and the spread gets narrower when p values are close to both ends.

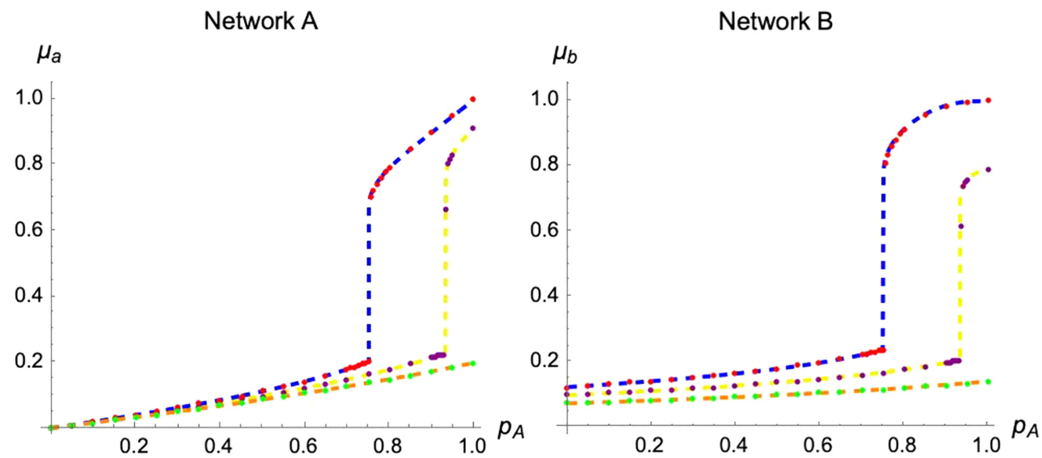

FIG. 12. Surviving fraction of nodes when the in- and out-degree distributions all follow the Poisson distribution $P_{\text{in},X}(k) = \mathscr{P}_5$ and the threshold function is $r_X(j, k) = \mathscr{G}_2$. The dashed lines represent theoretical results, and the dots represent the average of ten simulations. The x axis for both networks shows the fraction p_A of survived nodes in network A after the initial attack. The three different curves correspond to three different fractions p_B of nodes survived the initial attack on network B: $p_B = 1$, $p_B = 0.8$, and $p_B = 0.6$, respectively, corresponding to blue (theory) and red (simulation), yellow (theory) and purple (simulation), and orange (theory) and green (simulation).

follow the theoretical predictions but with a large spread which increases as $\alpha \to 1+$ when the first moment of the Pareto distribution diverges.

VI. ATTACKS ON BOTH NETWORKS

So far, we have discussed only the directional bipartite network with the initial attack limited to a single network. It is not difficult to generalize our model to the case when the initial attacks are made on both networks, such that a fraction of nodes p_A remains functional in network A and a fraction of nodes p_B remains fractional in network B. In this case, the original recursive relations must be modified, as follows.

For stage $n > 1$:

$$f_{A,n} = p_A W_A(f_{B,n}),$$
$$f_{B,n} = p_B W_B(f_{A,n-1}).$$

The fraction of surviving nodes at stage n of the cascade are

$$\mu_{A,n} = f_{A,n},$$
$$\mu_{B,n} = f_{B,n},$$

where

$$f_{A,1} = p_A,$$
$$f_{B,1} = p_B.$$

See Fig. 12 for how changing p_B can change the fraction of surviving nodes.

If the networks are identical and $p_A = p_B = p$, the recurrent equations dramatically simplify and become just $f_n = pW(f_{n-1})$ for both networks, with odd values of n representing network A end even values of n representing network B. Moreover, networks A and B can be understood as the same network because, due to a low degree of dependency links, the chance that a node will become dependent on itself is practically zero. This case is equivalent to the Watts opinion model [14] with directional links. In the case of a Poisson degree distribution and $r(j, k) = \mathscr{G}_m$, the equations (16) for the critical point λ_c (such that for $\lambda > \lambda_c$, the catastrophic cascade of of failures emerges) reduce to simple algebraic equations which yield

$$\lambda_c = m + \frac{e^m \Gamma(1+m, m)}{m^m},$$
$$f_c = 1 - m/\lambda_c, \tag{18}$$
$$p_c = \frac{e^m m!}{m^m \lambda_c}.$$

For example, for $m = 1$, $\lambda_c = 3$, $f_c = 2/3$, and $p_c = e/3$, while for $m = 2$, $\lambda_c = 9/2$, $f_c = 5/9$, and $p_c = e^2/9$. Asymptotically, when $m \to \infty$, $\lambda = m + \sqrt{2\pi/m} + O(m^{-1/2})$, and $p_c \approx f_c = \sqrt{2\pi/m} + O(1/m)$, where $\sqrt{2\pi/m}$, comes from Stirling approximation of the factorial. These results qualitatively resemble the behavior of the critical parameters for the bipartite network, when only one network is attacked (Fig. 9).

VII. CONCLUSION

Our paper generalizes previous studies on bidirectional bipartite networks to directional ones. We have provided a set of recursion relations that can precisely describe the cascading process for large networks, and we have compared the theoretical results with simulations to show that they are in accordance with each other in most of the cases. Analytic proofs have been provided for the special cases of the gap functionality threshold, Eq. (3), and delta ingoing link distributions that the fraction of survived nodes at the end of each stage of the cascading process $\mu_{X,n}$ is greater for bidirectional systems than for directional systems, i.e., bidirectional $\mu_{X,n}$ $\geqslant$ directional $\mu_{X,n}$ for any stage $n > 1$; we did the same for the case of a geometric distribution with the gap threshold function, where we found the opposite behavior with regard to directional and bidirectional systems. We also study another interesting setting when the ingoing links follow a Poisson distribution and show that, in the case of the gap threshold function that we have introduced, a directional system behaves exactly the same way as a bidirectional system, $W_X = Z_X$, and both generating functions can be described using the incomplete gamma functions as

$$W(f) = Z(f) = \frac{\Gamma(l+1, (1-f)\lambda)}{\Gamma(l+1)}.$$

We found similar simplifications for the W and Z functions for the geometric distribution and gap threshold function.

The most prominent finding of our paper is the fact that the choice of out-degree distribution has no effect on the cascading process of directional interdependent networks, such as when they are Kronecker delta or Poisson. Even in the case of scale-free distributions, in which hub nodes can dominate survival of the entire network, we can still find high consistency between the theory and the simulations, although discrepancies exist, especially when close to the transition points.

We also presented the condition when the behavior of the network is continuous with respect to the change of the size of the initial attack versus where there is a transition point above which a catastrophic cascade of failures reduces the fraction of survived nodes to almost zero. We show that the phase diagram of this system is similar to the phase diagram of liquid and gas, for which the first-order transition ends at a critical point.

In the last section, we generalize our model to the case where random attacks are not isolated to a single network but are presented in both networks. We also show there that the simulation results are in agreement with the theoretical ones. Additionally, the case of two identical networks with equal initial attacks on the two interdependent networks becomes equivalent to a single network with supply links.

ACKNOWLEDGMENTS

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APPENDIX A: PROOF OF RECURSION RELATIONS FOR THE CASCADE OF FAILURES IN THE DIRECTIONAL CASE

The probability that node b in network B originally had k ingoing links is $P_{B,\text{in}}(k)$, In order to be functional, node b must have at least j_b ingoing functional links. The probability that an ingoing link is functional is $f = f_{A,n-1} = \mu_{A,n-1}$. Accordingly, the probability that exactly j out of k links are functional is given by the binomial formula

$$p_j = \binom{k}{j} f^j (1-f)^{k-j}. \tag{A1}$$

The probability that at least j_b links out of k links are functional is the sum

$$p(j \geqslant j_b) = \sum_{j=j_b}^{k} p_j. \tag{A2}$$

The probability that $j_b = j$ is by definition $r_B(j,k) - r_B(j-1,k)$, where $r_B(-1,k) \equiv 0$. Accordingly, the probability that the number of functional ingoing links is greater than or equal to the functional threshold for randomly selected node b of degree k is

$$P_k = \sum_{j_b=0}^{k} [r_B(j_b,k) - r_B(j_b-1,k)] \sum_{j=j_b}^{k} p_j. \tag{A3}$$

Changing the order of summation, we get

$$P_k = \sum_{j=0}^{k} p_j \sum_{j_b=0}^{j} [r_B(j_b,k) - r_B(j_b-1,k)] = \sum_{j=0}^{k} p_j r_B(j,k). \tag{A4}$$

Summing up the contributions from all k, we get

$$f_{B,n} = \sum_{k=0}^{\infty} P_{B,\text{in}}(k) P_k = W_B(f_{A,n-1}). \tag{A5}$$

.

APPENDIX B: PROOF OF RECURSION RELATIONS FOR THE CASCADE OF FAILURES IN THE BIDIRECTIONAL CASE

The problem arises if we want to apply a similar strategy to describe the cascading process when all of the links are bidirectional. If we select a random ingoing link of a node b in network B at the nth stage of the cascade in the bidirectional case, it also serves as an ingoing link of a node a in network A at the other end, and hence the functionality of a affects the functionality of b. Thus, it is not necessarily the case that $f_{B,n} = \mu_{B,n}$ and may instead be the case that $f_{B,n} \neq \mu_{B,n}$.

Hence, in order to derive the recursion relations for the bidirectional case, we must change the definition of functional links. We will call $f_{B,n}$ a conditional probability that a bidirectional link ends in a functional node in network B, provided that its other end is a functional node in network A at the nth stage of the cascade. This condition must be included because, at this stage of the cascade, the nodes in network A are assumed to be functional before we start to compute their new status. Analogously, $f_{A,n}$ is a conditional probability that a bidirectional link ends in a functional node in network A provided that its other end is connected to a functional node in network B at the nth stage of the cascade. Applying the same logic as in the directional case, we can say for the bidirectional case that $\mu_{A,n} = pW_A(f_{B,n})$ and $\mu_{B,n} = W_B(f_{A,n-1})$.

Now let us take a look at a randomly selected bidirectional link, assuming that the end of this link in network A is a functional node at the nth stage of the cascade. It connects with a node b in network B with a probability proportional to the in-degree of node b, which coincides in the bidirectional case with its out-degree, $P_{B,\text{in}}(k) = P_{B,\text{out}}(k) \equiv P_B(k)$. This probability is given by the excess degree distribution [10] $P_B(k)k/\langle k \rangle_B$, where $\langle k \rangle_B$ is the average degree distribution of network B. Thus, to find a recursive relation to represent f_B, the probability $P_{\text{in},B}(k)$ in the expression for $W_B(f)$ must be replaced by the excess degree distribution.

We now need to determine the probability that the node b is functional. By construction, the link by which we arrive at node b is functional, and hence the minimal number of other functional links coming to node b in order to keep it functional must be at least $j_b - 1$. The number of other functional links j in the sum over j in the expression for $W(f)$ must now go from $j = 0$ to $k - 1$, while the threshold function must be changed from $r_B(j, k)$, to $r_B(j + 1, k)$, because the actual number of functional links coming to node b is not j but $j + 1$. Additionally, k must be replaced by $k - 1$, as we are now dealing with the excess degree of b, or the degree in excess of the one link that we take to be functional. Note that these j links are functional with probability $f_{A,n-1}$ at the previous stage of the cascade. Combining all the changes in the function W, we conclude that

$$f_{B,n} = \sum_{k=1}^{\infty} \frac{P_B(k)k}{\langle k\rangle_B} \sum_{j=0}^{k-1} \binom{k-1}{j} \times r_B(j+1, k) f_{A,n-1}^{j} (1 - f_{A,n-1})^{k-1-j}. \tag{B1}$$

The right-hand side of Eq. (B1) coincides with the definition Eq. (2) function. Thus, $f_{B,n} = Z_B(f_{A,n-1})$. Analogously,

$$f_{A,n} = pZ_A(f_{B,n}). \tag{B2}$$

APPENDIX C: PROOF OF EQ. (6)

Starting with Eq. (5) and introducing the index $n = m - l$,

$$Z(f) - W(f) = \sum_{j=n}^{m} \binom{m-1}{j-1} f^{j-1}(1-f)^{m-j} - \sum_{j=n}^{m} \binom{m}{j} f^{j}(1-f)^{m-j}.$$

Utilizing Pascal's Triangle in the second line

$$\binom{m}{j} = \binom{m-1}{j} + \binom{m-1}{j-1}$$

and joining the terms with $\binom{m-1}{j-1}$, we get

$$Z(f) - W(f) = \sum_{j=n}^{m} \binom{m-1}{j-1} f^{j-1}(1-f)^{m-j+1} - \sum_{j=n}^{m} \binom{m-1}{j} f^{j}(1-f)^{m-j}.$$

Replacing the summation index $j - 1$ in the first line by j', we get

$$(Z - W)(f) = \sum_{j=n-1}^{m-1} \binom{m-1}{j} f^{j}(1-f)^{m-j} - \sum_{j=n}^{m} \binom{m-1}{j} f^{j}(1-f)^{m-j}.$$

We notice that all of the terms except for the first one in the first line and the last one in the second line cancel and that $\binom{m-1}{m} = 0$. Hence,

$$(Z - W)(f) = \binom{m-1}{n-1} f^{n-1}(1-f)^{m-n+1} = \binom{m-1}{m-l-1} f^{m-l-1}(1-f)^{l+1}.$$

Q.E.D.

APPENDIX D: PROOF OF EQ. (7)

Given a Poisson distribution

$$P(k) = \frac{\lambda^k}{k!} e^{-\lambda},$$

then

$$\langle k\rangle = \lambda.$$

Also, as we have noticed in the main text, if $r_X(j, k) = \mathscr{G}_m$, then $r_X(j + 1, k) = r_X(j, k - 1)$. Taking all of this into account in expression (2) for Z_X, we get

$$Z_X(f) = \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} e^{-\lambda} \sum_{j=0}^{k-1} \binom{k-1}{j} \times f^{j}(1-f)^{k-1-j} r_X(j, k-1). \tag{D1}$$

Changing index of summation $k - 1$ to k' we see that this equation is identical to Eq. (7). Q.E.D.

APPENDIX E: PROOF OF EQ. (8)

We have stated that

$$W(f) = \sum_{k=0}^{\infty} P_{\text{in}}(k) \sum_{j=0}^{k} \binom{k}{j} f^{j}(1-f)^{k-j} r(j, k) \tag{E1}$$

can be written as

$$W(f) = \frac{\Gamma(m+1, (1-f)\lambda)}{\Gamma(m+1)} \tag{E2}$$

if $P_{\text{in}}(k) = \mathscr{P}_\lambda(k)$ is a Poisson distribution and $r(j, k) = \mathscr{G}_m(j, k)$ is a gap step function. For integer values of m, using the definitions of the Gamma and incomplete Gamma functions, Eq. (E2) can be rewritten as

$$W(f) = \frac{\int_{\lambda(1-f)}^{\infty} e^{-t} t^m \, dt}{m!}.$$

Integrating by parts (by integrating the first factor and differentiating the second factor), we get

$$\frac{1}{m!}\left(-e^{-t} t^m \Big|_{\lambda(1-f)}^{\infty} + m \int_{\lambda(1-f)}^{\infty} e^{-t} t^{m-1} \, dt \right) = \frac{1}{m!}\left(e^{-\lambda(1-f)} (\lambda(1-f))^m + m \int_{\lambda(1-f)}^{\infty} e^{-t} t^{m-1} \, dt \right).$$

Repeating the integration by parts until the integral vanishes leaves us with a finite amount of terms:

$$W(f) = e^{-\lambda(1-f)} \sum_{l=0}^{m} \frac{[\lambda(1-f)]^l}{l!}. \tag{E3}$$

Going back to the formula in Eq. (E1), we can rewrite it to replace the threshold function, which is a gap step function of gap m, with a change in index, along with plugging in the Poisson distribution of mean λ into the degree distribution:

$$\sum_{k=0}^{\infty} \frac{e^{-\lambda}\lambda^k}{k!} \sum_{j=\max(0,k-m)}^{k} \binom{k}{j} f^j (1-f)^{k-j},$$

where $\max(x, y)$ denotes the maximum value of x and y. Using the replacements of $k - j = l$ and $k - l = n$, and noting that the minimum and maximum values of $k - l$ are, by definition, 0 and ∞, we get

$$\begin{aligned}
&\sum_{k=0}^{\infty}\sum_{l=0}^{m} \frac{e^{-\lambda}\lambda^k}{k!} \frac{k!}{l!(k-l)!} f^{k-l}(1-f)^l \\
&= e^{-\lambda} \sum_{k=0}^{\infty}\sum_{l=0}^{m} \lambda^k \frac{f^{k-l}}{(k-l)!} \frac{(1-f)^l}{l!} \\
&= e^{-\lambda} \sum_{l=0}^{m}\sum_{n=0}^{\infty} \lambda^{n+l} \frac{f^n}{(n)!} \frac{(1-f)^l}{l!}, \\
&= e^{-\lambda} \sum_{l=0}^{m} \frac{[\lambda(1-f)]^l}{l!} \sum_{n=0}^{\infty} \lambda^n \frac{f^n}{(n)!} \\
&= e^{-\lambda} \sum_{l=0}^{m} \frac{[\lambda(1-f)]^l}{l!} e^{\lambda f} \\
&= e^{-\lambda(1-f)} \sum_{l=0}^{m} \frac{[\lambda(1-f)]^l}{l!}.
\end{aligned} \tag{E4}$$

Recognizing that Eqs. (E3) and (E4) are equivalent completes the proof. Q.E.D.

APPENDIX F: PROOF OF EQ. (9)

To begin the proof, let us write out both functions:

$$W(f) = \sum_{k=0}^{\infty} \theta^k (1-\theta) \sum_{j=0}^{k} \binom{k}{j} f^j (1-f)^{k-j} r_X(j,k)$$

and

$$\begin{aligned}
Z(f) &= \sum_{k=1}^{\infty} \frac{\theta^k(1-\theta)k}{\langle k\rangle} \sum_{j=0}^{k-1} \binom{k-1}{j} \\
&\quad \times f^j (1-f)^{k-j-1} r_X(j+1,k) \\
&= \sum_{k=1}^{\infty} \frac{\theta^k(1-\theta)k}{\frac{\theta}{(1-\theta)}} \sum_{j=0}^{k-1} \binom{k-1}{j} \\
&\quad \times f^j (1-f)^{k-j-1} r_X(j+1,k) \\
&= \sum_{k=1}^{\infty} \theta^k k \frac{(1-\theta)^2}{\theta} \sum_{j=0}^{k-1} \binom{k-1}{j} \\
&\quad \times f^j (1-f)^{k-j-1} r_X(j+1,k),
\end{aligned}$$

where we took into account that $\langle k\rangle = \frac{\theta}{1-\theta}$.

Using the fact that for the gap step function $r_X(j,k) = r_X(j+1,k+1)$ and replacing $k-1$ with k' (and then rewriting k' as k), we can rewrite $Z(f)$ as

$$(1-\theta)^2 \sum_{k=0}^{\infty} \theta^k (k+1) \sum_{j=0}^{k} \binom{k}{j} f^j (1-f)^{k-j} r_X(j,k).$$

Noting that $\frac{\partial}{\partial\theta}\theta^{k+1} = \theta^k(k+1)$, we can rewrite $Z(f)$ as

$$(1-\theta)^2 \sum_{k=0}^{\infty} \frac{\partial}{\partial\theta} \theta^{k+1} S(k),$$

where we made the substitution

$$S(k) = \sum_{j=0}^{k} \binom{k}{j} f^j (1-f)^{k-j} r_X(j,k).$$

Exchanging the partial derivative with the infinite sum, we get

$$\begin{aligned}
&(1-\theta)^2 \frac{\partial}{\partial\theta} \sum_{k=0}^{\infty} \theta^{k+1} S(k) \\
&= (1-\theta)^2 \frac{\partial}{\partial\theta}\left[\frac{\theta}{1-\theta} W(f)\right] \\
&= (1-\theta)^2 \left[\frac{1}{(1-\theta)^2} W(f) + \frac{\theta}{1-\theta}\frac{\partial W}{\partial\theta}\right] \\
&= W(f) + \theta(1-\theta)\frac{\partial W}{\partial\theta}.
\end{aligned}$$

Solving for $W(f) - Z(f)$, we get

$$W(f) - Z(f) = -\theta(1-\theta)\frac{\partial W}{\partial\theta}.$$

Solving for $\frac{\partial W}{\partial\theta}$, we get

$$\begin{aligned}
&\frac{\partial}{\partial\theta} \sum_{k=0}^{\infty} \theta^k (1-\theta) S(k) \\
&= \frac{\partial}{\partial\theta} \sum_{k=0}^{\infty} (\theta^k - \theta^{k+1}) S(k) \\
&= \sum_{k=0}^{\infty} \frac{\partial}{\partial\theta} [(\theta^k - \theta^{k+1}) S(k)] \\
&= \sum_{k=0}^{\infty} (k\theta^{k-1} - (k+1)\theta^k) S(k).
\end{aligned}$$

Splitting this sum into two parts, we get

$$\begin{aligned}
&= \sum_{k=0}^{\infty} k\theta^{k-1} S(k) - \sum_{k=0}^{\infty} (k+1)\theta^k S(k) \\
&= 0 + \sum_{k=1}^{\infty} k\theta^{k-1} S(k) - \sum_{k=0}^{\infty} (k+1)\theta^k S(k).
\end{aligned}$$

Shifting k in the first sum by 1, we get

$$\sum_{k=0}^{\infty}(k+1)\theta^k S(k+1) - \sum_{k=0}^{\infty}(k+1)\theta^k S(k)$$
$$= \sum_{k=0}^{\infty}(k+1)\theta^k[S(k+1) - S(k)].$$

Plugging this back into our formula for $W(f) - Z(f)$, we get

$$\sum_{k=0}^{\infty}(k+1)\theta^{k+1}(1-\theta)[S(k) - S(k+1)].$$

Let us simplify the factor $S(k) - S(k+1)$ and rewrite the function $r_X(j,k)$ as changing the index of j by spanning from $k-m$ to k:

$$\sum_{j=k-m}^{k}\binom{k}{j}f^j(1-f)^{k-j} - \sum_{j=k-m+1}^{k+1}\binom{k+1}{j}f^j(1-f)^{k-j+1}.$$

Replacing in the second sum j with $j+1$ and appropriately shifting the index gives us

$$\sum_{j=k-m}^{k}\binom{k}{j}f^j(1-f)^{k-j} - \sum_{j=k-m}^{k}\binom{k+1}{j+1}f^{j+1}(1-f)^{k-j}.$$

Utilizing Pascal's Triangle in the second sum

$$\binom{k+1}{j+1} = \binom{k}{j+1} + \binom{k}{j}$$

and joining the terms with $\binom{k}{j}$, we get

$$\sum_{j=k-m}^{k}\binom{k}{j}f^j(1-f)^{k-j+1} - \sum_{j=k-m}^{k}\binom{k}{j+1}f^{j+1}(1-f)^{k-j}.$$

Replacing in the first sum j with $j+1$ and appropriately shifting the index gives us

$$\sum_{j=k-m-1}^{k-1}\binom{k}{j+1}f^{j+1}(1-f)^{k-j}$$
$$- \sum_{j=k-m}^{k}\binom{k}{j+1}f^{j+1}(1-f)^{k-j}.$$

We notice that all of the terms except for the first one in the first sum and the last one in the second sum cancel and that $\binom{k}{k+1} = 0$. Hence

$$S(k) - S(k+1) = \binom{k}{k-m}f^{k-m}(1-f)^{m+1}.$$

Plugging this back into our formula, we get

$$W(f) - Z(f) = \sum_{k=0}^{\infty}(k+1)\theta^{k+1}$$
$$\times (1-\theta)\binom{k}{k-m}f^{k-m}(1-f)^{m+1}.$$

Upon inspection, it is clear that each factor in our sum is positive, as both θ and f range between 0 and 1. We therefore find that $W(f) \geqslant Z(f)$. Q.E.D.

APPENDIX G: PROOF OF EQUATIONS (11) AND (12)

Continuing our notation from Appendix F, and noting that $S(0) = 1$, we can split W into two parts, shift the index in the first sum, and recombine them, as follows:

$$W(f) = \sum_{k=0}^{\infty}(\theta^k - \theta^{k+1})S(k)$$
$$= \sum_{k=0}^{\infty}\theta^k S(k) - \sum_{k=0}^{\infty}\theta^{k+1}S(k)$$
$$= 1 + \sum_{k=1}^{\infty}\theta^k S(k) - \sum_{k=0}^{\infty}\theta^{k+1}S(k)$$
$$= 1 + \sum_{k=0}^{\infty}\theta^{k+1}S(k+1) - \sum_{k=0}^{\infty}\theta^{k+1}S(k)$$
$$= 1 + \sum_{k=0}^{\infty}\theta^{k+1}[S(k+1) - S(k)]$$
$$= 1 - \sum_{k=0}^{\infty}\theta^{k+1}[S(k) - S(k+1)].$$

Plugging in our previous result for $S(k) - S(k+1)$, we get

$$1 - \sum_{k=0}^{\infty}\theta^{k+1}\binom{k}{k-m}f^{k-m}(1-f)^{m+1}.$$

Since k cannot be less than m, we can start the sum at $k = m$, and then replace $k-m$ with k' (and rewrite k' as k):

$$1 - \sum_{k=0}^{\infty}\theta^{k+m+1}\binom{k+m}{k}f^k(1-f)^{m+1}$$
$$= 1 - (\theta(1-f))^{m+1}\sum_{k=0}^{\infty}\frac{(k+m)!}{k!m!}(\theta f)^k.$$

To solve for the infinite sum, note that we can express $(\frac{1}{1-\theta f})^{m+1}$ through a Taylor series around $\theta = 0$:

$$\left(\frac{1}{1-\theta f}\right)^{m+1} = 1 + (m+1)\theta f$$
$$+ \frac{(m+2)(m+1)}{2!}\theta^2 f^2 + \cdots,$$

which is exactly the same as our infinite sum. Therefore, we can express $W(f)$ as a very simple expression:

$$W(f) = 1 - \left(\frac{\theta(1-f)}{1-\theta f}\right)^{m+1}.$$

Solving for $Z(f)$ in Eq. (10), we get

$$Z(f) = 1 - \left(\frac{\theta(1-f)}{1-\theta f}\right)^{m+1}\left(1 + \frac{(m+1)(1-\theta)}{1-f\theta}\right).$$

APPENDIX H: GRAPH-GENERATING ALGORITHM

Let the bipartite graph consist of networks A and B with number of nodes N_A and N_B, respectively. In the directional case each in-link supporting a node in network A is an out-link of a node in network B. Hence, the number of in-links supporting nodes in network A must be equal to the number of out-links emanating out of nodes of network B: $n_{A,\text{in}} = n_{B,\text{out}}$ and vice versa $n_{A,\text{out}} = n_{B,\text{in}}$. Moreover,

$$\sum_{i=1}^{N_A} k_{i,\text{in}} = \sum_{j=1}^{N_B} k_{j,\text{out}} = n_{A,\text{in}} \tag{H1}$$

and

$$\sum_{i=1}^{N_A} k_{i,\text{out}} = \sum_{j=1}^{N_B} k_{j,\text{in}} = n_{B,\text{in}}, \tag{H2}$$

where $k_{i,\text{in}}$ and $k_{i,\text{out}}$ are respectively in and out-degrees of node i in network A, and $k_{j,\text{in}}$ and $k_{j,\text{out}}$ are respectively in and out-degrees of node j in network B.

Let $N_{A,\text{out}}(k)$ be the number of nodes with out-degree k in network A. We analogously introduce notations $N_{A,\text{in}}(k)$, $N_{B,\text{out}}(k)$ and $N_{B,\text{in}}(k)$. If we want to construct a graph with given degree distributions $P_{X,\text{out}}(k)$ and $P_{X,\text{in}}(k)$, where $X = A$ or $X = B$, we must assume that $N_{X,\text{in}}(k)/N_X \to P_{X,\text{in}}(k)$ and $N_{X,\text{out}}(k)/N_X \to P_{X,\text{out}}(k)$ for $N_X \to \infty$. Thus, the distributions must satisfy conditions: $N_A \sum_k kP_{A,\text{in}}(k) = N_B \sum_k kP_{B,\text{out}}(k)$ for $N_A \to \infty, N_B \to \infty$ or

$$\sum_k N_A \langle k \rangle_{A,\text{in}} = \sum_k N_B \langle k \rangle_{B,\text{out}} \tag{H3}$$

and

$$\sum_k N_A \langle k \rangle_{A,\text{out}} = \sum kN_B \langle k \rangle_{B,\text{in}}. \tag{H4}$$

It is obvious that it is impossible to satisfy these conditions with arbitrary degree distributions, even if N_A/N_B is a free parameter.

If our goal is to demonstrate that the results of the cascade of failures are independent of the out-degree distribution, we must find the number of nodes $N(k)$ with degree $k = 0, 1, 2, \dots$ such that $\sum_k N(k) = N$, $\sum_k kN(k) = n$ and $N(k)/N \approx P(k)$, where $P(k)$ is a given distribution. The sign $\approx$ means that the error is comparable with the value predicted by the central limit theorem: $|N(k)/[P(k)N] - 1| < \sqrt{P(k) - P^2(k)}/\sqrt{N} < 1/\sqrt{N(k)}$. This condition is always satisfied if one changes $N(k)$ by 1.

To achieve our goal, we first generate $N'(k) = \text{int}(NP(k) + 0.5)$, where $\text{int}(x)$ is the integer part of real value x. Then we find a preliminary number of links based on our choice of $N'(k)$, $n' = \sum_k kN'(k)$. This number may not be exactly equal to the desired value of n, so the values $N'(k)$ must be slightly modified in order to satisfy the condition $n' = n$ exactly. If $n' > n$, we find the value of k for which $N'(k)$ is maximal and change $N'(k-1) = N'(k-1) + 1$ and $N'(k) = N'(k) - 1$. This operation reduces n' by one. Then we repeat this process until $\sum_k N(k) = n$. If $n' < n$ then we do the opposite operation.

Once $N(k)$ is known, we assign degree k for $N(k)$ randomly selected nodes. We repeat these procedure for both networks A and B using their in-degree and out-degree distributions, and for both networks we assign the degree 1 to nodes $1, 2, \dots, N(1)$, degree 2 to the next $N(2)$ nodes $1 + N(1), 2 + N(1), \dots, N(2) + N(1)$, and so on. Then for both networks we construct a sequence of n numbers $1, 1, \dots, 1, 2, 2, \dots, 2, \dots, i, i, \dots, i, \dots, N, N, \dots, N$ in which the number of repeating elements i is the degree of node i. Finally, one of these sequences is randomly shuffled, and the corresponding elements in both sequences, representing nodes in networks A and B, are connected.

The difficulty comes for the Pareto, or scale-free, distribution, which is characterized with the exponent α and whose minimal degree and maximal degree cutoffs are such that it will have a given $\langle k \rangle$. In principle, for demonstration of the effect, any distribution with Pareto tail: $P(k) \sim 1/k^{\alpha+1}$ for $k \to \infty$ will work. However, for finite networks consisting of N nodes, k has a natural cutoff $k \leqslant N^{1/\alpha}$ [8]. Accordingly, at the beginning, we generate a preliminary sequence of degrees $k_i' = \text{int}[(N/k)^{1/\alpha}]$, for $i = 1, 2, \dots, N$. The degree distribution of such a sequence $P(k)$ scales as $C(1/k)^{\alpha+1}$ where C is a normalization constant.

First, we find the number of bonds that this degree distribution would produce: $n' = \sum_{i=1}^{N} k_i'$. Then we define the desired number of bonds: $n = \langle k \rangle N$, and define the new degree distribution as $k_i'' = \text{int}[(1 - \epsilon)(n/n')(N/k)^{1/\alpha}]$, where ϵ is a small number selected in a such a way that $n'' = \sum_{i=0}^{N} k_i''$ is slightly less than n. One can select $\epsilon = 0.1$. Note that k_i'' is a monotonically decreasing function of i, so that if we keep assigning the degrees of nodes according to our preliminary choice, the sum of all the degrees will be less than n, so at certain j, instead of k_j'' we select a constant value, which can be found, knowing the current sum the degrees, and the number of nodes left unassigned. Thus, the final degrees for all N nodes starting from node $j = 1$ can be chosen as $k_j = \max(k_j'', m_j)$, where m_j is the estimate of the degree needed to satisfy the condition

$$\sum_{i=1}^{N} k_i = n, \tag{H5}$$

assuming that the rest of the degrees are equal to

$$m_j = \text{int}\left[\left(n - \sum_{i=1}^{j-1} k_i\right) \Big/ (N - j + 1)\right]. \tag{H6}$$

This method guarantees that the condition of Eq. (35) is satisfied and the major part of the degree distribution follows the Pareto tail, while only very small degrees depending on the value of ϵ are replaced by the larger value m_j.

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