Simple evolving random graphs

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We study random graphs densifying by adding edges. In each step, two vertices are randomly chosen, and an edge between these vertices is created if the vertices belong to trees. An edge is added with probability p if only one vertex belongs to a tree and an attempt fails otherwise. Simple random graphs generated by this procedure contain only trees and unicycles. In the thermodynamic limit, the fraction of vertices in unicycles exhibits a phase transition resembling a percolation transition in classical random graphs. In contrast to classical random graphs, where a giant component born at the transition point eventually engulfs all finite components and densifies forever, the evolution of simple random graphs freezes when trees disappear. We quantify simple random graphs in the supercritical phase and the properties of the frozen state.

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I. INTRODUCTION

Classical evolving random graphs [1] are built by drawing edges between pairs of randomly chosen vertices. In the most interesting sparse regime, when the number of edges is comparable with the number of vertices, components (i.e., maximal connected subgraphs) are predominantly trees. Unicyclic components (shortly unicycles) occasionally appear, yet the number of unicycles remains finite even in the thermodynamic limit when the size of the graph (the number of vertices N) diverges. The only exception is the percolation point where the number of unicycles is infinite; when $N \gg 1$, the transition occurs in a scaling window where the number of unicycles scales logarithmically with N. The giant component arising in the supercritical phase is neither a tree nor a unicycle [2].

Classical random graphs constitute a developed and growing field of research with applications to computer science, mathematics, and natural sciences [3–7]. The probabilistic treatment of classical random graphs is popular in mathematical literature [3–5]. A dynamical treatment was already used by Erdös and Rényi [1]. Earlier studies of gelation by Flory and Stockmayer employed other techniques and explored somewhat different random graph models [8–10]; see Refs. [11,12] for comparison of the Flory and Stockmayer approaches. We employ a dynamical treatment, a version that we call a kinetic theory approach [6,13]. Originally, such an approach was applied to finding the distribution of trees [14,15]. The distribution of the average number of unicycles is also amenable to the kinetic theory treatment [16,17].

We distinguish the complexity of components by their Euler characteristics. For a graph with V vertices and E edges, the Euler characteristic is

$$\chi = V - E. \tag{1}$$

The identities $\chi(\text{tree}) = 1$ and $\chi(\text{unicycle}) = 0$ illustrate the topological nature of the Euler characteristic. Trees and unicycles are known as simple components; components with $\chi < 0$ are complex [18]. Simple graphs, by definition, are disjoined unions of trees and unicycles.

Bicyclic components are born when unicycles merge or after adding an edge in a unicycle. In a classical random graph process, such events are rare. The giant component appearing in the postpercolation phase is complex. The probability that, throughout the evolution, there never is more than a single complex component is $5\pi/18$. Up to the percolation point, the evolving graph consists entirely of trees, unicycles, and bicycles with probability $\sqrt{2/3}\cosh(\sqrt{5/18}) = 0.932\,548\ldots$. See Refs. [2,18–20] for these and other subtle behaviors. A few complex components arise during the evolution of the classical random graph in the sparse regime. This property suggests exploring graph evolutions in which the formation of complex components is strictly forbidden—attempts to draw edges between unicycles and inside the same unicycle are rejected, so components are either trees or unicycles.

We examine a one-parametric class of simple random graph (SRG) processes. Initially, the graph is disjoint collection of N vertices. In each step, a pair of vertices is randomly selected. An edge between these vertices is potentially created according to the following procedure:

(1) An attempt to draw an edge between vertices belonging to trees is always successful.

(2) An attempt to draw an edge between a vertex from a tree and a vertex from a unicycle is successful with gluing probability p.

(3) An attempt to draw an edge between vertices from unicycles is rejected.

As in classical random graphs, we adopt the continuous time formulation, namely, we postulate that pairs of vertices are chosen at a constant rate which is conveniently set to $(2N)^{-1}$. The N^{-1} scaling ensures that in the thermodynamic limit, $N \to \infty$, the phase transition occurs at finite time; the precise choice implies $t_c = 1$.

There are no restrictions on the identity of pairs: (i) the same pair can be chosen multiple times and (ii) a pair consisting of two identical vertices can be chosen. Thus, up to

two edges between two different vertices are allowed (but no more than two to ensure that a component containing vertices is simple). A loop, that is, an edge from a vertex to itself, is possible (but not a bouquet of loops, i.e., more than one edge from a vertex to itself).

We defined SRG processes by constraining the random graph process to be free of complex components. There are other constrained random graph processes in which the formation of some subgraphs is forbidden [5,21,22]. In these processes, the evolution eventually stops, and the graph freezes. In the triangle-free process [23–26], a new edge is randomly chosen and added if it is not present and not creating triangles. The triangle-free process provides a lower bound for the Ramsey number $R(3,k)$. The triangle-free process ends with a dense frozen graph [27,28]; sparse graphs emerge at the end of the SRG processes. The sparsity is a simplifying feature; constrained random graph processes creating dense jammed graphs tend to be more challenging for analyses.

Random graphs with various constraints have been mostly studied in static frameworks [29–32]. We employ an evolution framework.

We now outline a few features of SRGs derived below. The fraction of mass $s(t)$ in unicycles in the supercritical phase, $t > t_c = 1$, is implicitly determined by

$$t = 1 + \frac{1}{p\, s^{\frac{1}{p}}} \int_0^s dz\, \frac{z^{\frac{1}{p}}}{1-z}. \tag{2}$$

The quantity $s(t)$ plays the role of the order parameter. Using the implicit solution (2), we extract explicit asymptotic behaviors of $s(t)$:

$$s = \begin{cases} 0 & t < 1 \\ (1+p)(t-1) + \mathcal{O}[(t-1)^2] & t \downarrow 1 \\ 1 - \mathcal{E} + \mathcal{O}[t\mathcal{E}^2] & t \uparrow \infty, \end{cases} \tag{3}$$

with

$$\mathcal{E}(t) = e^{-pt+p-\gamma-\psi(1+1/p)}. \tag{4}$$

Here $\gamma = 0.577215\ldots$ is the Euler-Masceroni constant, $\psi(z) = \Gamma'(z)/\Gamma(z)$ the digamma function, and $\Gamma(z)$ the gamma function.

In the supercritical phase, SRGs are significantly different from classical random graphs. In Sec. III, we show that SRGs undergo a continuous phase transition, determine the fraction of mass in unicycles, and derive the distribution of trees.

The classical random graph becomes connected at time $t_{\rm cond} \simeq \ln N$ and the densification continues ad infinitum. The SRG process, in contrast, reaches a jammed state once trees disappear. Using heuristic arguments, we estimate the jamming time:

$$t_{\rm jam} \simeq p^{-1} \ln N. \tag{5}$$

The average number of unicycles $U(t)$ increases until the system reaches a jammed state where

$$U_{\rm jam} \simeq \frac{1+p}{6p} \ln N. \tag{6}$$

Equations (2)–(6) do not apply when $p = 0$, i.e., to the model with frozen unicycles. Some results simplify in this extreme case, e.g., $s = 1 - 1/t$ in the supercritical phase, $t > 1$. The logarithmic scaling laws (5) and (6) valid for models with $p > 0$ are replaced by algebraic scaling behaviors (Sec. IV B).

In Sec. II, we outline several properties of classical random graphs needed for the analysis of SRGs. We also probe less-known features like the complexity of the giant component. In Secs. III–IV, we describe the phase transition in the SRGs, determine the densities of trees and the average number of unicycles in the supercritical phase, and probe the properties of the final jammed state. In Sec V, we step away from average characteristics and emphasize that fluctuations that could be particularly important for the SRG process with frozen unicycles ($p = 0$). We finish with concluding remarks (Sec. VI).

II. CLASSICAL RANDOM GRAPHS

It is customary to begin with a disjoint graph, i.e., a collection of N isolated vertices. Edges are drawn between any two vertices with rate $1/(2N)$. With this convention, an infinite system undergoes the percolation transition at time $t_c = 1$. If $N \gg 1$, almost all components are trees. The number $\mathfrak{T}_k$ of trees with k vertices (equivalently, of size k) is a self-averaging random quantity. More precisely, $\mathfrak{T}_k = Nc_k + \sqrt{N}\xi_k$ with random $\xi_k = O(1)$. Therefore, fluctuations are relatively small, so the deterministic densities c_k provide the chief insight. In the $N \to \infty$ limit, the densities evolve according to an infinite set of coupled nonlinear ordinary differential equations (ODEs)

$$\frac{dc_k}{dt} = \frac{1}{2} \sum_{i+j=k} ij\, c_i c_j - k c_k \tag{7}$$

from which [6,12]

$$c_k(t) = \frac{k^{k-2}}{k!}\, t^{k-1}\, e^{-kt} \tag{8}$$

The smoothness of densities throughout the entire evolution, $0 < t < \infty$, hides the emergence of the giant component when $t > t_c = 1$. Mass conservation, $\sum k c_k(t) = 1$, is manifestly obeyed in the subcritical (prepercolation) phase. In the supercritical phase, $\sum k c_k(t) = 1 - g(t)$, where $g(t)$ is the fraction of vertices belonging to the giant component, shortly the mass of the giant component. The mass $g(t)$ is implicitly determined by [6,12]

$$g = 1 - e^{-gt}. \tag{9}$$

For instance, the giant component comprises half mass of the system, $g = 1/2$, at time $t = 2\ln 2$.

The total cluster density $c(t) = \sum_{k\geqslant 1} c_k(t)$ reads

$$c(t) = \begin{cases} 1 - \frac{t}{2} & t \leqslant 1 \\ 1 - \frac{t}{2} + \frac{1}{2}\int_1^t d\tau\, g^2(\tau) & t > 1. \end{cases} \tag{10}$$

The integral can be expressed as a function of g:

$$\begin{aligned} \int_1^t g^2(\tau)\, d\tau &= \int_0^g h^2\, d[-h^{-1}\ln(1-h)] \\ &= \int_0^g \left[\ln(1-h) + \frac{h}{1-h}\right] dh \\ &= -2g - (2-g)\ln(1-g). \end{aligned} \tag{11}$$

Here $h = g(\tau)$ and we use $\tau = -h^{-1}\ln(1-h)$ following from (9). Thus, in the supercritical phase ($t > 1$):

$$c(t) = 1 - \frac{t}{2} - g - \left(1 - \frac{g}{2}\right)\ln(1-g). \tag{12}$$

Using this formula, we extract more explicit results for the cluster density $c(t)$ just beyond the critical point and in the long time limit:

$$c = \begin{cases} 1 - t/2 + 2(t-1)^3/3 + \dots & t \downarrow 1 \\ e^{-t} + (t/2)\,e^{-2t} + \dots & t \uparrow \infty. \end{cases} \tag{13}$$

In the following, we shall also need the second moment $M_2 = \sum_{k\geqslant 1} k^2 c_k$. It can be expressed [6] as follows:

$$M_2(t) = \begin{cases} (1-t)^{-1} & t < 1 \\ \frac{1-g}{1-t(1-g)} & t > 1. \end{cases} \tag{14}$$

In the supercritical (postpercolation) phase, the second moment accounts only for finite components. (In the leading order, the second moment is Ng^2 if we include the contribution of the giant component.) The second moment rapidly decays with time as the giant component quickly engulfs finite components.

The giant component has more edges than vertices. The ratio of the number of edges to the number of vertices in the giant component is [16]

$$\frac{\text{No. (edges)}}{\text{No. (vertices)}} = 1 + \frac{1}{2g(t)}\int_1^t d\tau\, g^2(\tau). \tag{15}$$

where the integral accounts that the number of edges within the giant component increases with rate $g^2/2$. The integral on the right-hand side of (15) is given by (11), so Eq. (15) simplifies to

$$\frac{\text{No. (edges)}}{\text{No. (vertices)}} = -\frac{2-g}{2g}\ln(1-g). \tag{16}$$

By definition, the number of vertices in the giant component is No. (vertices) $= gN$, and from (1) and (16) we find the Euler characteristic of the giant component:

$$\begin{aligned} \chi_{\text{giant}} &= N\left[g + \frac{2-g}{2}\ln(1-g)\right] \\ &= -N\sum_{m\geqslant 3}\frac{m-2}{2m(m-1)}\,g^m. \end{aligned} \tag{17}$$

The giant component is complex as its Euler characteristic is negative. Further, the Euler characteristic of the giant component is giant, viz. extensive in system size.

As an illustration, in Fig. 1, we show a random graph with $N = 80$ vertices and $E = 65$ edges. Such a graph is generated, on average, at time $t = 13/8$. So, the graph is in the supercritical phase and, indeed, it has a complex component, $\chi = -5$, that is significantly larger than other components. The fraction of mass in the giant component of the graph in Fig. 1 is $\frac{54}{80} = 0.675$, a little larger than the expected $g \approx 0.655$ predicted by (9) and realized when $N \to \infty$. The average total number of isolated vertices is $N_1 = Ne^{-t} \approx 15.753$; the graph shown in Fig. 1 has $N_1 = 18$ isolated vertices. From (20b), we get $U \approx 0.41$; the graph in Fig. 1 has no finite unicycles. The ratio of the number of edges to the number of vertices in the giant component is $\frac{59}{54} = 1.0925925\dots$ for the graph shown in Fig. 1; the predicted ratio (16) realized in the $N \to \infty$ limit is $1.0927078\dots$.

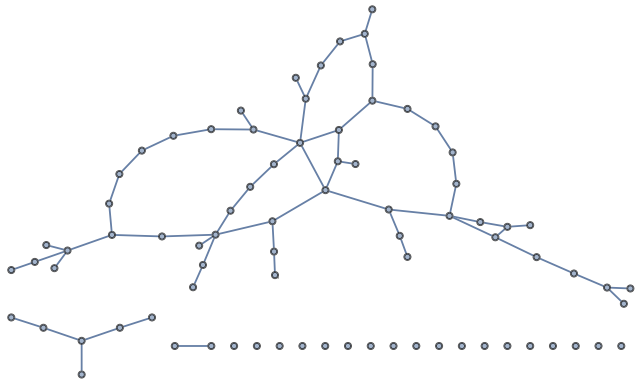

FIG. 1. An example of a random graph with $N = 80$ vertices and $E = 65$ edges. The average number of edges in an evolving random graph is $Nt/2$. Hence, it is natural to interpret the above graph as a possible outcome of the random graph process at time $t = 2E/N = 13/8$ and compare with theoretical predictions in the supercritical phase at $t = 13/8$.

Unicycles in classical random graphs have been investigated using probabilistic and combinatorial techniques [2,18–20,33], kinetic theory approach [16,17], and other methods, some of which are more suitable to random graph models different from classical random graphs, see, e.g., Refs. [34–37]. The number $\mathfrak{U}_k$ of unicycles of size k is a non-self-averaging random quantity, i.e., fluctuations of $\mathfrak{U}_k$ are comparable with the average. Still, the average $U_k = \langle \mathfrak{U}_k \rangle$ sheds light on unicycles. The kinetic theory approach [16,17] leads to the infinite set of ODEs:

$$\frac{dU_k}{dt} = \frac{1}{2}\,k^2 c_k + \sum_{i+j=k} iU_i\, jc_j - k\,U_k. \tag{18}$$

These equations are linear and, in this respect, simpler than Eq. (7) for the densities of trees. An extra source term, the first term on the right-hand side, accounting for turning a tree into a unicycle by creating an edge inside the tree, is a complication.

Summing Eq. (18), we deduce an equation

$$\frac{dU}{dt} = \frac{1}{2}M_2 \tag{19}$$

for the average total number of unicycles $U = \sum U_k$ in the subcritical phase. Using $M_2 = 1/(1-t)$, we find

$$U(t) = \frac{1}{2}\ln\frac{1}{1-t} \tag{20a}$$

in the subcritical phase. Similarly, the average number of finite unicycles is [16,17]

$$U(t) = \frac{1}{2}\ln\frac{1}{1-(1-g)t} \tag{20b}$$

in the supercritical phase.

For a finite system, the critical point broadens into the scaling window, $|1-t| \sim N^{-1/3}$. Using this and either (20a) or (20b), we find that the total number of unicycles remains finite

but scales logarithmically with the total number of vertices in the critical regime:

$$U(1) = \tfrac{1}{6} \ln N. \tag{21}$$

Solving (18) subject to $U_k(0) = 0$ recurrently for small k, one guesses

$$U_k(t) = B_k\, t^k\, e^{-kt}. \tag{22}$$

Substituting (22) into (18), one fixes the amplitudes:

$$U_k(t) = \frac{1}{2}\, t^k\, e^{-kt} \sum_{n=0}^{k-1} \frac{k^{n-1}}{n!}. \tag{23}$$

At the critical point, the average number of unicycles decays algebraically with the cycle, viz.

$$U_k(1) = \frac{1}{2}\, e^{-k} \sum_{n=0}^{k-1} \frac{k^{n-1}}{n!} \simeq \frac{1}{4k} \tag{24}$$

for $k \gg 1$. For finite graphs, the largest components have size $O(N^{2/3})$ at the critical point [2–6]. Summing (24) up to $k \sim N^{2/3}$, one obtains (21).

We now turn to the SRG evolution processes in which, by definition, complex components cannot be created.

III. SIMPLE RANDOM GRAPHS

In the SRG evolution processes, the merging of unicycles is forbidden. The merging of two trees, say a tree of size i and a tree of size j, proceeds at the same rate,

$$[T_i] \oplus [T_j] \to [T_{i+j}] \quad \text{rate} \quad ij/(2N), \tag{25a}$$

as for the classical random graph process. The merging of a tree and a unicycle proceeds via

$$[T_i] \oplus [U_j] \to [U_{i+j}] \quad \text{rate} \quad pij/(2N). \tag{25b}$$

Drawing an edge inside a tree turns it into a unicycle:

$$[T_k] \to [U_k] \quad \text{rate} \quad k^2/(2N). \tag{25c}$$

Two unicycles never merge and drawing an edge inside a unicycle is also forbidden—such processes would have generated complex components. Thus, the total number of unicycles increases via the reaction channel (25c).

A. Trees and the phase transition

The influence of the reaction channel (25b) on the evolution of trees is asymptotically negligible in the subcritical phase since the number of unicycles is N times smaller than the number of trees. Therefore, Eq. (7) are applicable as in the classical random graph, and the densities of trees are given by the same Eq. (8) when $t \leqslant 1$. Below, we show that the influence of unicycles on the evolution of trees is also negligible in the supercritical phase. Only in the proximity of the jammed state do unicycles affect the evolution of trees.

In classical random graphs, the components have a size up to $O(N^{2/3})$ at the critical point. The maximal merging rate is $N^{2/3} \times N^{2/3}/N = N^{1/3}$, explaining why the width of the scaling window is $|1 - t| \sim N^{-1/3}$. When $t - 1 \gg N^{-1/3}$, the most massive components merge, and a single giant component emerges. This giant component progressively engulfs finite components. Eventually, only the giant component remains.

In SRGs, trees of size $O(N^{2/3})$ merge with the rate $O(N^{1/3})$, and any such tree turns into a unicycle with the same rate. Thus, trees of size $O(N^{2/3})$ become unicycles in the scaling window.

The nature of the phase transition is captured by the behavior of $s(t) = N^{-1} \sum_{k\geqslant 1} k\mathfrak{U}_k(t)$, the fraction of vertices belonging to unicycles, shortly the mass of unicycles. This quantity plays the role of an order parameter. For finite systems, $s(t) \sim N^{-1}$ in the subcritical phase and $s \sim N^{-1/3}$ in the critical regime [see Sec. III B for details]. In the supercritical phase, $s(t)$ is finite and growing with time.

1. The densities of trees

The densities of trees satisfy

$$\frac{dc_k}{dt} = \frac{1}{2} \sum_{i+j=k} ij\; c_i c_j - kc_k \left(\sum_{\ell\geqslant 1} \ell c_\ell + ps \right). \tag{26}$$

The second sum on the right-hand side of (26) goes over all trees, so

$$\sum_{\ell\geqslant 1} \ell c_\ell = 1 - s \tag{27}$$

by the definition of s. Thus, Eq. (26) simplifies to

$$\frac{dc_k}{dt} = \frac{1}{2} \sum_{i+j=k} ij\; c_i c_j - kc_k(1 - qs), \tag{28}$$

with $q = 1 - p$. In the subcritical phase, $s = 0$ in the leading order, so (28) and (7) coincide, and the solution (8) remains valid. Finding $s(t) > 0$ in the supercritical phase is part of the solution.

We solve Eq. (28) by employing the same approach [6] as for classical random graphs. The emerging solution depends on yet-unknown s, but by inserting this formal solution into relation (27), we will fix s. The approach used in Ref. [6] relies on the generating function technique [38]. It is convenient to introduce an exponential generating function based on the sequence kc_k:

$$\mathcal{C}(y, t) = \sum_{k\geqslant 1} k\, c_k(t)\, e^{yk}. \tag{29}$$

This allows us to recast an infinite set of ODEs, Eq. (28), into a single partial differential equation:

$$\partial_t \mathcal{C} = (\mathcal{C} - 1 + qs)\partial_y \mathcal{C}. \tag{30}$$

Let us consider $y = y(\mathcal{C}, t)$ instead of $\mathcal{C} = \mathcal{C}(y, t)$. This transformation recasts (30) into

$$\partial_t y + \mathcal{C} - 1 + qs = 0, \tag{31}$$

which is integrated to give

$$y + (\mathcal{C} - 1)t + q \int_0^t d\tau\, s(\tau) = f(\mathcal{C}). \tag{32}$$

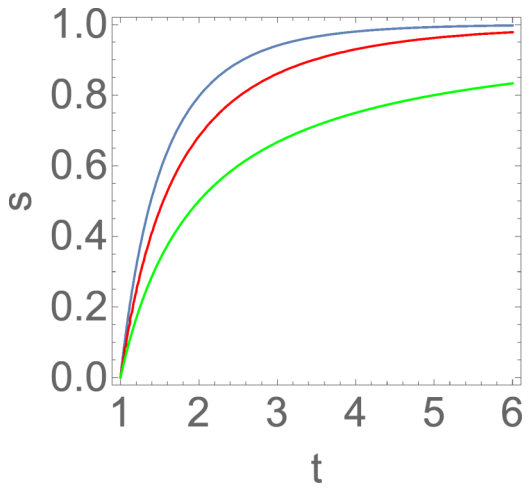

FIG. 2. The mass $s(t)$ of unicycles determined by Eq. (2) in the supercritical phase, $t > 1$. We show $s(t)$ for $p = 1, \frac{1}{2}, 0$ (top to bottom). For $p = 1$, the mass of unicycles coincides with the mass $g(t)$ of the giant component in the classical random graph process. When $p = 0$, the mass of unicycles is particularly simple: $s = 1 - 1/t$.

Initially, $c_k(0) = \delta_{k,1}$, and therefore $\mathcal{C}(y, 0) = e^y$. This fixes $f(\mathcal{C}) = \ln \mathcal{C}$, and (32) leads to

$$\mathcal{C}\, e^{-\mathcal{C}t} = e^{y-t+q\int_0^t d\tau\, s(\tau)}. \tag{33}$$

We rewrite Eq. (33) as $Ze^{-Z} = Y$ with $Z = \mathcal{C}t$ and $Y = te^{y-t+q\int_0^t d\tau\, s(\tau)}$. The power series $Y = Y(Z)$ directly follows from $Ze^{-Z} = Y$. The inverse power series $Z = Z(Y)$ can be deduced using the Lagrange inversion formula. One gets [6,38]

$$Z = \sum_{k\geqslant 1} \frac{k^{k-1}}{k!}\, Y^k. \tag{34}$$

Substituting $Z = \mathcal{C}t$ and $Y = te^{y-t+q\int_0^t d\tau\, s(\tau)}$ into (34) and comparing with the definition (29) of the generating function, we arrive at a formal solution:

$$c_k(t) = \frac{k^{k-2}}{k!}\, t^{k-1}\, \exp\left[-kt + kq\int_0^t s(\tau)\, d\tau\right]. \tag{35}$$

2. Phase transition

To complete the solution (35), we must determine $s(t)$. This can be accomplished by specializing (33) to $y = 0$ and using relation (27), i.e., $\mathcal{C}(0, t) = 1 - s(t)$. We obtain

$$q\int_1^t d\tau\, s(\tau) = st + \ln(1 - s), \tag{36}$$

which implicitly determines $s(t)$. Differentiating the integral equation (36) with respect to time gives an ODE:

$$(1 - t + st)\frac{ds}{dt} = ps(1 - s). \tag{37}$$

Changing $s = s(t)$ to $t = t(s)$, we obtain

$$p\,\frac{dt}{ds} = \frac{1 - t}{s} + \frac{1}{1 - s}. \tag{38}$$

Integrating Eq. (38) subject to $t(0) = 1$ we arrive at the announced result (2).

The quantity $s(t)$ plays a role of an order parameter. The phase transition is continuous (see Fig. 2) and mean field in nature as manifested by the asymptotic behavior of the order parameter, $s \propto (t - 1)$, near the transition. More precisely,

$$s = (1 + p)T - \tfrac{(1+p)^3}{1+2p}T^2 + \tfrac{(1+p)^4(1+4p+2p^2)}{(1+2p)^2(1+3p)}T^3 + \dots \tag{39a}$$

when $T = t - 1 \downarrow 0$. When $t \to \infty$,

$$s = 1 - \mathcal{E} - (t - 1)\mathcal{E}^2 + \dots, \tag{39b}$$

with $\mathcal{E}(t)$ defined by Eq. (4).

We compute the integral in (35) using the same trick as in the computation of the integral (11), namely, we treat $\sigma = s(\tau)$ as an integration variable and use

$$p\,\frac{d\tau}{d\sigma} = \frac{1 - \tau}{\sigma} + \frac{1}{1 - \sigma} \tag{40}$$

following from Eq. (38). We get

$$\begin{aligned} p\int_0^t \sigma\, d\tau &= \int_0^s \sigma\left[\frac{1 - \tau}{\sigma} + \frac{1}{1 - \sigma}\right]d\sigma \\ &= \int_0^s \left[\frac{1}{1 - \sigma} - \tau\right]d\sigma \\ &= -\ln(1 - s) - ts + \int_0^t \sigma\, d\tau, \end{aligned}$$

from which we deduce

$$q\int_0^t \sigma\, d\tau = ts + \ln(1 - s). \tag{41}$$

Using (41), we recast (35) into

$$c_k(t) = \frac{k^{k-2}}{k!}\, t^{k-1}\, (1 - s)^k\, e^{-kt(1-s)}. \tag{42}$$

The densities (8) in classical random graphs remain smooth throughout the evolution, thereby hiding the phase transition. In SRGs, the densities (42) undergo a jump in the second derivative $\frac{d^2 c_k}{dt^2}$ at $t = 1$ for all $k \geqslant 1$ when $p < 1$. For instance ($T = t - 1$),

$$c_1 = e^{-1} \times \begin{cases} 1 - T + (1 - p^2/2)T^2 + \dots & T \downarrow 0 \\ 1 - T + T^2/2 + \dots & T \uparrow 0. \end{cases} \tag{43}$$

3. Moments $M_0 = c$ and M_2

The total density of trees is given by

$$c(t) = \begin{cases} 1 - \frac{t}{2} & t \leqslant 1 \\ 1 - \frac{1}{2}(1 + s^2)t + (t - 1)s & t > 1. \end{cases} \tag{44}$$

In the subcritical phase, the total density of trees is the same as in classical random graphs as the governing equations coincide (see Fig. 3). To derive $c(t)$ in the supercritical phase, we sum Eq. (28) and use (27) to find

$$\frac{dc}{dt} = -\frac{1}{2}\,(1 - s)[1 + (2p - 1)s]. \tag{45}$$

Integrating this equation, we obtain

$$c = 1 - \frac{t}{2} + q\int_0^t \sigma\, d\tau + \left(p - \frac{1}{2}\right)\int_0^t \sigma^2\, d\tau \tag{46}$$

in the supercritical phase. We already computed the first integral on the right-hand side of (46), see Eq. (41). Employing

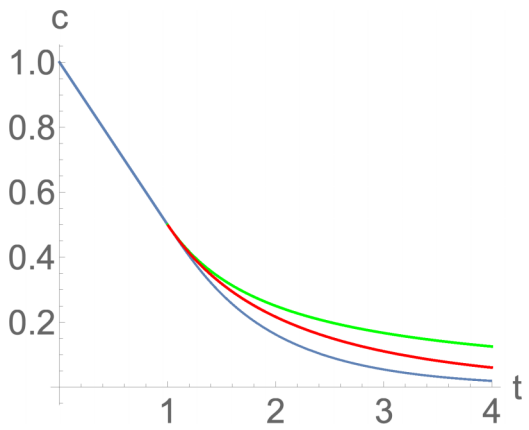

FIG. 3. Time dependence of the density of trees $c(t)$ determined by Eq. (44). We plot $c(t)$ in the supercritical phase for $p = 1, \frac{1}{2}, 0$ (bottom to top). When $p = 0$, the total density of trees is particularly simple: $c = \frac{1}{2t}$. In the subcritical phase, $c = 1 - t/2$ independently on p.

the same approach, we determine the second integral. Combining these results yields the total density (44). In the long time limit, the density decays exponentially, namely, as

$$c = \mathcal{E} + \left(\frac{t}{2} - 1\right)\mathcal{E}^2 + \ldots, \tag{47}$$

with $\mathcal{E}$ given by (4).

Interestingly, the ratio of the total number of edges E to the total number of vertices N can be expressed via the density of trees (see Fig. 4):

$$\frac{E}{N} = \sum_{k\geqslant 1}(k-1)c_k + s = 1 - c. \tag{48}$$

We now consider the second moment which will be needed for the determination of the average total number of unicycles. The second moment $M_2 = \sum_{k\geqslant 1} k^2 c_k$ can be expressed via

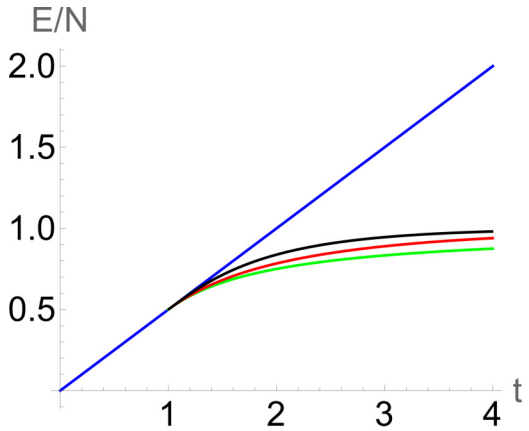

FIG. 4. Time dependence of the ratio of the total number of edges to the total number of vertices. For classical random graphs, $\frac{E}{N} = \frac{t}{2}$ for all $t > 0$, and in the subcritical phase, the ratio is the same for SRGs independently on the parameter p. In the supercritical phase, the ratio depends on p as illustrated in the figure for $p = 1, \frac{1}{2}, 0$ (top to bottom). When $p = 0$, the ratio is particularly simple: $\frac{E}{N} = 1 - \frac{1}{2t}$.

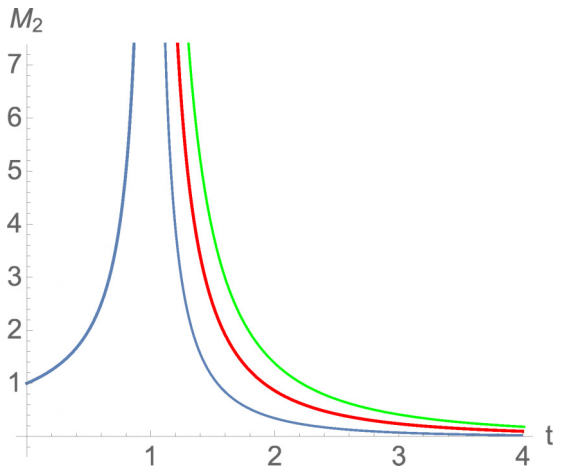

FIG. 5. Time dependence of the second moment determined by Eq. (52). We plot the second moment in the supercritical phase for $p = 1, \frac{1}{2}, \frac{1}{3}$ (bottom to top). In the subcritical phase, $M_2 = 1/(1-t)$.

the generating function:

$$M_2 = \partial_y \mathcal{C}|_{y=0}. \tag{49}$$

Using (41), we rewrite (33) as

$$\mathcal{C}\, e^{-\mathcal{C}t} = (1-s)\, e^{y - t(1-s)}. \tag{50}$$

We take the logarithm of (50), differentiate with respect to y, and set $y = 0$ to give

$$M_2(t) = \left[\frac{1}{\mathcal{C}(0,t)} - t\right]^{-1}. \tag{51}$$

Recalling that $\mathcal{C}(0,t) = 1$ in the subcritical phase and $\mathcal{C}(0,t) = 1 - s$ in the supercritical phase, we obtain

$$M_2(t) = \begin{cases} (1-t)^{-1} & t < 1 \\ \frac{1-s}{1-t(1-s)} & t > 1. \end{cases} \tag{52}$$

Note the similarity with Eq. (14), giving M_2 for classical random graphs. When $p = 1$, we have $s = g$ and (52) reduces to (14). In Fig. 5, we plot the second moment at three values $p = 1, \frac{1}{2}, \frac{1}{3}$ of the parameter.

Using (39a)–(39b), one finds extremal behaviors

$$M_2(t) = \begin{cases} \frac{1}{p(t-1)} & t \downarrow 1 \\ \mathcal{E}(t) & t \uparrow \infty \end{cases} \tag{53}$$

in the supercritical phase.

B. Unicycles

The number of trees decreases and at some moment, t_{jam}, the last tree disappears, and only unicycles remain. At this moment, the graph freezes. The average number of trees $Nc(t)$ approaches $N\mathcal{E}(t)$ in the large time limit [cf. (47)]. Combining the criterion $Nc(t_{\text{jam}}) \sim 1$ with (4), we arrive at the asymptotic (5) for the jamming time. The logarithmic scaling with N is certainly correct (when $p > 0$), while the amplitude in Eq. (5) is a bit questionable. Indeed, the number of trees is a random variable. Fluctuations are significant when the average number of trees is small. If fluctuations are comparable to the average, the asymptotic (5) holds. Recalling that fluctuations

in the number of trees scale $\sqrt{N}$ when $t = O(1)$ suggest $\sqrt{N}$ as an upper bound for fluctuations near the jamming time, and then from $Nc(t_{\rm jam}) \sim \sqrt{N}$ we get a lower bound, $(2p)^{-1} \ln N$, for the jamming time. These heuristic arguments suggest the asymptotic bounds:

$$(2p)^{-1} \ln N \leqslant t_{\rm jam} \leqslant p^{-1} \ln N. \tag{54}$$

We conjecture that the upper bound is the true asymptotic, so we used it in Eq. (5). Justifying or disproving Eq. (5) is left for the future.

1. Unicycles in the jammed state

Arguments in favor of the scaling law (6) for the final number of unicycles are also heuristic but stronger than the arguments in favor of Eq. (5). Indeed, the dominant contribution to $U(t_{\rm jam})$ is gathered at times $t = O(1)$ far below the jamming time where fluctuations are asymptotically negligible. Hence, the arguments are more solid, and the amplitude in Eq. (6) appears exact. We begin by noting that for the SRG process, Eq. (19) remains valid in the supercritical phase: Unicycles cannot disappear, and they are born with rate $\frac{1}{2}M_2$. The exact expressions (52) for the second moment are valid for infinite graphs. For finite graphs, we do not know $M_2(t)$ in the scaling window, $|t-1| \sim N^{-1/3}$, but we know the magnitude of the second moment: $M_2 \sim N^{1/3}$. To gain insight, let us take a test expression for the second moment,

$$M_2^{\rm test}(t) = \begin{cases} \frac{1}{\sqrt{(1-t)^2+\epsilon^2}} & t < 1 \\ \frac{1}{\sqrt{p^2(t-1)^2+\epsilon^2}} & 2 > t > 1, \end{cases} \tag{55}$$

with $\epsilon \sim N^{-1/3}$. This function is continuous, has the correct width of the scaling window, and provides an excellent approximation in the $t < 1$ regime away from the scaling window and in the $\epsilon \ll t-1 \ll 1$ regime. The gain of U during the time interval $2 < t < t_{\rm jam}$ is of the order of unity as $M_2(t)$ decays exponentially with time [see Eq. (53) and Fig. 5]. The choice $t = 2$ of the cutoff time is asymptotically irrelevant; any cutoff time exceeding $t = 1$ suffices—the dominant contribution comes from a small region containing the scaling window. Thus,

$$\frac{1}{2}\int_0^2 dt\, M_2^{\rm test}(t) \tag{56}$$

estimates $U_{\rm jam}$. Using (55) and computing the integral in (56) gives (6) in the leading order.

The specific form (55) plays little role—the important feature is the logarithmic divergence

$$\int_0^t dt'\, M_2(t') \simeq \ln\frac{1}{1-t}, \quad \int_t^{t_{\rm jam}} dt'\, M_2(t') \simeq \frac{1}{p}\ln\frac{1}{t-1}$$

of the integrals near $t = 1$ which follows from (52) and (53). For finite N, we must use $|t-1| \sim N^{-1/3}$ to find the integrals in the (0,1) and $(1, t_{\rm jam})$ regions. Thanks to the logarithmic divergence, we obtain an asymptotically exact leading behavior (6).

2. Intermediate SRG process

As another piece of evidence in favor of the prediction (6), we notice that it reduces to

$$U_{\rm jam} \simeq \frac{1}{2}\ln N \tag{57}$$

for $p = \frac{1}{2}$. An intermediate SRG process with $p = \frac{1}{2}$ appears in a recent study of a parking process on Cayley trees [39]. This SRG process has an intriguing connection with random maps. A map of a set to itself, $f : S \to S$, can be represented by a graph with edges $(x, f(x))$ for all $x \in S$. This graph decomposes into maximal connected components (communities). If maps are uniformly chosen among N^N possible maps ($N = |S|$), the distributions of the number of communities and the number of unicycles in the jammed state of the intermediate SRG process coincide [39]. In particular, the average number of communities in random maps is well-known [40–43], and it grows with N according to (57).

Other results about communities in random maps can be restated in terms of unicycles in the SRG process with $p = \frac{1}{2}$. Restating the prediction for the probability to have a single community [44], we obtain the probability to end up with one unicycle:

$${\rm Prob}[U_{\rm jam} = 1] = \frac{(N-1)!}{N^N}\sum_{n=0}^{N-1}\frac{N^n}{n!} \simeq \sqrt{\frac{\pi}{2N}}. \tag{58a}$$

The probability to have the maximal number of communities is

$${\rm Prob}[U_{\rm jam} = N] = \frac{1}{N^N}. \tag{58b}$$

Let K be the size of the largest community. The size distribution of the largest community was probed numerically in Ref. [45]. The average size of the largest community is analytically known [43]:

$$\langle K\rangle = \lambda N, \tag{59a}$$

$$\lambda = 2\int_0^\infty dy\big[1 - e^{-\Gamma(0,y)/2}\big] = 0.757\,823\ldots. \tag{59b}$$

(Here $\Gamma(0,y) = \int_y^\infty dz\, z^{-1}e^{-z}$ is the incomplete gamma function.)

3. Size distribution of the largest unicycle

We return to the SRG processes with arbitrary $p > 0$ and denote by K the size of the largest unicycle at jamming. (For the intermediate SRG process, K is the same random quantity as the size of the largest community.) The linear scaling (59a) of the average size in the intermediate SRG process suggests that the same scaling holds generally when $p > 0$ and that the distribution $\Psi_p(K,N)$ acquires the scaling form

$$\Psi_p(K,N) = N^{-1}\psi_p(\kappa) \tag{60}$$

when $K \to \infty$ and $N \to \infty$ with $\kappa = K/N$ kept finite. The distribution $\psi_p(\kappa)$ has singularities at $\kappa = 1/m$ with $m = 2, 3, \ldots$, weakening as m increases. These singularities were noticed [45] in the special case of $p = \frac{1}{2}$, and they are present in the general case. The origin of singularities is easy to appreciate: One unicycle may have size exceeding $N/2$, i.e., in the

$\frac{1}{2} < \kappa < 1$ range; two unicycles may have size in the $\frac{1}{3} < \kappa < \frac{1}{2}$ range; etc. Similar singularities appear in several random processes ranging from fragmentation and random walks to spin glasses and genealogical trees (see Refs. [45–51]).

An infinite set of singularities prevents an analytical determination of the scaled distribution $\psi_p(\kappa)$. For the intermediate SRG process, Eqs. (58a) and (58b) imply the asymptotic behaviors:

$$\psi_{\frac{1}{2}}(\kappa) \propto \begin{cases} (1-\kappa)^{-1/2} & \kappa \uparrow 1 \\ \kappa^{1/\kappa} & \kappa \downarrow 0. \end{cases} \tag{61}$$

The scaled distribution $\psi_{\frac{1}{2}}(\kappa)$ vanishes faster than any power law in the $\kappa \downarrow 0$ limit. This feature is a generic outcome [45–51] of the accumulation of singularities in the $\kappa \downarrow 0$ limit. One thus anticipates a similar essential singularity of $\psi_p(\kappa)$ in the $\kappa \downarrow 0$ limit for all SRG processes with $p > 0$.

The scaling form (60) implies a linear scaling with N of the average size of the largest unicycle,

$$\langle K \rangle = \lambda(p) N, \quad \lambda(p) = \int_0^1 d\kappa\, \kappa \psi_p(\kappa), \tag{62}$$

with $\lambda(\frac{1}{2})$ given by (59b).

In Sec. IV B below, we show that different behaviors occur for the SRG process with frozen unicycles ($p = 0$).

4. Evolution of the distribution of unicycles

For SRG processes with arbitrary $p > 0$, it is difficult to probe the evolution of the average number of unicycles $U_k(t)$ analytically. Even in the subcritical phase where the governing equations

$$\frac{dU_k}{dt} = \frac{1}{2} k^2 c_k + p \sum_{i+j=k} i U_i\, j c_j - p\, k\, U_k \tag{63}$$

differ from Eq. (18) only by the factor of p in the last two terms on the right-hand side, Eq. (63) does not admit a simple general solution like the solution (23) of Eq. (18) in the case of classical random graphs.

Equation (63) is recursive, so one can solve them one by one. The average number of smallest unicycles is

$$U_1 = \frac{e^{-pt} - e^{-t}}{2(1-p)} \tag{64}$$

in the subcritical phase. For larger unicycles, the formulas quickly become cumbersome:

$$U_2 = \frac{(2+p)e^{-2pt} - 2pe^{-(1+p)t} - (2-p+4qt)e^{-2t}}{8(1-p)^2},$$

etc. There is no simple ansatz like (22) fixing the temporal behavior of $U_k(t)$.

To shed light on the distribution $U_k(t)$, let us look at the behavior of the moments. We have examined the behavior of the zeroth moment, $U(t) = \sum_{k\geqslant 1} U_k(t)$, and know that in the subcritical phase, $U(t) = \frac{1}{2} \ln \frac{1}{1-t}$. The first moment, i.e., the average mass of unicycles

$$S(t) = \sum_{k\geqslant 1} k U_k(t) \tag{65}$$

satisfies

$$\frac{dS}{dt} = \sum_{k\geqslant 1} k \left[\frac{1}{2} k^2 c_k + p \sum_{i+j=k} i U_i\, j c_j - p\, k\, U_k \right] \tag{66}$$

in the subcritical phase. Massaging the sums in (66), we arrive at a neat formula:

$$\frac{dS}{dt} = \frac{1}{2} M_3 + p M_2 S. \tag{67}$$

We already know $M_2 = \sum_{k\geqslant 1} k^2 c_k = (1-t)^{-1}$. Similarly, one finds $M_3 = \sum_{k\geqslant 1} k^3 c_k = (1-t)^{-3}$ in the subcritical phase. Combining these results with (67) gives

$$S = \frac{1}{2(2-p)} \left[\frac{1}{(1-t)^2} - \frac{1}{(1-t)^p} \right], \tag{68}$$

applicable in the subcritical phase. Combining (68) with $1 - t \sim N^{-1/3}$ describing the scaling window, we deduce the scaling behavior $S(1) \sim N^{2/3}$ of the average mass of unicycles in the critical regime. In the supercritical phase, $S(t) = N s(t)$ by definition of $s(t)$.

Higher moments can be probed in a similar way. The exact expressions are already cumbersome for the second moment, so we just mention the leading asymptotic in the $t \uparrow 1$ limit,

$$\sum_{k\geqslant 1} k^n U_k(t) \simeq \frac{A_n}{(1-t)^{2n}}, \tag{69}$$

with amplitudes

$$A_1 = \frac{1}{2(2-p)}, \quad A_2 = \frac{p}{[2(2-p)]^2} + \frac{3}{4(2-p)}, \tag{70}$$

etc. From (69), we conclude that in the scaling window, the nth moment diverges as $N^{2n/3}$.

IV. EXTREME SIMPLE RANDOM GRAPHS

Some of the above formulas [e.g., Eqs. (5) and (6)] become singular when $p = 0$, so they are valid when $p > 0$. The model with $p = 0$ in which unicycles are frozen exhibits different behaviors than the models with $p > 0$. We now outline some behaviors for the SRG processes with extreme values of gluing probability.

A. $p = 1$

When $p = 1$, Eq. (63) coincide with Eq. (18). (A singularity in the solution (64) at $p = 1$ disappears if one carefully takes the $p \to 1$ limit.) Thus, in the subcritical phase, the solution is given by (23).

In the supercritical phase,

$$\frac{dU_k}{dt} = \frac{1}{2} k^2 c_k + \sum_{i+j=k} i U_i\, j c_j - k\, U_k (1-g), \tag{71}$$

where we have taken into account that $s = g$ when $p = 1$.

Consider the smallest unicycles. Solving

$$\frac{dU_1}{dt} + U_1(1-g) = \frac{1}{2} c_1 = \frac{1}{2} e^{-t} \tag{72}$$

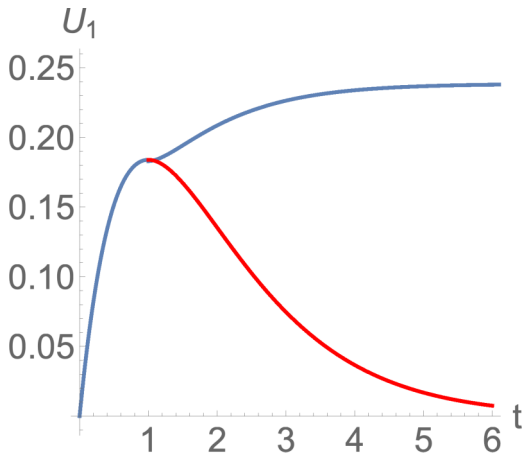

FIG. 6. Time dependence of the average number of smallest unicycles $U_1(t)$ for SRGs with $p = 1$. In the subcritical phase, $U_1 = \frac{1}{2}te^{-t}$, the same as for the classical random graphs. In the supercritical phase, $U_1(t)$ is determined by (76). For comparison, we also show (bottom curve) $U_1(t)$ for classical random graphs.

yields

$$2U_1 e^{t-G(t)} = 1 + \int_1^t dt'\, e^{-G(t')}, \tag{73}$$

where we shortly write

$$G(t) = \int_1^t d\tau\, g(\tau). \tag{74}$$

To compute this integral, we treat again G as a function of g rather than t. Repeating the steps used in the computation of the integral (11), we obtain

$$G = -\ln(1-g) - \mathrm{Li}_2(g), \tag{75}$$

where $\mathrm{Li}_2(g) = \sum_{n\geqslant 1} \frac{g^n}{n^2}$ is the dilogarithm.

Similarly, we simplify the integral on the right-hand side of (73) and arrive at the solution

$$2U_1 e^{t(1-g)+\mathrm{Li}_2(g)} = 1 + \int_0^g dh\, H(h)\, e^{\mathrm{Li}_2(h)} \tag{76}$$

in the supercritical phase with

$$H(h) = \frac{1}{h} + \frac{(1-h)\ln(1-h)}{h^2}. \tag{77}$$

Figure 6 shows that the average number of smallest unicycles is an increasing function of time over the entire evolution, $0 < t < t_{\rm jam}$. In particular, the final average number of smallest unicycles is

$$U_1(t_{\rm jam}) = \frac{1 + \int_0^1 dh\, H(h)\, e^{\mathrm{Li}_2(h)}}{2e^{\pi^2/6}} \approx 0.23898433. \tag{78}$$

Therefore, at least one smallest unicycle arises with a finite probability in the thermodynamic limit. In contrast, the average number of smallest unicycles at the same time $t_{\rm jam} \simeq \ln N$ is inverse of the number of vertices, $U_1(t_{\rm jam}) \sim N^{-1}$, for classical random graphs.

B. $p = 0$

The densities in the subcritical phase are given by (8) independently on p. The densities in the supercritical phase are given by (42) with $s = 1 - 1/t$, the latter result follows from (2) in the $p \to 0$ limit. Thus, in the supercritical phase ($t > 1$), we arrive at

$$c_k(t) = \frac{k^{k-2}e^{-k}}{k!}\, t^{-1}, \tag{79}$$

known as the Stockmayer solution [9]; see Refs. [11,12]. The total cluster density is particularly simple:

$$c(t) = \begin{cases} 1 - \frac{t}{2} & t \leqslant 1 \\ \frac{1}{2t} & t > 1. \end{cases} \tag{80}$$

The governing equations

$$\frac{dU_k}{dt} = \frac{1}{2}\, k^2 c_k \tag{81}$$

for the average number of unicycles are valid throughout the entire evolution. In the subcritical phase, we insert the densities of trees (8) into (81) and integrate to give

$$U_k(t) = \frac{1}{2k}\, e^{-kt} \sum_{n\geqslant k} \frac{(kt)^n}{n!}. \tag{82}$$

This solution is different from the solution (23) for classical random graphs. At the critical point,

$$U_k(1) = \frac{1}{2k}\, e^{-k} \sum_{n\geqslant k} \frac{k^n}{n!} \tag{83}$$

differs from $U_k(1)$ for classical random graphs, Eq. (24), but the large k behaviors are the same: $U_k(1) \simeq \frac{1}{4k}$ when $k \gg 1$.

In the supercritical phase, the densities of trees are given by (79), which we insert into (81) and deduce

$$U_k(t) = U_k(1) + \frac{(k/e)^k}{2k!}\, \ln t, \tag{84}$$

with $U_k(1)$ given by (83).

We now outline the chief properties of the jammed state. Using $c = (2t)^{-1}$ together with criterion $Nc(t_{\rm jam}) \sim 1$, we arrive at a linear scaling

$$t_{\rm jam} \sim N \tag{85}$$

of the jamming time. (For models with $p > 0$, the scaling is logarithmic.) Using Eqs. (84) and (85) and the Stirling formula, we deduce the large k asymptotic

$$U_k(t_{\rm jam}) \simeq (8\pi k)^{-\frac{1}{2}}\, \ln N \tag{86}$$

at the jamming time. There are only unicycles at the jamming time. Therefore,

$$N = \sum_{k=1}^{N} k U_k(t_{\rm jam}) \sim \ln N \sum_{k\lesssim K} k^{\frac{1}{2}} \sim K^{\frac{3}{2}} \ln N, \tag{87}$$

implying that Eq. (86) is applicable when $1 \ll k \lesssim K$ with

$$K \sim \left(\frac{N}{\ln N}\right)^{\frac{2}{3}}. \tag{88}$$

Similarly to the estimating of the sum in (87), we deduce

$$U_{\rm jam} = \sum_{k=1}^{N} U_k(t_{\rm jam}) \sim \sum_{k \lesssim K} k^{-\frac{1}{2}} \ln t_{\rm jam} \sim K^{\frac{1}{2}} \ln N, \qquad (89)$$

which we combine with (88) to obtain

$$U_{\rm jam} \sim N^{\frac{1}{3}} (\ln N)^{\frac{2}{3}}. \qquad (90)$$

Thus, in the model with frozen unicycles ($p = 0$), the average number of unicycles in the jammed state exhibits a peculiar scaling (90). The size K is smaller by $(\ln N)^{2/3}$ than the maximal size of unicycles in the jammed state: The components of size $N^{2/3}$ at the phase transition point are unicycles or will become unicycles.

V. FLUCTUATIONS

Fluctuations are often relatively small in large systems. Whenever this is true, fluctuations are traditionally investigated in the realm of the van Kampen expansion [52]. Van Kampen expansions have been applied to various reaction processes, see, e.g., Refs. [52–59]. In our problem, we expect the total number of trees to be the sum of the linear in N deterministic contribution and proportional to $\sqrt{N}$ stochastic contribution:

$$T(t) = Nc(t) + \sqrt{N}\xi(t). \qquad (91)$$

The criterion $Nc(t_{\rm jam}) \sim 1$ for estimating the jamming time is rather naive. The criterion

$$Nc(t_{\rm jam}) \sim \sqrt{N} v(t_{\rm jam}), \quad v(t) = \sqrt{\langle \xi^2(t) \rangle} \qquad (92)$$

is better, albeit still nonrigorous as it assumes that the $\sqrt{N}$ scaling of fluctuations holds till the very end.

For the SRG processes with $p > 0$, the density of trees decays exponentially, and the criterion (92) leads to the logarithmic scaling, and only the amplitude can differ. For instance, if $v(t_{\rm jam}) = O(1)$, the amplitude is twice smaller than predicted by naive criterion. As we argued in Sec. III B, these two estimates provide asymptotic bounds (54).

The naive criterion is difficult to justify, but it often leads to asymptotically exact results. For classical random graphs, the naive criterion, $Nc(t_{\rm cond}) \sim 1$, gives the leading behavior with correct amplitude, $t_{\rm cond} \simeq \ln N$, of the condensation time when the graph becomes connected [3–5].

For the model with frozen unicycles, $p = 0$, fluctuations are potentially more important than for models with $p > 0$, where they only affect the amplitude (5). Indeed, the linear scaling (85) of the jamming time is based on the naive criterion. If $v(t_{\rm jam}) = O(1)$, the criterion (92) leads to diffusive, $t_{\rm jam} \sim \sqrt{N}$, rather than linear scaling of the jamming time.

The jammed states in the model with frozen unicycles are remarkably similar to jammed states [60–62] discovered in a few addition-fragmentation processes. These jammed states are known as supercluster states, as clusters tend to be large and the total number of clusters is nonextensive (scales sublinearly with N).

In the model with frozen unicycles, we have not computed the variance $v(t)$, so the precise scaling of the jamming time remains unknown. The formula (84) remains valid before the jamming, and (86) with $\ln t_{\rm jam}$ instead of $\ln N$. We anticipate $t_{\rm jam} \sim N^b$, which would modify (86) by a numerical factor. This factor is irrelevant in deriving (88)–(90). The scaling laws (88)–(90) suggest a similar scaling law

$$t_{\rm jam} \sim N^b (\ln N)^\beta \qquad (93)$$

for the jamming time.

VI. CONCLUDING REMARKS

Classical random graphs undergo a percolation transition at $t_c = 1$ when the giant component is born. The mass of the giant component, i.e., the fraction of vertices belonging to the giant component, plays the role of the order parameter in the supercritical phase $t > t_c = 1$. Classical random graphs subsequently condense into a single component, i.e., the giant component engulfs all finite components. This condensation transition occurs at $t_{\rm cond} \simeq \ln N$. The densification of classical random graphs continues forever or until the graph becomes complete if loops and multiple edges are forbidden.

SRGs evolve similarly to classical random graphs with the constraint that the formation of complex components is forbidden. (A complex component has a negative Euler characteristic.) The SRGs undergo a phase transition at $t_c = 1$. When $t \leqslant 1$, the evolution of SRGs is asymptotically identical to classical random graphs where only a few complex components may arise. In the supercritical phase, $t > 1$, the mass of unicycles is an order parameter behaving similarly to the mass of the giant component for classical random graphs. A crucial difference between classical random graphs and SRGs is that for $t > 1$, the former contain a single giant component, while SRGs contain many macroscopic unicycles (i.e., unicycles with size proportional to N). Moreover, the size of the giant component is a self-averaging random quantity—fluctuations around the average gN scale as $\sqrt{N}$. For the SRGs, the size of the largest unicycle at jamming is a non-self-averaging random quantity with nontrivial distribution (60).

We have studied a class of SRG processes depending on the gluing probability $p \in [0, 1]$. Some properties of random graphs emerging in models with $p = 0$ and $p = 1$ resemble classical results about aggregation processes with the merging rate of components proportional to the product of their masses. The interpretation of mathematically identical formulas is different, however. For instance, the mass of unicycles for the SRG process with frozen unicycles ($p = 0$) coincides with the gel mass in the Stockmayer model of gel formation [9]. The mass of unicycles for the SRG process with $p = 1$ is the same as the gel mass in the Flory model of gelation [8,10] or, equivalently, the mass of the giant component in classical random graphs.

The SRG processes with arbitrary p were mentioned in Ref. [39] as the generalization of the intermediate SRG process with $p = \frac{1}{2}$, which was examined in Ref. [39] as the tool to understand a parking process on Cayley trees. The behavior of the intermediate SRG process, particularly the behavior in the scaling window around the phase transition point, has been analyzed in Ref. [39]. A remarkable connection [39] between jammed unicycles in the intermediate SRG process and communities in classical random maps [40–43] makes the intermediate SRG process most tractable. Seeking a class

of random maps similarly connected to the class of SRG processes would be an interesting avenue for future work.

Jamming is a distinctive feature of the SRG processes. The evolution of a simple random graph freezes when trees disappear, and only unicycles remain. An analytical determination of the distribution of the unicycles at the jamming time for models with $p > 0$ is an open issue. We argued that the jamming time and the average number of unicycles in a jammed state scale logarithmically with system size, Eqs. (5) and (6). The arguments leading to these scaling laws are heuristic since we ignored fluctuations. The logarithmic scaling laws seem correct. The amplitude in (5) is a bit questionable and could be just an upper bound, see (54). Arguments in favor of the amplitude in (6) are more solid.

The SRG process with frozen unicycles ends up in intriguing jammed states resembling supercluster jammed states [60–62] found in aggregation-fragmentation processes. To compute the jamming time, one should know the variance $\langle \xi^2(t) \rangle$. We argued, however, that the scaling laws (88)–(90) derived without knowing the jamming time are asymptotically exact. Relying on the form of the scaling laws (88)–(90), we conjectured the scaling law (93) for the jamming time. Needless to say, (93) is just the functional form as we do not know the exponents b and β in (93).

Apart from nonextensive supercluster jammed states [60–62], nonextensive steady states also appear in some aggregation-fragmentation processes [63]. The emergence of such nonthermodynamic behaviors often requires tuning the parameters, like setting $p = 0$ in the SRG process. Still, nonthermodynamic behaviors appear widespread. The theoretical analysis is challenging since fluctuations often play a decisive role, and computing fluctuations in a system with numerous interacting cluster species tends to be impossible.

Simple and classical random graphs undergo a mean-field continuous phase transition. More exotic phase transitions can also arise in random graph processes. It would be interesting to modify these processes, forbidding the creation of complex components or imposing other constraints. One can then examine the robustness of the phase transition. For instance, many growing networks [64–72] exhibit an infinite-order percolation transition of the Berezinskii-Kosterlitz-Thouless type [73,74]. Unicycles in such networks are also worth investigating.

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