Statistics of quantum heat in the Caldeira-Leggett model

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Nonequilibrium fluctuation relation lies at the heart of the quantum thermodynamics. Many previous studies have demonstrated that the heat exchange between a quantum system and a thermal bath initially prepared in their own Gibbs states at different temperatures obeys the famous Jarzynski-Wójcik fluctuation theorem. However, this conclusion is obtained under the assumption of Born-Markovian approximation. In this paper, going beyond the Born-Markovian limitation, we investigate the statistics of quantum heat in an exactly non-Markovian relaxation process described by the well-known Caldeira-Leggett model. It is revealed that the Jarzynski-Wójcik fluctuation theorem breaks down in the strongly non-Markovian regime. Moreover, we find the steady-state quantum heat within the non-Markovian framework can be widely tunable by using the quantum reservoir-engineering technique. These results are sharply contrary to the common Born-Markovian predictions. Our results presented in this paper may update the understanding of the quantum thermodynamics in strongly coupled and low-temperature systems. Moreover, the controllable heat may have some potential applications in improving the performance of a quantum heat engine.

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I. INTRODUCTION

Recently, a lot of interest has been generated in studying the fluctuation relations for thermodynamic quantities, such as work [1–6], heat [7–11], and entropy production [12–15], of small-scale systems away from equilibrium. As a consequence, the study of the traditional thermodynamics has been extended to the nonequilibrium and the quantum regimes [16–20]. On the one hand, from the theoretical perspective, these fluctuation theorems provide an alternative viewpoint to reexamine the laws of thermodynamics at the microscopic level [21–23]. On the other hand, from the application perspective, fluctuation theorems may aid in the development of quantum heat machines beyond the classical constraints, due to the fact that fluctuations are inherent in the thermodynamic cycles of microscopic machines [24–27]. Thus, the investigation of fluctuation theorems, going beyond the linear response regime, lies at the heart of the nonequilibrium thermodynamics.

The fluctuation theorem for heat exchange between two quantum systems, respectively prepared in their own Gibbs states at different temperatures, was originally proposed by Jarzynski and Wójcik in Ref. [7]. The original Jarzynski-Wójcik fluctuation theorem was derived with a severe assumption that the coupling between the two quantum systems is very weak. Under such an assumption, the correctness of the Jarzynski-Wójcik fluctuation theorem has been experimentally verified in nuclear-magnetic-resonance setups [28–30]. In several subsequent studies [31–34], the authors reexamined the validity of the Jarzynski-Wójcik fluctuation theorem in a relaxation process, in which a quantum harmonic oscillator exchanges the heat with its surrounding thermal bath via the system-bath coupling. They found that the Jarzynski-Wójcik fluctuation theorem can be still satisfied if the Born-Markovian approximation is employed [31–33]. The Born-Markovian approximation is a combination of the Born approximation and the Markovian approximation. The Born approximation neglects the correlations, which are included in the expression of the density operator, between the system and the bath. The Markovian approximation ignores the memory effect embedded in the convolution kernels which leads to a time-local master equation. The Born-Markovian approximation is acceptable when the system-bath coupling is weak and the characteristic timescale of the bath is much smaller than that of the system [35]. Situations beyond the Born-Markovian treatment still remain indistinct. An interesting question naturally arises here: Does the Jarzynski-Wójcik fluctuation theorem continue to hold in a non-Markovian relaxation process?

To address the above question, we investigate the statistics of heat exchange in a non-Markovian relaxation process described by the Caldeira-Leggett model, which is exactly solvable [36–38]. The exact solvability of the Caldeira-Leggett model can provide a reliable insight for the characteristics of the quantum heat beyond unwanted approximations. By using the exact solution of the famous Hu-Paz-Zhang equation [37] in terms of the reduced Wigner function [38,39] as well as the semiclassical phase-space formulation approach [40,41], we derive the analytical expression of the characteristic function

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for the quantum heat. It is found that the Jarzynski-Wójcik fluctuation theorem generally breaks down due to the non-Markovian effect. We also find the non-Markovian average quantum heat is sensitive to the details of the spectral density, which suggests it can be controlled by using the quantum reservoir-engineering technique [42–44]. Such an engineered tunable quantum heat can be useful in optimizing the output work or the efficiency of a thermodynamic cycle. All these results are in sharp contrast to previous Born-Markovian studies [31,32].

This paper is organized as follows. In Sec. II, we first recall the definition and the fluctuation theorems for quantum heat in an open system. In Sec. III, we introduce the Caldeira-Leggett model and give its exact solution in terms of the reduced Wigner function. The analytical expressions of the characteristic function and the average quantum heat are presented in Sec. IV. Using these expressions, we discuss the influence of the non-Markovian effect on the statistics of quantum heat. Some related discussions and the main conclusions of this paper are given in Sec. V. In several Appendices, we provide some additional materials about the main text. Throughout this paper, for the sake of convenience, we set $\hbar = k_B = 1$, and the inverse temperature is accordingly rescaled as $\beta = 1/T$.

II. HEAT IN AN OPEN QUANTUM SYSTEM AND ITS STATISTICS

A. Two-point-measurement-based definition

In this paper, we concentrate on the heat transferred between a quantum system and a thermal bath. The whole Hamiltonian of this open system reads $\hat{H}_{\text{tot}} = \hat{H}_s + \hat{H}_b + \hat{H}_{\text{int}}$, where $\hat{H}_s$ and $\hat{H}_b$ are, respectively, the Hamiltonian of the quantum system and the thermal bath and $\hat{H}_{\text{int}}$ denotes the system-bath interaction. Following previous studies [33,45,46], we regard the quantum heat as the energy change of the bath plus the system-bath interaction. Such an energy exchange can be detected by measuring the system if the whole Hamiltonian is time dependent [34]. When the system-bath coupling strength is so weak that the energy of the interaction Hamiltonian becomes negligible, our results naturally reduce to the situations considered in Refs. [31,32].

In the two-point-measurement-based definition [8,20,47,48], the quantum heat is defined as an energy difference of $\mathcal{Q}_{l'l} = E_{l'} - E_l$, where $E_{l'} = \langle l'|\hat{H}_s|l'\rangle$ and $E_l = \langle l|\hat{H}_s|l\rangle$ are the energies of the quantum system at the initial time $t = 0$ and the final time $t = \tau$, respectively. Here $|l\rangle$ and $|l'\rangle$ are the eigenstates of the quantum system. We assume the whole Hamiltonian is initially prepared in a product state $\rho_{\text{tot}}(0) = \rho_s(0) \otimes \rho_b(0)$, then the probability of obtaining the energy E_l at the initial time $t = 0$ is $P_l^0 = \text{Tr}_s[\rho_s(0)|l\rangle\langle l|]$. After the first measurement, the system instantaneously collapses to $|l\rangle\langle l|$. Thus, the conditional transition probability of obtaining the energy $E_{l'}$ at the final time $t = \tau$ is given by

$$\begin{aligned} P_{l'l}^{\tau} &= \text{Tr}[|l'\rangle\langle l'|\hat{U}(\tau)|l\rangle\langle l| \otimes \rho_b(0)\hat{U}^{\dagger}(\tau)] \\ &= \text{Tr}_s[|l'\rangle\langle l'|\hat{\Phi}_{\tau}(|l\rangle\langle l|)], \end{aligned} \tag{1}$$

where $\hat{U}(\tau) = \exp(-i\tau\hat{H}_{\text{tot}})$ is the time evolution operator of the whole Hamiltonian, and

$$\hat{\Phi}_{\tau}[\rho_s(0)] \equiv \text{Tr}_b[\hat{U}(\tau)\rho_s(0) \otimes \rho_b(0)\hat{U}^{\dagger}(\tau)], \tag{2}$$

is introduced as a dynamical mapping operator (superoperator) of the system from a given initial state $\rho_s(0)$ to the reduced density operator at $t = \tau$. From Eq. (1), one can see that as long as the reduced dynamics of the quantum system or the mapping operator $\hat{\Phi}_{\tau}$ is known, the conditional transition probability $P_{l'l}^{\tau}$ can be accordingly determined in principle.

With P_l^0 and $P_{l'l}^{\tau}$ at hand, the corresponding quantum heat distribution is derived as [31–34]

$$P_{\tau}(\mathcal{Q}) \equiv \sum_{l,l'} P_{l'l}^{\tau} P_l^0 \delta(\mathcal{Q} - \mathcal{Q}_{l'l}). \tag{3}$$

The characteristic function with respect to $P_{\tau}(\mathcal{Q})$ reads

$$\begin{aligned} \chi_{\tau}(\nu) &= \int_{-\infty}^{+\infty} d\mathcal{Q} e^{-\nu\mathcal{Q}} P_{\tau}(\mathcal{Q}) \\ &= \text{Tr}[e^{-\nu\hat{H}_s}\hat{U}_{\tau}e^{\nu\hat{H}_s}\rho_s(0) \otimes \rho_b(0)\hat{U}_{\tau}^{\dagger}]. \end{aligned} \tag{4}$$

The kth moment (cumulant) of the quantum heat can be computed by using the following formula:

$$\langle \mathcal{Q}^k(\tau)\rangle = (-1)^k \frac{\partial^k}{\partial \nu^k}\chi_{\tau}(\nu)\bigg|_{\nu=0}. \tag{5}$$

Equations (4) and (5) fully characterize the statistics of heat in an open quantum system.

B. Fluctuation theorems for quantum heat

Assuming the system and the bath are initially prepared in their own Gibbs states at different temperatures, namely $\rho_{s,b}(0) = \rho_{\beta_{s,b}}^G$, where

$$\rho_{\beta_{\varsigma}}^G \equiv \frac{e^{-\beta_{\varsigma}\hat{H}_{\varsigma}}}{\text{Tr}(e^{-\beta_{\varsigma}\hat{H}_{\varsigma}})} = \frac{e^{-\beta_{\varsigma}\hat{H}_{\varsigma}}}{\mathcal{Z}_{\varsigma}(\beta_{\varsigma})}, \tag{6}$$

with $\varsigma = \{s, b\}$, one can demonstrate that the characteristic function satisfies the following symmetry relation:

$$\chi_{\tau}(\nu) = \chi_{\tau}(\beta_s - \beta_b - \nu), \tag{7}$$

when (i) the system-bath coupling is so weak that the energy of the interaction can be neglected [7] or (ii) the reduced dynamics of the system is purely Markovian [31,32]. From Eq. (7), it is easy to prove that the heat distribution satisfies the following fluctuation theorem in the differential form [7]:

$$\frac{P_{\tau}(\mathcal{Q})}{P_{\tau}(-\mathcal{Q})} = e^{-(\beta_b-\beta_s)\mathcal{Q}}. \tag{8}$$

By setting $\nu = 0$, one can immediately find $\chi_{\tau}(0) = \chi_{\tau}(\beta_b - \beta_s) = 1$, from which the heat exchange fluctuation theorem in the integral form [7],

$$\langle e^{(\beta_b-\beta_s)\mathcal{Q}}\rangle = 1, \tag{9}$$

can be derived as well. Equations (8) and (9) establish the universal fluctuation theorems for quantum heat in the weak-coupling and the Born-Markovian regimes.

III. CALDEIRA-LEGGETT MODEL

A. The Hamiltonian and the quantum Langevin equation

The Caldeira-Leggett model is widely used to describe the dissipative dynamics of a quantum harmonic oscillator interacting with a bosonic bath via a linear position-position coupling. This model can be exactly solved by making use of various methods, for example, the path-integral approach [37,49], and the stochastic decoupling dynamical scheme [50–52], the non-Markovian quantum-state-diffusion-equation technique [53–55], as well as the normal mode transformation [56–59]. In this paper, we employ the path-integral approach from which an exact non-Markovian master equation, which is named as the Hu-Paz-Zhang equation, can be derived and exactly solved [37–39].

The whole Hamiltonian of the Caldeira-Leggett model is given by

$$\hat{H}_{\rm CL} = \frac{1}{2}\hat{p}^2 + \frac{1}{2}\omega_0^2\hat{q}^2 + \sum_{j=1}^{N}\left(\frac{1}{2}\hat{p}_j^2 + \frac{1}{2}\omega_j^2\hat{q}_j^2\right) - \hat{q}\sum_{j=1}^{N}c_j\hat{q}_j + \sum_{j=1}^{N}\frac{c_j^2}{2\omega_j^2}\hat{q}^2, \tag{10}$$

where $\hat{q}$ ($\hat{q}_j$) and $\hat{p}$ ($\hat{p}_j$) are the position and momentum operators of the quantum system (the thermal bath) with the corresponding frequency ω_0 (ω_j), respectively. Parameters c_j quantify the system-bath coupling strengths. The last term in Eq. (10) is a counterterm that can compensate the frequency shift induced by the system-bath interaction and can guarantee the thermodynamically stable of the Caldeira-Leggett model [60]. All the masses of these quantum harmonic oscillators are weighted as $m_0 = m_j = 1$ in this paper.

Commonly, the coupling strengths are further characterized by the so-called spectral density, which is defined by

$$J(\omega) \equiv \sum_{j=1}^{N}\frac{c_j^2}{\omega_j}\delta(\omega-\omega_j). \tag{11}$$

In this paper, we assume that $J(\omega)$ explicitly takes the following Ohmic form:

$$J(\omega) = \frac{2}{\pi}\frac{\kappa\omega\omega_c^2}{\omega^2+\omega_c^2}, \tag{12}$$

where κ denotes the dimensionless system-bath coupling constant and ω_c is the cutoff frequency. It is necessary to point out that our results presented in this paper can be generalized to other spectral densities without difficulties.

Assuming the initial state of the whole Caldeira-Leggett model is given by $\rho_{\rm tot}(0) = \rho_s(0)\otimes\rho_{\beta_b}^G$, the following quantum Langevin equation can be derived from the Heisenberg equation of motion [61–63]:

$$\ddot{\hat{q}}(t) + \int_0^t dt'\mu(t-t')\dot{\hat{q}}(t') + \omega_0^2\hat{q}(t) + \mu(t)\hat{q}(0) = \hat{f}(t), \tag{13}$$

where

$$\mu(t) \equiv \sum_{j=1}^{N}\frac{c_j^2}{\omega_j^2}\cos(\omega_j t), \tag{14}$$

is introduced as the memory function and

$$\hat{f}(t) \equiv \sum_{j=1}^{N}c_j\left[\hat{q}_j(0)\cos(\omega_j t) + \hat{p}_j(0)\frac{\sin(\omega_j t)}{\omega_j}\right] \tag{15}$$

is the so-called stochastic force operator. One easily finds that $\langle\hat{f}(t)\rangle = 0$, the correlation function, and the commutator of $\hat{f}(t)$ are [39]

$$\frac{1}{2}\langle\hat{f}(t)\hat{f}(t') + \hat{f}(t')\hat{f}(t)\rangle = \frac{1}{\pi}\int_0^\infty d\omega\omega\,{\rm Re}[\tilde{\mu}(\omega+i0^+)]\coth\left(\frac{\omega\beta_b}{2}\right) \times\cos[\omega(t-t')], \tag{16}$$

$$[\hat{f}(t),\hat{f}(t')] = \frac{2}{i\pi}\int_0^\infty d\omega\omega{\rm Re}[\tilde{\mu}(\omega+i0^+)]\sin[\omega(t-t')], \tag{17}$$

where the bracket $\langle\cdot\rangle$ stands for the quantum expectation value with respect to $\rho_{\beta_b}^G$, i.e., $\langle\hat{\mathcal{O}}\rangle \equiv {\rm Tr}(\hat{\mathcal{O}}\rho_{\beta_b}^G)$, and $\tilde{\mu}(\omega)$ is defined as the modified Laplace transform of the memory function:

$$\tilde{\mu}(\omega) = \int_0^\infty dt\mu(t)e^{i\omega t}. \tag{18}$$

Using the skills proposed in Ref. [39], the general solution to the quantum Langevin equation reads

$$\hat{q}(t) = \dot{G}(t)\hat{q}(0) + G(t)\hat{p}(0) + \hat{x}(t), \tag{19}$$

$$\hat{p}(t) = \ddot{G}(t)\hat{q}(0) + \dot{G}(t)\hat{p}(0) + \dot{\hat{x}}(t), \tag{20}$$

where $G(t)$ is the Green function determined by

$$\ddot{G}(t) + \int_0^t dt'\mu(t-t')\dot{G}(t') + \omega_0^2G(t) + \mu(t)G(t) = 0, \tag{21}$$

with the initial conditions $G(0) = 0$ and $\dot{G}(0) = 1$. And $\hat{x}(t)$ is introduced as the fluctuating position operator,

$$\hat{x}(t) = \int_0^t dt'G(t-t')\hat{f}(t'). \tag{22}$$

Based on Eqs. (19) and (20), the analytical expression of the reduced Wigner function, which is an exact solution of the Hu-Paz-Zhang equation, can be derived (see Ref. [39] and Appendix B for details).

B. The reduced Wigner function

By tracing out the degrees of freedom of the thermal bath, the evolution of the reduced density operator of the system is governed by the famous Hu-Paz-Zhang equation [37]. This Hu-Paz-Zhang equation can be exactly solved in the form of an expression for the Wigner function as (see Refs. [38,39] or Appendix B for details)

$$W_t(q,p) = [\rho_s(t)]_w(q,p) = \int_{-\infty}^{+\infty}dq'\int_{-\infty}^{+\infty}dp'\mathcal{P}_t(q,p,q',p')W_0(q',p'), \tag{23}$$

where $\{q,p\}$ represent the point in the classical phase space. The notation $[\hat{\mathcal{O}}]_w(q,p)$ denotes the Weyl symbol of a given operator $\hat{O}$, which is defined by [40,41]

$$[\hat{O}]_w(q,p) \equiv \int dv \left\langle q - \frac{1}{2}v \middle| \hat{O} \middle| q + \frac{1}{2}v \right\rangle e^{ipv}. \tag{24}$$

And $W_0(q,p) = [\rho_s(0)]_w$ is the initial Wigner function of the system, $\mathcal{P}_t(q,p,q',p')$ can be regarded as a propagator, which is given by

$$\mathcal{P}_t(q,p,q',p') = \frac{1}{2\pi} \frac{1}{\sqrt{\mathrm{Det}(\mathbb{A}_t)}} e^{-\frac{1}{2}\mathbb{R}_t^T \mathbb{A}_t^{-1} \mathbb{R}_t}, \tag{25}$$

where

$$\mathbb{R}_t = \begin{pmatrix} q \\ p \end{pmatrix} - \mathbb{G}_t \begin{pmatrix} q' \\ p' \end{pmatrix} = \begin{pmatrix} q \\ p \end{pmatrix} - \begin{bmatrix} \dot{G}(t) & G(t) \\ \ddot{G}(t) & \dot{G}(t) \end{bmatrix} \begin{pmatrix} q' \\ p' \end{pmatrix}, \tag{26}$$

and $\mathbb{A}_t$ is the covariant matrix with respect to the fluctuating position operator $\hat{x}(t)$, i.e.,

$$\mathbb{A}_t = \begin{bmatrix} \langle \hat{x}^2(t) \rangle & \frac{1}{2}\langle \hat{x}(t)\dot{\hat{x}}(t) + \dot{\hat{x}}(t)\hat{x}(t) \rangle \\ \frac{1}{2}\langle \hat{x}(t)\dot{\hat{x}}(t) + \dot{\hat{x}}(t)\hat{x}(t) \rangle & \langle \dot{\hat{x}}^2(t) \rangle \end{bmatrix}. \tag{27}$$

For the Ohmic spectral density considered in this paper, the explicit expressions of $\mathbb{G}_t$ and $\mathbb{A}_t$ are given in the Appendix A.

If the initial state of the system is a Gibbs state, i.e., $\rho_s(0) = \rho^G_{\beta_s}$, then the exact expression of the reduced Wigner function can be derived from Eq. (25) using the formula of the Gaussian integral,

$$\begin{aligned} W_t(q,p) = {} & \frac{2\tanh\left(\frac{1}{2}\omega_0\beta_s\right) e^{-\frac{1}{2}z^T \mathbb{A}_t^{-1} z}}{\sqrt{\mathrm{Det}(\mathbb{A}_t)\mathrm{Det}[\Pi(\beta_s) + \mathbb{G}_t^T \mathbb{A}_t^{-1}\mathbb{G}_t]}} \\ & \times \exp\left\{ \frac{1}{2} z \mathbb{A}_t^{-1} \mathbb{G}_t \left[\Pi(\beta_s) + \mathbb{G}_t^T \mathbb{A}_t^{-1} \mathbb{G}_t\right]^{-1} \right. \\ & \left. \times\ \mathbb{G}_t^T \mathbb{A}^{-1} z^T \right\}, \end{aligned} \tag{28}$$

where $z = (q,p)^T$ and

$$\Pi(\beta_s) = \frac{2}{\omega_0} \tanh\left(\frac{1}{2}\omega_0\beta_s\right) \begin{pmatrix} \omega_0^2 & 0 \\ 0 & 1 \end{pmatrix}. \tag{29}$$

One can check the normalization condition $\frac{1}{2\pi}\int dq \int dp W_t(q,p) = 1$ is satisfied. Equation (28) fully determines the reduced dynamics of the quantum system. Next, we use Eq. (28) to derive the analytical expression of the characteristic function for the quantum heat.

IV. STATISTICS OF QUANTUM HEAT

A. Phase-space formulation approach

To analytically derive the characteristic function of quantum heat in the Caldeira-Leggett model, we employ the phase-space formulation approach reported in Refs. [40,41]. To this aim, we first rewrite Eq. (4) into

$$\begin{aligned} \chi_\tau(\nu) = {} & \frac{1}{\mathcal{Z}_s(\beta_s)} \mathrm{Tr}\left[e^{-\nu\hat{H}_s}\hat{U}(\tau)e^{-(\beta_s-\nu)\hat{H}_s} \otimes \rho^G_{\beta_b}\hat{U}^\dagger(\tau)\right] \\ = {} & \frac{\mathcal{Z}_s(\beta_s-\nu)\mathcal{Z}_s(\nu)}{\mathcal{Z}(\beta_s)} \mathrm{Tr}\left[\rho^G_\nu \hat{U}(\tau)\rho^G_{\beta_s-\nu} \otimes \rho^G_{\beta_b}\hat{U}^\dagger(\tau)\right] \\ = {} & \frac{\mathcal{Z}_s(\beta_s-\nu)\mathcal{Z}_s(\nu)}{\mathcal{Z}_f(\beta_s)} \mathrm{Tr}_s\left[\rho^G_\nu \Phi_\tau\left(\rho^G_{\beta_s-\nu}\right)\right], \end{aligned} \tag{30}$$

where $\mathcal{Z}_s(\beta_s) = 1/2\sinh(\omega_0\beta_s/2)$ is the partition function of the quantum harmonic oscillator. Then, using the Weyl symbol, one can express Eq. (30) in the language of the phase-space formulation of quantum mechanics as

$$\begin{aligned} \mathrm{Tr}_s\left[\rho^G_\nu \Phi_\tau\left(\rho^G_{\beta_s-\nu}\right)\right] = {} & \frac{1}{2\pi}\int_{-\infty}^{+\infty} dq \int_{-\infty}^{+\infty} dp \left[\rho^G_\nu\right]_w(q,p) \\ & \times \left[\Phi_\tau\left(\rho^G_{\beta_s-\nu}\right)\right]_w(q,p). \end{aligned} \tag{31}$$

Thus, as long as the Weyl symbols of $\rho^G_s(\nu)$ and $\Phi_\tau(\rho^G_{\beta_s-\nu})$ are obtained, the expression of the characteristic function $\chi_\tau(\nu)$ can be accordingly derived by simply performing the integrals over the classical variables $\{q,p\}$.

The Weyl symbol of the Gibbs-like state ρ^G_ν is given by (see Ref. [64] and Appendix C for details)

$$\begin{aligned} \left[\rho^G_\nu\right]_w(q,p) = {} & 2\tanh\left(\frac{\omega_0\nu}{2}\right) \\ & \times \exp\left[-\frac{\tanh\left(\frac{1}{2}\omega_0\nu\right)}{\omega_0}\left(p^2 + \omega_0^2 q^2\right)\right]. \end{aligned} \tag{32}$$

On the other hand, the Weyl symbol of $\Phi_\tau(\rho^G_{\beta_s-\nu})$ can be obtained from Eq. (28) via simply replacing β_s by $\beta_s - \nu$. With these results at hand, we finally find

$$\begin{aligned} \chi_\tau(\nu) = {} & \frac{\mathcal{Z}_s(\beta_s-\nu)\mathcal{Z}_s(\nu)}{\mathcal{Z}(\beta_s)} \\ & \times \frac{1}{\sqrt{\mathrm{Det}\left[\mathbb{A}_\tau + \Lambda(\nu) + \mathbb{G}_\tau\Lambda(\beta_s-\nu)\mathbb{G}_\tau^T\right]}}, \end{aligned} \tag{33}$$

where

$$\Lambda(\nu) = \frac{1}{2\omega_0}\coth\left(\frac{1}{2}\nu\omega_0\right)\begin{pmatrix} 1 & 0 \\ 0 & \omega_0^2 \end{pmatrix}. \tag{34}$$

The corresponding expression of the average quantum heat is then given by

$$\begin{aligned} \langle \mathcal{Q}(\tau) \rangle = {} & \frac{1}{2}\left[\omega_0^2\mathbb{A}_{11}(\tau) + \mathbb{A}_{22}(\tau)\right] \\ & + \frac{1}{2}\omega_0\coth\left(\frac{\omega_0\beta_s}{2}\right)\left[\frac{1}{2}\omega_0^2 G^2(\tau) + \dot{G}^2(\tau) \right. \\ & \left. + \frac{1}{2\omega_0^2}\ddot{G}^2(\tau) - 1\right]. \end{aligned} \tag{35}$$

Equations (33) and (35) are the main results of this paper, which is beyond the usual Born-Markovian approximation. Next, we shall use them to analyze the non-Markovian effects on the statistics of quantum heat in the relaxation process describe by the Caldeira-Leggett model.

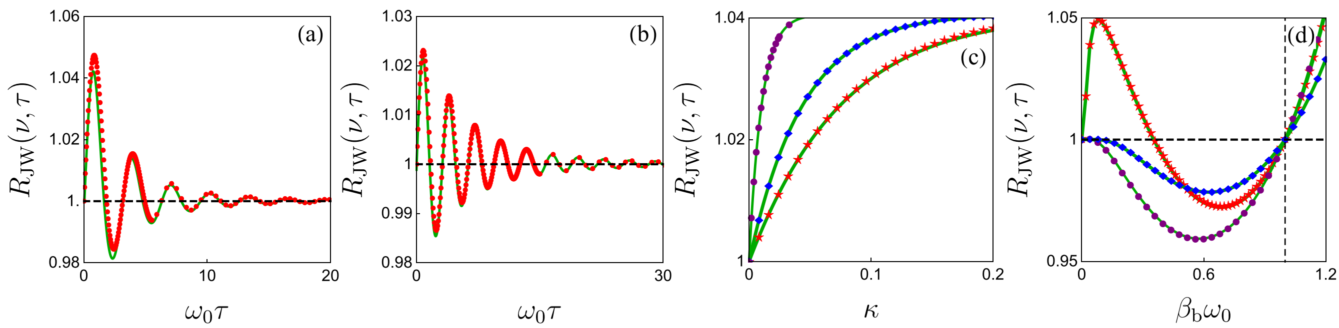

FIG. 1. The ratio of $R_{\rm JW}(\nu,\tau)$ is plotted as a function of $\omega_0\tau$ with (a) $\nu = 5$ and (b) $\nu = 6$. Other parameters are chosen as $\omega_0 = 1$ cm^{-1}, $\omega_0\beta_s = 2$, $\omega_0\beta_b = 0.1$, and $\kappa = 0.05$. (c) The ratio of $R_{\rm JW}(\nu,\tau)$ versus the coupling strength κ with different evolution times: $\omega_0\tau = 1$ (red stars), $\omega_0\tau = 5$ (blue rhomboids), and $\omega_0\tau = 100$ (purple circles). Other parameters chosen are $\omega_0 = 1$ cm^{-1}, $\omega_0\beta_s = 1$, and $\omega_0\beta_b = 0.1$. (d) The ratio of $R_{\rm JW}(\nu,\tau)$ versus the temperature of the bath $\beta_b\omega_0$ with different evolution times: $\omega_0\tau = 1$ (red stars), $\omega_0\tau = 5$ (blue rhomboids), and $\omega_0\tau = 100$ (purple circles). Other parameters chosen are $\omega_0 = 1$ cm^{-1}, $\omega_0\beta_s = 1$, and $\kappa = 0.2$. The circles, rhomboids, and five-pointed stars are exact results from Eq. (33) and the green solid lines present the numerical results predicted by the normal mode transformation method proposed in Ref. [33]. The vertical dotted line in (d) highlights the position where $\omega_0\beta_b = \omega_0\beta_s$.

B. The long-time limit

In the exact results of Eqs. (33) and (35), the expressions of $\chi_\tau(\nu)$ and $\langle\mathcal{Q}(\tau)\rangle$ are complicated. We leave them to exact calculations. However, by analyzing their asymptotic behaviours in the long-time limit, we can obtain some qualitative results, which is beneficial for us to establish a clear physical picture.

In the long-time limit $\omega_0\tau \to \infty$, we have $\mathbb{G}_t = \mathbf{0}_2$, which leads to

$$\chi_\infty(\nu) = \frac{\mathcal{Z}_s(\beta_s-\nu)\mathcal{Z}_s(\nu)}{\mathcal{Z}_s(\nu)}\frac{1}{\sqrt{\mathrm{Det}[\mathbb{A}_\infty+\Lambda(\nu)]}}. \quad (36)$$

The corresponding average quantum heat is given by

$$\langle\mathcal{Q}(\infty)\rangle = \frac{1}{2}\big[\omega_0^2\mathbb{A}_{11}(\infty)+\mathbb{A}_{22}(\infty)\big] - \frac{1}{2}\omega_0\coth\left(\frac{\beta_s\omega_0}{2}\right). \quad (37)$$

The explicit expressions of $\mathbb{A}_{11}(\infty)$ and $\mathbb{A}_{22}(\infty)$ are given in Appendix A. Equations (36) and (37) show that the steady-state characteristic function and the average quantum heat heavily depend on the details of the spectral density, which are involved in the expression of $\mathbb{A}_\infty$. This result is in sharp contrast to Markovian results reported in Refs. [31,32,34].

However, in the weak-coupling limit, one has (see Appendix A)

$$\lim_{\kappa\to 0^+}\chi_\infty(\nu) = \frac{\mathcal{Z}_s(\beta_s-\nu)\mathcal{Z}_s(\beta_b+\nu)}{\mathcal{Z}_s(\beta_s)\mathcal{Z}_s(\beta_b)}, \quad (38)$$

which becomes independent of the details of the relaxation dynamics and recovers the previous result of canonical thermalization in Ref. [34]. Moreover, from Eq. (38), one can immediately demonstrate the symmetry relation of $\chi_\infty(\beta_s-\beta_b-\nu) = \chi_\infty(\nu)$ is satisfied in this weak-coupling case, which means the fluctuation theorem in the differential form holds in the Markovian regime.

The corresponding average quantum heat in the same weak-coupling limit is given by

$$\lim_{\kappa\to 0^+}\langle\mathcal{Q}(\infty)\rangle = \frac{1}{2}\omega_0\left[\coth\left(\frac{\beta_b\omega_0}{2}\right) - \coth\left(\frac{\beta_s\omega_0}{2}\right)\right]. \quad (39)$$

The above expression is simply the energy difference for the quantum harmonic oscillator at temperatures $1/\beta_b$ and $1/\beta_s$. It can be rewritten in terms of the thermal occupation number, namely $\lim_{\kappa\to 0^+}\langle\mathcal{Q}(\infty)\rangle = \omega_0[\bar{n}(\beta_b,\omega_0) - \bar{n}(\beta_s,\omega_0)]$ with $\bar{n}(\beta_\alpha,\omega_0) \equiv (e^{\omega_0\beta_\alpha}-1)^{-1}$. This result is consistent with that of Ref. [32] and is physically reasonable when the quantum system experiences a canonical thermalization under the usual Born-Markovian approximation [31–34].

V. EXACT RESULTS

In this section, we provide the exact results predicted by Eqs. (33) and (35). To evaluate the validity of the Jarzynski-Wójcik fluctuation theorem in the deep non-Markovian regimes, we define the following ratio of the characteristic function:

$$R_{\rm JW}(\nu,\tau) \equiv \frac{\chi_\tau(\beta_s-\beta_b-\nu)}{\chi_\tau(\nu)}. \quad (40)$$

If and only if $R_{\rm JW}(\nu,\tau) = 1$, then one can conclude that the Jarzynski-Wójcik fluctuation theorem holds to be valid. However, as long as $R_{\rm JW}(\nu,\tau) \neq 1$, the Jarzynski-Wójcik fluctuation theorem breaks down.

Alternatively, we also provide the purely numerical simulations from the normal mode transformation method reported in Ref. [33] as a comparison. As shown below, these two different methods produce consistent results in physics. This comparison commendably confirms the reliability of our conclusions.

In Figs. 1(a) and 1(b), we plot the time evolution of $R_{\rm JW}(\nu,\tau)$. Our exact results are in good agreement with these of the normal mode transformation method in Ref. [33]. The results from these two approaches jointly demonstrate $R_{\rm JW}(\nu,\tau) \neq 1$ in the short-time transient regime, which means the breakdown of the Jarzynski-Wójcik fluctuation theorem. In the long-time limit, $R_{\rm JW}(\nu,\infty) = 1$ if the coupling is very weak [say, $\kappa = 0.05$ in Figs. 1(a) and 1(b)], which justifies our previous long-time results discussed in Sec. IV B.

It is well accepted that the non-Markovian characteristic for the reduced dynamics of the quantum system becomes negligible when (i) the system-bath coupling is sufficiently

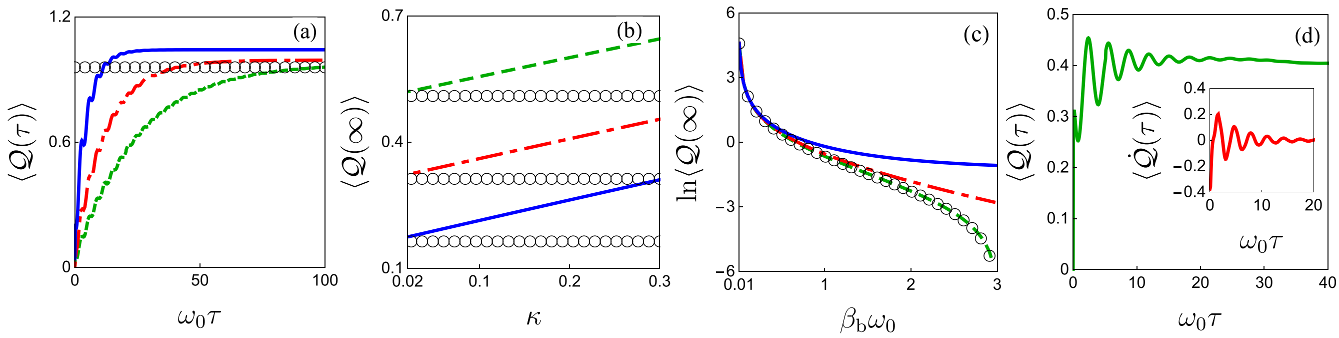

FIG. 2. (a) The time evolution of the average quantum heat $\langle \mathcal{Q}(\tau)\rangle$ with different coupling strengths: $\kappa = 0.04$ (green dashed line), $\kappa = 0.08$ (red dot-dashed line), and $\kappa = 0.2$ (blue solid line). Other parameters are as follows: $\omega_0 = 1\ \text{cm}^{-1}$, $\omega_0\beta_b = 0.5$, and $\omega_0\beta_s = 1$. (b) The steady-state quantum heat $\langle \mathcal{Q}(\infty)\rangle$ is plotted as a function of the coupling strength with different temperatures: $\omega_0\beta_b = 0.65$ (green dashed line), $\omega_0\beta_b = 0.75$ (red dot-dashed line), and $\omega_0\beta_b = 0.85$ (blue solid line). The temperature of the system is chosen as $\omega_0\beta_s = 3$ with $\omega_0 = 1\ \text{cm}^{-1}$. (c) $\ln\langle \mathcal{Q}(\infty)\rangle$ versus the bath temperature with different coupling strengths: $\kappa = 0.01$ (green dashed line), $\kappa = 0.1$ (red dot-dashed line), and $\kappa = 0.6$ (blue solid line). The temperature of the system is chosen as $\omega_0\beta_s = 3$ with $\omega_0 = 1\ \text{cm}^{-1}$. (d) The dynamics of the average quantum heat $\langle \mathcal{Q}(\tau)\rangle$. Other parameters are chosen as follows: $\omega_0 = 1\ \text{cm}^{-1}$, $\omega_0\beta_b = 0.1$, $\omega_0\beta_s = 1$, and $\kappa = 0.2$. All the gray circles in (a)–(c) are the Born-Markovian results presented by Eq. (39).

weak or (ii) the temperature of the bath is very high. In these two cases, the dynamics of heat exchange becomes Markovian which validates the Jarzynski-Wójcik fluctuation theorem. To check this conclusion, we display $R_{\text{JW}}(\nu, \tau)$ versus $\beta_b\omega_0$ in Fig. 1(c) and κ in Fig. 1(d). From these figures, one can find $R_{\text{JW}}(\nu, \tau)$ approaches to 1 if $\omega_0\beta_b \to 0$ or $\kappa \to 0$. These exact results meet our expectations. Exceptions can occur in the vicinities of $\omega_0\beta_b = \omega_0\beta_s$, which corresponds to the trivial case: There is no heat exchange if the system and the bath are at the same temperature. The above exact results decidedly demonstrate the non-Markovian effects can result in the invalidation of the Jarzynski-Wójcik fluctuation theorem.

In Fig. 2(a), we display the dynamics of $\langle \mathcal{Q}(\tau)\rangle$ with different coupling strengths. One sees that, when the system-bath coupling is very weak, the long-time result from the exact simulation is in qualitative agreement with that of Eq. (39). However, in the strong-coupling regimes [see Fig. 2(b)] or at low temperature [see Fig. 2(c)], the exact result shows a relatively large deviation from Eq. (39). Such a difference from the prediction of the canonical thermalization theory is induced by the non-Markovian effect.

The oscillations appeared in the dynamics of $\langle \mathcal{Q}(\tau)\rangle$ can be viewed as an evidence of non-Markovianity [65]. As demonstrated in many previous articles [66–69], in the Markovian regime, the quantum heat unidirectionally flows between the system and the bath; in the non-Markovian regime, an energy backflow may occur and leads to these oscillations in the dynamics of the quantum heat. In Fig. 2(d), we clearly observe such oscillations of during the dynamics of the quantum heat. This result convinces us that our method truly captures the non-Markovian behavior of the quantum heat exchange in the relaxation process described by the Caldeira-Leggett model.

VI. DISCUSSIONS AND CONCLUSIONS

Before concluding our paper, three important remarks shall be made here.

(i) Our exact results demonstrate Jarzynski-Wójcik fluctuation theorem breaks down in a non-Markovian relaxation process. This naturally brings about another interesting question: Does a generalized fluctuation theorem for quantum heat exchange exist in strong-coupling or low-temperature systems? Such a problem has been discussed in our recent article [70]. In this article, we find the invalidation of the Jarzynski-Wójcik fluctuation theorem can be traced back to the violation of the detailed balance condition [71,72] in non-Markovian regimes, which generally results in an absence of time-reversal symmetry [72] and an appearance of noncanonical thermalization [70,73,74]. Regardless of the disappearance of time-reversal symmetry [7] or the occurrence of noncanonical thermalization [70,75,76] shall contribute to the breakdown of the Jarzynski-Wójcik fluctuation theorem. In this non-Markovian case, by introducing an effective temperature and recovering the detailed balance condition, we establish a generalized fluctuation theorem for quantum heat being valid for arbitrary system-bath coupling strengths at arbitrary temperatures [70]. However, it is still an unknown question whether the scheme proposed in Ref. [70] can be feasible to the Caldeira-Leggett model considered in this paper.

(ii) From Eq. (37), one sees the non-Markovian steady state of the quantum heat is sensitive to the details of the spectral density, which is in sharp contrast to the Markovian result [Eq. (39)]. This result means the steady-state quantum heat can be widely tunable by using the well-developed quantum reservoir-engineering technique [42–44], in which the primary parameters or the structure of the spectral density are experimentally controllable. For example, the coupling strength of a quantum emitter coupled to a surface-plasmon polariton as an environment is adjustable by changing the distance between them [77,78]; and the structure of the spectral density for a reservoir, consisting of ultracold atomic gases, can be adjusted by regulating the scattering lengths of these atomic gases via the Feshbach resonance [79]. The rich nonequilibrium character of quantum heat in the non-Markovian regime may be beneficial for realizing an engineered quantum heat engine. In fact, several previous studies have demonstrated that a controllable quantum heat during thermodynamic cycles is able to boost the performance of quantum heat engines [80–83].

(iii) Although the results from our present paper are in qualitative agreement with those of the normal mode transformation method used in Ref. [33], they are totally different from a technical point of view. In Ref. [33], one needs to diagonalize the Caldeira-Leggett Hamiltonian. While, in this paper, one needs to solve the evolution of the Hu-Paz-Zhang equation in terms of the Wigner function, which is a pure dynamics problem. Compared with that of Ref. [33], the scheme in this paper is much simpler. As displayed in Appendix A of the Ref. [33], one needs to handle a large number of numerical integrations and summations of series using the normal mode transformation method, which may lead to problems of convergence. These numerical difficulties are effectively avoided in this paper. In this sense, our present scheme have an advantage of high efficiency over that of the normal mode transformation method. On the other hand, the normal mode transformation is applicable only for a quadratic Hamiltonian, say, the Caldeira-Leggett model and the model of an externally dragged harmonic oscillator linearly coupled to an assembly of harmonic oscillators [84]. It is generally difficult to apply the normal mode transformation to the anharmonic case. In contrast, as demonstrated in Refs. [85–87], the master equation approach in terms of the Wigner formalism can be generalized to the general potential cases. In this sense, the master equation approach enables a more extensive scope of applicability. It would be interesting to extend our present study to the anharmonic case in the future. Anyway, we believe that any alternative analytical and simple approach can be of importance for a better understanding of the statistics of quantum heat in a non-Markovian relaxation process.

In summary, by making use of the exact solution of the Hu-Paz-Zhang equation in terms of the reduced Wigner function, we investigate the statistics of quantum heat during a non-Markovian relaxation process described by the Caldeira-Leggett model. The exactly analytical expressions of the characteristic function and the average quantum heat are derived. With these results at hand, in the non-Markovian regime, we find (i) the Jarzynski-Wójcik fluctuation theorem for quantum heat breaks down and (ii) the steady-state average quantum heat becomes sensitive to the details of the spectral density, which leads to a rich nonequilibrium characteristics. These results are completely different from those of the Born-Markovian results in many previous studies. Our work may improve the understanding of the quantum thermodynamics in the strongly coupled or the low-temperature systems. Moreover, the highly tunable quantum heat may have potential applications in optimizing the efficiency of quantum heat engines.

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APPENDIX A: THE EXPRESSIONS OF $G(t)$ AND $\mathbb{A}(t)$

For the Ohmic spectral density given by Eq. (12), the memory function reads

$$\mu(t) = \int_0^\infty d\omega \frac{J(\omega)}{\omega} \cos(\omega t) = \omega_c \kappa e^{-\omega_c t}. \tag{A1}$$

For the sake of convenience, in this paper, we concentrate on the case of $\omega_c/\kappa \gg 1$, which leads to $\mu(t) \simeq 2\kappa\delta(t)$ [33,88,89]. The corresponding expression of $G(t)$ is then given by

$$G(t) = \mathcal{L}^{-1}\left(\frac{1}{s^2+\kappa s+\omega_0^2}\right) = \frac{\sin(\Omega t)}{\Omega} e^{-\frac{1}{2}\kappa t}, \tag{A2}$$

where $\mathcal{L}^{-1}$ denotes the inverse Laplace transformation and $\Omega = \sqrt{\omega_0^2 - \frac{1}{4}\kappa^2}$ is the renormalized frequency. From the above expression, one finds

$$G(-t') - G(t-t') = G(-t') - G(t)\big[\dot{G}(-t') + \tfrac{1}{2}\kappa G(-t')\big] - G(-t')\big[\dot{G}(t) + \tfrac{1}{2}\kappa G(t)\big]. \tag{A3}$$

Using the above equality, one sees

$$\begin{aligned} \hat{x}(t) &= \int_0^t dt' G(t-t')\hat{f}(t') \\ &= \int_{-\infty}^t dt' G(t-t')\hat{f}(t') - \int_{-\infty}^0 dt' G(-t')\hat{f}(t') + \int_{-\infty}^0 dt'[G(-t') - G(t-t')]\hat{f}(t') \\ &= \hat{F}(t) - [\dot{G}(t) + \kappa G(t)]\hat{F}(0) - G(t)\dot{\hat{F}}(t), \end{aligned} \tag{A4}$$

where

$$\hat{F}(t) = \int_{-\infty}^t dt' G(t-t')\hat{f}(t'). \tag{A5}$$

Using the fluctuation-dissipation theorem [39], one can find the correlation function of $\hat{F}(t)$ is

$$\Theta(t-t') = \langle \hat{F}(t)\hat{F}(t') + \hat{F}(t')\hat{F}(t)\rangle = \frac{2\kappa}{\pi}\int_0^\infty d\omega \frac{\omega}{\left(\omega_0^2-\omega^2\right)^2+\kappa^2\omega^2}\coth\left(\frac{\beta_b\omega}{2}\right)\cos[\omega(t-t')]. \tag{A6}$$

With Eq. (A4) and Eq. (A6) at hand, the elements of $\mathbb{A}_t$ are given by

$$\mathbb{A}_{11}(t) = \tfrac{1}{2}\Theta(0) - \tfrac{1}{2}\ddot{\Theta}(0)G^2(t) + \tfrac{1}{2}\Theta(0)[\kappa G(t)+\dot{G}(t)]^2 - \Theta(t)[\kappa G(t)+\dot{G}(t)] + \dot{\Theta}(t)G(t), \tag{A7}$$

$$\mathbb{A}_{22}(t) = -\tfrac{1}{2}\ddot{\Theta}(0) - \tfrac{1}{2}\ddot{\Theta}(0)\dot{G}^2(t) + \tfrac{1}{2}\Theta(0)[\kappa \dot{G}(t)+\ddot{G}(t)]^2 - \dot{\Theta}(t)[\kappa \dot{G}(t)+\ddot{G}(t)] + \ddot{\Theta}(t)\dot{G}(t), \tag{A8}$$

$$\mathbb{A}_{12}(t) = \mathbb{A}_{21}(t) = -\tfrac{1}{2}\ddot{\Theta}(0)G(t)\dot{G}(t) + \tfrac{1}{2}\Theta(0)[\kappa G(t)+\dot{G}(t)][\kappa \dot{G}(t)+\ddot{G}(t)] - \tfrac{1}{2}\Theta(t)[\kappa \dot{G}(t)+\ddot{G}(t)] - \tfrac{1}{2}[\kappa\dot{\Theta}(t) - \ddot{\Theta}(t)]G(t). \tag{A9}$$

In the long-time limit, we have $G(\infty) = \dot{G}(\infty) = \ddot{G}(\infty) = 0$ which results in

$$\mathbb{A}_{11}(\infty) = \frac{1}{2}\Theta(0) = \frac{\kappa}{\pi}\int_0^\infty d\omega\frac{\omega}{(\omega_0^2-\omega^2)^2+\kappa^2\omega^2}\coth\left(\frac{\beta_b\omega}{2}\right), \tag{A10}$$

$$\mathbb{A}_{22}(\infty) = -\frac{1}{2}\ddot{\Theta}(0) = \frac{\kappa}{\pi}\int_0^\infty d\omega\frac{\omega^3}{(\omega_0^2-\omega^2)^2+\kappa^2\omega^2}\coth\left(\frac{\beta_b\omega}{2}\right), \tag{A11}$$

and $\mathbb{A}_{12}(\infty) = \mathbb{A}_{21}(\infty) = 0$. From Eqs. (A10) and (A11), it is clear to see that $\mathbb{A}_\infty$ heavily depends on the system-bath coupling strength.

Notice that

$$\lim_{\kappa\to 0^+}\frac{\kappa}{\pi}\frac{1}{\left(\omega_0^2-\omega^2\right)^2+\kappa^2\omega^2} = \frac{1}{\omega^2}\delta(\omega-\omega_0), \tag{A12}$$

then, in the weak-coupling limit, we have

$$\lim_{\kappa\to 0^+}\mathbb{A}_{11}(\infty) = \int_0^\infty d\omega\frac{\omega}{\omega^2}\coth\left(\frac{\beta_b\omega}{2}\right)\delta(\omega-\omega_0) = \frac{1}{\omega_0}\coth\left(\frac{\beta_b\omega_0}{2}\right), \tag{A13}$$

$$\lim_{\kappa\to 0^+}\mathbb{A}_{22}(\infty) = \int_0^\infty d\omega\frac{\omega^3}{\omega^2}\coth\left(\frac{\beta_b\omega}{2}\right)\delta(\omega-\omega_0) = \omega_0\coth\left(\frac{\beta_b\omega_0}{2}\right). \tag{A14}$$

Using the above two equations, one can easily finds Eq. (36) and Eq. (37) reduces to Eq. (38) and Eq. (39), respectively.

APPENDIX B: THE REDUCED WIGNER FUNCTION OF THE CALDEIRA-LEGGETT MODEL

In this Appendix, we show the details of deriving the reduced Wigner function of the Caldeira-Leggett model. By tracing out the degrees of freedom of the thermal bath, the time-dependent reduced Wigner function of the system reads

$$W(q_t,p_t) = \frac{1}{(2\pi)^N}\int_{-\infty}^{+\infty}d\boldsymbol{q}_t\int_{-\infty}^{+\infty}d\boldsymbol{p}_t W_{\rm tot}(q_t,p_t,\boldsymbol{q}_t,\boldsymbol{p}_t), \tag{B1}$$

where $W_{\rm tot}(q_t,p_t,\boldsymbol{q}_t,\boldsymbol{p}_t)$ denotes the Wigner function of the total Hamiltonian, $\boldsymbol{q}_t = [q_1(t),q_2(t),...,q_N(t)]$ and $\boldsymbol{p}_t = [p_1(t),p_2(t),\ldots,p_N(t)]$ are the time-dependent coordinates and the momenta of the thermal bath, respectively. Due to the fact that the evolution of the total system is determined by the Liouville equation, we have $W_{\rm tot}(q_t,p_t,\boldsymbol{q}_t,\boldsymbol{p}_t) = W_{\rm tot}(q_0,p_0,\boldsymbol{q}_0,\boldsymbol{p}_0)$, which leads to

$$\begin{aligned} W(q_t,p_t) &= \frac{1}{(2\pi)^N}\int_{-\infty}^{+\infty}d\boldsymbol{q}_t\int_{-\infty}^{+\infty}d\boldsymbol{p}_t W_{\rm tot}(q_0,p_0,\boldsymbol{q}_0,\boldsymbol{p}_0)\\ &= \frac{1}{(2\pi)^N}\int_{-\infty}^{+\infty}d\boldsymbol{q}_t\int_{-\infty}^{+\infty}d\boldsymbol{p}_t W(q_0,p_0)\prod_{j=1}^N W_G[q_j(0),p_j(0)], \end{aligned} \tag{B2}$$

with

$$W_G[q_j(0),p_j(0)] = 2\tanh\left(\frac{\beta_b\omega_0}{2}\right)\exp\left\{-\frac{\tanh\left(\frac{1}{2}\beta_b\omega_0\right)}{\omega_0}[p_j(0)^2+\omega_0 q_j(0)^2]\right\} \tag{B3}$$

being the Wigner function of the Gibbs state of the jth environmental mode (see Appendix C for the derivation).

Next, we transform the integration variables from the time-dependent coordinates of the thermal bath $d\boldsymbol{q}_t d\boldsymbol{p}_t$ to the initial coordinates of the thermal bath $dq_0 dp_0$ with fixed $\{q_t, p_t\}$ as follows [39]

$$\begin{aligned} d\boldsymbol{q}_t d\boldsymbol{p}_t &= \frac{\partial(q_t, p_t, \boldsymbol{q}_t, \boldsymbol{p}_t)}{\partial(q_t, p_t, \boldsymbol{q}_0, \boldsymbol{p}_0)} d\boldsymbol{q}_0 d\boldsymbol{p}_0 \\ &= \frac{\partial(q_t, p_t, \boldsymbol{q}_t, \boldsymbol{p}_t)}{\partial(q_0, p_0, \boldsymbol{q}_0, \boldsymbol{p}_0)} \frac{\partial(q_0, p_0, \boldsymbol{q}_0, \boldsymbol{p}_0)}{\partial(q_t, p_t, \boldsymbol{q}_0, \boldsymbol{p}_0)} d\boldsymbol{q}_0 d\boldsymbol{p}_0 \\ &= \left[\frac{\partial(q_t, p_t, \boldsymbol{q}_0, \boldsymbol{p}_0)}{\partial(q_0, p_0, \boldsymbol{q}_0, \boldsymbol{p}_0)}\right]^{-1} d\boldsymbol{q}_0 d\boldsymbol{p}_0 \\ &= \left(\frac{\partial q_t}{\partial q_0}\frac{\partial p_t}{\partial p_0} - \frac{\partial q_t}{\partial p_0}\frac{\partial p_t}{\partial q_0}\right)^{-1} d\boldsymbol{q}_0 d\boldsymbol{p}_0. \end{aligned} \tag{B4}$$

The relation between $\{q_t, p_t\}$ and $\{q_0, p_0\}$ can be obtained from Eq. (19) and Eq. (20) by performing the Weyl-Wigner transformation, which transforms the operators $\{\hat{q}(t), \hat{p}(t), \hat{q}(0), \hat{p}(0), \hat{x}(t)\}$ to classical variables $\{q_t, p_t, q_0, p_0, x_t\}$ as

$$q_t = \dot{G}_t q_0 + G_t p_0 + x_t, \quad p_t = \ddot{G}_t q_0 + \dot{G}_t p_0 + \dot{x}_t. \tag{B5}$$

Together with the above equations, we finally have

$$\begin{aligned} W(q_t, p_t) = &\frac{1}{(2\pi)^N} \frac{1}{\dot{G}_t^2 - G_t\ddot{G}_t} \\ &\times \int_{-\infty}^{+\infty} d\boldsymbol{q}_0 \int_{-\infty}^{+\infty} dp_0 W(q_0, p_0) \\ &\times \prod_{j=1}^{N} W_G[q_j(0), p_j(0)], \end{aligned} \tag{B6}$$

where $G_t \equiv G(t)$. By introducing the Fourier transformation of $W(q_0, p_0)$

$$\begin{aligned} W(q_0, p_0) = &\frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} d\tilde{\boldsymbol{q}}_0 \\ &\times \int_{-\infty}^{+\infty} d\tilde{p}_0 \tilde{W}(\tilde{q}_0, \tilde{p}_0) e^{i(\tilde{q}_0 p_0 + \tilde{p}_0 q_0)}, \end{aligned} \tag{B7}$$

and using the relations

$$q_0 = \frac{\dot{G}_t(q_t - x_t) - G_t(p_t - \dot{x}_t)}{\dot{G}_t^2 - G_t\ddot{G}_t}, \tag{B8}$$

$$p_0 = \frac{-\ddot{G}_t(q_t - x_t) + \dot{G}_t(p_t - \dot{x}_t)}{\dot{G}_t^2 - G_t\ddot{G}_t}, \tag{B9}$$

one finds

$$\begin{aligned} W(q_t, p_t) = &\frac{1}{(2\pi)^2} \frac{1}{\dot{G}_t^2 - G_t\ddot{G}_t} \int_{-\infty}^{+\infty} d\tilde{q}_0 \\ &\times \int_{-\infty}^{+\infty} d\tilde{p}_0 \tilde{W}(\tilde{q}_0, \tilde{q}_0) \langle e^{i(\tilde{q}_0 p_0 + \tilde{p}_0 q_0)} \rangle. \end{aligned} \tag{B10}$$

Next, we define two new variables ξ and ζ as

$$\tilde{q}_0 = \dot{G}_t\xi + G_t\zeta, \quad \tilde{p}_0 = \ddot{G}_t\xi + \dot{G}_t\zeta, \tag{B11}$$

which leads to $d\tilde{q}_0 d\tilde{p}_0 = (\dot{G}_t^2 - G_t\ddot{G}_t) d\xi d\zeta$ and $\tilde{q}_0 p_0 + \tilde{p}_0 q_0 = \xi(p_t - \dot{x}_t) + \zeta(q_t - x_t)$, so that

$$\begin{aligned} W(q_t, p_t) &= \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} d\xi \int_{-\infty}^{+\infty} d\zeta \tilde{W}(\dot{G}_t\xi + G_t\zeta, \ddot{G}_t\xi + \dot{G}_t\zeta) e^{i(\xi p_t + \zeta q_t)} \langle e^{-i(\xi \dot{x}_t + \zeta x_t)} \rangle \\ &= \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} d\xi \int_{-\infty}^{+\infty} d\zeta \tilde{W}(\dot{G}_t\xi + G_t\zeta, \ddot{G}_t\xi + \dot{G}_t\zeta) e^{i(\xi p_t + \zeta q_t)} e^{-\frac{1}{2}(\langle \dot{x}_t^2 \rangle \xi^2 + \langle x_t^2 \rangle \zeta^2 + \langle x_t \dot{x}_t + \dot{x}_t x_t \rangle \xi\zeta)}, \end{aligned} \tag{B12}$$

where we have used the Gaussian property of x_t. Inserting the inverse Fourier transform of $\tilde{W}(\tilde{q}_0, \tilde{p}_0)$, namely

$$\tilde{W}(\tilde{q}_0, \tilde{p}_0) = \int_{-\infty}^{+\infty} dq_0 \int_{-\infty}^{+\infty} dp_0 W(q_0, p_0) e^{-i(\tilde{q}_0 p_0 + \tilde{p}_0 q_0)} \tag{B13}$$

into Eq. (B12), we have

$$\begin{aligned} W(q_t, p_t) = &\frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} dq_0 \int_{-\infty}^{+\infty} dp_0 W(q_0, p_0) \\ &\times \int_{-\infty}^{+\infty} d\xi \int_{-\infty}^{+\infty} d\zeta e^{-i[(\dot{G}_t\xi + G_t\zeta)p_t' + (\ddot{G}_t\xi + \dot{G}_t\zeta)q_t']} e^{i(\xi p_t + \zeta q_t)} e^{-\frac{1}{2}(\langle \dot{x}_t^2 \rangle \xi^2 + \langle x_t^2 \rangle \zeta^2 + \langle x_t \dot{x}_t + \dot{x}_t x_t \rangle \xi\zeta)} \\ = &\frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} dq_0 \int_{-\infty}^{+\infty} dp_0 W(q_0, p_0) \int_{-\infty}^{+\infty} d\boldsymbol{y} e^{-\frac{1}{2}\boldsymbol{y}\mathbb{A}_t\boldsymbol{y}^T + i\mathbb{R}_t\boldsymbol{y}}, \end{aligned} \tag{B14}$$

where $\boldsymbol{y} = (\xi, \zeta)^T$. Then, using the following Gaussian integral formula:

$$\int_{-\infty}^{+\infty} d\boldsymbol{y} e^{-\frac{1}{2}\boldsymbol{y}\mathbb{A}_t\boldsymbol{y}^T + i\mathbb{R}_t\boldsymbol{y}} = \sqrt{\frac{(2\pi)^{\mathrm{Dim}(\mathbb{A}_t)}}{\mathrm{Det}(\mathbb{A}_t)}} e^{-\frac{1}{2}\mathbb{R}_t\mathbb{A}_t^{-1}\mathbb{R}_t}, \tag{B15}$$

to work out the integrals over ξ and ζ, we finally arrive at

$$\begin{aligned} W(q_t, p_t) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} dq_0 \int_{-\infty}^{+\infty} dp_0 \frac{1}{\sqrt{\mathrm{Det}(\mathbb{A}_t)}} e^{-\frac{1}{2}\mathbb{R}_t \mathbb{A}_t^{-1}\mathbb{R}_t} W(q_0, p_0) \\ &= \int_{-\infty}^{+\infty} dq_0 \int_{-\infty}^{+\infty} dp_0 \mathcal{P}_t(q_t, p_t, q_0, p_0) W(q_0, p_0). \end{aligned} \tag{B16}$$

The above expression recovers Eq. (23) in the main text by using $\{q_t, p_t, q_0, p_0\}$ in place of $\{q, p, q', p'\}$.

APPENDIX C: THE WELY SYMBOL OF THE GIBBS STATE

In this Appendix, we show how to derive the Wely symbol of the Gibbs state. According to the definition of the Weyl symbol (Eq. (24) in the main text), one has

$$\begin{aligned} [\rho_s^G]_w(q,p) &= \int_{-\infty}^{\infty} dv \Big\langle q - \frac{1}{2}v \Big| \rho_s^G \Big| q + \frac{1}{2}v \Big\rangle e^{ipv} \\ &= \frac{1}{\mathcal{Z}_s(\beta_s)} \sum_{n=0}^{\infty} e^{-\beta_s\omega_0(n+\frac{1}{2})} \int_{-\infty}^{\infty} dv \Big\langle q - \frac{1}{2}v \Big| n \Big\rangle \Big\langle n \Big| q + \frac{1}{2}v \Big\rangle e^{ipv} \\ &= \frac{1}{\mathcal{Z}_s(\beta_s)} \sum_{n=0}^{\infty} e^{-\beta_s\omega_0(n+\frac{1}{2})} \int_{-\infty}^{\infty} dv u_n\Big(q - \frac{1}{2}v\Big) u_n\Big(q + \frac{1}{2}v\Big) e^{ipv}, \end{aligned} \tag{C1}$$

where $u_n(x)$ is the wave function of the Fock state in the position representation, i.e.,

$$u_n(x) = \left(\frac{\omega_0}{\pi}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n(\sqrt{\omega_0}x) e^{-\frac{1}{2}\omega_0 x^2}, \tag{C2}$$

with $H_n(x)$ being the Hermite polynomials. By introducing a new variable $\gamma = \sqrt{\omega_0}v$, Eq. (C1) can be reexpressed as

$$[\rho_s^G]_w(q,p) = \frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{1}{2^n n!} e^{-\beta_s\omega_0(n+\frac{1}{2}) - \omega_0 q^2} \int_{-\infty}^{\infty} d\gamma H_n\Big(\sqrt{\omega_0}q + \frac{1}{2}\gamma\Big) H_n\Big(\sqrt{\omega_0}q - \frac{1}{2}\gamma\Big) e^{-\frac{ip\gamma}{\sqrt{\omega_0}} - \frac{1}{4}\gamma^2}. \tag{C3}$$

Notice that

$$-\Big(\frac{\gamma}{2}\Big)^2 - \frac{ip\gamma}{\sqrt{\omega_0}} - \Big(\frac{ip}{\sqrt{\omega_0}}\Big)^2 - \Big(\frac{p}{\sqrt{\omega_0}}\Big)^2 = -\Big(\frac{\gamma}{2} + \frac{ip}{\sqrt{\omega_0}}\Big)^2 - \Big(\frac{p}{\sqrt{\omega_0}}\Big)^2, \tag{C4}$$

and by introducing another new variable $\lambda = \gamma/2 + ip/\sqrt{\omega_0}$, Eq. (C3) becomes

$$\begin{aligned} [\rho_s^G]_w(q,p) &= \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{1}{2^n n!} e^{-\beta_s\omega_0(n+\frac{1}{2}) - \omega_0 q^2 - \frac{1}{\omega_0}p^2} \int_{-\infty}^{\infty} d\lambda H_n\Big(\sqrt{\omega_0}q + \lambda - \frac{ip}{\sqrt{\omega_0}}\Big) H_n\Big(\sqrt{\omega_0}q - \lambda + \frac{ip}{\sqrt{\omega_0}}\Big) e^{-\lambda^2} \\ &= \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} e^{-\beta_s\omega_0(n+\frac{1}{2}) - \omega_0 q^2 - \frac{1}{\omega_0}p^2} \int_{-\infty}^{\infty} d\lambda H_n\Big(\lambda + \sqrt{\omega_0}q - \frac{ip}{\sqrt{\omega_0}}\Big) H_n\Big(\lambda - \sqrt{\omega_0}q - \frac{ip}{\sqrt{\omega_0}}\Big) e^{-\lambda^2}, \end{aligned} \tag{C5}$$

where we have used the symmetry relation of the Hermite polynomials $H_n(-x) = (-1)^n H_n(x)$.

Next, by using the following integral formula of the Hermite polynomials:

$$\frac{1}{\sqrt{\pi}2^n n!} \int_{-\infty}^{\infty} d\lambda H_n\Big(\lambda + \sqrt{\omega_0}q - \frac{ip}{\sqrt{\omega_0}}\Big) H_n\Big(\lambda - \sqrt{\omega_0}q - \frac{ip}{\sqrt{\omega_0}}\Big) e^{-\lambda^2} = L_n\Big[\frac{2}{\omega_0}(p^2 + \omega_0^2 q^2)\Big], \tag{C6}$$

where $L_n(x)$ denotes the nth Laguerre polynomial and the generating function of the Laguerre polynomials,

$$(1-\theta)\sum_{n=0}^{\infty} L_n(x)\theta^n = e^{x\theta/(\theta-1)}, \tag{C7}$$

we finally arrive at

$$[\rho_s^G]_w(q,p) = 2\tanh\Big(\frac{\beta_s\omega_0}{2}\Big) \exp\Big[-\frac{\tanh\big(\frac{1}{2}\beta_s\omega_0\big)}{\omega_0}(p^2 + \omega_0^2 q^2)\Big], \tag{C8}$$

which reproduces Eq. (32) in the main text.

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