Qubit reset with a shortcut-to-isothermal scheme

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(Received 9 November 2023; revised 21 March 2024; accepted 16 May 2024; published 13 June 2024)

Landauer's principle shows that the minimum energy cost to reset a classical bit in a bath with temperature T is $k_B T \ln 2$ in the infinite time. However, the task to reset the bit in finite time has posted a new challenge, especially for quantum bit (qubit) where both the operation time and controllability are limited. We design a shortcut-to-isothermal scheme to reset a qubit in finite time τ with limited controllability. The energy cost is minimized with the optimal control scheme with and without bound. This optimal control scheme can provide a reference to realize qubit reset with minimum energy cost for the limited time.

DOI: 10.1103/PhysRevE.109.064132

I. INTRODUCTION

Quantum information and quantum computation, a frontier interdisciplinary field of quantum mechanics and computer or information science, has been developing rapidly in recent years. One of its key goals is to realize the quantum computer to complete tasks that cannot be completed by traditional classical methods [1]. Quantum bit (qubit) is the basic unity for quantum computer to store, process, and transmit information [1]. Different from classical computers, the number of available qubits is typically limited due to the difficulties of producing a single qubit. In consideration of the difficulties and costs of producing qubits, resetting qubits for reuse is therefore an inevitable step [1–6] for the qubit-demanding tasks.

The process of bit resetting is to restore its state irreversibly to one particular state regardless of its initial state. Such an irreversibility process leads to unavoidable energy cost. Landauer derived the famous Landauer's Principle in 1961 [2], which states that a minimum bound of energy cost of $k_B T \ln 2$ is required to reset one bit in a heat bath with temperature T [2,7–11]. Such bound is reached only in an infinite-time process. However, we are typically limited by the available operation time in the quantum computation process, which should be completed within the coherence time of the qubit [1]. Such new scenario has posted a quest for the quantum generalization of the Landauer's principle for the finite-time reset processes [1,11–25].

One of the possible protocols is to shorten the reset time with the shortcut-to-isothermal scheme, where the system is driven with an auxiliary Hamiltonian H_{a} in addition to the original Hamiltonian H_{o} to evolve along the instantaneous equilibrium state of H_{o} within a finite-time process [26,27]. Such scheme has been applied into many fields to reduce the time to drive the system from one state to another [28–30], to control biological evolution [31,32], to construct finite-time engines [33–36], and to improve the accuracy of free-energy estimation [37]. Such finite-time process typically accompanies additional energy consumption due to irreversibility produced in the finite-time thermodynamical process. In the current application into qubit reset, the controllability of the quantum devices also limits the available auxiliary Hamiltonian H_{a} in the shortcut-to-isothermal scheme. Therefore, we consider the shortcut-to-isothermal reset process of qubits for the condition with and without the limits of the controllability, referred as bounded and unbounded control condition in the later discussion.

The rest of the current paper is organized as follows. In Sec. II, we introduce the concept of the qubit reset, design a shortcut-to-isothermal scheme on the qubit reset, and find the optimal control protocol to minimize the extra work in the unbounded control condition. In Sec. III, we obtain the optimal scheme for the bounded control condition, and show the cases where the experimental conditions are regarded as bounded or unbounded. We also show the existence of the inaccessible region where the desired reset state is not achievable due to experimental limitations. The extra work is also calculated for the unbounded control condition and compared with the bounded control condition. In Sec. IV, we conclude the paper with additional discussions.

II. MINIMUM ENERGY COST FOR UNBOUNDED CONTROL

For the quantum computing devices, qubit is the basic element whose two quantum states are encoded as the logical states zero and one, respectively. The two-state system is described by the Hamiltonian

$$H_{\mathrm{o}}[\lambda(t)] = \tfrac{1}{2}\lambda(t)\sigma_z, \tag{1}$$

where the excited state $|e\rangle$ represents the logical state one and the ground state $|g\rangle$ represents the logical state zero. $\sigma_z = |e\rangle\langle e| - |g\rangle\langle g|$ is the Pauli operator, and the energy difference $\lambda(t)$ between the two states is tuned by an outside agent under a given protocol to realize the reset process.

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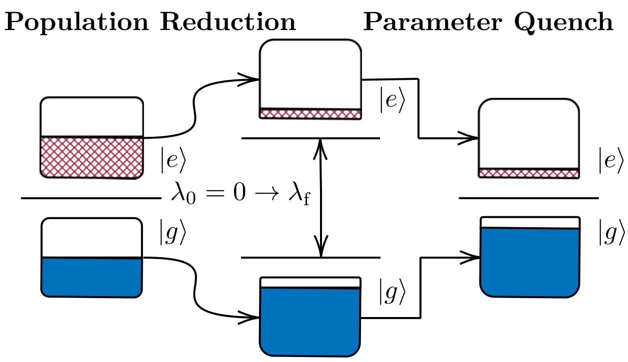

FIG. 1. Two steps to reset a qubit. In the first step, the energy difference is raised from $\lambda_0 = 0$ to $\lambda_{\rm f}$ within the heat bath at the temperature T to reduce the population of excited state $|e\rangle$ with the precision ϵ. In the second step, we reset the energy difference and keep the population unchanged.

The process to reset the qubit is to drive its state evolution to the ground state $|g\rangle$ regardless of its initial state. A straightforward scheme is to increase the energy difference to drive the major population to the ground state. Here, we denote $p_e(t)(p_g(t))$ as the population of the excited (ground) state. The goal of the reset process typically lies in two aspects. The first one is to reduce the population p_e on the excited state to a tolerable precision, i.e., $p_e(\tau) = \epsilon$ in the finite time τ. The second one is to reset the control parameter to its original value, i.e., $\lambda(\tau) = \lambda_0$, for subsequent operations. To achieve the goal above, we design a two-step scheme, illustrated in Fig. 1.

(i) *Population reduction with the shortcut-to-isothermal scheme.* Raise the energy difference $\lambda(t)$ from λ_0 to $\lambda_{\rm f}$ to reduce the population of the excited state $|e\rangle$. Such population reduction is done by the shortcut-to-isothermal scheme presented later. It is worth mentioning that our shortcut scheme ensures that the bit is reset to the logical zero with the required precision

$$\epsilon = \frac{e^{-\beta\lambda_{\rm f}}}{1 + e^{-\beta\lambda_{\rm f}}}. \tag{2}$$

(ii) *Parameter quench.* In this step, the energy difference of the system is reset to λ_0 and the population remains unchanged.

By the end of the two steps, not only is the population reset to the logical state zero, but also the system's parameter is reset to the initial value. In the normal control process, the evolution of the system typically has a lag, which prevents the qubit to reach the desired control precision. To overcome such lag, we introduce the shortcut to isothermal to escort the system evolution along the designed path. In the shortcut-to-isothermal scheme, an auxiliary Hamiltonian $H_{\rm a}$ is added into the original Hamiltonian to escort the system evolution as the instantaneous equilibrium states $\rho^{(1)}_{\rm sc,eq}[\lambda(t)] = \exp(-\beta H_{\rm o}(t))/\text{tr}(\exp(-\beta H_{\rm o}(t)))$ of the original Hamiltonian $H_{\rm o}$. Here $\beta = (k_B T)^{-1}$ is the inverse temperature with Boltzmann constant k_B. Normally, we ensure the vanish of the current auxiliary Hamiltonian $H_{\rm a}(0_-) = H_{\rm a}(\tau_+) = 0$ to remove the additional control, where $0_- = 0 - \delta$, $\tau_+ = \tau + \delta$, and δ is infinitesimal in the standard mathematical definition.

A straightforward auxiliary Hamiltonian for the qubit is

$$H_{\rm a}[\lambda_{\rm a}(t)] = \tfrac{1}{2}\lambda_{\rm a}(t)\sigma_z. \tag{3}$$

The total Hamiltonian is $H_{\rm tot} = 1/2\lambda_H(t)\sigma_z$, where $\lambda_H(t) = \lambda(t) + \lambda_{\rm a}(t)$ is an effective energy difference. The master equation for the two-level system is [38]

$$\begin{aligned}\frac{d}{dt}\rho(t) = &-\frac{\gamma}{2}(\sigma_-\sigma_+\rho(t) - 2\sigma_+\rho(t)\sigma_- + \rho(t)\sigma_-\sigma_+)n(\lambda_H)\\ &-\frac{\gamma}{2}(\sigma_+\sigma_-\rho(t) - 2\sigma_-\rho(t)\sigma_+ + \rho(t)\sigma_+\sigma_-)\\ &\times (n(\lambda_H) + 1),\end{aligned} \tag{4}$$

where $\sigma_+ = |e\rangle\langle g|$, $\sigma_- = |g\rangle\langle e|$ and γ is the decay rate. Here $n(\lambda_H) = (\exp(\beta\lambda_H) - 1)^{-1}$ is the average number for the thermal bath mode with the frequency λ_H. With the requirement of the instantaneous equilibrium state $\rho(t) = \rho^{(1)}_{\rm sc,eq}[\lambda(t)]$ in the short cut scheme, the evolution of the excited state population is given by the following equation:

$$\frac{dp_e}{dt} = \gamma\frac{e^{-\beta\lambda_H}(1 - p_e) - p_e}{1 - e^{-\beta\lambda_H}}. \tag{5}$$

To find the optimal control, we first calculate the energy cost for the two-step reset process. The energy cost in the first step $W^{(1)}_{\rm sc} = \int_0^\tau \text{tr}(\rho^{(1)}_{\rm sc,eq}\dot{H}_{\rm tot})dt$ [33,39–41] is obtained explicitly as

$$W^{(1)}_{\rm sc} = \frac{1}{\beta}J + \lambda_H(\tau)\left(\epsilon - \frac{1}{2}\right), \tag{6}$$

where $J = -\gamma\beta\int_0^\tau (e^{-\beta\lambda_H}(1 - p_e) - p_e)/(1 - e^{-\beta\lambda_H})\lambda_H dt$. In the second step, the state of the system remains unchanged $\rho^{(2)}_{\rm qa} = \rho^{(1)}_{\rm sc,eq}[\lambda_{\rm f}]$. The energy cost is obtained as [13]

$$W^{(2)}_{\rm qa} = -\lambda_H(\tau)\left(\epsilon - \frac{1}{2}\right), \tag{7}$$

and the total energy cost in our shortcut-to-isothermal scheme $W_{\rm sc} = W^{(1)}_{\rm sc} + W^{(2)}_{\rm qa}$ is

$$W_{\rm sc} = \frac{1}{\beta}J. \tag{8}$$

The detailed derivations are in Appendix A.

The energy cost to reset the bit for a quasistatic process $W_{\rm qs}$ is the change of the free energy [13]

$$W_{\rm qs} = \Delta F = \frac{1}{\beta}(\ln 2 - S(\epsilon)), \tag{9}$$

where $S(\epsilon) = -\epsilon\ln\epsilon - (1-\epsilon)\ln(1-\epsilon)$ is the Shannon entropy of the final state. For ideal reset $\epsilon \to 0$, $S(\epsilon) \to 0$, the energy cost reached the Landauer's limit.

Here we aim to determine the extra energy cost $W_{\rm ex} = W_{\rm sc} - W_{\rm qs}$ to reset the qubit due to finite-time operation as

$$W_{\rm ex} = \frac{1}{\beta}(J - (\ln 2 - S(\epsilon))). \tag{10}$$

For the fixed reset precision ϵ, the task to minimize the extra work $W_{\rm ex}$ is converted to the question to find the minimum of

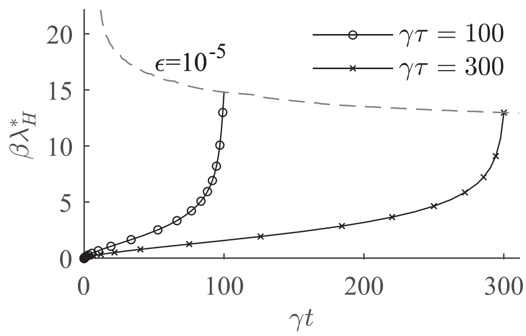

FIG. 2. The optimal control scheme λ_H^* (in units of β^{-1}) as the function of time t (in units of γ^{-1}) with different reset times τ (in units of γ^{-1}). The numerical simulation is performed by the shooting method and the boundary condition is $p_e(0) = 1/2$ and $p_e(\tau) = \epsilon$. Different markers represent different reset times $\gamma\tau = 100$ (circle) and 300 (cross) and the reset precision is set as $\epsilon = 10^{-5}$. The gray dashed lines show the final control amplitude $\lambda_H(\tau)$. It is clear to observe that for the optimal control scheme, the control parameter $\lambda_H^*(t)$ monotonically increases, and for fixed reset precision larger control amplitude $\lambda_H(\tau)$ is needed for shorter reset time τ.

the objective function J with constraints as follows:

$$\text{equation of motion:} \quad \frac{dp_e}{dt} = \gamma \frac{e^{-\beta\lambda_H}(1-p_e) - p_e}{1 - e^{-\beta\lambda_H}},$$

$$\text{objective function:} \quad J[p_e(t); \lambda_H(t)] = -\gamma\beta \int_0^\tau \frac{e^{-\beta\lambda_H}(1-p_e) - p_e}{1 - e^{-\beta\lambda_H}} \lambda_H dt,$$

$$\text{boundary conditions:} \quad \begin{cases} p_e(0) = \frac{1}{2} \\ p_e(\tau) = \epsilon. \end{cases}$$

To find such minimum, we introduce an effective Lagrange $L(p_e; \lambda_H) = -\gamma\beta(e^{-\beta\lambda_H}(1-p_e) - p_e)/(1 - e^{-\beta\lambda_H})\lambda_H$. The cost function is rewritten as $J = \int_0^\tau L dt$, and the minimum is obtained by solving the Euler-Lagrange equation $\partial L/\partial p_e - d(\partial L/\partial \dot{p}_e)/dt = 0$, which yields an ordinary differential equation for $p_e(t)$ as follows:

$$\ddot{p}_e = \frac{\gamma^2(1 - 2p_e + 2p_e^2)\dot{p}_e^2 + 2\gamma\dot{p}_e^3 + 2\dot{p}_e^4}{\gamma(1 - 2p_e)(2\gamma p_e(1-p_e) + \dot{p}_e)}. \quad (11)$$

Here, we have used the equation of motion in Eq. (5) to replace the parameter λ_H. The solution of the above equation is denoted as $p_e^*(t)$. Substituting $p_e^*(t)$ back into the equation of motion, we can get the corresponding control scheme $\lambda_H^*(t)$.

In Fig. 2, we show the optimal control scheme $\lambda_H^*(t)$ with different reset time τ. The numerical simulation is performed by solving Eq. (11) with the shooting method [27] for different reset times $\gamma\tau = 100$ and 300 under the reset precision $\epsilon = 10^{-5}$. The boundary conditions are $p_e(0) = 1/2$ and $p(\tau) = \epsilon$. The gray dashed lines show the final control amplitude $\lambda_H(\tau)$.

With these curves, we observe two facts as follows:

(i) For the optimal control scheme, the control parameter $\lambda_H^*(t)$ increases monotonically with time t.

(ii) For fixed reset precision, larger control amplitude $\lambda_H(\tau)$ is needed for shorter reset time τ.

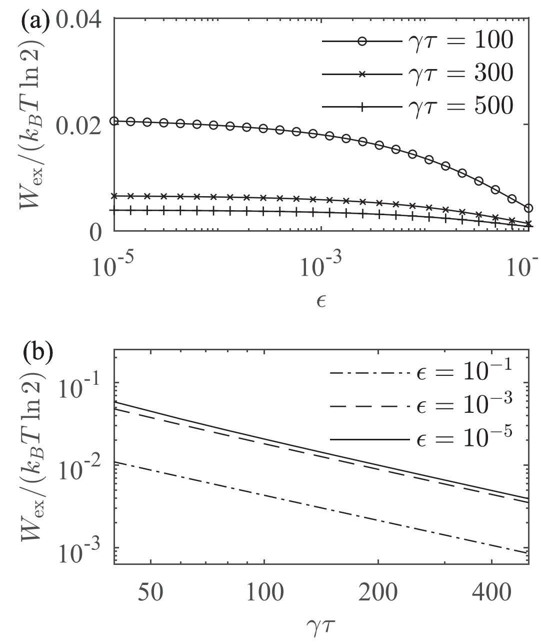

FIG. 3. (a) The minimum extra work $W_{\rm ex}$ (in units of $\beta^{-1}\ln 2$) vs the reset precision ϵ in linear-log plot. Different colors and markers represent different reset times $\gamma\tau = 100$ (circle), 300 (cross), and 500 (plus sign), respectively. We can see that the lower reset precision we desire, the higher extra work we need to down. And when the reset precision $\epsilon \to 0$, the extra work approaches a constant. (b) The minimum extra work $W_{\rm ex}$ (in units of $\beta^{-1}\ln 2$) vs the reset time τ (in units of γ^{-1}) in the log-log plot. A line with a slope of -1, shows the inverse relation between the minimum extra work $W_{\rm ex}$ and the reset time τ. Different colors and markers represent different reset precisions $\epsilon = 10^{-1}$ (dashdot), 10^{-3} (dashed), and 10^{-5} (solid), respectively.

With the optimal control scheme λ_H^*, we calculate and show the extra work $W_{\rm ex}$ as functions of the reset precision ϵ in Fig. 3(a) and the control time τ in Fig. 3(b). In Fig. 3(a), different markers represent difference reset times $\gamma\tau = 100$ (circle), 300 (cross), and 500 (plus sign), respectively. For a fixed reset time, the lower the reset precision we desire, the larger the extra work is needed. In Fig. 3(b), different markers represent different reset precisions $\epsilon = 10^{-1}$ (dashdot), 10^{-3} (dashed), and 10^{-5} (solid), respectively. The curves show that the extra work is inversely proportional to reset time τ for fixed reset precision. The choice of parameters ensures the visible difference in the reset process.

Noticing the second fact, we find the control parameter λ_H may reach the maximum value of the bounded control condition, i.e., $\lambda_H \in [0, \lambda_m]$ in some situations, which is discussed in the next section.

III. MINIMUM ENERGY COST FOR BOUNDED CONTROL

In this section, we consider the energy minimization in the bounded control condition $\lambda_H \in [0, \lambda_m]$, which is determined by the detailed condition of the experimental setup. In some of the experimental setup, the energy difference λ_H cannot be raised to infinity. For example, considering a

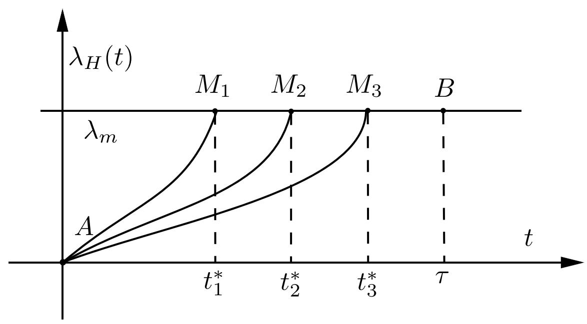

FIG. 4. The optimization of the touch time t^*. All the curves AM_1, AM_2, and AM_3 are the solution of Euler-Lagrange equation $\lambda_H^{(\mathrm{opt})}(t) = \lambda_H^*(t)$ and all the lines M_1B, M_2B, and M_3B are on the boundary $\lambda_H^{(\mathrm{opt})}(t) = \lambda_m$. Different points M_1, M_2, and M_3 represent different touch times t_1^*, t_2^*, and t_3^*, respectively. It is proved that the touch time t^* is optimized by the continuity of $\lambda_H^{(\mathrm{opt})}$ and $p_e^{(\mathrm{opt})}$in Appendix D.

transmon superconducting qubit in an LC resonant circuit with a nonlinear Josephson junction, the energy difference is controlled by changing the Josephson energy E_J, the capacitive energy E_C, and the inductive energy E_L [42–49]. Naturally, the technological limits of the Josephson junction, capacitor, and inductor, such as materials choice, junction area of the Josephson junction, and insulator thickness, limit the energy difference [44,46]. A specific example of energy difference control is to replace the single Josephson junction with the so-called dc-SQUID loop. "SQUID" stands for superconducting quantum interference device and a dc-SQUID loop is a superconducting loop with two junctions in parallel. By applying a magnetic field on the dc-SQUID loop, the critical current I_C of the Josephson junction can be controlled. In other words, the Josephson energy E_J can be tuned by the magnetic flux on the dc-SQUID, since for the Josephson junction we have $E_J = \Phi_0 I_C/(2\pi)$ where $\Phi_0 = 2\pi\hbar/(2e)$ is the magnetic flux quantum [43,50]. In this example, the limit of magnetic flux on dc-SQUID limits the energy difference. Under such constraint, the typical experimental parameter is

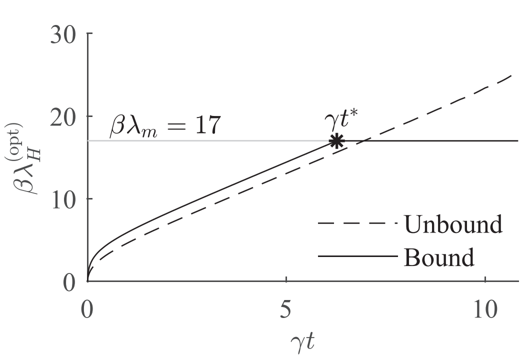

FIG. 5. The optimal control scheme with (solid line) and without (dashed line) bound as the function of time t. The gray line is the bound $\beta\lambda_m = 17$. Reset precision is set to $\epsilon = 10^{-5}$. In this situation, $\gamma\tau_{c1} = 10.8239$ and $\gamma\tau_{c2} = 38.2281$. The reset time is set as $\gamma\tau = 10.8321$. At the bound control condition, the optimal scheme stays on the boundary once it touches the boundary and stays on it after the touch time $t^* = 6.2715/\gamma$.

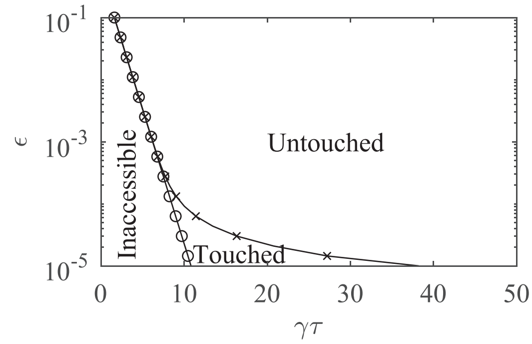

FIG. 6. Diagram of cases for the bounded control problem. The circle and the cross lines represent the boundary lines τ_{c1} and τ_{c2} divided three cases. The boundary is set to $\beta\lambda_m = 17$. The reset tasks which is represented by the point in inaccessible region cannot be accomplished. The untouched region means that those reset tasks can be accomplished and do not touch the boundary. The touched region represents those reset tasks which can be accomplished but touch the boundary.

$\lambda_m \sim 2\pi \times 10$ GHz or $\beta\lambda_m \sim 5$ for superconducting qubit's typical working temperature $T = 10$mK [46].

To simplify the discussion for the bounded control condition, we first introduce a proposition with its proof presented in Appendix B.

Proposition. For the optimal reset scheme, if λ_H touches the upper boundary λ_m at the touch time t^*, then λ_H will remain λ_m for later time $t^* < t < \tau$ in the control protocol.

With this proposition, there are two critical reset times τ_{c1} and τ_{c2}, by which the reset time τ is categorized into three cases as inaccessible, untouched, and touched for given reset precision ϵ as in Table I [51]. The first reset time τ_{c1} is determined by the extreme optimal control $\lambda_H(t) = \lambda_m$, and the second reset time τ_{c2} is determined by the boundary condition $\lambda_H(\tau) = \lambda_m$. Detailed derivations are presented in Appendix C. The three cases are described as follows.

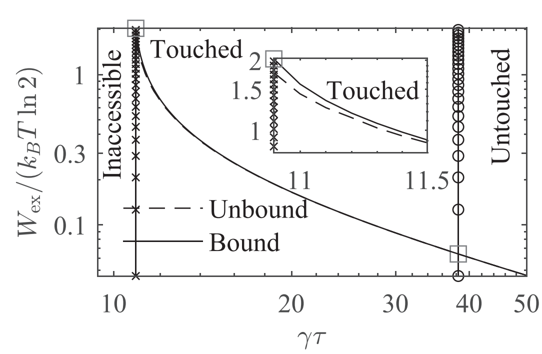

FIG. 7. Minimum extra work done in the reset process as the function of reset time τ in the log-log plot with (solid line) and without (dashed line) bound. The parameters are the same as Fig. 5. The circle and the cross lines represent τ_{c1} and τ_{c2}, respectively. In the untouched region, the solid and the dashed lines are coincident. In the touched region, the bound minimum extra work is larger than the unbound minimum extra work with a deviation from the inverse relation. Their difference increases with the decrease of the reset time.

TABLE I. Reset time τ is categorized into three cases for given reset precision ϵ with two critical reset times τ_{c1} and τ_{c2}. $\lambda_H^*(t)$ is the solution of Eq. (11). The calculations of τ_{c1} and τ_{c2} are presented in Appendix C. The touch time t^* is discussed in Fig. 4.

Case	Reset time τ	Reset process
Inaccessible	$\tau < \tau_{c1}$	not accomplish
Touched	$\tau_{c1} < \tau < \tau_{c2}$	$\lambda_H^{(\text{opt})}(t) = \begin{cases} \lambda_H^*(t) & t < t^* \\ \lambda_m & t > t^* \end{cases}$
Untouched	$\tau > \tau_{c2}$	$\lambda_H^{(\text{opt})}(t) = \lambda_H^*(t)$

(i) *Inaccessible*, where $\tau < \tau_{c1}$. In the inaccessible case, one cannot drive the system to the final state with the precision ϵ with any possible reset process, so the reset task is not accomplished when $\tau < \tau_{c1}$.

(ii) *Touched*, where $\tau_{c1} < \tau < \tau_{c2}$. In the touched case, the energy difference λ_H reaches the boundary λ_m at touch time t^*, which is an additional parameter variable to be optimized as illustrated in Fig. 4. Before the touch time, the optimal reset scheme follows the solution of Eq. (11) with $\lambda_H^{(\text{opt})}(t) = \lambda_H^*(t)$. After the touch time, the optimal reset scheme is on the boundary with $\lambda_H^{(\text{opt})}(t) = \lambda_m$. It is proved that the touch time t^* is optimized by the continuity of $\lambda_H^{(\text{opt})}$ and $p_e^{(\text{opt})}$ at the touch time t^* in Appendix D.

(iii) *Untouched*, where $\tau > \tau_{c2}$. In the untouched case, the reset process is the same with the unbounded control condition with $\lambda_H^{(\text{opt})}(t) = \lambda_H^*(t)$.

Figure 5 shows the difference in optimal control schemes with and without bound in the touched case. The parameters are set as $\beta\lambda_m = 17$ and $\epsilon = 10^{-5}$. In this situation, we obtain the two critical times as $\gamma\tau_{c1} = 10.8239$ and $\gamma\tau_{c2} = 38.2281$. We show the touched case with $\tau = 10.8321/\gamma$. The solid line and dashed line represent the optimal control scheme with and without bound, respectively. For the bounded control, the optimal scheme (the solid line) $\lambda_H^{(\text{opt})}(t)$ touches the bound (the gray line) and stays on it after the touch time $t^* = 6.2715/\gamma$.

Figure 6 shows the diagram of cases for the bounded condition with $\beta\lambda_m = 17$ on the plane of $[\tau, \epsilon]$. The circle and cross lines represent the boundary lines τ_{c1} and τ_{c2} divided the three cases. The choice of parameters is based on the experimental conditions of superconducting qubits. With the choice of these parameters, the difference between unbound and bound conditions is clearly demonstrated. The extra work is calculated via equation Eq. (10). For the touched case, the objective function is calculated in the two time intervals $[0, t^*]$ and $[t^*, \tau]$ as

$$J = J^{(1)} + J^{(2)}. \tag{12}$$

In the time interval $[0, t^*]$, the optimal control scheme is $\lambda_H^{(\text{opt})}(t) = \lambda_H^*(t)$, $p_e^{(\text{opt})}(t) = p_e^*(t)$, and the function is explicitly obtained as

$$J^{(1)} = -\beta\gamma \int_0^{t^*} \frac{\mathrm{e}^{-\beta\lambda_H^*}(1 - p_e^*) - p_e^*}{1 - \mathrm{e}^{-\beta\lambda_H^*}} \lambda_H^* dt. \tag{13}$$

In the time interval $[t^*, \tau]$, the optimal control scheme is $\lambda_H^{(\text{opt})}(t) = \lambda_m$. Substituting it into the motion equation, we get

$$p_e^{(\text{opt})}(t) = \frac{n(\lambda_m)}{2n(\lambda_m) + 1} + \left(\epsilon - \frac{n(\lambda_m)}{2n(\lambda_m) + 1}\right) \mathrm{e}^{-\gamma(2n(\lambda_m)+1)(t-\tau)}, \tag{14}$$

which is calculated in Appendix D. And $J^{(2)}$ is computed analytically

$$J^{(2)} = -\beta\gamma \int_{t^*}^{\tau} \frac{\mathrm{e}^{-\beta\lambda_m}\left(1 - p_e^{(\text{opt})}\right) - p_e^{(\text{opt})}}{1 - \mathrm{e}^{-\beta\lambda_m}} \lambda_m dt. \tag{15}$$

For the untouched case, the extra work is the same as that in the unbound control condition.

Figure 7 shows the extra work with and without bound. In the untouched case, the energy cost in the bound control condition is the same as in the unbound control condition. In the touched case, $W_{\text{ex,b}}$ is larger than $W_{\text{ex,ub}}$ with a deviation from the inverse relation, and their difference increases with the decrease of the reset time. In the inaccessible case, the reset task cannot be accomplished.

IV. CONCLUSION

In this paper, we design a finite-time reset scheme based on the shortcut-to-isothermal approach, and find the optimal control scheme for the minimum extra energy cost with and without bound $\lambda_H \in [0, \lambda_m]$. The scheme is a two-step scheme including population reduction and parameter quench. We find out the optimal reset scheme $\lambda_H^*(t)$ and the extra energy cost W_{ex} as the function of the reset precision ϵ and the reset time τ. The extra energy cost W_{ex} follows the inverse proportional relationship as $W_{\text{ex}} \propto 1/\tau$.

The bounds on the controllability make a difference to the problem of minimizing the extra work. In this condition, the system has three possible cases: inaccessible, untouched, and touched. We show the existence of the first critical reset time τ_{c1} and the second critical reset time τ_{c2}. When $\tau < \tau_{c1}$, the reset task is inaccessible. When $\tau > \tau_{c2}$, the reset task can be accomplished without being any different from the unbound control condition. When $\tau_{c1} < \tau < \tau_{c2}$, the reset task can be accomplished but the optimal reset scheme $\lambda_H^{(\text{opt})}$ is not the same as $\lambda_H^*(t)$. There exists an additional parameter touch time t^* to be optimized, dividing the optional reset scheme $\lambda_H^{(\text{opt})}$ into two stages. The first stage is $\lambda_H^{(\text{opt})}(t) = \lambda_H^*(t)$, while $t < t^*$. The second stage is the boundary value $\lambda_H^{(\text{opt})}(t) = \lambda_m$, while $t > t^*$.

ACKNOWLEDGMENTS

This work is supported by the National Natural Science Foundation of China (NSFC) (Grants No. 12088101, No. U2230203, and No. U2230401).

APPENDIX A: DETAILED DERIVATIONS OF THE ENERGY COST

In the first step, the system is on the instantaneous equilibrium states of the original Hamiltonian H_o as

$$\rho^{(1)}_{\mathrm{sc,eq}}[\lambda(t)] = \begin{pmatrix} \frac{e^{-\beta\lambda(t)}}{1+e^{-\beta\lambda(t)}} & 0 \\ 0 & \frac{1}{1+e^{-\beta\lambda(t)}} \end{pmatrix}, \tag{A1}$$

which is the basic requirement of the shortcut scheme. The energy cost in the first step is

$$\begin{aligned} W^{(1)}_{\mathrm{sc}} &= \int_0^\tau \mathrm{tr}\big(\rho^{(1)}_{\mathrm{sc,eq}}\dot{H}_{\mathrm{tot}}\big)dt \qquad \text{(A2)} \\ &= \int_0^\tau \frac{1}{2}\dot{\lambda}_H(2p_e-1)dt \\ &= \frac{1}{2}\lambda_H(2p_e-1)\Big|_{t=0}^{t=\tau} - \int_0^\tau \frac{1}{2}\lambda_H d2p_e - 1 \\ &= \lambda_H(\tau)\Big(\epsilon-\frac{1}{2}\Big) - \gamma\int_0^\tau \frac{e^{-\beta\lambda_H}(1-p_e)-p_e}{1-e^{-\beta\lambda_H}}\lambda_H dt \\ &= \frac{1}{\beta}J + \lambda_H(\tau)\Big(\epsilon-\frac{1}{2}\Big), \qquad \text{(A3)} \end{aligned}$$

where $J = -\gamma\beta\int_0^\tau(e^{-\beta\lambda_H}(1-p_e)-p_e)/(1-e^{-\beta\lambda_H})\lambda_H dt$. This is Eq. (6) in the main context. The first term J/β is the driving energy and the second term $\lambda_H(\tau)(\epsilon-1/2)$ is the change of free energy. The first term vanishes ($J\to 0$) when the reset time approaches to infinity $\tau\to+\infty$.

In the second step, the state of the system remains unchanged

$$\rho^{(2)}_{\mathrm{qa}} = \begin{pmatrix} \frac{e^{-\beta\lambda_f}}{1+e^{-\beta\lambda_f}} & 0 \\ 0 & \frac{1}{1+e^{-\beta\lambda_f}} \end{pmatrix}. \tag{A4}$$

The energy cost in the second step is

$$\begin{aligned} W^{(2)}_{\mathrm{qa}} &= \frac{1}{2}\frac{e^{-\beta\lambda_f}-1}{1+e^{-\beta\lambda_f}}\int_{\lambda_H(\tau)}^0 d\lambda \qquad \text{(A5)} \\ &= -\frac{\lambda_H(\tau)}{2}\frac{e^{-\beta\lambda_f}-1}{1+e^{-\beta\lambda_f}} \\ &= -\lambda_H(\tau)\Big(\epsilon-\frac{1}{2}\Big), \qquad \text{(A6)} \end{aligned}$$

where $\varepsilon = e^{-\beta\lambda_f}/(1+e^{-\beta\lambda_f})$ is the reset precision. This is Eq. (7) in the main context. It corresponds to the change of free energy. For the total energy cost $W_{\mathrm{sc}} = W^{(1)}_{\mathrm{sc}} + W^{(2)}_{\mathrm{qa}}$, the change in free energy of the system in Eq. (6) and Eq. (7) cancel each other and only the driving energy J/β remains as

$$W_{\mathrm{sc}} = \frac{1}{\beta}J. \tag{A7}$$

APPENDIX B: THE PROOF OF THE PROPOSITION

By rewriting the motion equation as the equation of λ_H, we get

$$\lambda_H = -\beta\ln\frac{\dot{p}_e+\gamma p_e}{\dot{p}_e+\gamma(1-p_e)}. \tag{B1}$$

By differentiating Eq. (B1), we get

$$\frac{d\lambda_H}{dt} = \frac{2(1-p_e(1+e^{-\beta\lambda_H}))(p_e-e^{-\beta\lambda_H}(1-p_e))}{\beta(1-2p_e)(p_e(1-e^{-\beta\lambda_H})+e^{-\beta\lambda_H})(1-e^{-\beta\lambda_H})}. \tag{B2}$$

Noticing $0<e^{-\beta\lambda_H}<1$, $0<p_e<1/2$, we observe that the following factors in the right side of Eq. (B2) are lager than zero

(i) $(1-p_e(1+e^{-\beta\lambda_H}))>0$,
(ii) $(1-2p_e)>0$,
(iii) $(p_e(1-e^{-\beta\lambda_H})+e^{-\beta\lambda_H})>0$,
(iv) $(1-e^{-\beta\lambda_H})>0$.

And with the factor $(p_e-e^{-\beta\lambda_H}(1-p_e))>0$ noticing $\lambda_H>\lambda$, we prove that $\lambda^*_H(t)$ is a monotonically increasing function with $d\lambda_H/dt>0$.

If λ^*_H touches the upper boundary λ_m at the time t^*, it stays larger than λ_m in the interval $[t^*,\tau]$. Thus, for the optional reset scheme, if the optimal control $\lambda^{(\mathrm{opt})}_H$ touches the upper boundary λ_m at touch time t^*, λ_H stay on it for all times $t>t^*$.

APPENDIX C: THE TWO CRITICAL RESET TIMES

The extreme optimal control is $\lambda_H(t)=\lambda_m$ for the whole control process $t\in[0,\tau_{c1}]$ with the touch time $t^*=0$. With this control scheme, the motion equation becomes

$$\frac{dp_e}{dt} = \gamma(-(2n(\lambda_m)+1)p_e+n(\lambda_m)), \tag{C1}$$

whose solution is

$$p_e(t) = \frac{n(\lambda_m)+\frac{1}{2}e^{-\gamma(2n(\lambda_m)+1)t}}{2n(\lambda_m)+1} \stackrel{\mathrm{def}}{=} \phi(t). \tag{C2}$$

The first critical reset time is obtained by setting $p_e(\tau_{c1})=\epsilon$ as

$$\tau_{c1} = \frac{1}{\gamma(2n(\lambda_m)+1)}\ln\frac{1}{2(\epsilon(2n(\lambda_m)+1)-n(\lambda_m))}. \tag{C3}$$

For any $\tau<\tau_{c1}$, the reset task with precision ϵ cannot be accomplished in the reset time τ.

According to the proposition in the main content, the condition for λ^*_H not to touch the boundary is $\lambda^*_H(\tau)<\lambda_m$. The critical condition is that λ^*_H touches λ_m at the end of the control process $t=\tau$. And the second critical reset time τ_{c2} is given by $\lambda^*_H(\tau_{c2})=\lambda_m$.

APPENDIX D: USING CONTINUITY CONDITION TO CALCULATE TOUCH TIME t^*

In Fig. 4, all the curves AM_1, AM_2, and AM_3 are the solution of Euler-Lagrange equation $\lambda^{(\mathrm{opt})}_H(t)=\lambda^*_H(t)$ and all the lines M_1B, M_2B, and M_3B are on the boundary $\lambda^{(\mathrm{opt})}_H(t)=\lambda^*_H(t)$. Different points M_1, M_2, and M_3 represent different touch times t^*_1, t^*_2, and t^*_3, respectively. In this section, we derive the condition for the optimal control in the touched case.

We show that the $p^{(\mathrm{opt})}_e(t)$ and $\lambda^{(\mathrm{opt})}_H(t)$ is continuous at the optimal touch time t^*. The continuity of $p^{(\mathrm{opt})}_e(t)$ is natural because of the physical reason that the population is continuous. The motion equation Eq. (B1) ensures the continuity of the control scheme $\lambda_H^{(\mathrm{opt})}$ once $\dot{p}_e^{(\mathrm{opt})}$ is continuous in the whole process. The continuity of $\dot{p}_e^{(\mathrm{opt})}$ is proved as the result of the so-called one-sided variational problem as follows.

We write the objective function into two parts with respect to the touch time t^*:

$$J[p_e(t)] = \int_0^{t_-^*} L(p_e, \dot{p}_e)dt + \int_{t_+^*}^{\tau} L(\phi, \dot{\phi})dt. \tag{D1}$$

In the second part, $p_e(t)$ is replaced with the defined function in Eq. (C2) as $p_e(t) = \phi(t)$ to avoid the misunderstanding. The variation of the functional Eq. (D1) is obtained as

$$\delta J = L(p_e, \dot{p}_e)|_{t_-^*}\delta t^* + \int_0^{t_-^*}\left(\frac{\partial L}{\partial p_e}\delta p_e + \frac{\partial L}{\partial \dot{p}_e}\delta \dot{p}_e\right)dt - L(\phi, \dot{\phi})|_{t_+^*}\delta t^*. \tag{D2}$$

With the integration by parts, we get

$$\delta J = (L(p_e, \dot{p}_e)|_{t_-^*} - L(\phi, \dot{\phi})|_{t_+^*})\delta t^* + \frac{\partial L}{\partial \dot{p}_e}\delta p_e\bigg|_{t_-^*} + \int_0^{t_-^*}\left(\frac{\partial L}{\partial p_e} - \frac{d}{dt}\frac{\partial L}{\partial \dot{p}_e}\right)\delta p_e dt. \tag{D3}$$

The continuous condition $p_e(t_-^*) = \phi(t_-^*)$ results in the $\delta p_e(t_-^*) = \dot{\phi}|_{t_-^*}\delta t^*$. Noticing $\delta p_e(t_-^*) = \delta p_e|_{t_-^*} + \dot{p}_e|_{t_-^*}\delta t^*$, we get $\delta p_e|_{t_-^*} = (\dot{\phi} - \dot{p}_e)|_{t_-^*}\delta t^*$. And the variation is simplified as

$$\delta J = \int_0^{t_-^*}\left(\frac{\partial L}{\partial p_e} - \frac{d}{dt}\frac{\partial L}{\partial \dot{p}_e}\right)\delta p_e dt \tag{D4}$$

$$+\left(\left(L(p_e, \dot{p}_e) + (\dot{\phi} - \dot{p}_e)\frac{\partial L}{\partial \dot{p}_e}\right)\bigg|_{t_-^*} - L(\phi, \dot{\phi})|_{t_+^*}\right)\delta t^*. \tag{D5}$$

In the time interval $[0, t_-^*]$, the optimal control scheme ensures $p_e(t) = p_e^*(t)$ with the the Euler-Lagrange equation from Eq. (D4):

$$\frac{\partial L}{\partial p_e} - \frac{d}{dt}\frac{\partial L}{\partial \dot{p}_e} = 0. \tag{D6}$$

At the time point t^*, we get the so-called transversality condition from Eq. (D5) to connect the two parts and point out the optional t^*:

$$-L(\phi(t^*), \dot{\phi}(t^*)) + L(p_e^*(t^*), \dot{p}_e^*(t^*)) + (\dot{\phi}(t^*) - \dot{p}_e^*(t^*))\frac{\partial L}{\partial \dot{p}_e}(\dot{p}_e^*(t^*), \dot{p}_e^*(t^*)) = 0. \tag{D7}$$

With the mean value theorem on the first two terms, we get

$$(\dot{\phi}(t^*) - \dot{p}_e^*(t^*))\left(\frac{\partial L}{\partial \dot{p}_e}(p_e^*(t^*), k) - \frac{\partial L}{\partial \dot{p}_e}(p_e^*(t^*), \dot{p}_e^*(t^*))\right) = 0, \tag{D8}$$

where k is a value satisfying the condition $\min(\dot{p}_e^*(t^*), \dot{\phi}(t^*)) < k < \max(\dot{p}_e^*(t^*), \dot{\phi}(t^*))$. Using the mean value theorem again, we get

$$(\dot{\phi}(t^*) - \dot{p}_e^*(t^*))(k - \dot{p}_e^*(t^*))\frac{\partial^2 L}{\partial \dot{p}_e^2}(p_e^*(t^*), l) = 0, \tag{D9}$$

where l is a value satisfying the condition $\min(k, \dot{p}_e^*(t^*)) < k < \max(k, \dot{p}_e^*(t^*))$. It is clear that

$$\dot{\phi}(t^*) = \dot{p}_e^*(t^*). \tag{D10}$$

Therefore, $\lambda_H^{(\mathrm{opt})}$ is also continuous $\lambda_H^*(t^*) = \lambda_m$ because of the motion equation Eq. (B1).

Explicitly, we get the optimal control as

$$p_e^{(\mathrm{opt})}(t) = \begin{cases} p_e^*(t) & 0 < t < t^*, \\ \phi(t) & t^* < t < \tau, \end{cases} \tag{D11}$$

and the optional t^* is obtained with the above condition $\lambda_H^*(t^*) = \lambda_m$.

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