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Dynamic redundancy as a mechanism to optimize collective random searches

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We explore the case of a group of random walkers looking for a target randomly located in space, such that the number of walkers is not constant but new ones can join the search, or those that are active can abandon it, with constant rates r_b and r_d, respectively. Exact analytical solutions are provided both for the fastest-first-passage time and for the collective time cost required to reach the target, for the exemplifying case of Brownian walkers with $r_d = 0$. We prove that even for such a simple situation there exists an optimal rate r_b at which walkers should join the search to minimize the collective search costs. We discuss how these results open a new line to understand the optimal regulation in searches conducted through multiparticle random walks, e.g., in chemical or biological processes.

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I. INTRODUCTION

In recent years, there has been an increasing interest about the so-called *redundancy principle* and its role in biochemical and biological processes [1–10]. This principle is based on the notion that *"many copies of a single object such as molecules, cells, etc... is not a waste, but it has a specific function in living systems"* [11]. This has been seen as a facilitation mechanism to activation rates at the cellular level through the presence and creation of a large (redundant) number of activators. The corresponding statistical properties of the activation process will be then governed by the fastest activator to reach the receptor, providing an attractive link to extreme value statistics [11–17].

There is actually a rich literature on the problem of the fastest-first-passage time (FFPT) of a group of N random particles to a target located at a distance $\mathbf{x}_0$ from their initial position (see, e.g., Refs. [18–36]). For Brownian particles in one-dimensional (1D) infinite domains, for example, it is known that the mean fastest-first-passage time (MFFPT) becomes finite for $N \geqslant 3$ [24] (while the classical case $N = 1$ yields a divergent mean first-passage time), and its distribution has a leading term $\langle T \rangle = \mathbf{x}_0^2/(4D \ln N)$ for $N \to \infty$ [19,23], with D the diffusion coefficient. In two and higher dimensions, instead, the MFFPT remains infinite for any value of N [18,24]. While some approximations have been proposed for the corresponding FFPT distribution [23,37,38], exact results are extremely difficult to obtain.

The fact that redundant walkers/trajectories can modify the first-passage statistics of a process is reminiscent of the case of random walks with stochastic resetting and similar processes [39–43]. There, the possibility to restart the trajectory anew gives the walker the option to rectify strong departures from the target location. This has been applied to different fields including molecular biophysics [44,45], stochastic thermodynamics [46,47], Brownian gases [48], quantum computing [49,50], optimal control [51], or photon dynamics [52], to name a few. Likewise, stochastic resetting has also been studied for the multiparticle case (the corresponding survival probability was actually explored in the seminal work in Ref. [39]), but most previous approaches have been limited to simplified or approximated results (see Refs. [41,53] and references therein). Also, a particular case of interest is that given in Ref. [54], where a process of random searchers disappearing and being replaced by new ones (so N is kept constant) was explored both analytically and numerically, reaching a formal expression for the mean search time given in terms of the probability density function of the searcher position.

In this article we explore an extended version of this problem through the notion of *dynamic redundancy*, which denotes the case where the number of searchers N is not fixed, so searchers can join/abandon the foraging process. This is inspired in the case of eusocial biological species, like honeybees or ants, where the rate at which foragers emerge out of the nest is regulated internally by the colony according to some communication rules, but it extends to other situations of interest like signal transduction [55], cell selection/recognition [56,57], and the dynamics of deformable mediums [58] or quantum particles [59]. Surprisingly, the case of a variable number of searchers has received low attention in the literature, with a few exceptions in which random walkers are assumed to become inactivated and vanished after some time, see, e.g., Refs. [17,30,35,54], or aggregate [60]. However, considering a continuous release of particles and/or a decay of particles represent natural extensions of these problems. For the cases of signal transduction and cell selection and recognition, for example, where activation and deactivation of certain processes is governed by the arrival of the fastest particle to an activation and recognition site, a birth-death dynamics can be a natural way of considering a continuous release of particles and/or an absorption process which removes the particles during their transit from the release point to the activation site, respectively. The case of multiparticle processes in deformable mediums would be another useful example, though this corresponds to the more difficult situation where explicit interactions between particles and the surrounding media are at play. Finally, the multiparti-

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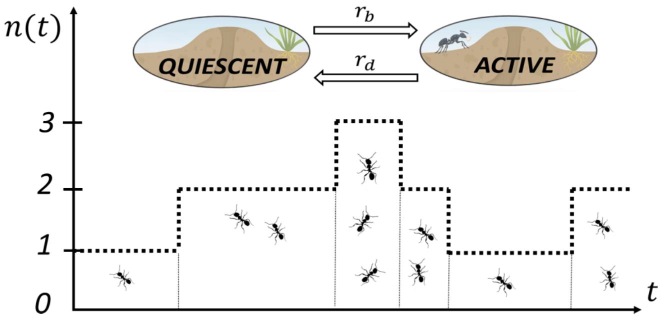

FIG. 1. Schematic representation of the dynamic redundancy problem, where the number of walkers searching for a random target, $n(t)$, evolves in time according to a birth-death dynamics between a *quiescent* (where the individual stays quiet in the nest) and an *active* (random-walk) state.

cle quantum random-walk has been traditionally employed as a search algorithm in different contexts [59], so it would be reasonable to explore whether creation-annihilation operators representing a birth-death dynamics may have some significant influence on the search efficiency within this context.

II. DYNAMIC REDUNDANCY PROBLEM

We consider a group of walkers initially located at position $\mathbf{x}_0$ (which represents their "nest") in a *quiescent* state, so they cannot move from there. At $t = 0$ one walker suddenly switches to the *active* state, which means that it starts a random walk trajectory from $\mathbf{x}_0$. From then on, the number of walkers in the *active* state, $n(t)$, is assumed to follow a birth-death stochastic dynamics so new *active* individual can emerge from the nest with a certain probability, or those *active* can become *quiescent* again (and then its position is instantaneously reset to $\mathbf{x}_0$). For simplicity, we will assume that the transition between the two states is Markovian, so the birth-death process is governed by constant rates r_b and r_d (see Fig. 1 for a schematic visualization of the idea).

Our specific aim in the present work is twofold. First, we want to understand the properties of the FFPT (denoted by T) that it takes for the group to reach a target located at $\mathbf{x} = 0$, as a function of the birth-death dynamics described above. Second, we want to check whether there is an optimal shape of $n(t)$ (that is, optimal rates r_b and r_d) that can be used to reach the target with a minimum cost. For this, we introduce the concept of collective time cost (CTC) as

$$T_c(t) = \int_0^t n(t')\, dt'. \tag{1}$$

This can be interpreted as the sum of the individual search times spent by every *active* walker up to time t. Then, $T_c(t = T)$ represents a first-passage functional the CTC up to the FFPT (formally, $T_c(T)$ is a first-passage functional [61,62] subordinated to the birth-death process $n(t)$. Note that T itself depends explicitly on $n(t)$, so this makes the computation of $T_c(T)$ nontrivial. Typically, minimizing the CTC will require a balance between keeping both T and $n(t)$ small; we will focus here on understanding the properties of that balance.

A. Connection to previously studied cases

The dynamic redundancy problem for $r_b = r_d = 0$ reduces to the classical first-passage problem for a single walker, while the case $r_b = 0$, $r_d \neq 0$ would correspond to the case of mortal random walkers (see Refs. [17,28,63–66]). Also, the situation $r_b \ll r_d$ can be mapped into the case of stochastic resetting, with r_d representing the reset rate. In this case, the birth-death process will satisfy most of the time $n(t) = 0$, and occasionally a single *active* walker will emerge from the nest ($n(t) > 1$ being extremely unlikely). Since the *quiescent* periods where $n(t) = 0$ do not contribute to the integral in Eq. (1), the statistics of $T_c(T)$ will then correspond to that of the first-passage time for the stochastic resetting case. So, its mean value will tend to $\langle T_c \rangle = (e^{x_0\sqrt{r_d/D}} - 1)/r_d$ for Brownian walkers in one dimension [39].

III. EXACT SOLUTION FOR $r_d = 0$

In the following we focus in an exemplifying case where search trajectories are never terminated ($r_d = 0$), and then $n(t)$ is a strictly increasing process. As we shall see, this particular case admits an exact analytical treatment and it suffices to investigate the aforementioned balance necessary for minimizing the CTC.

We define $f(t)$ as the probability distribution for the FFPT, and the corresponding survival probability is denoted as $S(t) = \int_t^\infty f(t')dt'$. For convenience, we will separate all the realizations of the stochastic process that lead to the target detection at time t into those which correspond to $n(T) = 1$, $n(T) = 2$, and so on. So that we define the survival probability for the case $n(T) = N$ as $S(t|N)$, so this corresponds to the probability that the target has not been reached yet at time t, conditioned to the fact that there are N *active* walkers by that time. Note that the expression for $N = 1$ will satisfy $S(t|1) = S_{\mathrm{sw}}(t)e^{-r_b t}$, where $S_{\mathrm{sw}}(t)$ represents the survival probability for the classical case of a single walker (it is, for $r_b = r_d = 0$), and the exponential factor $e^{-r_b t}$ is the probability that no additional *active* walkers have emerged in the interval $(0, t)$.

Through these definitions, we can then write the recurrence relation [67]

$$S(t|N) = r_b e^{-r_b t} \int_0^t S_{\mathrm{sw}}(t) S(t - t_1|N - 1) dt_1, \tag{2}$$

where the integration variable t_1 represents the random time at which the second walker becomes *active*, it is, the time at which $n(t) = 1$ switches to $n(t) = 2$. So that, in Eq. (2) the term $S_1(t) = S_{\mathrm{sw}}(t)e^{-r_b t}$ is being multiplied by $r_b \int_0^t S(t - t_1|N - 1)dt_1$, where the latter represents the survival probability for the remaining $N - 1$ *active* walkers integrated over all possible values of t_1. The previous relation allows us to find recurrently the general expression for $S(t|N)$, which takes the form

$$S(t|N) = e^{-r_b t} S_{\mathrm{sw}}(t) \frac{[r_b g(t)]^{N-1}}{(N-1)!}, \tag{3}$$

where we have introduced $g(t) \equiv \int_0^t S_{\mathrm{sw}}(t')dt'$. Now, the general expression for the survival probability in our dynamic redundancy problem is easily derived as

$$S(t)=\sum_{N=1}^{\infty}S(t|N)=S_{\mathrm{sw}}(t)e^{-r_b(t-g(t))}, \tag{4}$$

and the corresponding first-passage distribution is simply $f(t)=-dS(t)/dt$ [67].

One can follow a similar procedure for T_c. So, we will define $\overline{T_c}(t|N)$ as the contribution to the mean collective time cost (MCTC) that comes from all the realizations of the search process for which $n(T)=N$. Hence, the overall MCTC (computed over all possible values of N and all possible values of the FFPT) is simply

$$\langle T_c\rangle=\int_0^{\infty}dT\,\overline{T_c}(T)=\int_0^{\infty}dT\sum_{N=1}^{\infty}\overline{T_c}(T|N). \tag{5}$$

It is possible to check that the expressions for $\overline{T_c}(t|N)$ also satisfy a recurrence relation which allows us to write them in terms of $S(t|N-1)$ and $\overline{T_c}(t|N)$. Using that recurrent relation, one can derive again the expression for the integrand in the MCTC (5) [67]:

$$\begin{aligned}\overline{T_c}(T)={}&e^{-r_b[T-g(T)]}\{r_bS_{\mathrm{sw}}(T)[T+g(T)-2TS_{\mathrm{sw}}(T)]\\&+r_b^2h(T)S_{\mathrm{sw}}(T)[1-S_{\mathrm{sw}}(T)]\\&+f_{\mathrm{sw}}(T)[T+r_bh(T)]\}\end{aligned} \tag{6}$$

with $h(t)\equiv\int_0^t t'S_{\mathrm{sw}}(t')dt'$, and where we define $f_{\mathrm{sw}}(t)=-dS_{\mathrm{sw}}/dt$ as the first-passage time distribution for the classical (single-walker) case.

A. Brownian walkers in one dimension

While we have reached exact expressions for the survival probability and the CTC, it is not trivial in general to compute their average properties. Let us consider that *active* walkers behave as Brownian particles in a 1D domain with diffusion coefficient D. For this, it is well-known that $S_{\mathrm{sw}}(t)=\mathrm{erf}(\frac{x_0}{2\sqrt{Dt}})$, with $\mathrm{erf}(\cdot)$ representing the error function [68]. However, introducing this expression into (4) or (6) does not allow to reach an exact expression neither for the mean-first passage time $\langle T\rangle=\int_0^{\infty}S(t)dt$ nor for the MCTC (5). So, we will take the two timescales driving the search process, i.e. the diffusive scale $x_0^2/4D$ and the birth scale r_b^{-1}, and will focus on the behavior for the limit cases where the dimensionless parameter $\chi\equiv x_0^2r_b/4D$ becomes either small or large.

Integrating by parts and using Eq. (4), the MFFPT $\langle T\rangle$ can be written in the form $\langle T\rangle=x_0^2\phi(\chi)/2D$, where the scaling function $\phi(\chi)$ is defined in the Supplemental Material [67]. Using this scaling, the leading term of $\langle T\rangle$ can be properly computed in the limits $\chi\ll1$ and $\chi\gg1$ (the corresponding derivation is also provided in the SI file). This leads to

$$\langle T\rangle\simeq\begin{cases}\dfrac{x_0}{\sqrt{Dr_b}}, & r_b\ll4D/x_0^2,\\[2ex] \dfrac{x_0^2}{4D\ln\left(\sqrt{\frac{2}{243\pi}}\frac{x_0^2r_b}{D}\right)}, & r_b\gg4D/x_0^2.\end{cases} \tag{7}$$

Similarly, the CTC also admits a scaling form $\langle T_c\rangle=x_0^2\phi_c(\chi)/2D$, where $\phi_c(\chi)$ is also defined in the SI file. Its leading order behavior in the two limits $\chi\ll1$ and $\chi\gg1$ above reads

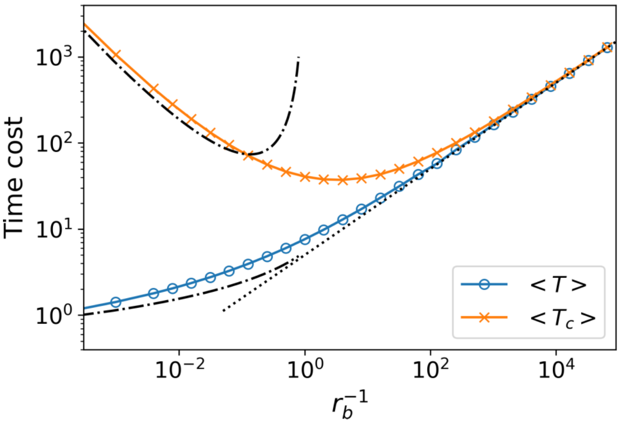

FIG. 2. MFFPT (in blue) and the MCTC (in orange) as a function of the birth timescale r_b^{-1}. Solid lines correspond to the exact values computed from numerical integration, and symbols correspond to random-walk simulations. Dotted and dotted-dashed lines represent the analytical asymptotic behavior predicted in Eqs. (7) and (8). All the results shown correspond to the case $x_0=5$, $D=1$.

$$\langle T_c\rangle\sim\begin{cases}\frac{x_0}{\sqrt{Dr_b}}, & r_b\ll4D/x_0^2,\\ \frac{\chi}{(\ln\chi)^2}, & r_b\gg4D/x_0^2.\end{cases} \tag{8}$$

Equations (7) and (8), together with the general expressions (4) and (6) from which they are derived, represent the main results of our work. To visualize these results, in Fig. 2 we plot both the MFFPT and the MCTC as a function of the birth scale. There, the solid lines correspond to the exact values of these two quantities obtained from numerical integration of our exact expressions above, and the symbols correspond to random-walk simulations that we have carried out (averaging over 10^4–10^5 realizations of the collective search process) to confirm the validity of our derivations. From the plots obtained, we see how the asymptotic results indicated above for the limits $\chi\gg1$ and $\chi\ll1$ (represented through dotted and dashed-dotted lines, respectively) are recovered in the appropriate regimes.

It is interesting to note that in the regime $\chi\ll1$ one finds $\langle T\rangle\approx\langle T_c\rangle$. This is because in most of the realizations the target will be reached by the first *active* walker before the second one becomes *active*. There is still a relatively small probability that the first *active* walker departs from the target as time goes by. For $r_b=0$ this would lead to the divergence of both $\langle T\rangle$ and $\langle T_c\rangle$, while for r_b small the target is found in a finite time thanks to the new *active* walkers eventually appearing. From the properties of a diffusive flux to an absorbing boundary in a 1D semi-infinite domain (see, e.g., Ref. [69]), we know that the fraction of the trajectories that lead the first searcher far away from the target decreases as $\sim t^{-1/2}$, so this explains the scaling $\langle T\rangle\sim1/\sqrt{r_b}$ obtained in Eq. (7).

For the situation $\chi\gg1$, a very large number of *active* walkers will emerge in a time much smaller than the diffusive scale x_0^2/D. Such a situation is then reminiscent of the case of a set of N walkers searching from the target from the same initial position, where the MFFPT satisfies $\langle T\rangle\sim(\ln N)^{-1}$ in the limit $N\to\infty$, as stated above. Since the average number

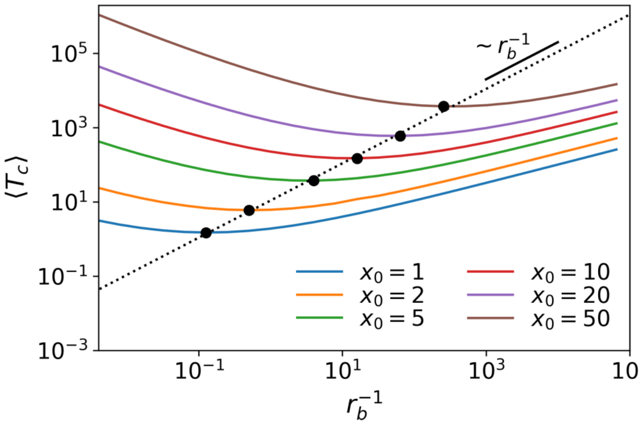

FIG. 3. MCTC as a function of r_b^{-1} for different values of the distance between nest and target x_0. Full symbols denote the minimum search intensity for each case and the dotted line provides a linear fit between them to confirm the scaling $\langle T_c \rangle^* \sim (r_b^*)^{-1}$. In all cases $D = 1$ is used.

of *active* walkers will be proportional to r_b, this explains the scaling $\langle T \rangle \sim (\ln r_b)^{-1}$ we observe. Also, for this regime we can approximate $\langle T_c \rangle = \langle \frac{1}{2} N T \rangle \sim \langle N \rangle \langle T \rangle$, where $\langle N \rangle$ is the mean number of *active* walkers we have at $t = T$ (the factor $\frac{1}{2}$ appears from the fact that any walker will be *active* on average a time $T/2$). Then, $\langle N \rangle = r_b \langle T \rangle$ is satisfied. So that, we realize that $\langle T_c \rangle \approx \frac{1}{2} r_b \langle T \rangle^2$, and combining this with Eq. (7) we finally obtain $\langle T_c \rangle \sim \chi/(\ln \chi)^2$, which confirms intuitively the formal result in Eq. (8).

Anyway, the most relevant feature in Fig. 2 is that while $\langle T \rangle$ decays monotonically with r_b, there is instead an optimum rate which minimizes $\langle T_c \rangle$. To understand better this search optimization mechanism, in Fig. 3 we show again $\langle T_c \rangle$ as a function of r_b^{-1}, but now for different values of x_0^2/D. If we denote by $(r_b^*, \langle T_c \rangle^*)$ the coordinates of the point at which the MCTC reaches its minimum, then we observe that (i) r_b^* increases linearly with the diffusive timescale, such that $r_b^* \approx 7.536 D/x_0^2$, and (ii) the minimum value of the MCTC scales as $\langle T_c \rangle^* \approx 1.489 x_0^2/D$, so this means that $\langle T_c \rangle^* \sim (r_b^*)^{-1}$ (note that additional justification for these relations can be mathematically provided [67]).

B. Brownian walkers in higher dimensions

The continuous generation of new *active* trajectories in the dynamic redundancy mechanism above makes that the target will be eventually found always in a finite time even for the case of nonrecurrent random walks, so both $\langle T \rangle$ and $\langle T_c \rangle$ will be finite for Brownian particles in arbitrary dimensions. In Fig. 4 we show the results for $d = 2$ and $d = 3$ to check whether the properties found above for the 1D case remain valid. This confirms the existence of an optimum in $\langle T_c \rangle$ as a function of r_b. For this case a simple closed expression for the survival probability $S_{\rm sw}(T)$ is not available, so our efforts to predict analytically the behavior in the limits $r_b \gg 4D/x_0^2$ and $r_b \ll 4D/x_0^2$ from Eq. (4) have been fruitless. However, generalizing our intuitive arguments above for 1D we observe that the former still satisfies $\langle T \rangle \sim 1/\ln N$ for the case of N fixed, and so the scalings $\langle T \rangle \sim 1/\ln r_b$ and $\langle T_c \rangle \sim \chi/(\ln N)^2$ derived for the 1D case should persist. On the other side, for $r_b \ll 4D/x_0^2$ the same reasoning above should lead to $\langle T \rangle \approx \langle T_c \rangle \sim r_b^{-d/2}$ (taking into account again the properties of a diffusive flux to an absorbing trap in d dimensions [69]).

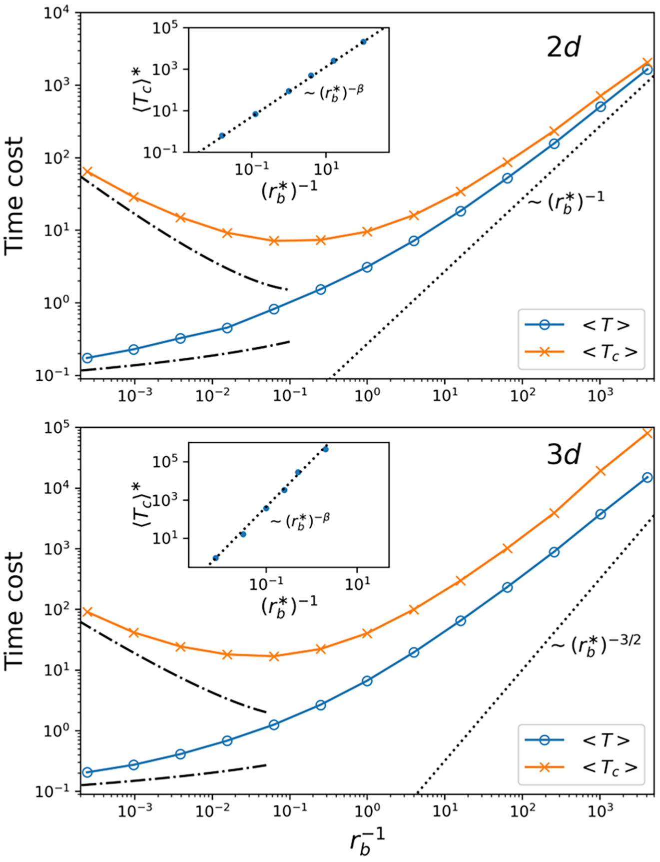

FIG. 4. MFFPT (in blue) and the MCTC (in orange) for two and three dimensions (upper and lower panels, respectively) as a function of the birth timescale r_b^{-1}. Symbols correspond to random-walk simulations, and the solid lines in this case are only provided as a visual guide. Dotted and dashed-dotted lines are the limit behavior expected for $\chi \ll 1$ and $\chi \gg 1$, respectively. All results shown correspond to the case $x_0 = 2$, $D = 1$. (Insets) Minimum of the MCTC when computed for different values of x_0, as in Fig. 3. The fitted scaling $\langle T_c \rangle^* \sim (r_b^*)^{-\beta}$ is provided, with $\beta = 1.20 \pm 0.01$ and $\beta = 2.43 \pm 0.02$ for 2D and 3D, respectively.

From Fig. 4 we observe that the convergence to the asymptotic regimes $\chi \ll 1$ and $\chi \gg 1$ becomes extremely slow, so it is not possible at practice to fully observe the agreements with the scalings above. Indeed, the scaling between $\langle T_c \rangle^*$ and $(r_b^*)^{-1}$ becomes now nontrivial; in the insets of Fig. 4 we provide a power-law fit $\langle T_c \rangle^* \sim (r_b^*)^{-\beta}$ to the numerical results, which yield $\beta = 1.20 \pm 0.01$ (for 2D) and $\beta = 2.43 \pm 0.02$ (for 3D) for the region of parameters explored.

IV. DISCUSSION

While our results here are restricted to the particular case $r_d = 0$ for the sake of clarity, they show the potential interest that the concept of dynamic redundancy has in terms of optimizing collective random searches, and they pave the way to consider more general situations. For instance, some preliminary work with Brownian trajectories in the case $r_d > 0$ we have carried out suggests that the MCTC can reach minimum values for nontrivial combinations of r_b, r_d (this will be presented in a forthcoming publication). Also, the extension to non-Markovian birth-death mechanisms or movement processes (e.g., persistent random-walks, Lévy flights,...) and/or different geometries, offers a vast range of possibilities to explore.

Furthermore, our predictions for $\langle T \rangle$ and $\langle T_c \rangle$ could be experimentally tested in real scenarios, e.g., in collective animal foraging or searches through networks, where it is reasonable to expect that regulation of $n(t)$ could represent a plausible mechanism that groups of searchers could use to minimize search costs and efforts. In group animal foraging, for instance, it would be reasonable to ask whether the recruitment of new foragers can significantly reduce the search time, and/or increase the average energy intake for the group. Regulation of $n(t)$ then seems to be a simple mechanism that groups could use to orchestrate their searches and minimize their search efforts, and one could hypothesize this should actually play a role in the adaptation of such groups to their specific environmental conditions. Similarly, optimal control of random searches on activation processes or through physical and social networks could be carried out through these mechanisms, all in all showing the potential interest of the concept of dynamic redundancy.

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