Triadic interaction in the background of a pairwise spin-glass

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(Received 31 January 2024; accepted 8 May 2024; published 3 June 2024)

Developing an equilibrium solution for a pairwise spin-glass with a quenched random infinite range shows a continuous phase transition. Models with p-spin interactions have been studied and the exact solution was provided that shows a continuous phase transition for $p = 2$ and a first-order one for $p > 2$. Although the p-spin interactions were studied individually without considering lower-order interactions, is it always feasible to ignore the lower ones? Here, we are interested in finding an analytical solution for considering a triadic interaction as a perturbation in the background of a pairwise interaction in the Sherrington-Kirkpatrick spin-glass model. Our results indicate a sudden phase transition as a consequence of considering triadic interactions that signal a switch from a continuous to an explosive phase transition.

DOI: 10.1103/PhysRevE.109.064105

I. INTRODUCTION

Over the years, great effort has gone into understanding the behavior of systems of spins interacting via quenched random couplings, so-called spin-glasses. In 1975 Sherrington and Kirkpatrick (SK) proposed an idealized model of a spin-glass [1,2] which is the infinite-range version of the Edwards-Anderson model [3]. Methods for solving the SK model include generalizations to models involving p-spin interactions [4–7]. A solution for p-spin interactions was provided. It showed that in $p = 2$ a continuous phase transition occurs, and for $p > 2$, there exists a first-order phase transition [5,6]. Also, Derrida showed that under $p \to \infty$ the SK model is identical to a random energy model and is exactly solvable [8,9]. Thouless *et al.* represented a solution to the SK model via the mean-field equation, the so-called Thouless-Anderson-Palmer (TAP) equation [10]. Ultimately, in 1979–1980 Parisi proposed a solution with an interpretation of the structure of valleys of free energy [11–13]. Despite years of effort and focus on the spin-glass [14–20], there is still no analytical solution to illustrate the consequences of considering higher-order interactions in the background of a pairwise spin-glass.

The effects of higher-order interactions are summarized in Refs. [21–25]. Studies show that going beyond the pairwise interactions and considering higher-order ones can change the transition from continuous to discontinuous [22,23]. Considering higher-order interactions and studying the collective behavior of the system by representing an analytical solution, a discrete phase transition has been reported [21,26–28]. Research on triadic interactions confirms the occurrence of abrupt critical behavior in a system including three-body interactions [29–33]. Approaching the higher-order interactions from a different perspective, Refs. [34–36] address the topological aspects of higher-order connectivity which introduces a class of nanonetwork evolution by the self-assembly of simplexes with a focus on triangle-based interactions. They indicated how considering higher-order interactions could cause an abrupt transition. Although the physics often relies on pairwise interactions and the higher-order ones are weaker, they cannot be ignored. In addition, in much research considering higher-order interactions, they exist entirely independent of the lower-order ones, so here we study the triadic interactions in the background of pairwise ones for the quest of how the phase transition may change as a consequence.

An order parameter is required to study the phases of a complex system. In the normal ferromagnet Ising model, the magnetization is the order parameter which is zero for high temperatures and the system has only one equilibrium state, whereas it has two equilibrium states in low temperatures. For a spin-glass, the situation is quite different. There are many equilibrium states [2] and the order parameter would be sensitive to the existence of those which is a characteristic feature of the glassy phase [11–13]. Following the Edwards-Anderson order parameter [3], the correlation between two spins, called an overlap, is defined as an order parameter for the spin-glass [37]. This parameter indicates that a pairwise spin-glass experiences a continuous phase transition. However, the phase transition of triadic interactions in the background of the SK spin-glass model cannot be easily recognized. By considering weak triadic interactions in the background of the pairwise SK spin-glass model, our results show that derivatives of the order parameter (overlap) with respect to temperature have a strange behavior around the critical point.

Two fundamental properties of the SK spin-glass are thought to be disorder and frustration, and the most important consequence of frustration is that it leads to a high degeneracy

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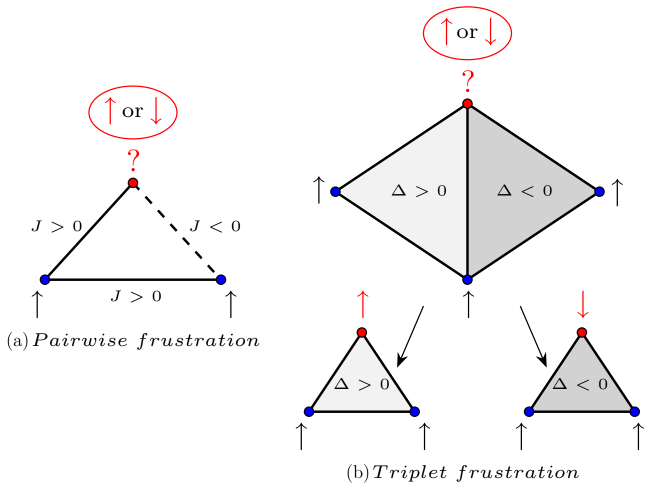

FIG. 1. (a) Spin frustration in a pairwise interacting spin-glass system. The frustrated node (red node) cannot choose a state under the effect of a relationship with the other nodes. (b) schematically displays the frustrated state in a triadic interaction when two triadics are placed together. Each triadic is not frustrated individually.

of the ground state of the system [38]. By considering triadic interactions we need to define the frustration for three interacting spins. We introduce the frustration of triadic interactions when two triplets are placed together (see Fig. 1). Figure 1(a) demonstrates the concept of a frustrated state in the pairwise interactions. The frustrated node (red node) cannot choose a state under the effect of a relationship with other nodes. In Fig. 1(b), we schematically display triadic frustration where the red node in Fig. 1(b) chooses to be upward in the left triadic of the bottom line and downward in the right triadic of the bottom line while the juxtaposition of two triadics results in a frustrated state that the red node cannot determine its direction.

II. MODEL

In this section, we look at the triadic interaction in the SK spin-glass model's background via the perturbative approach. Considering the Hamiltonian for the SK spin-glass model with pairwise interactions,

$$H_J = -\sum_{i<j} J_{ij}\, S_i\, S_j - h \sum_i S_i, \tag{1}$$

where the J_{ij}'s are independently random values taken from Gaussian probability distribution $\mathcal{P}(\mu_J, \sigma_J^2)$. The mean and the variance of pairwise interactions are proportional to $1/N$ because the energy needs to be extensive. To pursue our goal of considering higher-order interactions, we introduce the triadic interaction Hamiltonian as follows,

$$H_\Delta = -\sum_{i<j<k} \Delta_{ijk}\, S_i\, S_j\, S_k, \tag{2}$$

where the summation runs over all triadics of spins, and the combination of $\binom{N}{3}$, and each triadic is assigned by an independent random value, Δ_{ijk}, from Gaussian probability distribution $\mathcal{P}(\mu_\Delta, \sigma_\Delta^2)$. The mean and the variance of this distribution should be proportional to $1/N^2$ to have Hamiltonian Eq. (2) of order N.

The primary goal of this study is to address the triadic spin interaction in the background of a pairwise one to investigate how it modifies the apparent nature of the transition and may affect the spin-glass behavior of the SK model. However, the general solution of p-spin interactions is obtained in Refs. [5,6] but the additive effect of three interacting spins besides the pairwise is not considered yet. Our approach to exploring this effect is to take advantage of a perturbation trick which provides us an opportunity to track the effectiveness of three interacting spins. We aim to figure out what happens when we add a triadic interaction as a perturbation term to the SK spin-glass Hamiltonian. Our proposed perturbation trick satisfies the following conditions,

$$\mu_\Delta \ll \mu_J, \quad \sigma_\Delta^2 \ll \sigma_J^2, \tag{3}$$

which means the mean and the variance of the triadic's random values are smaller than the pairwise ones. To solve the triadic interaction as a perturbation term in the background of the SK spin-glass model, we write the total Hamiltonian including the background and the perturbation term,

$$H = H_J + H_\Delta. \tag{4}$$

Based on the proposed perturbation trick Eq. (3), the value of H_Δ is smaller than the value of H_J.

III. ANALYSIS

In the following, we represent the analytical solution for this system and our findings confirm the statement of Refs. [21–23,26] that considering higher-order interactions leads to an explosive transition [39]. Now, the Hamiltonian (4) can be used to calculate the average free energy of this system,

$$[F] = -T[\log Z] = -T \int \prod_{i<j} dJ_{ij} P(J_{ij}) \prod_{i<j<k} d\Delta_{ijk} P(\Delta_{ijk}) \log Z.$$

The dependence of $\log Z$ on the interaction coefficients, J's, and triadic's random values, Δ's, is very complicated and it is difficult to average on it. Therefore, we apply the well-known replica method to calculate the configurational average of $\log Z$ [40–42],

$$[\log Z] = \lim_{n\to 0} \frac{[Z^n] - 1}{nN}.$$

According to the replica method, we first replicate the system n times, calculate the configurational average of the nth power of the partition function, and then take the limit of $n \to 0$. From the partition function, we manage to derive statistical quantities to describe the statistical properties of the system. The total partition function is

$$[Z^n] = \iint \left(\prod_{i<j} dJ_{ij} P(J_{ij})\right)\left(\prod_{i<j<k} d\Delta_{ijk} P(\Delta_{ijk})\right) \times \mathrm{Tr}\left[\exp\left\{\beta \sum_{i<j} J_{ij} \sum_{\alpha=1}^{n} S_i^\alpha S_j^\alpha + \beta h \sum_{i=1}^{N}\sum_{\alpha=1}^{n} S_i^\alpha\right\} \exp\left\{\beta \sum_{i<j<k} \Delta_{ijk} \sum_{\alpha=1}^{n} S_i^\alpha S_j^\alpha S_k^\alpha\right\}\right], \tag{5}$$

where α is the index variable for the replica. The configurational integral can be calculated for each J_{ij} and Δ_{ijk} separately. Since for each J_{ij} and Δ_{ijk} the quadratic and cubic polynomials appear in the exponential, by using the Gaussian integral the solution for each pairwise interaction coefficient and triadic's random value can be calculated and finally the average of the partition function of replicas for large N is obtained as follows,

$$[Z^n] = \exp\left\{\frac{N\beta^2\sigma_J^2 n}{4}\right\} \exp\left\{\frac{N\beta^2\sigma_\Delta^2 n}{12} - \frac{\beta^2\sigma_\Delta^2 n}{4} - \frac{\beta^2\sigma_\Delta^2 n}{12N}\right\} \times \mathrm{Tr}\left[\exp\left\{\frac{\beta^2\sigma_J^2}{2N}\sum_{\alpha<\beta}\left(\sum_i S_i^\alpha S_i^\beta\right)^2 + \frac{\beta\mu_J}{2N}\sum_\alpha\left(\sum_i S_i^\alpha\right)^2 + \beta h \sum_i\sum_\alpha S_i^\alpha\right\} \times \exp\left\{\frac{\beta^2\sigma_\Delta^2}{6N^2}\sum_{\alpha<\beta}\left(\sum_i S_i^\alpha S_i^\beta\right)^3 + \frac{\beta\mu_\Delta}{6N^2}\sum_\alpha\left(\sum_i S_i^\alpha\right)^3 - \left(\frac{\beta^2\sigma_\Delta^2}{2N} + \frac{\beta^2\sigma_\Delta^2}{6N^2}\right) \times \sum_{\alpha<\beta}\sum_i S_i^\alpha S_{i\beta} - \left(\frac{\beta\mu_\Delta}{2N} + \frac{\beta\mu_\Delta}{6N^2}\right)\sum_\alpha\sum_i S_i^\alpha\right\}\right], \tag{6}$$

where we use a generalized version of the Hubbard-Stratonovich transformation (see Appendix) and linearize the terms of the sum of squared and cubic powers on the spins in one replica by introducing the quantities $q_{\alpha\beta}$ for the term $(\sum_i S_i^\alpha S_i^\beta)^2$ and m_α for $(\sum_i S_i^\alpha)^2$. Then the average of the partition function can be written as

$$[Z^n] = \exp\left\{\frac{Nn\beta^2\sigma_J^2}{4} + \frac{Nn\beta^2\sigma_\Delta^2}{12}\right\} \int \prod_{\alpha<\beta} dq_{\alpha\beta} \int \prod_\alpha dm_\alpha \times \mathrm{Tr}\left[\exp\left\{-\left(\frac{N\beta^2\sigma_J^2}{2} - \gamma\sum_i S_i^\alpha S_i^\beta\right)\sum_{\alpha<\beta} q_{\alpha\beta}^2 - \left(\frac{N\beta\mu_J}{2} - \gamma'\sum_i S_i^\alpha\right)\sum_\alpha m_\alpha^2\right\} \times \exp\left\{\beta^2\sigma_J^2 \sum_{\alpha<\beta} q_{\alpha\beta}\sum_i S_i^\alpha S_i^\beta + \beta\sum_\alpha(\mu_J m_\alpha + h)\sum_i S_i^\alpha\right\} \times \exp\left\{-\left(\frac{\beta^2\sigma_\Delta^2}{2N} + \frac{\beta^2\sigma_\Delta^2}{6N^2}\right)\sum_{\alpha<\beta}\sum_i S_i^\alpha S_i^\beta - \left(\frac{\beta\mu_\Delta}{2N} + \frac{\beta\mu_\Delta}{6N^2}\right)\sum_\alpha\sum_i S_i^\alpha\right\}\right]. \tag{7}$$

After calculating integrals for $q_{\alpha\beta}$ and m_α individually, by considering γ and γ' as small parameters, we are allowed to expand the expressions in the exponents. In this expansion, the zeroth orders of γ and γ' are pairwise and the first orders are related to triadic interactions (see Appendix). Notice that due to our perturbative approach, the mean value of the random pairwise interactions is supposed to be nonzero. This is a necessary constraint for proceeding with the perturbative technique [see Eq. (A2) in the Appendix]. Also, according to Eq. (3), we must select the mean and the variance of random couplings and triadic interactions in a way that satisfies those criteria. In Fig. 3, we discuss the effect of the mean and the variance on critical temperature in more detail.

To find the coefficients γ and γ', we compare the partition function derived by Eq. (7) up to the first order of expansion in terms of γ and γ' at the exponents with ones directly calculated from integration over the Gaussian distributions in Eq. (5). Then we can find that

$$\gamma = \frac{\beta^2\sigma_\Delta^2}{6}, \quad \gamma' = \frac{\beta\mu_\Delta}{6}, \tag{8}$$

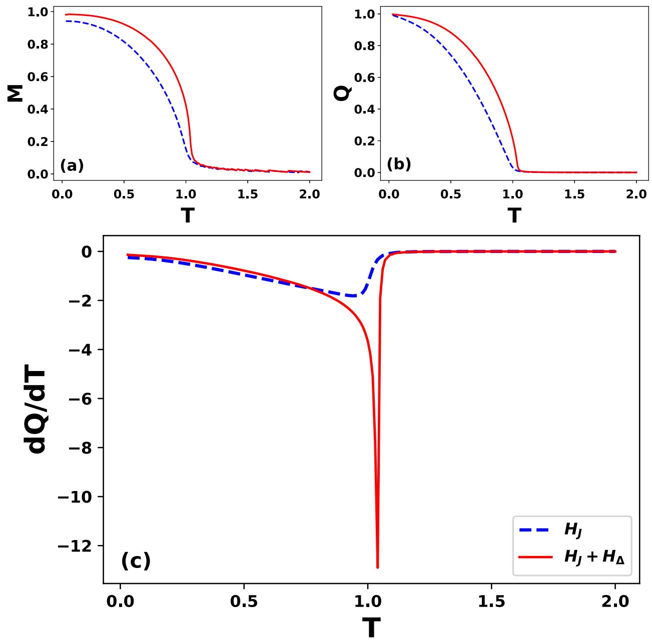

FIG. 2. The dashed blue line is for the SK spin-glass model with a pairwise interaction and the red solid line is by considering a triadic interaction with constrainrs $\mu_J = 1$, $\mu_\Delta = 0.5$, $\sigma_J^2 = 0.5$, and $\sigma_\Delta^2 = 0.25$. (a) Magnetization as a function of temperature displays that considering a triadic interaction causes the M changes to be sharper. (b) Overlap vs temperature is indicated. The critical temperature is altered under considering triadic interactions and this is the result of nonzero mean values of pairwise and triadic interactions. Also, an explosive transition is seen by adding triadic interactions. (c) Derivative of the overlap with respect to temperature. In critical temperature, it has a limited value for the SK spin-glass while considering triadic interactions sharpens the temperature derivation of the order parameter.

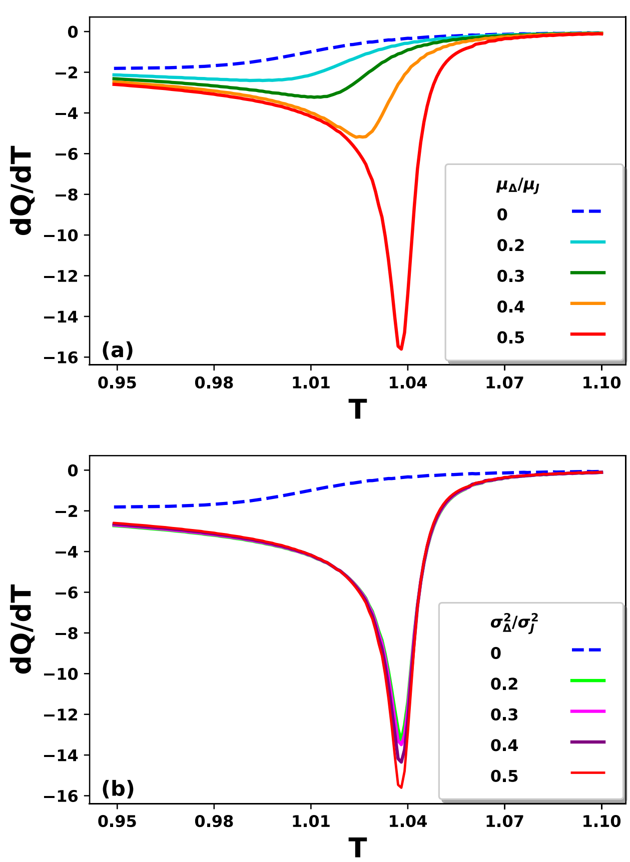

FIG. 3. (a) shows how increasing the ratio of mean values causes the change of the phase transition. A larger ratio of mean values makes a bigger shift in critical temperature. (b) It indicates the effect of variance on the depth of the line around the critical temperature. As we can see, the critical temperature is the same for all cases.

which perfectly agrees with our assumption that γ, γ' are small values due to their dependence on $\mu_\Delta, \sigma_\Delta^2$. This is under our the perturbation assumption Eq. (3) that the mean value and the variance of the triadic interactions are smaller than those of pairwise interactions. Therefore, our perturbation trick enables us to study the perturbation effect of triadic interactions in the background of the SK spin-glass model. In Eq. (7) the terms proportional to n and inverse of N are ignorable by tending n to zero and considering N to be large. So, the coefficients of the terms $(\sum_{\alpha<\beta}\sum_i S_i^\alpha S_i^\beta)$ and $(\sum_\alpha \sum_i S_i^\alpha)$ can be ignored in the thermodynamic limit. In Eq. (7) the exponent of the integrand is proportional to N, so in the thermodynamic limit that $N \to \infty$ the integral can be evaluated by the steepest descent method. Also, we have represented the sum $\sum_i$ over a single site and considered the statement to the power of N,

$$[Z^n] \simeq \exp\left\{-\frac{N\beta^2\sigma_J^2}{2}\sum_{\alpha<\beta} q_{\alpha\beta}^2 - \frac{N\beta\mu_J}{2}\sum_\alpha m_\alpha^2 + N\log \operatorname{Tr} e^{L'} + \frac{Nn\beta^2\sigma_J^2}{4} + \frac{Nn\beta^2\sigma_\Delta^2}{12}\right\}$$
$$\simeq 1 + Nn\left\{-\frac{\beta^2\sigma_J^2}{2n}\sum_{\alpha<\beta} q_{\alpha\beta}^2 - \frac{\beta\mu_J}{2n}\sum_\alpha m_\alpha^2 + \frac{1}{n}\log \operatorname{Tr} e^{L'} + \frac{\beta^2\sigma_J^2}{4} + \frac{\beta^2\sigma_\Delta^2}{12}\right\}. \tag{9}$$

In the last expression, the limit $n \to 0$ has been taken with N kept very large but finite. Now, based on the replica method the free energy would be derived,

$$-\beta[f] = \lim_{n\to 0}\frac{[Z^n]-1}{nN} = \lim_{n\to 0}\left\{-\frac{\beta^2\sigma_J^2}{4n}\sum_{\alpha\neq\beta} q_{\alpha\beta}^2 - \frac{\beta\mu_J}{2n}\sum_\alpha m_\alpha^2 + \frac{1}{n}\log \operatorname{Tr} e^{L'} + \frac{\beta^2\sigma_J^2}{4} + \frac{\beta^2\sigma_\Delta^2}{12}\right\}, \tag{10}$$

where

$$L' = \beta^2\sigma_J^2 \sum_{\alpha<\beta} q_{\alpha\beta} S^\alpha S^\beta + \beta\sum_\alpha (\mu_J m_\alpha + h) S^\alpha + \gamma \sum_{\alpha<\beta} q_{\alpha\beta}^2 S^\alpha S^\beta + \gamma' \sum_\alpha m_\alpha^2 S^\alpha. \tag{11}$$

The values of $q_{\alpha\beta}$ and m_α should be chosen to extremize the quantity within the braces {} of Eq. (10). Hence, regarding the saddle-point condition, maximizing free energy Eq. (10) with respect to $m_\alpha, q_{\alpha\beta}$ results in the self-consistent equations,

$$m_\alpha = \frac{\operatorname{Tr}\left(S^\alpha + \frac{2\gamma'}{\beta\mu_J} m_\alpha S^\alpha\right)e^{L'}}{\operatorname{Tr} e^{L'}},$$
$$q_{\alpha\beta} = \frac{\operatorname{Tr}\left(S^\alpha S^\beta + \frac{2\gamma}{\beta^2\sigma_J^2} q_{\alpha\beta} S^\alpha S^\beta\right)e^{L'}}{\operatorname{Tr} e^{L'}}. \tag{12}$$

The variables $q_{\alpha\beta}$ and m_α that have been introduced as integration variables are the order parameters of the SK spin-glass model. The definition of order parameters for our suggested system are related to the variables $q_{\alpha\beta}$ and m_α and in the normalized versions they are

$$M_\alpha \equiv \frac{\operatorname{Tr} S^\alpha e^{L'}}{\operatorname{Tr} e^{L'}} = \frac{m_\alpha}{1 + \frac{2\gamma'}{\beta\mu_J} m_\alpha},$$
$$Q_{\alpha\beta} \equiv \frac{\operatorname{Tr} S^\alpha S^\beta e^{L'}}{\operatorname{Tr} e^{L'}} = \frac{q_{\alpha\beta}}{1 + \frac{2\gamma}{\beta^2\sigma_J^2} q_{\alpha\beta}}, \tag{13}$$

where these order parameters are explicitly dependent on replica indices. To derive the replica symmetric solution it has been assumed that $q_{\alpha\beta} = q$, $m_\alpha = m$ to discover that the replica indices should not affect the physics of the system. The symmetric solution for the free energy, Eq. (10), is

$$-\beta[f] = -\frac{\beta^2\sigma_J^2}{4n}\{n(n-1)q^2\} - \frac{\beta\mu_J}{2n} n m^2 + \frac{1}{n}\log \operatorname{Tr} e^{L'} + \frac{1}{4}\beta^2\sigma_J^2 + \frac{1}{12}\beta^2\sigma_\Delta^2, \tag{14}$$

so the third term including L', on the right-hand side of Eq. (14), can be calculated by using its definition from Eq. (11) and a Gaussian integral. Inserting its result into Eq. (14) and replacing the value of γ, γ' from Eq. (8), then taking the limit $n \to 0$, we have the free energy as

$$-\beta[f] = \frac{\beta^2\sigma_J^2}{4}(1-q)^2 + \frac{\beta^2\sigma_\Delta^2}{12}(1-q^2) - \frac{\beta\mu_J}{2}m^2 + \int Dz \log\{2\cosh[\beta\tilde{H}(z)]\}, \tag{15}$$

where $Dz = dz \exp(\frac{-z^2}{2})\frac{1}{\sqrt{2\pi}}$ and $\beta\tilde{H}(z) = \sqrt{\beta^2\sigma_J^2 q + \gamma q^2}\, z + (\beta\mu_J m + \beta h + \gamma' m^2)$.

Extremizing the free energy concerning m, q results in

$$m = \left(1 + \frac{2\gamma' m}{\beta\mu_J}\right)\int Dz \tanh[\beta\tilde{H}(z)],$$
$$q = \frac{1}{\left(1 - \frac{\sigma_\Delta^2}{3\sigma_J^2}\right)}\left\{1 - \frac{\beta^2\sigma_J^2 + 2\gamma q}{\beta^2\sigma_J^2}\int Dz \frac{1}{\cosh^2[\beta\tilde{H}(z)]}\right\}. \tag{16}$$

These self-consistent equations need to be renormalized based on the definition of the order parameters M, Q [Eq. (13)]. They satisfy our expectation that at the low temperature, the system goes to a ferromagnetic state where the magnetization and overlap equal one. In addition, in the limit of $\mu_\Delta \to 0$,

$\sigma_\Delta \to 0$, equations for M, Q tend to their correspondences m, q in the SK spin-glass model.

IV. RESULTS AND DISCUSSION

In this section, we draw the normalized order parameters M, Q, and dQ/dT (derivative of overlap versus temperature) to illustrate how the system's behavior changes by considering triadic interactions. Figure 2 shows the temperature dependence of the order parameters M, Q, and dQ/dT for pairwise interactions alone as well as in the presence of triadic interactions. While considering triadic interactions, we adjust the mean and variance of the Gaussian probabilities to follow our perturbation assumptions Eq. (3). They are chosen to be $\mu_J = 1$, $\mu_\Delta = 0.5$, $\sigma_J^2 = 0.5$, and $\sigma_\Delta^2 = 0.25$. Figure 2(a) shows the temperature dependence of M and its behavior in the presence of triadics. Continuous changes in the value of M at each temperature occur with a steeper slope under consideration of triadic interactions. In Fig. 2(b), we show the temperature dependence of the overlap, Q. The figure demonstrates a forward shift in the critical temperature and a sharp transition by triadic interactions. Notice that the higher critical temperature of the glass by considering triadic interaction with respect to the SK model is due to the nonzero mean value of both pairwise and triadic interaction. Figure 3 highlights how the ratio of means and variances might affect the critical temperature. In Fig. 2(c), we show the overlap derivative with respect to temperature. While Fig. 2(b) emphasizes the explosive transition brought about by the inclusion of triadics, Fig. 2(c) highlights this transition. Whereas the derivative of dQ/dT is limited for the SK spin-glass, taking into account the triadic interactions sharpens the derivation of the overlap.

Figure 3 indicates how the variance and mean value of the coupling interactions, J_{ij}, and the random value of triadic interactions, Δ_{ijk}, might affect the critical temperature. When we set a nonzero mean value for the triadic interactions it can be interpreted as a random triadic interaction with a zero mean value plus a deterministic term added to the Hamiltonian. The additive deterministic term may induce a shift in critical temperature. As seen in Fig. 3(a), raising the ratio of mean values clearly indicates a shift in critical temperature when the variance ratio is constant, $\sigma_\Delta^2/\sigma_J^2 = 0.5$.

In Ref. [43] a random field type of disorder was considered in addition to the random pairwise coupling in a class of materials called ferroelectric glasses. They studied the effect of a linear random field beside a random coupling term which modeled a relaxor ferroelectric, and their results show the ratio of the variance of the random field to the variance of random coupling can reveal the changes in the transition. However, we use the term random triadic interaction which is nonlinear and we investigate its effect on the background of a pairwise spin-glass, perturbatively. Figure 3(b) displays that as we fix the ratio of mean values $\mu_\Delta/\mu_J = 0.5$, increasing the ratio of variances causes no shift in critical transition. This indicates that by fixing the ratio of mean values and changing the variance ratio, only the depth of the line around the critical temperature is being affected and there is no shift in T_c. In all figures the dashed blue line is for the SK spin-glass model with pairwise interactions and the solid red line is for the condition of $\mu_\Delta/\mu_J = 0.5$ and $\sigma_\Delta^2/\sigma_J^2 = 0.5$.

V. CONCLUSION

It has been found that frustration and disorder are the two most crucial characteristics of having a spin-glass system and a better understanding of a heterogeneous system has roots in the statistical physics of disordered systems [44]. The Ising spin-glass with only a nearest-neighbor interaction was introduced by Edwards and Anderson. Then the infinite-range interacting pairs of spins which is an exactly solvable model of a spin-glass was introduced by Sherrington and Kirkpatrick. Later, Parisi discovered the model's equilibrium solution by using the replica approach in 1979. An exact solution for p-spin interactions were done and the exact solution has obtained in the cases of $p = 2$ and $p \to \infty$. The limited cases, $p > 2$, show a discontinuous phase transition in contrast to $p = 2$. Although the problem is solved in the general case of p which takes an arbitrary value according to the problem, considering higher-order interactions in the presence of lower-order ones has yet to be studied. This question motivates us to address triadic interacting spins in the background of the SK spin-glass model to study how it affects the behavior of the system. Our results shed light on the following issues: (I) Generalizing the frustration concept in the case of triadic interactions, where two triangles are placed together. (II) How the presence of random valued triadic interactions can make changes in the SK spin-glass model; for this purpose we employed a perturbation trick which abled us to control the effect of triadic interactions. (III) The temperature dependence of order parameters M, Q indicates a forward shifting in the critical temperature which is due to nonzero mean values of pairwise and triadic interactions. (IV) The three-spin interactions play an important role in the system's dynamics. In comparison to the SK spin-glass model, the slope of the overlap is sharper due to triadic interactions. This means the type of transition is projected to change from second order to an explosive transition when the triadic interactions are applied.

ACKNOWLEDGMENTS

The authors would like to express their deepest appreciation to Professor Marc Mézard for his constructive comments which greatly enhanced the depth and quality of our research outcomes. Also, the authors would like to extend their sincere thanks to Professor Muhammad Sahimi, Professor Mohammad Khorrami, Professor Farhad Shahbazi, Professor M. Reza Rahimi Tabar, Professor Saman Moghimi Araghi, and Professor Shahin Rouhani for reading the manuscript and their valuable insights and constructive feedback.

APPENDIX: GENERALIZATION OF HUBBARD-STRATONOVICH TRANSFORMATION

The Hubbard-Stratonovich transformation is a mathematical technique used to transform quadratic terms in exponentials into Gaussian integrals by an auxiliary variable or field. The main idea can be understood from the following identity,

$$e^{ax^2/2} = \sqrt{\frac{1}{2\pi a}} \int_{-\infty}^{\infty} dq e^{-q^2/2a + xq}, \tag{A1}$$

where a is a real and positive constant. In this sense, x was linearized with the help of variable q. In this sense, x is linearized at the cost of adding a continuous variable q to the problem.

One can modify this transformation to linearize x of higher order in a perturbative approach. Here, γ is the perturbation parameter. For the first order, we have

$$\begin{aligned} e^{ax^2/2+bx^3} &= \sqrt{\frac{1}{2\pi a}} \int_{-\infty}^{\infty} dq e^{-(\frac{1}{2a}-\gamma x)q^2 + xq - a\gamma x} \\ &= \sqrt{\frac{1}{2\pi a}} \sqrt{\frac{\pi}{(\frac{1}{2a} - \gamma x)}} \exp\left\{\frac{x^2}{4(\frac{1}{2a} - a\gamma x)}\right\} \\ &= (1 - 2a\gamma x)^{-1/2} \exp\left\{\frac{ax^2}{2(1-2a\gamma x)} - a\gamma x\right\} \\ &= e^{a\gamma x} e^{\frac{ax^2}{2}(1+2a\gamma x)} e^{-a\gamma x}. \end{aligned} \tag{A2}$$

In this expression, $\gamma x q^2$ is added to create x in the third order on the left-hand side of the identity, and $-a\gamma x$ is added to remove the extra term after integration over q. In the end, we can set $\gamma = b/a^2$. In other words, the generalized Hubbard-Stratonovich transformation gives us a triadic interaction when the state-dependent parts of Gaussian weight (γ and γ') are small.

Notice that corresponding to Eq. (6), $a = \frac{1}{N\beta^2\sigma_J^2}$ is proportional to N^{-1}; so in the case of ferromagnet state although $x = \sum_{\alpha<\beta} \sum_i S_i^\alpha S_i^\beta$ (the summation of states of spins) is proportional to $\sqrt{N}$, the second term of the coefficient of q^2, γx, is smaller that the first term $\frac{1}{2a}$ so the expansion is allowed. In addition, the last term in the exponential, $a\gamma x$, can be ignored in our following calculation. It is proportional to $\sqrt{N}$ due to our preceding explanation. We apply this transformation to solve the integrals in Eq. (6).

Taking the advantage of Eq. (A2), the integral dq can be solved with the corresponding placement of $a = \frac{1}{N\beta^2\sigma_J^2}$, $x = \sum_{\alpha<\beta} \sum_i S_i^\alpha S_i^\beta$, and $q = q_{\alpha\beta}$,

$$\begin{aligned} &\int \prod_{\alpha\beta} dq_{\alpha\beta} \exp\left\{-\left(\frac{N\beta^2\sigma_J^2}{2} - \gamma \sum_i S_i^\alpha S_i^\beta\right) \sum_{\alpha<\beta} q_{\alpha\beta}^2 \right. \\ &\quad + \beta^2\sigma_J^2 \sum_{\alpha<\beta} q_{\alpha\beta} \sum_i S_i^\alpha S_i^\beta \\ &\quad \left. -\left(\frac{\beta^2\sigma_\Delta^2}{2N} + \frac{\beta^2\sigma_\Delta^2}{6N^2}\right) \sum_{\alpha<\beta} \sum_i S_i^\alpha S_i^\beta \right\} \\ &= \exp\left\{ \frac{\left(\beta^2\sigma_J^2 \sum_i S_i^\alpha S_i^\beta\right)^2}{4\left(\frac{N\beta^2\sigma_J^2}{2} - \gamma \sum_i S_i^\alpha S_i^\beta\right)} \right. \\ &\quad \left. - \left(\frac{\beta^2\sigma_\Delta^2}{2N} + \frac{\beta^2\sigma_\Delta^2}{6N^2}\right) \sum_{\alpha<\beta} \sum_i S_i^\alpha S_i^\beta \right\}, \end{aligned} \tag{A3}$$

and with the expansion of the denominator under the assumption of a small value for γ, our calculation results in

$$\begin{aligned} &\exp\left\{ \frac{\beta^2\sigma_J^2}{2N} \left(\sum_i S_i^\alpha S_i^\beta\right)^2 + \frac{\gamma}{N^2} \left(\sum_i S_i^\alpha S_i^\beta\right)^3 \right. \\ &\quad \left. -\left(\frac{\beta^2\sigma_\Delta^2}{2N} + \frac{\beta^2\sigma_\Delta^2}{6N^2}\right) \sum_{\alpha<\beta} \sum_i S_i^\alpha S_i^\beta \right\}. \end{aligned} \tag{A4}$$

The same can be done for calculating the integral dm under the corresponding placement of $a = \frac{1}{N\beta\mu_J}$, $x = \sum_\alpha \sum_i S_i^\alpha$, and $m = m_\alpha$. The result will be

$$\begin{aligned} &\exp\left\{ \frac{\beta\mu_J}{2N} \left(\sum_i S_i^\alpha\right)^2 + \frac{\gamma'}{N^2} \left(\sum_i S_i^\alpha\right)^3 \right. \\ &\quad \left. -\left(\frac{\beta\mu_\Delta}{2N} + \frac{\beta\mu_\Delta}{6N^2}\right) \sum_\alpha \sum_i S_i^\alpha + \beta h \sum_\alpha \sum_i S_i^\alpha \right\}. \end{aligned} \tag{A5}$$

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