Full-record statistics of one-dimensional random walks

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We develop a comprehensive framework for analyzing full-record statistics, covering record counts $M(t_1), M(t_2), \ldots$, their corresponding attainment times $T_{M(t_1)}, T_{M(t_2)}, \ldots$, and the intervals until the next record. From this multiple-time distribution, we derive general expressions for various observables related to record dynamics, including the conditional number of records given the number observed at a previous time and the conditional time required to reach the current record given the occurrence time of the previous one. Our formalism is exemplified by a variety of stochastic processes, including biased nearest-neighbor random walks, asymmetric run-and-tumble dynamics, and random walks with stochastic resetting.

I. INTRODUCTION

The statistics of records in the time series $(X_t)_{t=0,1,\ldots}$ is a longstanding topic in probability theory [1–4], finding applications in various fields such as hydrology [5], finance [6], climatology [7], and sports [8]. In a time series $(X_t)_{t\geqslant0}$, we have a record at time t if X_t exceeds all preceding values $X_{t'}$ with $t' = 0, 1, \ldots, t-1$. While the topic was first introduced in the context of independent and identically distributed random variables X_t [2], recent works have considered records when $(X_t)_{t\geqslant0}$ represents the successive positions of a random walk (RW), i.e., $X_{t+1} - X_t = \eta_t$, where the η_t denote the steps of the RW. Records of RWs have been studied when the steps η_t are independent [4,9–12] or correlated [13–15], including cases with resetting of the position X_t [16–18]. The existing results on record dynamics are however essentially restricted to the single-time distribution of the number $M(t)$ of records reached at time t as well as the joint distribution of both the number of records and the time at which the current running record was achieved. These results have been obtained for Brownian motion with and without bias [1,4,19,20], run-and-tumble particles [17,21,22], a random acceleration process [13], fractional Brownian motion [23], resetting Brownian motion [16,18,24], and resetting run-and-tumble particles [25–28]. For most studies on records, the long-time properties are described using the continuous-in-time asymptotic limit process $dX_t/dt = \eta_t$ [13]. The continuum counterpart of the number of records is then the maximum of the walk, which coincides with the number of records as well as with the value (or position) of the record in the case of nearest-neighbor random walks (every time a new record is reached, the maximum position is incremented by one). Characterizing the record dynamics at long times is then equivalent to characterizing the maximum dynamics.

Since record dynamics are non-Gaussian (see the single-time distribution [29]) and non-Markovian [knowledge of $M(t')$ at time t' is insufficient to determine the properties of $M(t)$ for time $t > t'$ because the position of the random walk at time t' is not known], determining multiple-time quantities is essential to fully characterize the process. A first step in this

direction was taken in Refs. [12,30], where two-time distributions were determined. However, these works concerned only two-time quantities and were limited to Brownian motion and Brownian bridges.

The pathway to the generalization of these results lies in the fact that the times when new maxima are achieved form a Markov process [31]: Indeed, at the moment when the maximum m is reached, the position of the RW is known and given by m. Here we develop a general formalism based on renewal theory [13,32,33], utilizing the record ages τ_k, defined as the time elapsed between the kth and $(k+1)$th records, as renewal times. Indeed, we rely on the simple but important remark that the number of records can be seen as the number of renewal events of length given by record ages. Note that our approach connects the number of records of a discrete RW and the maximum of its continuous counterpart. In the continuous case, the number of records is not clearly defined, as it tends to infinity at every timescale [31,34]. To establish this connection, we employ a discretization step that determines the additional length the RW must traverse to be considered to have surpassed its previous record. By setting this length scale to unity, the expressions derived for the maximum and number of records align [13] in the asymptotic limit.

The merits of our approach are the following. (i) It allows us to (re)obtain very naturally the distributions of standard observables such as the number of records at time t or the time at which this record was reached, i.e., the time of last record, but also that of the time of next record, which has not been considered yet. (ii) It applies to both discrete (records) and continuous (maxima) processes. (iii) Importantly, it gives access to *all n time distributions*, n arbitrarily large, associated with these observables. (iv) It provides results not limited to Brownian motion but which hold for any nearest-neighbor Markovian RW, i.e., explicit expressions for representative examples such that the asymmetric Brownian motion (ABM) and the asymmetric run-and-tumble particle (ARTP) are obtained. (v) It can be extended to cover the case of non-nearest-neighbor RWs, as illustrated by the example of resetting RWs.

In terms of limitations and potential future directions, it is important to note that our study does not encompass processes characterized by long jumps, such as Lévy flights [11,35], as the number of records and the maximum position exhibit fundamental differences. Similarly, our analysis does not extend to processes with strong non-Markovian properties, such as fractional Brownian motion [36] or self-repulsive RWs [37–39], where each step is influenced by the entire past history of the trajectory.

The article is structured as follows. We start by deriving the single-time distribution of the number of records, the time of occurrence of the last record, the backward record time, and the forward record time as an illustration of the method. Then we derive the multiple-time distributions of the same observables and provide explicit expressions for the example of the distribution of the times at which the last record events occurred at two different times. Finally, we generalize our approach to aging record times by considering the example of resetting RWs.

II. SINGLE-TIME DISTRIBUTIONS

We consider a nearest-neighbor Markovian one-dimensional (1D) lattice RW [40]. In this case, the running maximum of the RW and the number of records coincide [41]. Successive positions are denoted by $(X_t)_{t=0,1,\dots}$. To introduce our formalism, we first show how to recover the classic asymptotic distribution of the number of records at time t, $M(t)$. Our approach relies on the random variable τ_k, defined as the time elapsed between the breaking of record k and $k+1$. Since the RW is Markovian and a nearest neighbor, the τ_k are independent and identically distributed random variables with probability distribution $F(t) = \mathbb{P}(\tau_k = t)$, corresponding to the renewal times [32,33], as they represent the times between two record events. This simple remark makes $M(t)$ a renewal process. In turn, as we show below, it allows us to use the powerful methodology of renewal processes to easily obtain the statistical properties of $M(t)$ (including n time quantities).

A. Number of records

We first relate the statistics of $M(t)$ to that of the times τ_k,

$$\mathbb{P}(M(t) \geqslant m) = \mathbb{P}(\tau_0 + \cdots + \tau_{m-1} \leqslant t)$$
$$= \mathbb{E}(H(t - \tau_0 - \cdots - \tau_{m-1})), \quad (1)$$

where H is the Heaviside function and $\mathbb{E}(\cdots)$ the average operator. This equality states that at least m records have been reached by time t if and only if there were m record events before time t. To compute the distribution of $M(t)$, we use the discrete Laplace transform of a probability distribution function $f(\tau)$ of the random variable τ (having realizations $t = 0, 1, 2, \dots$):

$$\mathcal{L}\{f(t)\} = \hat{f}(s) \equiv \sum_{t=0}^{\infty} f(t) e^{-st} = \mathbb{E}(e^{-s\tau}). \quad (2)$$

Starting from the distribution $F(t)$ of record ages τ_k, we derive the Laplace transform expression for the distribution of the number $M(t)$ of records at time t by Laplace transforming

Eq. (1),

$$\mathcal{L}\{\mathbb{P}(M(t) \geqslant m)\}$$
$$= \mathbb{E}\left(\sum_{t=0}^{\infty} e^{-st} H(t - \tau_0 - \cdots - \tau_{m-1})\right)$$
$$= \mathbb{E}\left(\sum_{t=\tau_0+\cdots+\tau_{m-1}}^{\infty} e^{-st}\right)$$
$$= \frac{1}{1 - e^{-s}} \mathbb{E}(e^{-s(\tau_0+\cdots+\tau_{m-1})})$$
$$= \frac{1}{1 - e^{-s}} \hat{F}(s)^m. \quad (3)$$

In particular, Eq. (3) provides a simple expression for the moments of the number of records, given by

$$\mathcal{L}\{\mathbb{E}(M^k(t))\} = \sum_{m=0}^{\infty} m^k \mathcal{L}\{\mathbb{P}(M(t) = m)\}. \quad (4)$$

Then considering the long-time small-s limit for which $\hat{F}(s)$ approaches one [see Eq. (2) with $s = 0$], one can replace the sum by an integral to obtain the following simple expression:

$$\mathcal{L}\{\mathbb{E}(M^k(t))\} \sim \int_0^{\infty} dm\, m^k (-\partial_m) \mathcal{L}\{\mathbb{P}(M(t) \geqslant m)\}$$
$$\sim \frac{\Gamma(k+1)}{s[-\ln \hat{F}(s)]^k}. \quad (5)$$

Finally, obtaining record statistics is reduced to the determination of the statistics of the τ_k or, in other words, to solving a first-passage-time (FPT) problem from the origin to a point at distance one. In Appendix A we recall for self-consistency how to obtain these FPT distributions for various models, such as the ABM and the ARTP in both discrete and continuous time (see also [42]). It results in the expressions in the long-time small-s limit [17,26,29,43,44]

$$-\ln \hat{F}(s) \sim w(s) \equiv \begin{cases} \sqrt{s/D} & \text{(BM)} \\ -\frac{\mu}{2D} + \sqrt{\frac{\mu^2}{4D^2} + \frac{s}{D}} & \text{(ABM)} \\ \sqrt{\frac{s^2}{v^2} + \frac{s}{v^2 T}} & \text{(RTP)} \\ \frac{\sqrt{4sTv^2(sT+1)+\mu^2}-\mu(2sT+1)}{2T(v^2-\mu^2)} & \text{(ARTP)}, \end{cases} \quad (6)$$

where D is the diffusive constant, μ is the bias, T is the tumbling rate, and v is the average absolute speed during a run of the RTP (see Appendix A for precise definition of these models and extension of the ARTP with a state-dependent tumbling rate). As we proceed to show, numerous record-type observables have generic and simple expressions as functions of $w(s)$.

B. Time of last record

Using renewal theory, one can easily obtain the joint distribution of the number m of records at time t and the time $T_m = \tau_0 + \cdots + \tau_{m-1}$ at which this last record was reached, as well as the backward (forward) record times $B_t \equiv t - T_{M(t)}$
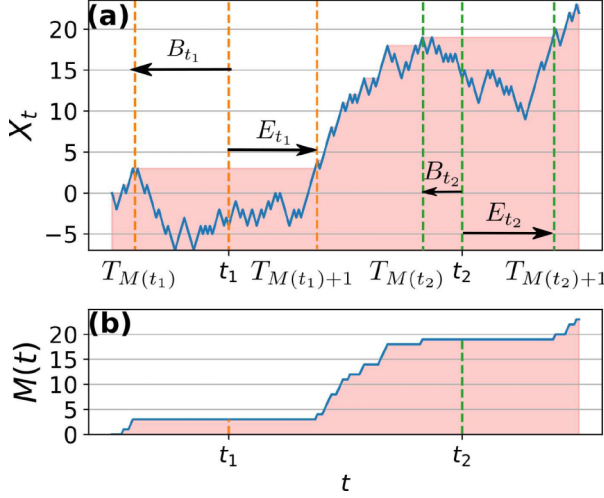
FIG. 1. Representation of the record process. (a) Space and time trajectory of a 1D symmetric nearest-neighbor random walk and (b) the dynamics of its number of records, which coincides with its maximum. We are interested in the joint statistics of the number of records at different times $M(t_1)$ and $M(t_2)$ as well as the time of occurrence of the last (next) record event $T_{M(t_1)} = t_1 - B_{t_1}$ ($T_{M(t_1)+1} = t_1 + E_{t_1}$).

($E_t \equiv T_{M(t)+1} - t$) corresponding to the duration of the current record (time left until observation of a new record), represented in Fig. 1. We note that, while the time at which the last record is reached is a standard observable (see, for instance, [4,21,24,45]), the forward record time has not been studied in the context of records.

To obtain the joint distribution of $(M(t), T_{M(t)})$, we use that having observed $M(t) = m$ records at time t with the last record occurring at time $T_{M(t)} = T_m = y$ is exactly the event of observing m records at time y shorter than t and observing the $(m + 1)$th record after t, i.e., $T_{m+1} > t$. By taking the Laplace transform with respect to both t and y of this event, we obtain that

$$\mathcal{L}\{\mathbb{P}(T_m = y, M(t) = m)\}$$

$$= \sum_{t,y=0}^{\infty} e^{-uy-st} \mathbb{E}(H(T_{m+1} - t) H(t - y) \delta(y - T_m))$$

$$= \frac{1}{1 - e^{-s}} \mathbb{E}(e^{-(s+u)(\tau_0 + \cdots + \tau_{m-1})}(1 - e^{-s\tau_m}))$$

$$= \hat{F}(s + u)^m \frac{1 - \hat{F}(s)}{1 - e^{-s}} \tag{7}$$

$$\sim e^{-w(s+u)m} \frac{w(s)}{s}. \tag{8}$$

While Eq. (7) corresponds to the exact expression of the number of records at any time for the discrete process, Eq. (8) corresponds to the long-time asymptotic limit which also describes exactly the maximum of the continuous asymptotic process (see Appendix B for details). Equation (8) provides the Laplace transform of the joint distribution of the maximum and the time to reach the maximum, which has been studied for a variety of continuous stochastic processes [20,21,23,28,46].

C. Backward record time

Then we compute the joint distribution of the backward record time and the number of records $(B_t, M(t))$. To do so, we consider the event of having reached $M(t) = m$ records with the time elapsed since the last record $B_t = t - T_m = y$. This event corresponds to having reached m records at time $t - y$ and not having reached any new record in the time y left (with $y \leqslant t$), that is,

$$\mathcal{L}\{\mathbb{P}(B_t = y, M(t) = m)\}$$

$$= \sum_{t,y=0}^{\infty} e^{-uy-st} \mathbb{E}(H(T_{m+1} - t) H(t - y) \delta(t - y - T_m))$$

$$= \frac{1}{1 - e^{-s-u}} \mathbb{E}(e^{-s(\tau_0 + \cdots + \tau_{m-1})}(1 - e^{-(s+u)\tau_m})),$$

$$\mathcal{L}\{\mathbb{P}(B_t = y, M(t) = m)\} = \hat{F}(s)^m \frac{1 - \hat{F}(s + u)}{1 - e^{-s-u}} \tag{9}$$

$$\sim e^{-w(s)m} \frac{w(s + u)}{s + u}. \tag{10}$$

Once again, Eq. (9) is the exact discrete expression and Eq. (10) is the asymptotic continuum limit (see Appendix B for details). It is noteworthy that the random variables $T_{M(t)}$ and B_t essentially represent the same observables; however, the former is commonly utilized in extreme value statistics [4,24,45,47], while the latter is more prevalent in renewal theory [33,48].

One can perform explicit inverse Laplace transforms of Eqs. (8) and (10) (see Appendix A 3) to obtain the expressions derived in [4,20,21,46,49] for the specific cases of ABM and the RTP. We stress that these results were obtained by a different method, relying on path decomposition technique.

D. Forward record time

Finally, we compute the joint distribution of the forward record time and current number of records $(E_t, M(t))$. To do so, we consider the event of reaching $M(t) = m$ records with the time until the next one $E_t = T_{m+1} - t = T_m + \tau_m - t = y$. This event corresponds to achieving m records at a time less than t and $m + 1$ records at exactly $t + y$. Thus,

$$\mathcal{L}\{\mathbb{P}(E_t = y, M(t) = m)\}$$

$$= \sum_{t,y=0}^{\infty} e^{-uy-st} \mathbb{E}(\delta(y + t - T_m - \tau_m) H(t - T_m))$$

$$= \mathbb{E}\left(e^{-s(\tau_0 + \cdots + \tau_{m-1})} \frac{e^{-u\tau_m} - e^{-s\tau_m}}{1 - e^{-s+u}}\right)$$

$$= \hat{F}(s)^m \frac{\hat{F}(u) - \hat{F}(s)}{1 - e^{-s+u}} \tag{11}$$

$$\sim e^{-w(s)m} \frac{w(s) - w(u)}{s - u}, \tag{12}$$

where Eq. (11) is the exact discrete expression and Eq. (12) its asymptotic continuum limit. This general result can be illustrated by the explicit examples of the ABM and the ARTP (see Fig. 2). In the continuum setting, one can take the Laplace
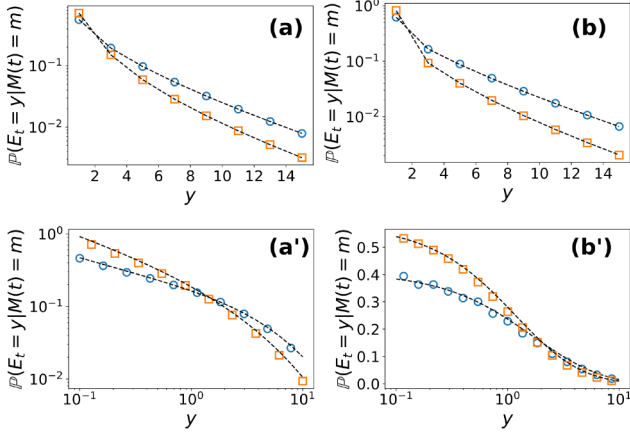
FIG. 2. Conditional forward record time. The distribution of E_t conditioned on the number of records $M(t) = 2$ (blue circles) or 8 (orange squares) for the asymmetric walks at time $t = 10$ is shown for (a) the discrete ABM (probability to go to the right, 3/4), (b) the discrete ARTP (probability to go to the right coming from the right, 7/8, and to the left coming from the left, 3/8), (a′) ABM (diffusion constant $D = 1$ and bias $\mu = 0.5$), and (b′) ARTP (tumbling rate $T = 1/2$, average absolute speed $v = 1$, and bias $\mu = 0.5$). The black dashed lines in (a) and (b) stand for the series expansion of the discrete expression (11). The black dashed lines in (a′) and (b′) stand for the expressions (13) and (14).

inverse of Eq. (12) and get, for the ABM the joint distribution,

$$\mathbb{P}(M(t) = m, E_t = y)$$
$$= \int_0^t dt' \frac{m \exp\left(-\frac{\mu^2(y+t')}{4D} - \frac{[m-\mu(t-t')]^2}{4D(t-t')}\right)}{4D\pi \left[(y+t')(t-t')\right]^{3/2}}, \quad (13)$$

and for the ARTP

$$\mathbb{P}(M(t) = m, E_t = y)$$
$$= \int_0^t \frac{(2T)^{-1}dt'}{t - t' + y} \frac{e^{-(t+y)/2T}}{\sqrt{v^2 - \mu^2}} I_1\left((t - t' + y)\frac{\sqrt{v^2 - \mu^2}}{2vT}\right)$$
$$\times \left[\delta\left(t' - \frac{m}{v + \mu}\right)\right.$$
$$+ \frac{m(2T)^{-1}H[t' - m/(v+\mu)]}{\sqrt{[t'(v-\mu)+m][t'(v+\mu)-m]}}$$
$$\left. \times I_1\{\sqrt{[t'(v-\mu)+m][t'(v+\mu)-m]}/2vT\}\right]. \quad (14)$$

Here D, T, μ, and v are as defined in Eq. (6).

We represent these two analytical expressions in Fig. 2 as well as their discrete counterpart. We compare the distributions conditioned on the number $M(t) = m$ of records at time t for different values of m. In particular, we observe that having a high number of records increases the probability of observing the next record in a short amount of time. This highlights the importance of correlations in record processes, as well as the necessity of quantifying multiple-time distributions to fully understand record dynamics. This is the object of the next section.

III. MULTIPLE-TIME POINT DISTRIBUTIONS

In this section we show that the methodology introduced above allows us to derive straightforwardly multiple-time distributions. For simplicity, we focus on the long time and large number of record asymptotics, even though our method gives the exact results at all times for the description of the maximum of the continuous process, as shown in the preceding section. In particular, we use continuous Laplace transforms and the record ages of the corresponding continuous processes.

A. Two-time distribution

We start with the two-time distribution of the number of records, before generalizing it to an arbitrary number of time points and considering other record-type observables.

As for the single-time case, we make use of the record ages τ_k, as

$$\mathbb{P}(M(t_1) \geqslant m_1, M(t_2) \geqslant m_2)$$
$$= \mathbb{P}(\tau_0 + \cdots + \tau_{m_1-1} \leqslant t_1, \tau_0 + \cdots + \tau_{m_2-1} \leqslant t_2)$$
$$= \mathbb{E}(H(t_1 - \tau_0 - \cdots - \tau_{m_1-1})H(t_2 - \tau_0 - \cdots - \tau_{m_2-1})). \quad (15)$$

By Laplace transforming with respect to both time variables t_1 and t_2, we obtain ($m_1 \leqslant m_2$)

$$\mathcal{L}\{\mathbb{P}(M(t_1) \geqslant m_1, M(t_2) \geqslant m_2)\}$$
$$= \iint_0^\infty dt_1 dt_2 e^{-s_1 t_1 - s_2 t_2} \mathbb{P}(M(t_1) \geqslant m_1, M(t_2) \geqslant m_2)$$
$$= \frac{\mathbb{E}(e^{-(s_1+s_2)(\tau_0 + \cdots + \tau_{m_1-1}) - s_2(\tau_{m_1} + \cdots + \tau_{m_2-1})})}{s_1 s_2}$$
$$= \frac{e^{-w(s_2+s_1)m_1 - w(s_2)(m_2-m_1)}}{s_1 s_2}. \quad (16)$$

Thus, by deriving Eq. (16) for $m_1 < m_2$, we get the density

$$\mathcal{L}\{\mathbb{P}(M(t_1) = m_1, M(t_2) = m_2)\}$$
$$= \frac{e^{-w(s_2+s_1)m_1 - w(s_2)(m_2-m_1)}}{s_1 s_2} w(s_2)[w(s_1 + s_2) - w(s_2)]. \quad (17)$$

Additionally, one gets the Laplace transform of the probability of having a maximum $M(t_1)$ of value m at time t_1 and $M(t_2)$ of the same value m at time t_2,

$$\mathcal{L}\{\mathbb{P}(M(t_1) = M(t_2) = m)\}$$
$$= \mathcal{L}\{\mathbb{P}(M(t_1) = m)\}$$
$$- \int_{m_2 > m} dm_2 \mathcal{L}\{\mathbb{P}(M(t_1) = m, M(t_2) = m_2)\}$$
$$- \int_{m > m_2} dm_2 \mathcal{L}\{\mathbb{P}(M(t_2) = m_2, M(t_1) = m)\}$$
$$= \frac{e^{-w(s_2+s_1)m}}{s_1 s_2}[w(s_1) + w(s_2) - w(s_1 + s_2)]. \quad (18)$$

Note that here the Laplace transform involves both t_1 and t_2 and thus the Laplace transform of $\mathbb{P}(M(t_1) = m)$ includes an additional prefactor of $1/s_2$ compared to Eq. (3), as it remains

independent of t_2. This results in a compact formula for the Laplace transform of the covariance of the number of records at two distinct times,

$$\mathcal{L}\{\text{Cov}[M(t_1), M(t_2)]\} = \frac{w(s_1) + w(s_2) - w(s_1 + s_2)}{s_1 s_2 w(s_1) w(s_2) w(s_1 + s_2)}, \quad (19)$$

in the limit of long times and a large number of records.

Equation (16) extends the findings of [12] to arbitrary Markovian processes with nearest-neighbor jumps. In particular, it allows the computation of various two-time functionals for the process. An illustrative example is provided by the conditional survival probability of a Rosenstock trapping problem [50], where traps, present in concentration q, are exclusively located on the positive axis [51]. The conditional survival probability $S(t_2|t_1)$, defined as the probability of survival up to time t_2 given that the random walk survived up to time t_1, is given by the probability that there is no trap in the newly visited domain of extension $M(t_2) - M(t_1)$. This implies that $S(t_2|t_1) = \mathbb{E}((1-q)^{M(t_2)-M(t_1)})$. This quantity is fully determined by the knowledge of the two-time distribution (16).

B. The n-time distributions

The previous calculations can be extended to determine n-time distributions. For the sake of simplicity, we focus here on the case $0 = m_0 < m_1 < \cdots < m_n$, where at least one new record occurs at each observation time.

First, for the joint distribution of having at least m_k records at times t_k for $k = 1$ to n, we get, in the Laplace domain,

$$\mathcal{L}\{\mathbb{P}(M(t_1) \geqslant m_1, \ldots, M(t_n) \geqslant m_n)\}$$

$$= \mathbb{E}\left(\prod_{k=1}^{n} \int_0^\infty dt_k H(t_k - T_{m_k}) e^{-s_k t_k}\right)$$

$$= \frac{1}{s_1 \cdots s_n} \mathbb{E}\left(\prod_{k=1}^{n} e^{-s_k T_{m_k}}\right)$$

$$= \frac{\mathbb{E}(e^{-(s_1+\cdots+s_n)T_{m_1}})}{s_1 \cdots s_n} \mathbb{E}\left(\prod_{k=2}^{n} e^{-s_k(T_{m_k}-T_{m_1})}\right)$$

$$= \frac{1}{s_1 \cdots s_n} e^{-w(s_1+\cdots+s_n)m_1} \mathbb{E}\left(\prod_{k=2}^{n} e^{-s_k T_{m_k-m_1}}\right)$$

$$= \frac{1}{s_1 \cdots s_n} \prod_{k=1}^{n} \exp\left[-w\left(\sum_{j=k}^{n} s_j\right)(m_k - m_{k-1})\right]. \quad (20)$$

We recall that the result is exact for the maximum, but asymptotic for describing the records, such that for every k, the timescales s_k approach zero (with their ratio remaining finite) and the differences $m_k - m_{k-1}$ behave inversely proportional to $w(s_k)$.

Then, for the joint distribution of the successive times $T_{M(t_k)}$ at which the records were last reached and their associated number of records

$$\mathcal{L}\{\mathbb{P}(M(t_k) = m_k, T_{m_k} = y_k; 1 \leqslant k \leqslant n)\}$$

$$= \mathbb{E}\left(\prod_{k=1}^{n} \iint_0^\infty dt_k dy_k H(t_k - T_{m_k}) H(T_{m_k+1} - t_k) \delta(y_k - T_{m_k}) e^{-u_k y_k - s_k t_k}\right)$$

$$= \mathbb{E}\left(\prod_{k=1}^{n} e^{-(u_k+s_k)T_{m_k}} \frac{1 - e^{-s_k \tau_{m_k}}}{s_k}\right)$$

$$= \mathbb{E}\left[\exp\left(-T_{m_1}\sum_{j=1}^{n}(u_j+s_j)\right) \exp\left(-\tau_{m_1}\sum_{j=2}^{n}(u_j+s_j)\right) \frac{1 - e^{-s_1 \tau_{m_1}}}{s_1}\right] \mathbb{E}\left(\prod_{k=2}^{n} e^{-(u_k+s_k)(T_{m_k}-T_{m_1}-\tau_{m_1})} \frac{1 - e^{-s_k \tau_{m_k}}}{s_k}\right)$$

$$= \frac{1}{s_1 \ldots s_n} \prod_{k=1}^{n} \left\{\exp\left[-w\left(\sum_{j=k}^{n}(u_j+s_j)\right)(m_k - m_{k-1})\right]\left[w\left(s_k + \sum_{j=k+1}^{n}(u_j+s_j)\right) - w\left(\sum_{j=k+1}^{n}(u_j+s_j)\right)\right]\right\}. \quad (21)$$

Similarly, for the n consecutive backward record times B_{t_k} and their associated number of records,

$$\mathcal{L}\{\mathbb{P}(M(t_k) = m_k, B_{t_k} = y_k; 1 \leqslant k \leqslant n)\} = \mathbb{E}\left(\prod_{k=1}^{n} \iint_0^\infty dt_k dy_k H(t_k - T_{m_k}) H(T_{m_k+1} - t_k) \delta(t_k - y_k - T_{m_k}) e^{-u_k y_k - s_k t_k}\right)$$

$$= \mathbb{E}\left(\prod_{k=1}^{n} e^{-s_k T_{m_k}} \frac{1 - e^{-(s_k+u_k)\tau_{m_k}}}{s_k + u_k}\right)$$

$$= \prod_{k=1}^{n}\left\{\exp\left[-w\left(\sum_{j=k}^{n} s_j\right)(m_k - m_{k-1})\right]\frac{w\left(u_k + \sum_{j=k}^{n} s_j\right) - w\left(\sum_{j=k+1}^{n} s_j\right)}{s_k + u_k}\right\}. \quad (22)$$

Finally, for the n consecutive forward record times E_{t_k} and their associated number of records,

$$
\mathcal{L}\{\mathbb{P}(M(t_k) = m_k, E_{t_k} = y_k; 1 \leqslant k \leqslant n)\}
$$

$$
= \mathbb{E}\left(\prod_{k=1}^{n} \iint_0^\infty dt_k dy_k H(t_k - T_{m_k}) \delta(y_k + t_k - T_{m_k+1}) e^{-u_k y_k - s_k t_k} \right)
$$

$$
= \mathbb{E}\left(\prod_{k=1}^{n} e^{-s_k T_{m_k}} \frac{e^{-u_k \tau_{m_k}} - e^{-s_k \tau_{m_k}}}{s_k - u_k} \right)
$$

$$
= \mathbb{E}\left(e^{-(s_1 + \cdots + s_n)T_{m_1} - (s_2 + \ldots + s_n)\tau_{m_1}} \frac{e^{-u_1 \tau_{m_1}} - e^{-s_1 \tau_{m_1}}}{s_1 - u_1} \right) \mathbb{E}\left(\prod_{k=2}^{n} e^{-s_k(T_{m_k} - T_{m_1} - \tau_{m_1})} \frac{e^{-u_k \tau_{m_k}} - e^{-s_k \tau_{m_k}}}{s_k - u_k} \right)
$$

$$
= \prod_{k=1}^{n} \left\{ \exp\left[-w\left(\sum_{j=k}^{n} s_j \right)(m_k - m_{k-1}) \right] \frac{w\left(\sum_{j=k}^{n} s_j \right) - w\left(u_k + \sum_{j=k+1}^{n} s_j \right)}{s_k - u_k} \right\}. \tag{23}
$$

Note that we do not derive the formula when two successive m_k are equal: As a matter of fact, one can derive these values from the one with distinct successive m_k as for Eq. (18). The multitime joint distribution of $M(t)$, E_t, and B_t can similarly be obtained. It is explicitly given in Appendix C.

As an explicit example of a multitime distribution, one can compute the joint distribution of the time at which the last record event occurred at two different times t_1 and t_2. This can be done using Eq. (21) where $n = 2$. We integrate over m_1 and m_2 and consider the times $y_1 < t_1 < y_2 < t_2$ such that for the BM (see Fig. 3)

$$
\mathbb{P}(T_{M(t_2)} = y_2 | T_{M(t_1)} = y_1) = \frac{\sqrt{y_2 - t_1}}{\pi(y_2 - y_1)\sqrt{t_2 - y_2}}. \tag{24}
$$

This provides a generalization of the celebrated arcsine law [1] for the distribution of the time to reach the maximum in a time interval. In particular, we observe that having a time y_1 of observation of the last record in the interval $[0, t_1]$ close to t_1 increases the probability of having a long time of observation of the last record up to t_2. Thus, there are high
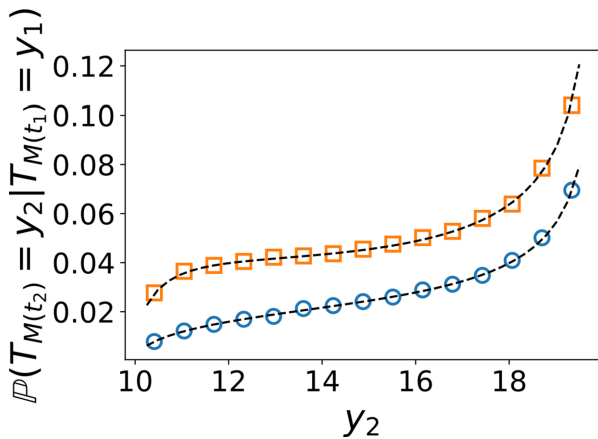

FIG. 3. Conditional last record time. The distribution of $T_{M(t_2)}$ conditioned on the time $T_{M(t_1)} = 2$ (blue circles) or 8 (orange squares) is shown for the symmetric BM of the parameters $D = 1$, $t_1 = 10$, and $t_2 = 20$. The black dashed line corresponds to Eq. (24).

correlations between successive last record times that cannot be accounted for by single-time expressions.

As an other example of application, for any n, we can compute the difference between the distribution of record events and that of independent record occurrences, akin to the methodology outlined in [51]. This investigation holds particular relevance when evaluating the probability of a maximum or record trajectory sampled at times $t_1, t_2, \ldots, t_n$ to discern properties of the RW, as explored in [30]. In the scenario where the time intervals between samplings are significantly large, $t_1 \ll t_2 \ll \cdots \ll t_n$, or equivalently $s_1 \gg s_2 \gg \cdots \gg s_n$ with $m_k w(s_k)$ held constant (considering only typical events), we have

$$
\mathcal{L}\{\mathbb{P}(M(t_k) \geqslant m_k; 1 \leqslant k \leqslant n)\} - \mathcal{L}\left\{ \prod_{k=1}^{n} \mathbb{P}(M(t_k) \geqslant m_k) \right\}
$$

$$
\sim \mathcal{L}\left\{ \prod_{k=1}^{n} \mathbb{P}(M(t_k) \geqslant m_k) \right\}
$$

$$
\times \left(-\sum_{k=1}^{n} w'(s_k) s_{k+1} m_k + w(s_k) m_{k-1} \right). \tag{25}
$$

Utilizing the asymptotics

$$
w'(s_k) s_{k+1} m_k \propto \frac{s_{k+1}}{s_k}, \qquad w(s_k) m_{k-1} \propto \frac{w(s_k)}{w(s_{k-1})}, \tag{26}
$$

we deduce that the process decorrelates when there are long-time intervals between samplings for processes where $w(s)$ approaches 0 as s tends to 0.[1] In such instances, the decorrelation follows an algebraic pattern with an exponent of $1/2$ for the symmetric case and 1 for the biased case. The rate of convergence is determined by the largest of the ratios s_k/s_{k-1}, or equivalently t_{k-1}/t_k. In essence, employing the distribution of maxima at n discrete times under the assumption of independence among maxima incurs an error that scales proportionally with the smallest ratio of sampling times, with

---

[1]Note that this condition is not always met for ABM and the ARTP, so correlations are not decaying in time.

the exponent contingent upon the symmetry properties of the walk.

We stress that, by determining these multiple-time distributions, we have provided a complete characterization of the record process.

IV. GENERALIZATION TO NONIDENTICAL RENEWAL TIMES

One can easily generalize the preceding results to the case of non-identically-distributed but still independent renewal times. An important explicit example involving aging renewal times is provided by the resetting RW [18,24,25,52,53] (see Appendix D 2 for the other important example of a RW with a reflecting boundary condition at the initial position).

In the resetting RW, at every time step, the RW returns to its initial position with probability $1 - \lambda = 1 - e^{-r}$ (r is

the resetting rate). These types of processes have attracted a great deal of attention in the past few years and are well documented. In particular, the record age distribution with resetting (where we reset both the position and velocity to the initial values), knowing that m records have been achieved, $\hat{F}_m^r(s)$, is given by (see Appendix D 1 for details)

$$1 - \hat{F}_{m-1}^r(s) = s \frac{1 - \hat{F}(s+r)}{s + r\hat{F}(s+r)^m}$$

$$= s \frac{w(s+r)}{s + re^{-w(s+r)m}} \equiv g(s, m), \qquad (27)$$

where $\hat{F}$ and w stand for the corresponding quantities without resetting.

Following the methodology developed in the previous sections, in particular following the steps described in Eqs. (20)–(23), we get, for the n records distributions,

$$\mathcal{L}\{\mathbb{P}(M(t_1) \geqslant m_1, \ldots, M(t_n) \geqslant m_n)\} = \frac{1}{s_1 \cdots s_n} \mathbb{E}(e^{-(s_1 + \cdots + s_n)T_{m_1}}) \mathbb{E}\left(\prod_{k=2}^{n} e^{-s_k(T_{m_k} - T_{m_1})}\right)$$

$$= \frac{1}{s_1 \cdots s_n} \exp\left(-\int_0^{m_1} g(s_1 + \cdots + s_n, m)dm\right) \mathbb{E}(e^{-s_2(T_{m_2} - T_{m_1})} \cdots e^{-s_n(T_{m_n} - T_{m_1})})$$

$$= \prod_{k=1}^{n} \left(\frac{\exp\left[-\int_{m_{k-1}}^{m_k} g\left(\sum_{j=k}^{n} s_j, m\right)dm\right]}{s_k}\right). \qquad (28)$$

Then, for the successive times at which the records were last reached and their current number,

$$\mathcal{L}\{\mathbb{P}(M(t_k) = m_k, T_{m_k} = y_k; 1 \leqslant k \leqslant n)\} = \prod_{k=1}^{n} \left\{ \exp\left[-\int_{m_{k-1}}^{m_k} dm\, g\left(\sum_{j=k}^{n}(u_j + s_j), m\right)\right] \right.$$

$$\left. \times \frac{g\left(s_k + \sum_{j=k+1}^{n}(u_j + s_j), m_k\right) - g\left(\sum_{j=k+1}^{n}(u_j + s_j), m_k\right)}{s_k} \right\}. \qquad (29)$$

Similarly, for the n consecutive backward record times and their associated number of records,

$$\mathcal{L}\{\mathbb{P}(M(t_k) = m_k, B_{t_k} = y_k; 1 \leqslant k \leqslant n)\} = \prod_{k=1}^{n} \left\{ \exp\left[-\int_{m_{k-1}}^{m_k} dm\, g\left(\sum_{j=k}^{n} s_j, m\right)\right] \frac{g\left(u_k + \sum_{j=k}^{n} s_j, m_k\right) - g\left(\sum_{j=k+1}^{n} s_j, m_k\right)}{s_k + u_k} \right\}. \qquad (30)$$

Finally, for the n consecutive forward record times and their associated number of records,

$$\mathcal{L}\{\mathbb{P}(M(t_k) = m_k, E_{t_k} = y_k; 1 \leqslant k \leqslant n) = \prod_{k=1}^{n} \left\{ \exp\left[-\int_{m_{k-1}}^{m_k} g\left(\sum_{j=k}^{n} s_j, m\right)dm\right] \right.$$

$$\left. \times \frac{g\left(s_k + \sum_{j=k+1}^{n} s_j, m_k\right) - g\left(u_k + \sum_{j=k+1}^{n} s_j, m_k\right)}{s_k - u_k} \right\}. \qquad (31)$$

As an example, we provide in Fig. 4 the conditional distribution of the number of records $M(t_2)$ at time t_2 knowing the number of records $M(t_1)$ at time t_1 for different values of $M(t_1)$, for the resetting BM. Numerically Laplace inverting Eq. (28) for the resetting BM, we observe that having a large number of records at early time decreases the probability of observing new records at later times. Indeed, if the RW reaches a large number of records (and thus a position far

from its origin) before resetting, it will become harder for later trajectories after reset to overcome a similar number of records.

Additionally, from Eqs. (30) and (31) one can derive the asymptotic distributions of B_t and E_t. We note that Eq. (29) gives the same formula as obtained in [24] for the single-time distribution and thus B_t is asymptotically uniformly distributed in the interval $[0, t]$ at long times t, independently
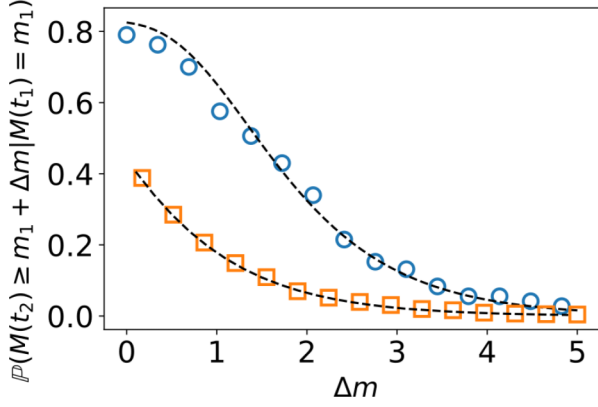
FIG. 4. Conditional tail distribution of $M(t_2)$ knowing that $M(t_1) = 1$ (blue circles) and $M(t_1) = 3$ (orange squares) at times $t_1 = 10$ and $t_2 = 20$ for the resetting BM of the parameters $r = 1$ and $D = 1$. Black dashed lines represent the result of numerical Laplace inversion of Eq. (28) for $n = 2$.

FIG. 5. Time E_t until the next record-breaking event from time $t = 100$ for BM (blue stars), ABM (orange circles), RTP (green squares), and ARTP (red triangles) of resetting rate $r = 1$, diffusion coefficient $D = 1$, $T = 1$, $v = 1$, and $\mu = 0.5$. The asymptotic expression of Eq. (33) is shown by the black dashed line.

of the process. In fact, one can find a similar result for the forward record time E_t. In the limit $t, y \gg 1/r$ corresponding to $s, u \ll r$, we have a universal form for the distribution of E_t,

$$
\mathcal{L}\{\mathbb{P}(E_t = y)\} \sim \int_0^\infty dm \frac{r}{r + se^{-w(r)m}} \frac{w(r)}{s - u}
$$
$$
\times \left(\frac{s}{s + re^{w(r)m}} - \frac{u}{u + re^{w(r)m}} \right)
$$
$$
= \int_0^1 dM \frac{r}{(r + sM)^2} \frac{1}{uM + r}, \tag{32}
$$

and by Laplace inversion,

$$
\mathbb{P}(E_t \geqslant y) = \int_0^1 dM \, rte^{-Mrt} e^{-Mry} \sim \frac{t}{t + y}. \tag{33}
$$

This is exactly the result one expects for the time to break the next record from a series of independent and identically distributed random variables at discrete times $(X_{t'})_{t' \geqslant 0}$ of cumulative $F(x) = \mathbb{P}(X \leqslant x)$:

$$
\mathbb{P}(E_t \geqslant y)
$$
$$
= \int_0^\infty dm \, \mathbb{P}(X_{t+k} \leqslant m; 1 \leqslant k \leqslant y) \mathbb{P}\left(\max_{t' \leqslant t}(X_{t'}) = m \right)
$$
$$
= \int_0^\infty dm \, t F(m)^{t+y-1} \frac{d}{dm} F(m) = \frac{t}{t + y}. \tag{34}
$$

Similarly to what was shown in [24] in the case of the backward time, this universal result holds for the general stochastic process [arbitrary $w(s)$] as it stems from the independence between reset trajectories, as it is shown in Fig. 5.

V. SUMMARY

We have developed a general approach based on renewal theory to derive multiple-time statistics for the number of records in one dimension for nearest-neighbor Markovian random walks, along with various observables associated with the record process. The formulas obtained are compact in the
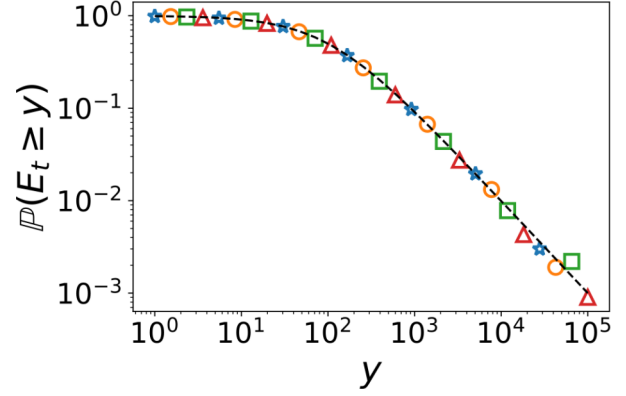
Laplace domain and can be easily used to gain insights into the statistics of records. These multitime distributions provide a complete characterization of the statistics of records. We have demonstrated that this method can be generalized to resetting random walks, thereby determining the complete statistics of record observables.

APPENDIX A: FIRST-PASSAGE-TIME DISTRIBUTIONS OF ABM AND ARTP

The FPT distribution is the key ingredient for that of the number of records [see Eq. (3)]. For self-consistency, we provide in this Appendix the standard computations [29,42,43] of the FPT distributions of both the ABM and the ARTP, in discrete and continuous time.

1. Asymmetric Brownian motion

a. Discrete time

In the discrete asymmetric nearest-neighbor RW model (or discrete ABM) on the 1D lattice, the random walker hops to the site on its right with probability p and the one on its left with probability $1 - p$. To obtain the FPT of the random walker to position m, we consider the backward master equation in discrete time, denoting by $P(x, t)$ the probability to reach position m for the first time starting at position $x < m$,

$$
P(x, t + 1) = pP(x + 1, t) + (1 - p)P(x - 1, t), \tag{A1}
$$

with boundary conditions $P(m, t) = \delta(t)$ and $P(-\infty, t) = 0$. The record age distribution is given by $P(0, t)$ for $m = 1$. By the discrete Laplace transform of Eq. (A1),

$$
\mathcal{L}\{f(t)\} = \hat{f}(s) = \sum_{t=0}^\infty e^{-st} f(t), \tag{A2}
$$

we obtain

$$
e^s \hat{P}(x, s) = p\hat{P}(x + 1, s) + (1 - p)\hat{P}(x - 1, s). \tag{A3}
$$

From the boundary conditions, we deduce that

$$\hat{P}(x, s) = \left(\frac{e^s}{2(1-p)} (1 - \sqrt{1 - 4p(1-p)e^{-2s}}) \right)^{m-x}. \quad \text{(A4)}$$

For $m = 1$ and $x = 0$, we deduce the record age distribution for the discrete ABM,

$$\hat{F}(s) = \frac{e^s}{2(1-p)} [1 - \sqrt{1 - 4p(1-p)e^{-2s}}]. \quad \text{(A5)}$$

b. Continuous time

In the ABM model on the 1D line, the time evolution of the position X_t of the random walker is given by the Langevin equation with constant external force μ (the bias),

$$\frac{dX_t}{dt} = \mu + \eta_t, \quad \text{(A6)}$$

where the stochastic process $(\eta_t)_{t \geqslant 0}$ is a Gaussian white noise,

$$\langle \eta_t \rangle = 0, \quad \langle \eta_t \eta_{t'} \rangle = 2D\delta(t - t'). \quad \text{(A7)}$$

To obtain the FPT to position m of the random walker, we consider the backward master equation in continuous time,

$$\partial_t p(x, t) = (D\partial_{x,x}^2 + \mu\partial_x)p(x, t), \quad \text{(A8)}$$

with boundary conditions $p(m, t) = \delta(t)$ and $p(-\infty, t) = 0$ such that $p(x, t)$ stands for the probability that the random walker initially at position x arrived at position $m > 0$ for the first time at time t. The record age distribution is given by $p(0, t)$ for $m = 1$.

To solve Eq. (A8), we make a Laplace transformation. The Laplace transform of a function $f(t)$ is defined in the continuum setting as

$$\mathcal{L}\{f(t)\} = \hat{f}(s) \equiv \int_0^\infty dt \, e^{-st} f(t). \quad \text{(A9)}$$

Using that $p(x < m, t = 0) = 0$, Eq. (A8) becomes, once Laplace transformed (in time),

$$(D\partial_{x,x}^2 + \mu\partial_x - s)\hat{p}(x, s) = 0 \quad \text{(A10)}$$

and the initial condition $\hat{p}(x = m, s) = 1$. We search an exponential solution of the form $\hat{p}(x, s) = e^{-w(s)(m-x)}$, where $w(s)$ is positive $[\hat{p}(-\infty, s) = 0]$. This results in the equation for $w(s)$,

$$Dw(s)^2 + \mu w(s) - s = 0, \quad \text{(A11)}$$

whose positive solution is

$$w(s) = -\frac{\mu}{2D} + \frac{\sqrt{\mu^2 + 4Ds}}{2D}. \quad \text{(A12)}$$

Finally, we have the Laplace transform of the first passage time at m starting at position x,

$$\hat{F}(s, x) \equiv \hat{p}(x, s) = e^{-w(s)(m-x)}, \quad \text{(A13)}$$

which provides the record age distribution for $m = 1$ (we set a scale needed to traverse in order to make a new record at 1; see Appendix B for details) and $x = 0$,

$$\hat{F}(s) = e^{-w(s)}. \quad \text{(A14)}$$

Note that the discrete model gives asymptotically the continuous model for $p = \frac{1+\mu}{2}$.

2. Asymmetric run-and-tumble particle

a. Discrete time

In the discrete ARTP model on the 1D lattice, the random walker has two states, the positive state in which it moves to the neighboring site on its right and a negative state in which it moves to the neighboring site on its left. At each time step, the random walker can change its state: It goes from the positive to the negative state with probability p_+ and from the negative to the positive state with probability p_-.

To obtain the FPT of the random walker to position m, we start from the backward master equation in discrete time,

$$P_+(x, t+1) = p_+ P_+(x+1, t) + (1-p_+)P_-(x-1, t),$$
$$P_-(x, t+1) = p_- P_-(x-1, t) + (1-p_-)P_+(x+1, t), \quad \text{(A15)}$$

with boundary conditions $P_+(m, t) = \delta(t)$ and $P_+(-\infty, t) = P_-(-\infty, t) = 0$ such that $P_+(x, t)$ $[P_-(x, t)]$ stands for the probability that the random walker initially at position x going to the right (going to the left) arrived at position $m > 0$ for the first time at time t (one can only arrive at position $m > x$ by doing a last step to the right). The record age distribution is given by $P_+(0, t)$ for $m = 1$. We take the discrete Laplace transform of both equations to obtain

$$e^s \hat{P}_+(x, s) = p_+ \hat{P}_+(x+1, s) + (1-p_+)\hat{P}_-(x-1, s),$$
$$e^s \hat{P}_-(x, s) = p_- \hat{P}_-(x-1, s) + (1-p_-)\hat{P}_+(x+1, s). \quad \text{(A16)}$$

Then, by expressing $\hat{P}_-$ as a function of $\hat{P}_+$ using the first of Eqs. (A16) and replacing it in the second, we have an equation solely for $\hat{P}_+$,

$$\hat{P}_-(x, s) = \frac{e^s \hat{P}_+(x+1, s) - p_+ \hat{P}_+(x+2, s)}{1 - p_+},$$

$$p_+ e^s \hat{P}_+(x+2, s) - [e^{2s} - 1 + (p_+ + p_-)]\hat{P}_+(x+1, s)$$
$$+ p_- e^s \hat{P}_+(x, s) = 0. \quad \text{(A17)}$$

This results in the expression for $\hat{P}_+(x, s)$,

$$\hat{P}_+(x, s) = \left(\frac{1 + (-1 + p_+ + p_-)e^{-2s} - \sqrt{-4p_- p_+ e^{-2s} + [1 + (-1 + p_- + p_+)e^{-2s}]^2}}{2p_- e^{-s}} \right)^{m-x}, \quad \text{(A18)}$$

and for $x = m - 1$ we deduce the record age distribution for the discrete ARTP,

$$\hat{F}(s) = \frac{1 + (-1 + p_+ + p_-)e^{-2s} - \sqrt{-4p_- p_+ e^{-2s} + [1 + (-1 + p_- + p_+)e^{-2s}]^2}}{2p_- e^{-s}}. \quad \text{(A19)}$$

b. Continuous time

In the ARTP model, the random walker has two states, the positive state in which it moves at speed v_+ towards the right side of the interval and a negative state in which it moves at speed v_- towards the left side of the interval. The random walker goes from the positive state to the negative state at time rate $1/2T_+$ and from the negative to the positive state at rate $1/2T_-$. These tumbling events occur independently. More specifically, the time evolution of the position X_t of the random walker is given by (we denote by μ the bias and by v the average absolute speed)

$$\frac{dX_t}{dt} = \frac{v_+ - v_- + \sigma(t)(v_+ + v_-)}{2} = \mu + \sigma(t)v, \quad \text{(A20)}$$

where $(\sigma(t))_{t \geqslant 0}$ is a Poisson process which switches between values -1 and 1 with rates $1/2T_+$ (for $+1$ to -1) and $1/2T_-$ (for -1 to $+1$) such that at any time t and for infinitesimal time step dt,

$$\sigma(t+dt) = \begin{cases} \sigma(t) & \text{with probability } 1 - \frac{dt}{2T_{\sigma(t)}} \\ -\sigma(t) & \text{with probability } \frac{dt}{2T_{\sigma(t)}}. \end{cases} \quad \text{(A21)}$$

To obtain the FPT of this model to position m, we consider the two coupled backward master equations

$$\partial_t p_+(x,t) = v_+ \partial_x p_+(x,t) + \frac{1}{2T_+}[p_-(x,t) - p_+(x,t)],$$

$$\partial_t p_-(x,t) = -v_- \partial_x p_-(x,t) + \frac{1}{2T_-}[p_+(x,t) - p_-(x,t)],$$

$$\text{(A22)}$$

with boundary condition $p_+(m,t) = \delta(t)$ such that $p_+(x,t)$ $[p_-(x,t)]$ stands for the probability that the random walker initially at position x with speed v_+ $(-v_-)$ arrived at position $m > 0$ for the first time at time t (one can only arrive at position m with positive speed). The record age distribution is given by $p_+(0,t)$ for $m = 1$. We Laplace transform both equations to obtain

$$s\hat{p}_+(x,s) = v_+ \partial_x \hat{p}_+(x,s) + \frac{1}{2T_+}[\hat{p}_-(x,s) - \hat{p}_+(x,s)],$$

$$s\hat{p}_-(x,s) = -v_- \partial_x \hat{p}_-(x,s) + \frac{1}{2T_-}[\hat{p}_+(x,s) - \hat{p}_-(x,s)].$$

$$\text{(A23)}$$

Then, by expressing $\hat{p}_-(x,s)$ as a function of $\hat{p}_+(x,s)$ and its derivatives using the first of Eqs. (A23) and replacing it in the second, we have an equation solely for $\hat{p}_+(x,s)$,

$$\hat{p}_-(x,s) = (1 + 2sT_+)\hat{p}_+(x,s) - 2v_+ T_+ \partial_x \hat{p}_+(x,s),$$

$$\left\{ s(T_- + T_+ + 2sT_- T_+) + [T_-(1 + 2sT_+)v_- - (1 + 2sT_-)T_+ v_+]\partial_x - 2T_- T_+ v_- v_+ \partial_{x,x}^2 \right\} \hat{p}_+(x,s) = 0. \quad \text{(A24)}$$

We have the boundary conditions $\hat{p}_+(m,s) = 1$ and $\hat{p}_\pm(-\infty, s) = 0$. This results in the following formula for $w(s)$:

$$w(s) = \frac{2sT_+ T_-(v_- - v_+) + T_- v_- - T_+ v_+ + \sqrt{[T_+ v_+ - T_- v_- - 2sT_+ T_-(v_- - v_+)]^2 + 8sT_- T_+ v_- v_+(2sT_- T_+ + T_- + T_+)}}{4T_- T_+ v_- v_+}.$$

$$\text{(A25)}$$

Because we are interested only in the case where the particle starts with positive velocity, we have the first passage at position m starting at position x with positive speed,

$$\hat{F}(x,s) = \hat{p}_+(x,s) = e^{-w(s)(m-x)}, \quad \text{(A26)}$$

and the record age distribution in the Laplace domain is obtained by taking $m = 1$ [similarly to the ABM; see Eq. (A13)] and $x = 0$,

$$\hat{F}(s) = e^{-w(s)}. \quad \text{(A27)}$$

Note that the discrete model gives asymptotically the continuous model for $v_+ = v_- = 1$, $p_+ = 1 - 1/2T_+$, and $p_- = 1 - 1/2T_-$. In the main text we presented the model with tumbling rate $T = T_+ = T_-$ and with external constant force $f = \mu$ such that the speed in each direction is $v_+ = v + \mu$ and $v_- = v - \mu$, as it is not described by the discrete ARTP model and has attracted more attention from the physics community [17,22,54].

3. Real-time expressions

We first recall the real-time expressions of the time to reach level m for the first time starting at position $x = 0$

[see (A13) and (A26)] in the case of the ABM and ARTP. For the ABM [29],

$$F(m,t) = \mathcal{L}^{-1}\{e^{-w(s)m}\}_{s \to t} \quad \text{(A28)}$$

$$= \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} ds \, e^{st} e^{-w(s)m} \quad \text{(A29)}$$

$$= \frac{m}{\sqrt{4\pi Dt^3}} \exp\left(-\frac{(m - \mu t)^2}{4Dt}\right) \quad \text{(A30)}$$

for the ARTP [17,44],

$$F(m,t) = e^{-t/2T_+}\delta\left(t - \frac{m}{v_+}\right) + mH\left(t - \frac{m}{v_+}\right)$$

$$\times \frac{\exp\left(-\frac{(T_+ v_+ + T_- v_-)t + (T_- - T_+)m}{(v_+ + v_-)2T_+ T_-}\right)}{\sqrt{4T_+ T_-(v_+ t - m)(v_- t + m)}}$$

$$\times I_1\left(\frac{\sqrt{(v_+ t - m)(v_- t + m)}}{(v_+ + v_-)\sqrt{T_+ T_-}}\right). \quad \text{(A31)}$$

From this we can obtain the inverse Laplace transform of $w(s)/s$ as

$$
\begin{aligned}
g(t) &= \mathcal{L}^{-1}\{w(s)/s\}_{s \to t} \\
&= \mathcal{L}^{-1}\left\{\lim_{m \to 0}(1 - e^{-w(s)m})/sm\right\}_{s \to t} \\
&= \lim_{m \to 0}\frac{1 - \int_0^t F(t', m)dt'}{m} = \lim_{m \to 0}\frac{\int_t^\infty F(t', m)dt'}{m}.
\end{aligned}
\tag{A32}
$$

For the ABM, it results in

$$
g(t) = \int_t^\infty \frac{dt'}{\sqrt{4\pi Dt'^3}} e^{-\mu^2 t'/4D},
\tag{A33}
$$

and for the ARTP,

$$
g(t) = \int_t^\infty dt' \frac{\exp\left(-\frac{(T_+ v_+ + T_- v_-)t'}{(v_+ + v_-)2T_+ T_-}\right)}{t'\sqrt{4T_+ T_- v_+ v_-}} I_1\left(\frac{t'\sqrt{v_+ v_-}}{(v_+ + v_-)\sqrt{T_+ T_-}}\right).
\tag{A34}
$$

From this we derive the inverse Laplace transform of $w(s)$, first for the ABM,

$$
f(t) = \mathcal{L}^{-1}\{w(s)\}_{s \to t} = \partial_t g(t)
\tag{A35}
$$

$$
= -\frac{1}{\sqrt{4\pi Dt^3}}\exp\left(-\frac{\mu^2 t}{4D}\right),
\tag{A36}
$$

and for the ARTP,

$$
f(t) = -\frac{\exp\left(-\frac{(T_+ v_+ + T_- v_-)t}{(v_+ + v_-)2T_+ T_-}\right)}{t\sqrt{4T_+ T_- v_+ v_-}} I_1\left(\frac{t\sqrt{v_+ v_-}}{(v_+ + v_-)\sqrt{T_+ T_-}}\right).
\tag{A37}
$$

Finally, using the standard properties of double inverse Laplace transforms, we have that [for $y, t > 0$, see Eq. (10)]

$$
\mathbb{P}(M(t) = m, B_t = y) = \mathcal{L}^{-1}\left\{e^{-w(s)m}\frac{w(s+u)}{s+u}\right\}_{s \to t, u \to y}
\tag{A38}
$$

$$
= \int_0^t \int_0^y dt' dy' \mathcal{L}^{-1}\{e^{-w(s)m}\}_{s \to t-t', u \to y-y'} \mathcal{L}^{-1}\left\{\frac{w(s+u)}{s+u}\right\}_{s \to t', u \to y'}
\tag{A39}
$$

$$
= \int_0^t \int_0^y dt' dy' F(m, t-t')\delta(y-y')g(t')\delta(t'-y')
\tag{A40}
$$

$$
= F(m, t-y)g(y)
\tag{A41}
$$

and for the forward record time [see Eq. (12)]

$$
\mathbb{P}(M(t) = m, E_t = y) = \mathcal{L}^{-1}\left\{e^{-w(s)m}\frac{w(s) - w(u)}{s - u}\right\}_{s \to t, u \to y}
\tag{A42}
$$

$$
= \int_0^t \int_0^y dt' dy' \mathcal{L}^{-1}\{e^{-w(s)m}\}_{s \to t-t', u \to y-y'} \mathcal{L}^{-1}\left\{\frac{w(s) - w(u)}{s - u}\right\}_{s \to t', u \to y'},
\tag{A43}
$$

which leads to

$$
\mathbb{P}(M(t) = m, E_t = y) = -\int_0^t \int_0^y dt' dy' F(m, t-t')\delta(y-y')f(t'+y') = -\int_0^t F(m, t')f(t'+y)dt',
\tag{A44}
$$

corresponding to Eqs. (13) and (14).

APPENDIX B: CORRESPONDENCE BETWEEN DISCRETE AND CONTINUOUS SETTINGS

In this Appendix we provide details on the passage from the discrete to the continuum descriptions. To this end, we start by discretizing the 1D line with spacings Δm and we consider directly $(X_t)_{t \geqslant 0}$ the continuous process, e.g., the ABM or the ARTP. The idea is that we now define a new record as occurring only when the continuous random walker exceeds its previous position by Δm. Consequently, if k records have occurred for the random walker, it precisely indicates that it has reached position $k\Delta m$ but has not yet reached $(k+1)\Delta m$. It is immediately evident that by setting $\Delta m = 1$, as done in Appendix A, the number of records and

the maximum position coincide (more precisely, the integer part of the maximum, which can be disregarded when the maximum is large). Then the τ_k correspond to the times for the continuous process having reached $k\Delta m = m$ to exceed the value $(k+1)\Delta m = m + \Delta m$ of density exactly given by $F(\Delta m, t)$ [see Eqs. (A30) and (A31) above]. This implies that the (discretized) maximum distribution is indeed given by a renewal formalism but with continuous record ages directly,

$$
\mathbb{P}(M(t) \geqslant m) = \mathbb{P}\left(\sum_{k=0}^{m/\Delta m - 1} \tau_k \leqslant t\right),
\tag{B1}
$$

from which, using the Laplace transform, we get

$$\mathcal{L}\{\mathbb{P}(M(t) \geqslant m)\} = \frac{1}{s} \prod_{k=0}^{m/\Delta m - 1} \hat{F}(\Delta m, s) \qquad (B2)$$

$$= \frac{1}{s} \exp\left(-\sum_{k=0}^{m/\Delta m - 1} w(s)\Delta m\right) \qquad (B3)$$

$$= \frac{1}{s} \exp[-w(s)m]. \qquad (B4)$$

Note that the sum of the τ_k, $\sum_{k=0}^{m/\Delta m} \tau_k = T_m$, is exactly the time to reach position m for the continuous process (no dependence on Δm as long as m lies on the lattice of step size Δm). For the backward record time, the probability that the maximum value $M(t)$ between m and $m + \Delta m$ (discretization step) occurred at time $t - y = t - B_t$ is given exactly by the probability that m was reached at time $T_m = t - y$ and the position $m + \Delta m$ was not reached in the time y left (with $y \leqslant t$). This implies that

$$\mathcal{L}\{\mathbb{P}(B_t = y, M(t) \in [m, m + \Delta m])\}$$
$$\times \iint dt\,dy\,e^{-uy-st}\mathbb{E}(H(T_{m+\Delta m} - t)H(t - y)$$
$$\times \delta(t - y - T_m))$$
$$= \frac{1}{s+u}\mathbb{E}(e^{-s(\tau_0 + \cdots + \tau_{m/\Delta m-1})}(1 - e^{-(s+u)\tau_{m/\Delta m}}))$$
$$= \hat{F}(\Delta m, s)^{m/\Delta m}\frac{1 - \hat{F}(\Delta m, s+u)}{s+u}$$
$$\sim e^{-w(s)m}\frac{w(s+u)}{s+u}\Delta m. \qquad (B5)$$

Similarly for the forward record time,

$$\mathcal{L}\{\mathbb{P}(E_t = y, M(t) \in [m, m + \Delta m])\}$$
$$= \iint dt\,dy\,e^{-uy-st}\mathbb{E}(\delta(y + t - T_m - \tau_{m/\Delta m})H(t - T_m))$$
$$= \mathbb{E}\left(e^{-sT_m}\frac{e^{-u\tau_{m/\Delta m}} - e^{-s\tau_{m/\Delta m}}}{s - u}\right)$$
$$= \hat{F}(\Delta m, s)^{m/\Delta m}\frac{\hat{F}(\Delta m, u) - \hat{F}(\Delta m, s)}{s - u}$$
$$\sim e^{-w(s)m}\frac{w(s) - w(u)}{s - u}\Delta m. \qquad (B6)$$

This emphasizes that all formulas given in the main text are exact for the continuous processes by taking the discretization (taken to $\Delta m = 1$ in the main text) to be infinitesimal.

APPENDIX C: NUMBER OF RECORDS AND FORWARD AND BACKWARD RECORD TIMES

1. Single-time distribution

For the sake of completeness, we provide here the full derivation of joint statistics of the number of records at time t as well as the backward and forward record time, $(M(t), B_t, E_t)$. To compute the distribution, we note that the event $\{M(t) = m, B_t = y, E_t = z\}$ is exactly the event of

reaching the mth record at time $t - y$ and the $(m + 1)$th record at time $t + z$. It results in the following identity:

$$\mathbb{P}(M(t) = m, B_t = y, E_t = z)$$
$$= \mathbb{E}(\delta(T_m - (t - y))\delta(T_{m+1} - (t + z))H(t - y)). \quad (C1)$$

Once we Laplace transform this equality, we get

$$\mathcal{L}\{\mathbb{P}(M(t) = m, B_t = y, E_t = z)\}$$
$$= \mathbb{E}\left(\sum_{t,y,z=0}^{\infty} e^{-st-uy-vz}\delta(T_m - (t - y))\right.$$
$$\times \left.\delta(T_{m+1} - (t + z))H(t - y)\right)$$
$$= \mathbb{E}\left(\frac{e^{-sT_m - v\tau_m} - e^{-sT_m - (s+u)\tau_m}}{1 - e^{-s-u+v}}\right)$$
$$= \hat{F}(s)^m\frac{\hat{F}(v) - \hat{F}(u+s)}{1 - e^{-s-u+v}}$$
$$\sim e^{-w(s)m}\frac{w(s+u) - w(v)}{s + u - v}. \qquad (C2)$$

2. The *n*-time distribution

We also provide the multiple-time distribution of the number of records and backward and forward record times for $m_0 = 0 < m_1 < \cdots < m_n$:

$$\mathcal{L}\{\mathbb{P}(M(t_k) = m_k, B_{t_k} = y_k, E_{t_k} = z_k; 1 \leqslant k \leqslant n)\}$$
$$= \mathbb{E}\left(\prod_{k=1}^{n}\iiint_0^{\infty} dt_k\,dy_k\,dz_k\,e^{-v_k z_k - u_k y_k - s_k t_k}\right.$$
$$\times \left.\delta(t_k - y_k - T_{m_k})\delta(t_k + z_k - T_{m_k+1})\right)$$
$$= \mathbb{E}\left(\prod_{k=1}^{n} e^{-s_k T_{m_k}}\frac{e^{-v_k \tau_{m_k}} - e^{-(s_k+u_k)\tau_{m_k}}}{s_k + u_k - v_k}\right)$$
$$= \prod_{k=1}^{n}\left\{\exp\left[-w\left(\sum_{j=k}^{n} s_j\right)(m_k - m_{k-1})\right]\right.$$
$$\times \left.\frac{w\left(u_k + \sum_{j=k}^{n} s_j\right) - w\left(v_k + \sum_{j=k+1}^{n} s_j\right)}{s_k + u_k - v_k}\right\}.$$
$$(C3)$$

APPENDIX D: RECORDS WITH AGING

1. Resetting RW

a. Discrete setting

We define the survival probabilities from the first-passage-time distribution at m starting at position x with (denoted by an r index) and without resetting (see Appendixes A 1 a

and A 2 a),

$$S^r(x,t) = \sum_{t'=t}^{\infty} P^r(x,t'), \tag{D1}$$

$$S(x,t) = \sum_{t'=t}^{\infty} P(x,t'), \tag{D2}$$

where $S^r(x,t)$ is the probability that the random walker which resets to 0 with probability $1 - \lambda = 1 - e^{-r}$ at every step and starting at position x (with positive speed in the case of the RTP) has not reached position m by time t and $S(x,t)$ is the probability that the random walker without resetting and starting at position x (with positive speed for the RTP) has not reached position m by time t. In terms of discrete Laplace transform, Eqs. (D1) and (D2) become

$$\hat{S}^r(x,s) = \frac{1 - \hat{P}^r(x,s)}{1 - e^{-s}}, \tag{D3}$$

$$\hat{S}(x,s) = \frac{1 - \hat{P}(x,s)}{1 - e^{-s}}. \tag{D4}$$

Then we use the following renewal equation, which states that the random walker under resetting starting at x has not reached position m either without having reset or with having reset at a given time t' and restarted at position 0 (with positive speed in the case of the RTP):

$$S^r(x,t) = \lambda^t S(x,t) + \sum_{t'=0}^{t-1} (1-\lambda)\lambda^{t'} S(x,t') S^r(0, t-t'-1). \tag{D5}$$

Once Laplace transformed, the renewal equation becomes

$$\hat{S}^r(x,s) = \hat{S}(x,s+r) + (1-\lambda)e^{-s}\hat{S}(x,s+r)\hat{S}^r(0,s). \tag{D6}$$

This results in the two equations (taking $x = 0$ and $x = m-1$)

$$\hat{S}^r(0,s) = \frac{\hat{S}(0,s+r)}{1 - (1-\lambda)e^{-s}\hat{S}(0,s+r)}, \tag{D7}$$

$$\hat{S}^r(m-1,s) = \hat{S}(m-1,s+r)$$
$$\times \left(1 + (1-\lambda)e^{-s}\frac{\hat{S}(0,s+r)}{1 - (1-\lambda)e^{-s}\hat{S}(0,s+r)}\right). \tag{D8}$$

From Eqs. (D3) and (D4) we get the discrete Laplace transform of the mth record age distribution $\hat{F}_m^r(s)$,

$$1 - \hat{F}_{m-1}^r(s) = 1 - \hat{P}^r(m-1,s)$$
$$= (1 - e^{-s})\frac{1 - \hat{P}(m-1,s+r)}{1 - e^{-s} + e^{-s}(1-\lambda)\hat{P}(0,s+r)}. \tag{D9}$$

Finally, using Eqs. (A4) and (A19), we deduce the Laplace transform of the record age of a discrete resetting ARTP and ABM,

$$1 - \hat{F}_{m-1}^r(s) = (1 - e^{-s})\frac{1 - \hat{F}(s+r)}{1 - e^{-s} + e^{-s}(1-\lambda)\hat{F}(s+r)^m}. \tag{D10}$$

b. Continuous setting

Here we reproduce the steps described in [18,25]. We define the survival probabilities from the first-passage-time distribution at m starting at position x with (denoted by the index r) and without resetting (see Appendixes A 1 b and A 2 b),

$$S^r(x,t) = \int_t^{\infty} F^r(x,t')dt', \tag{D11}$$

$$S(x,t) = \int_t^{\infty} F(x,t')dt', \tag{D12}$$

where $S^r(x,t)$ is the probability that the random walker under resetting at 0 (with positive speed in the case of the RTP) at rate r and starting at position x (with positive speed in the case of the RTP) has not reached position m by time t and $S(x,x)$ is the probability that the random walker without resetting and starting at position x (with positive speed for the RTP) has not reached position m by time t. In terms of Laplace transform, Eqs. (D11) and (D12) become

$$\hat{S}^r(x,s) = \frac{1 - \hat{F}^r(x,s)}{s}, \tag{D13}$$

$$\hat{S}(x,s) = \frac{1 - \hat{F}(x,s)}{s}. \tag{D14}$$

Then we use the following renewal equation, which states that the random walker under resetting starting at x has not reached position m because either it has not reset or it has reset at a given time t' and restarted at position 0 (with positive speed in the case of the ARTP):

$$S^r(x,t) = e^{-rt}S(x,t) + \int_0^t r\,dt'e^{-rt'}S(x,t')S^r(0, t-t'). \tag{D15}$$

Once Laplace transformed, the renewal equation becomes

$$\hat{S}^r(x,s) = \hat{S}(x,r+s) + r\hat{S}(x,r+s)\hat{S}^r(0,s). \tag{D16}$$

This results in the two equations (taking $x = 0$ and $x = m-1$)

$$\hat{S}^r(0,s) = \frac{\hat{S}(0,s+r)}{1 - r\hat{S}(0,s+r)}, \tag{D17}$$

$$\hat{S}^r(m-1,s) = \hat{S}(m-1,s+r)\left(1 + r\frac{\hat{S}(0,s+r)}{1 - r\hat{S}(0,s+r)}\right). \tag{D18}$$

From Eqs. (D13) and (D14) we get

$$1 - \hat{F}_{m-1}^r(s) = 1 - \hat{F}^r(m-1,s) = s\frac{1 - \hat{F}(m-1,s+r)}{s + r\hat{F}(0,s+r)}. \tag{D19}$$

Finally, using the expressions (A13) and (A26), we obtain Eq. (27),

$$1 - \hat{F}_{m-1}^r(s) = s\frac{1 - e^{-w(s+r)}}{s + re^{-w(s+r)m}}$$
$$\sim s\frac{w(s+r)}{s + re^{-w(s+r)m}} \equiv g(s,m). \tag{D20}$$

2. Reflecting boundary condition

We consider random walkers with a reflecting boundary condition at the initial position. In this case, the record age τ_m depends on m as the reflecting boundary gets further and further away from the record position. For the simpler symmetric case where $\mu = 0$, one can obtain the mth record age by solving Eqs. (A8) and (A22) on the domain $[-m, m]$ (using mirror symmetry at $x = 0$), starting at position $m - 1$ and absorbing boundary conditions at both ends of the interval. This results in the expressions

$$\hat{p}(x, s) = \frac{\sinh[\sqrt{s/D}(2m - x)] + \sinh(\sqrt{s/D}x)}{\sinh(\sqrt{s/D}2m)} \quad \text{(BM)}, \tag{D21}$$

$$\hat{p}_+(x, s) = \frac{sT \sinh\left(\frac{x\sqrt{s(s+\frac{1}{T})}}{v}\right) + \sqrt{sT(sT+1)}\cosh\left(\frac{x\sqrt{s(s+\frac{1}{T})}}{v}\right)}{sT \sinh\left(\frac{m\sqrt{s(s+\frac{1}{T})}}{v}\right) + \sqrt{sT(sT+1)}\cosh\left(\frac{m\sqrt{s(s+\frac{1}{T})}}{v}\right)} \quad \text{(RTP)}, \tag{D22}$$

which provide the expressions of $\hat{F}_{m-1}(s) = \hat{p}(s, m-1)$ and $g(s, m)$,

$$g(s, m) = \sqrt{s/D}\tanh\left(\sqrt{\frac{s}{D}}m\right) \quad \text{(BM)}, \tag{D23}$$

$$g(s, m) = \frac{\sqrt{s\left(s + \frac{1}{T}\right)}\left[\sqrt{sT(sT+1)}\sinh\left(\frac{m\sqrt{s(s+\frac{1}{T})}}{v}\right) + sT\cosh\left(\frac{m\sqrt{s(s+\frac{1}{T})}}{v}\right)\right]}{v\left[sT \sinh\left(\frac{m\sqrt{s(s+\frac{1}{T})}}{v}\right) + \sqrt{sT(sT+1)}\cosh\left(\frac{m\sqrt{s(s+\frac{1}{T})}}{v}\right)\right]} \quad \text{(RTP)}. \tag{D24}$$

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