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Perspective

Soliton gas: Theory, numerics, and experiments

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The concept of soliton gas was introduced in 1971 by Zakharov as an infinite collection of weakly interacting solitons in the framework of Korteweg–de Vries (KdV) equation. In this theoretical construction of a diluted (rarefied) soliton gas, solitons with random amplitude and phase parameters are almost nonoverlapping. More recently, the concept has been extended to dense gases in which solitons strongly and continuously interact. The notion of soliton gas is inherently associated with integrable wave systems described by nonlinear partial differential equations like the KdV equation or the one-dimensional nonlinear Schrödinger equation that can be solved using the inverse scattering transform. Over the last few years, the field of soliton gases has received a rapidly growing interest from both the theoretical and experimental points of view. In particular, it has been realized that the soliton gas dynamics underlies some fundamental nonlinear wave phenomena such as spontaneous modulation instability and the formation of rogue waves. The recently discovered deep connections of soliton gas theory with generalized hydrodynamics have broadened the field and opened new fundamental questions related to the soliton gas statistics and thermodynamics. We review the main recent theoretical and experimental results in the field of soliton gas. The key conceptual tools of the field, such as the inverse scattering transform, the thermodynamic limit of finite-gap potentials, and generalized Gibbs ensembles are introduced and various open questions and future challenges are discussed.

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I. INTRODUCTION

Random nonlinear waves in dispersive media have been the subject of intense research in nonlinear physics for more than half a century, most notably in the contexts of water wave dynamics and nonlinear optics. A significant portion of the work in this area has been centered around wave turbulence—the theory of out-of-equilibrium random weakly nonlinear dispersive waves in nonintegrable systems [1,2]. One of the most important results of the wave turbulence theory is the analytical determination in Ref. [3] of the power-law Fourier spectra analogous to the Kolmogorov spectra describing energy flux through scales in dissipative hydrodynamic turbulence.

More recently, a new theme in turbulence theory has emerged in connection with the dynamics of strongly nonlinear random waves described by integrable systems such as the Korteweg–de Vries (KdV) and one-dimensional (1D) nonlinear Schrödinger (NLS) equations. This kind of random wave motion in nonlinear conservative systems, dubbed *integrable turbulence* [4], has attracted significant attention from both the fundamental and applied perspectives. The interest in integrable turbulence is motivated by the inherent randomness of many real-life systems (due to random initial and boundary conditions or to complex interaction mechanisms) even though the underlying physical models may be amenable to the well-established mathematical techniques of integrable systems theory such as the inverse scattering transform or finite-gap theory [5,6].

The integrable turbulence framework is particularly pertinent to the description of modulationally unstable systems which can exhibit highly complex nonlinear behaviors that can be adequately described in terms of the turbulence theory concepts such as probability distribution functions, ensemble averages, Fourier spectra, etc. [7–12]. We stress that the term "turbulence" in this context is understood as complex spatiotemporal dynamics that require a probabilistic description and are not related to the energy cascades through scales, the prime feature of strong hydrodynamic and weak wave turbulence.

The main tool for the analysis of integrable nonlinear dispersive partial differential equations (PDEs) is the inverse scattering transform (IST) [13] which is based on the reformulation of a nonlinear PDE as a compatibility condition of two *linear* problems (the so-called Lax pair): a stationary spectral (scattering) problem and an evolution problem—for the same auxiliary function. Within the classical IST setting formulated for the wave fields decaying sufficiently rapidly as $|x| \to \infty$, the scattering spectrum consists of two components: discrete and continuous, corresponding to two contrasting types of the wave motion: solitary waves (solitons) and dispersive

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radiation, respectively. Importantly, integrable evolution preserves the IST spectrum in time.

Localized nonlinear solitary waves, termed solitons in the context of integrable systems, are a ubiquitous and fundamental feature of nonlinear dispersive wave propagation. They exhibit particle-like properties such as elastic, pairwise interactions accompanied by certain phase (position) shifts [14] and have been extensively studied both theoretically [5,15,16] and experimentally [17]. The particle-like properties of solitons suggest some natural questions pertaining to the realm of statistical mechanics, e.g., one can consider a *soliton gas* as an infinite ensemble of interacting solitons characterized by random amplitude and position distributions. Then, given the properties of the elementary, "microscopic," soliton interactions the next natural step is the determination of the emergent hydrodynamic, or kinetic, properties of a soliton gas. This consideration inspired the pioneering paper by V.E. Zakharov in 1971 [18], where an approximate kinetic equation for KdV solitons was introduced by evaluating the effective adjustment to the soliton's velocity in a *rarefied gas* due to the accumulation of the phase shifts acquired in individual soliton collisions. Despite potentially opening a new direction in the nonlinear wave research, the original Zakharov kinetic equation did not happen to attract much attention for more than three decades. The apparent reason is that, due to the inherent low-density assumption the (small) adjustment of the average tracer soliton velocity in the gas was the only tangible effect predicted by the original kinetic theory. The renewed interest in the theory of soliton gases came in the 2000s with the papers by El and Kamchatnov [19,20], where the generalization of Zakharov's kinetic equation to the case of dense soliton gases (KdV and NLS) was derived suggesting some nontrivial mathematical and physical implications [21–23] which only very recently came to fruition [24–34].

Early attempts to generate and observe soliton gases have been made in optical fiber experiments performed at the end of the 1990s [35–37]. The soliton gas was generated by the synchronous injection of laser pulses inside a passive ring cavity. Due to complexity of the dynamics of the ring resonator the nonlinear wave field observed in this fiber system included many "nonintegrable" features ranging from purely temporal chaos to spatiotemporal chaos or turbulence. More recently, analyzing ocean waves recordings, Costa *et al.* [38] have reported the observation of random solitary waves on shallow water that have been interpreted as a KdV soliton gas. One year later, large disordered ensembles of colliding KdV-like solitons have been observed on the surface of a water cylinder deposited on a heated channel and levitating on its own generated vapor film owing to the Leidenfrost effect [39]. There have been many other publications reporting observations of "stochastic" soliton ensembles in various physical systems including shallow-water waves [40,41], optical fibers [42], photofractive crystals [43], and Bose-Einstein condensates [44]. In this regard, what kind of insight into physics and mathematics of nonlinear wave phenomena can the "integrable" soliton gas theory provide? First of all, we stress the conceptual, paradigmatic, differences between classical (nonintegrable) and soliton (integrable) gases. Due to the presence of an infinite number of conserved quantities, integrable systems do not reach the thermal equilibrium state characterized by the so-called Rayleigh-Jeans distribution of modes (equipartition of energy). As a result, the statistical properties of soliton gases are drastically different compared with the properties of classical gases whose particle interactions are nonelastic. Specifically, instead of the relaxation to the classical Gibbs ensembles, soliton gases exhibit local nonthermal stationary states, the so-called generalized Gibbs ensembles, which play a fundamental role in the hydrodynamic theory of many-body quantum and classical integrable systems, dubbed generalized hydrodynamics [45,46]. Additionally, the dual particle-wave nature of solitons implies that the coarse-grained, hydrodynamic description of soliton gas should be complemented by the characterization of the associated nonlinear turbulent wave field in terms of the probability density function, power spectrum, autocorrelation, etc. It has transpired recently that soliton gas dynamics are instrumental in the understanding of such important nonlinear wave phenomena as spontaneous modulation instability and the rogue wave formation in one-dimensional wave propagation [34,47,48]. Moreover, it has been shown that such classical objects and phenomena of dispersive hydrodynamics as Whitham modulation equations [49], rarefaction and dispersive shock waves [50] and soliton-mean-field interactions [51] are naturally embedded in the soliton gas theory [30,52,53]. Soliton gases thus provide a conceptual bridge between the major areas of dispersive and generalized hydrodynamics, enriching both disciplines with complementary theoretical and experimental perspectives.

Returning to the question of the practical (numerical or experimental) realization of soliton gases, one can distinguish two basic mechanisms of the "spontaneous," uncontrollable generation of a soliton gas. One mechanism involves the process of soliton fission, where statistical soliton ensembles emerge as the asymptotic outcome of long-time evolution of the so-called "partially coherent waves," which can be viewed as collections of randomly distributed broad pulses, see Fig. 1 and Refs. [12,54,55]. Alternatively, soliton ensembles can be initially generated from a nonrandom (e.g., periodic) signal and then undergo effective randomization due to elastic reflections from the boundaries and subsequent multiple collisions, see Ref. [56] for the example of the soliton gas generation in a shallow-water wave tank. The second mechanism of the soliton gas generation is related to the already-mentioned phenomenon of modulation instability, where the basic coherent nonlinear mode of an unstable system– the plane wave—is subjected to a random perturbation (a noise), resulting in the development of large-amplitude small-scale fluctuations of the wave field and the establishment at $t \to \infty$ of a stationary integrable turbulence [7]. It was shown in Ref. [47] that for the wave systems described by the focusing nonlinear Schrödinger (fNLS) equation such integrable turbulence exhibits the properties of a dense bound-state (nonpropagating) soliton gas.

Soliton gas can also be synthesized in a controllable manner directly, e.g., by programming a water tank wavemaker according to the IST-prescribed random multisoliton solution of the relevant integrable equation, see Ref. [57].

The central object in the soliton gas theory is the density of states (DOS)—the function describing the distribution of

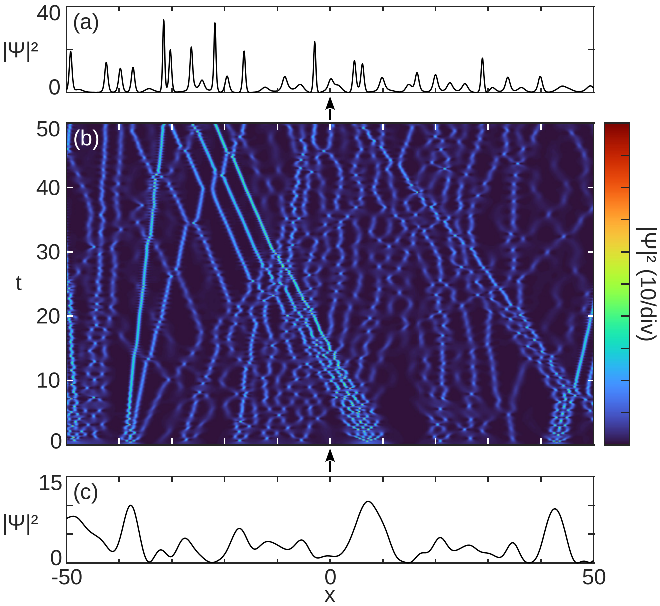

FIG. 1. Emergence of soliton gas in the long-time evolution of a partially coherent wave $\psi(x,t)$ in the focusing NLS equation [Eq. (6) with $\sigma = 1$]. (a) Intensity $|\psi(x,t=50)|^2$. (b) Spatiotemporal dynamics$|\psi(x,t)|^2$. (c) Intensity $|\psi(x,t=0)|^2$ (initial condition).

solitons with respect to the spectral parameter and the positions of the solitons' centers. When soliton gas is uniform (i.e., in a macroscopically equilibrium state) the DOS is stationary and space-independent. In a nonuniform (nonequilibrium) gas the spatiotemporal evolution of the DOS on a large (Eulerian) scale is described by the continuity equation that follows from the isospectrality of integrable dynamics.

Generally, DOS represents the fundamental object in the spectral theory of random linear operators, see, e.g., the classical monographs [58,59]. In the context of one-dimensional integrable wave equations associated with linear spectral problems within the IST framework there has been a large body of work on the analysis of the IST spectra of random KdV and NLS solutions, see, e.g., Refs. [60–62] and references therein. From an applied viewpoint such solutions correspond to the propagation of stochastic signals through a nonlinear dispersive medium, e.g., in fiber optics, see Refs. [63,64]. In the development of soliton gas theory one adopts the opposite, inverse problem, perspective by first introducing a solitonic spectral DOS and then performing the characterization of the associated nonlinear turbulent wave field.

In the original, rarefied soliton gas, construction by Zakharov [18] solitons are treated as isolated point-like quasiparticles which are subject to infrequent, short-range interactions accompanied by well-defined phase-shifts. In contrast, in a *dense* soliton gas the solitons exhibit significant overlap and, as a result, are continuously involved in a strong nonlinear interaction with each other. It is clear that, in a dense gas the particle interpretation of individual solitons becomes less transparent and the wave aspect of the collective soliton dynamics comes to the fore. Indeed, a consistent generalization of Zakharov's kinetic equation for KdV solitons to the case of a dense soliton gas has been achieved in Ref. [19] in the framework of the nonlinear wave modulation (Whitham) theory [49]. It was proposed in Ref. [19] that the KdV soliton gas can be modeled by the thermodynamic type solitonic limit of the multiphase, finite-gap KdV solutions and their modulations [65] (these solutions represent nontrivial generalization of solitons in problems with periodic boundary conditions). The resulting spectral kinetic equation has the form of a nonlinear integro-differential equation consisting of the continuity equation for the DOS [Eq. (15)] and the linear integral equation of state (16) relating the effective, average velocity of the "tracer" soliton in the gas with its DOS. The structure of the kinetic equation derived in Ref. [19] has motivated a fundamental conjecture that generally, in a dense gas, the net effect of soliton interactions can be formally evaluated using the same phase-shift argument that was used in the original rarefied gas theory [18]. This conjecture, termed the *collision rate ansatz,* has enabled an effective phenomenological theory of a dense soliton gas for the fNLS equation [20] and more recently, for the defocusing NLS and integrable shallow water waves equations supporting bidirectional soliton propagation [26]. The phenomenological soliton gas theory for the fNLS equation proposed in Ref. [20] has been analytically confirmed and substantially extended in Ref. [24] within the framework of the thermodynamic limit of spectral finite-gap solutions of the fNLS equation and their modulations. This latter work has revealed a number of new soliton gas phenomena due to a very different structure of the spectral phase space of the fNLS equation compared with the KdV equation. In particular, the generalization of soliton gas, termed a *breather gas*, was introduced by considering a special family of fNLS solitonic solutions on a nonzero unstable background [24,66]. Another peculiar type of soliton gas, termed in Ref. [24] a *soliton condensate,* can be viewed as the critically dense ensemble of solitons constrained by a given spectral domain. Properties of soliton condensates for the KdV equation and their relation to the fundamental coherent structures in dispersive hydrodynamics such as rarefaction and dispersive shock waves were investigated in Ref. [30].

Apart from the above line of research on soliton gases inspired by the Zakharov's 1971 work and summarized in the recent review [25] there have been many other developments—theoretical, numerical and experimental—exploring various aspects of soliton gas and soliton turbulence dynamics in both integrable and nonintegrable classical wave systems (see, e.g., Refs. [35,40,42,43,67–70]). In particular, recent numerical results [48,71] suggest that the soliton gas theory could be instrumental for the development of the statistical description of the of rogue wave formation. Additionally, soliton gases have been recently attracting a growing interest from the mathematical physics community. Various nontrivial algebraic and geometric properties of the kinetic equation for soliton gas were studied in Refs. [21,28,29,31,72,73]. Beyond the hydrodynamic, Euler scale, description, recent rigorous studies [52,74] were devoted to the construction of asymptotic solutions of the KdV and modified KdV equations, respectively, describing a special class of "regular" or "deterministic" soliton gases within the framework of *primitive potentials* [75], via the consideration of N-soliton solutions in the limit $N \to \infty$. These special soliton gases correspond to soliton condensates in the finite-gap approach to the soliton gas description [30].

Finally we note the recent major developments in the already mentioned closely related area of *generalized hydrodynamics* (GHD) (see Refs. [45,46,76,77] and references therein), where the equations analogous to those arising in the spectral kinetic theory of soliton gas became pivotal for the understanding of large-scale, emergent hydrodynamic properties of integrable quantum and classical many-body systems. The relation between spectral theory of KdV soliton gas and the GHD of KdV solitons has been recently established in Ref. [78] which enabled the formulation of the *thermodynamics* (free energy, entropy, temperature) of the KdV soliton gas. More generally, GHD provides an appropriate theoretical framework for the formulation of *statistical mechanics* of soliton gases via the fundamental notions of the thermodynamic Bethe ansatz and the generalized Gibbs ensemble.

The goal of this perspective article is to present the state of the art in the modern theoretical and experimental soliton gas research, highlighting the connections with other areas of nonlinear physics and mathematics and outlining the avenues for future investigations.

The structure of the article is as follows: In Sec. II we introduce the concept of soliton gas, from rarefied to dense, and present a straightforward phenomenological approach to the construction of the spectral kinetic equation for integrable systems with known two-soliton interactions. In Sec. III we proceed with outlining the results of the spectral theory of soliton gas based on the thermodynamic limit of finite-gap potentials and their modulations for the KdV and fNLS equations. In Sec. IV, we summarize the basic concept of IST and the recent progress allowing the numerical computation of N-soliton solutions with N large. In Sec. V, we review the experimental results on soliton gases. In Sec. VI, we show how soliton gas theory can be used to understand and predict integrable turbulence phenomena. In Sec. VII we review the key results of GHD and their links with soliton gases. Finally, in Sec. VIII, we review fundamental open questions and perspectives of this field of research.

II. THE CONCEPT OF SOLITON GAS

As mentioned in the Introduction, a soliton gas (SG) can be informally defined as an infinite ensemble of interacting solitons characterized by random amplitude and phase (position) distributions. Looking more closely at this intuitive concept, however, raises many questions, both conceptual and technical. In fact, at the moment there is no conventionally accepted rigorous mathematical definition of SG as a (genuine or probabilistic) solution to an integrable PDE, instead, we have several physics- or applied-mathematics-inspired SG models that focus on particular (wave vs particle) properties of solitons as the elementary SG constituents. The fact that these models provide consistent results that also agree with both direct numerical simulations of the corresponding integrable PDEs and physical experiments lends some confidence in their validity and also provides a strong motivation towards reconciliation of the existing models and the development of a comprehensive mathematical theory of SGs.

A. Solitons in integrable systems

We first outline the basic properties of solitons using the KdV equation as a prototypical example. We consider the KdV equation in the form

$$u_t + 6uu_x + u_{xxx} = 0. \quad (1)$$

Equation (1) belongs to the class of completely integrable equations and, for a broad class of initial conditions, its integrability is realized via the inverse scattering transform (IST) method [13] sometimes called a nonlinear Fourier transform. The inverse scattering theory associates a single soliton solution of the KdV equation with a point of discrete spectrum $\lambda = \lambda_1 < 0$ of the Schrödinger operator

$$\mathcal{L} = -\partial_{xx}^2 - u(x,t). \quad (2)$$

Assuming $u \to 0$ as $x \to \pm\infty$, the KdV soliton solution corresponding to the eigenvalue $\lambda_1 = -\eta_1^2$, $\eta_1 > 0$ is given by

$$u_s(x,t;\eta_1,x_1^0) = 2\eta_1^2 \text{sech}^2[\eta_1(x - 4\eta_1^2 t - x_1^0)], \quad (3)$$

where $2\eta_1^2$ is the soliton amplitude, $4\eta_1^2$ its speed, and x_1^0 its initial position or "phase." Note that soliton has finite width $\approx 1/\eta_1$, which affects the notion of the interaction range, particularly for small-amplitude solitons. In what follows we refer to η as a spectral parameter with the understanding that $\eta = \sqrt{-\lambda}$. Along with the simplest single-soliton solution (3), the KdV equation supports N-soliton solutions $u_N(x,t)$ characterized by N discrete spectral parameters $0 < \eta_N < \eta_{N-1} < \cdots < \eta_1$ and the set of the so-called norming constants that could be interpreted in terms of the initial positions of solitons—the analogs of $\{x_i^0 | i = 1, \ldots, N\}$ in (3) (note that the actual position of a soliton within the N-soliton solution depends nontrivially on all norming constants). Thus, N-soliton solution can be viewed as a nonlinear superposition of N single-soliton solutions, the notion supported by the asymptotic behavior at $t \to \pm\infty$, when $u_N(x,t)$ assumes the form of rank-ordered soliton trains, $u_N(x,t) \xrightarrow[t\to\pm\infty]{} \sum_i^N u_s(x,t;\eta_i,x_i^\pm)$, with appropriately chosen phases $x_i^\pm$ depending on the configuration at $t = 0$, see Refs. [5,15,79].

It should be stressed that general solutions to the KdV equation exhibit, along with solitons, a dispersive radiation component corresponding to the continuous spectrum of the Schrödinger operator (2). However, the soliton gas construction considered here involves only discrete spectrum.

The integrable structure of the KdV equation has profound implications for the dynamics of soliton interactions:

(1) The KdV evolution preserves the IST spectrum, $\partial_t \eta_j = 0$, implying that soliton collisions are "elastic" i.e., solitons remain unchanged [retaining their amplitude, speed and the waveform (3)] upon interactions. In other words, the solution exhibiting N solitons at $t \to -\infty$ will exhibit exactly the same N solitons (modulo their phases) at $t \to +\infty$.

(2) The collision of two solitons with spectral parameters η_i and η_j, $i \neq j$ results in the asymptotic shifts of their positions at $t \to +\infty$ relative to the respective free propagation trajectories from $t \to -\infty$. These position shifts correspond to the phase shifts of the discrete spectrum norming constants

and are given by

$$\Delta_{ij} \equiv \Delta(\eta_i, \eta_j) = \frac{\mathrm{sgn}(\eta_i - \eta_j)}{\eta_i} \ln \left| \frac{\eta_i + \eta_j}{\eta_i - \eta_j} \right|, \tag{4}$$

so that the taller soliton acquires shift forward and the smaller one—shift backwards.

(3) Solitons interact pairwise so that the resulting phase shift Δ_i of a given soliton with spectral parameter η_i after its interaction with M solitons with parameters η_j, $j \neq i$, is equal to the sum of the individual phase shifts,

$$\Delta_i = \sum_{j=1, j\neq i}^{M} \Delta_{ij}. \tag{5}$$

Thus the interaction of any N solitons can be factorized, with respect to the phase shifts, into superposition of two-soliton interactions, i.e., multiparticle effects are absent.

It is important to stress that the collision phase shifts are the far-field effects. Mathematically they are the artifacts of the asymptotic representation of the exact two-soliton solution of the KdV equation as a sum of two individual solitons: $u_2(x, t; \eta_1, \eta_2) \simeq u_s(x + \Delta_{12}, t; \eta_1) + u_s(x + \Delta_{21}, t; \eta_2)$, which is only valid if solitons are sufficiently separated (the long-time asymptotics). The interaction of solitons is a complex nonlinear process [80] and the resulting wave field $u(x, t)$ in the interaction region cannot be represented as a superposition of the phase-shifted one-soliton solutions. We note that the above properties of soliton collisions (the preservation of soliton parameters and pairwise phase shifts) are not exclusive to KdV but are generic features of other integrable systems supporting soliton propagation.

For the NLS equation

$$i\psi_t + \psi_{xx} + 2\sigma|\psi|^2\psi = 0, \quad \psi \in \mathbb{C}, \tag{6}$$

in the focusing regime, $\sigma = +1$, the single-soliton solution is characterized by a discrete complex eigenvalue $\lambda_1 = a + ib$ and c.c., of the linear scattering operator called the Zakharov-Shabat operator [81], the focusing NLS (fNLS) analog of the Schrödinger operator (2), see Sec. IV A. The fNLS soliton is given by

$$\psi_s(x, t) = 2b \frac{e^{-2i[ax + 2(a^2 - b^2)t] + i\phi_0}}{\cosh[2b(x + 4at - x_0)]}, \tag{7}$$

where x_0 is the initial position of the soliton and ϕ_0 the initial phase. One can see that the fNLS soliton represents a localized wave packet with the envelope propagating with the group velocity $c_g = -4a = -4\mathrm{Re}\lambda_1$ and the carrier wave having the phase velocity $c_p = 2(b^2 - a^2)/a = -2\mathrm{Re}(\lambda_1^2)/\mathrm{Re}\lambda_1$. In contrast with the KdV equation, the amplitude and velocity of the fNLS soliton are two independent parameters (as well as the position and phase).

Similar to other integrable models, solitons of the fNLS equation interact pairwise and experience position and phase shifts upon the interaction (note that for fNLS solitons the position and phase shifts are independent quantities). Unlike the KdV equation, the fNLS solitons are bidirectional but the position shifts in the overtaking and head-on soliton collisions are given by the same expression,

$$\Delta(\lambda, \mu) = \frac{\mathrm{sgn}[\mathrm{Re}(\mu - \lambda)]}{\mathrm{Im}\lambda} \ln \left| \frac{\mu - \lambda^*}{\mu - \lambda} \right| \tag{8}$$

(the associated phase shift expression can be found elsewhere, see, e.g., Ref. [81]). In some other bidirectional integrable systems such as the Kaup-Boussinesq equations describing shallow water waves and the resonant NLS equation having applications in magnetohydrodynamics of cold collisionless plasma the soliton collisions are anisotropic, i.e., the head-on and overtaking position shifts are described by different expressions, see Ref. [26].

B. Soliton gas: Phenomenological model

1. *Rarefied soliton gas*

Inspired by the historical Zakharov 1971 paper [18] we first introduce SG phenomenologically as an infinite random ensemble of well-separated KdV solitons distributed on $\mathbb{R}$ with some nonzero spatial density $\alpha \ll 1$. The solitons in such a gas are essentially considered as freely moving particles "dressed" with individual rapidly decaying wave fields (3) and involved in short-range pairwise interactions accompanied by the phase shifts (4). It turns out that this simple "flea-gas" model exhibits some nontrivial emergent, macroscopic features. To model a homogeneous rarefied SG on $\mathbb{R}$ we first introduce N-soliton ensemble as an approximate random KdV solution in the form of a "stochastic soliton train," a sum of $N \gg 1$ well-separated solitons

$$U_N\big(x, t; \{\eta_i\}, \{x_i^0\}\big) := \sum_{i=1}^{N} 2\eta_i^2 \mathrm{sech}^2\big[\eta_i\big(x - 4\eta_i^2 t - x_i^0\big)\big] \tag{9}$$

satisfying the periodicity condition $U_N(x + L, t) = U_N(x, t)$, where $L \propto N$, and equipped with certain probability distributions for the spectral parameters η_j and the soliton positions $x_j = x_j^0 + 4\eta_j^2 t$, $j = 1, \dots, N$ as specified below. (i) Let the soliton spectral parameters $0 < \eta_N < \eta_{N-1} < \cdots < \eta_1$ in (9) be the N values of the continuous random variable η distributed on a fixed, simply connected interval $\Gamma \in \mathbb{R}^+$ with smooth density $\phi(\eta) > 0$ defined via

$$N \to \infty: \quad \eta_{j+1} - \eta_j \sim \frac{1}{\phi(\eta_j)N}, \tag{10}$$

so that $\int_\Gamma \phi(\eta)d\eta = 1$. Hence $\phi(\eta)$ is the probability density for $\eta \in \Gamma$. (ii) Let $\{x_j^0\}_{j=1}^N$ be N independent random values each uniformly distributed on the period $[0, L]$, so that the spatial density of solitons (irrespective of the amplitude) is given by $\alpha = N/L$. (iii) For the rarefied ensemble approximation to be valid we require that $\alpha \ll \eta_0$, where η_0 is the typical value of the spectral parameter of the solitons in (9). Additionally we assume that the lower boundary of the spectral interval Γ is located not too close to the origin so that one can exclude the possibility of the presence of small-amplitude, wide, solitons. We propose that a homogeneous rarefied SG can be approximated on any sufficiently large interval of space $I(x_0, L) = [x_0 - L/2, x_0 + L/2]$ by N-soliton ensemble (9) with given $\phi(\eta)$ and $\alpha = N/L$. We note that, as $N \to \infty$, the classical result from the ergodic theory of ideal gas (see, e.g., Ref. [82]) implies that the uniform measure on $I(x_0, L)$ for the random initial soliton positions x_i^0 transforms under the above (thermodynamic) limit into the invariant Poisson measure for $x_i = x_i^0 + 4\eta_i^2 t$ on $\mathbb{R}$ [i.e., the probability of finding n solitons in the space interval $\Delta \subset \mathbb{R}$ at any t is given by $P_\Delta(n, \alpha) = e^{-\alpha|\Delta|}(\alpha|\Delta|)^n/n!$]. Thus we arrive at the model of a rarefied soliton gas as an infinite sequence of soliton pulses on $\mathbb{R}$, denoted $U_\infty(x, t; \{\eta_i\}, \{x_0^i\})$, with independent random parameters $\{\eta_i\}$ and $\{x_0^i\}$ distributed according to the probability measures $\phi(\eta)$ and $P_\Delta(n, \alpha)$ on Γ and $\mathbb{R}$, respectively. Due to the small spatial density $\alpha \ll \eta_0$, most of the individual solitons in a rarefied gas overlap only in the regions of their exponential tails, except for the rare events of soliton collisions. Thus, each realization of the rarefied SG U_∞ represents an approximate solution of the KdV Eq. (1) almost everywhere on $\mathbb{R}$. We supply the described phenomenological model with the "scattering shifts" (4) in two-soliton collisions and explore the emergent large-scale kinetics, or hydrodynamics, of such a SG. We now introduce the key aggregated spectral characteristic of SG—the density of states (DOS) defined as the number of solitons per unit interval of the spectrum η and unit interval of space (i.e., the density of "particles" in the spectral phase space $\mathfrak{S} = \Gamma \times \mathbb{R}$). In the above model of rarefied SG the DOS is given by

$$f(\eta) = \alpha\phi(\eta) > 0. \tag{11}$$

We call (11) the phenomenological DOS. We note that generally the spatial density α can depend on the spectral parameter, $\alpha = \tilde{\alpha}(\eta)$ (see Sec. III where SGs are introduced in a more general mathematical setting of finite-gap solutions), but the simplified phenomenological definition (11) with $\alpha = \text{const}$ is consistent with the way SGs are usually realized in practice (numerically or experimentally) via appropriately configured N-soliton ensembles, see Secs. IV and V. Importantly, for a homogeneous (equilibrium) SG the DOS does not depend on space and time. As a matter of fact, for a given DOS the spatial density of a SG (the total number of solitons per unit length irrespective of the amplitude) is evaluated by integrating the DOS over Γ:

$$\alpha = \int_\Gamma f(\eta) d\eta. \tag{12}$$

In a rarefied gas with $\alpha \ll \eta_0$ and in the absence of small-amplitude solitons the soliton interactions can be viewed as short-range (the width of the typical dominant interaction region is much less than the "free path" between soliton collisions). Then the total spatial shift of a soliton with spectral parameter η (we call it an η soliton) accumulated over a sufficiently large time interval dt, due to the interactions with "μ solitons" having spectral parameters $\mu \in \Gamma$, $\mu \neq \eta$, is approximately evaluated as

$$\Delta_\eta \approx \int_\Gamma [\Delta(\eta, \mu)|s_0(\eta) - s_0(\mu)|f(\mu)d\mu]dt, \tag{13}$$

where $s_0(\eta)$ is the speed of an isolated, noninteracting, η soliton. It is assumed in (13) that in a rarefied gas the collision rate is at leading order defined by the free soliton velocities. For the KdV equation $s_0(\eta) = 4\eta^2$ and $\Delta(\eta, \mu)$ is given by Eq. (4). Then the path covered by the η soliton over the time interval dt is given by $s_0(\eta)dt + \Delta_\eta$ and so the effective (average) velocity $s(\eta)$ of a soliton in a KdV soliton gas is given by [18]

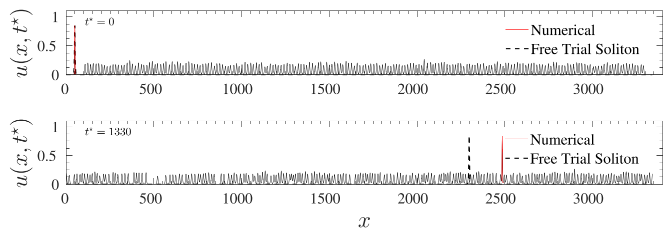

FIG. 2. Comparison for the propagation of a free soliton with the spectral parameter η in a void (black dashed line) with the propagation of the trial soliton with the same spectral parameter (red solid line) through a rarefied soliton gas with the DOS supported on a narrow spectral interval around some $\eta_0 < \eta$ (direct KdV simulations from Ref. [22]). One can see that the trial soliton propagates faster in the gas due to the interactions with smaller solitons. Reproduced with permission

$$s(\eta) \approx 4\eta^2 + \frac{1}{\eta}\int_\Gamma \ln\left|\frac{\eta + \mu}{\eta - \mu}\right| f(\mu)[4\eta^2 - 4\mu^2]d\mu. \tag{14}$$

Note that the value of η can be either inside or outside Γ distinguishing between the notions of the "tracer" ($\eta \in \Gamma$) and the "trial" ($\eta \notin \Gamma$) soliton. See Fig. 2 for the numerical simulations illustrating the effect of soliton interactions on the effective velocity of a trial soliton propagating through a soliton gas.

For a weakly nonhomogeneous (out of equilibrium) SG $f(\eta) \to f(\eta; x, t)$, $s(\eta) \to s(\eta; x, t)$, where (x, t) variations of f and s occur on a macroscopic "Euler" scale (i.e., x and t are slow variables here). More precisely, there are three spatiotemporal scales involved in weakly nonhomogeneous SG dynamics: (i) the microscopic scale $\varepsilon \sim \eta_0^{-1}$ associated with the KdV field variations within individual solitons; (ii) mesoscopic scale δ involving large numbers of solitons (i.e., $\delta \gg \alpha^{-1} \gg \varepsilon$ for rarefied gas) but characterized by an approximately x, t-independent DOS; and (iii) macroscopic (Euler) scale $\gamma \sim |f/f_x| \sim |f/f_t| \gg \delta \gg \varepsilon$ at which appreciable variations of $f(\eta)$ occur. We note that this scale separation is at heart of GHD, where the mesoscopic scale is associated with the notion of "fluid cells," where the entropy is locally maximized with respect to the infinite number of conserved quantities [45,76], see Sec. VII. Now, isospectrality of the KdV evolution within the IST framework implies the continuity equation (Euler-scale variations) for the spectral phase-space density (the DOS),

$$\partial_t f + \partial_x(sf) = 0, \tag{15}$$

which, together with (14), provides the spectral kinetic description of a rarefied KdV soliton gas. Equation (15) can be viewed as a modulation equation for SG. Indeed, there is a deep connection between the SG kinetic theory and the Whitham modulation theory for nonlinear multiperiodic waves [49,65], see Sec. III. Solution $f(\eta; x, t)$ of the kinetic Eqs. (15), (14) describes the evolution of the DOS and, via Eq. (12), the associated evolution of the Poisson probability measure $P_\Delta(n, \alpha)$ for the soliton positions in rarefied SG via the density parameter $\alpha(x, t)$ (assuming that the Poisson statistics remains valid in the weakly interacting gas at the

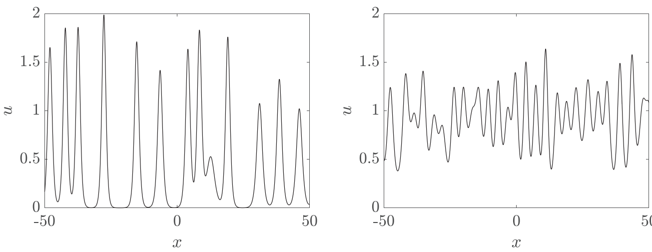

FIG. 3. Typical realizations of rarefied (left) vs dense (right) KdV soliton gases with the same spectral density $\phi(\eta) = \eta/(1-\eta^2)^{1/2}$ but different spatial densities: $\alpha \simeq 0.1$ (left) and $\alpha \simeq 0.3$ (right). (See Ref. [30] for the algorithm of the numerical synthesis of dense KdV SG used to produce this figure.)

mesoscopic scale). Furthermore, one can evaluate by (14) the average ("effective") velocity $s(\eta; x, t)$ of a trial soliton propagating through such a weakly nonuniform rarefied SG. The explicit representation of the SG wave field as a sum of the soliton pulses enables the evaluation of the KdV conserved quantities in terms of the spectral DOS moments (such expressions will be presented later in Sec. II C for the general case of a dense gas). One, however, should keep in mind that solutions to the approximate kinetic Eqs. (15) and (14) only make sense as long as the interaction term in the velocity expression (14) is small. In other words, the rarefied gas theory only allows for the evaluation of small corrections to the parameters of an "ideal" gas of noninteracting solitons, see Ref. [83].

2. Dense soliton gas

If the KdV SG is sufficiently dense, the simple heuristic construction of the previous section based on the assumption of short-range interactions between isolated solitons composing the gas becomes invalid as solitons in a dense gas strongly overlap and, hence, are involved in a continual nonlinear interaction so that the corresponding KdV solution can nowhere be represented as a linear superposition of individual solitons as in (9), cf. Fig. 3 left and right. In particular, the approximation (13) for the total phase shift based on the free soliton velocities ceases to be valid.

A natural way to approach the mathematical construction of a dense SG would be to consider N-soliton KdV solutions in the limit $N \to \infty$. Establishing such a limit, however represents a nontrivial mathematical problem even in a deterministic setting. We mention recent works [52,74,84] which use the Riemann-Hilbert problem techniques to construct special classes of infinite-soliton solutions for KdV, mKdV, and fNLS equations. These (deterministic) approaches hold promise for an extension to a random setting and the construction of a rigorous probabilistic model for SG (see also Refs. [62,85] for other promising directions). At the same time, it is clear that any practical (numerical or experimental) realization of soliton gas can only involve a finite number of solitons so we first attempt to define dense soliton gas phenomenologically, as a natural extension of the rarefied SG model in the previous section based on the properties of N-soliton solutions. To this end we assume that dense homogeneous SG can be approximated on any sufficiently large interval $I(x_0, L) = [x_0 - L/2, x_0 + L/2]$, $L \gg 1$ by a dense N-soliton ensemble by which we imply an ensemble of all *exact* N-soliton KdV solutions $u_N(x, t)$ localized on $I(x_0, L)$ (e.g., exponentially decaying outside of I) with $N \propto L \gg 1$ and characterized by a given spectral density $\phi(\eta)$ for $\eta \in \Gamma$ defined by (10) and the spatial density $\alpha = N/L$ (the assumption of the linear growth of L with N for large N is consistent with IST-based numerical simulations of KdV SGs in Refs. [30,78]; see also Sec. IV for the corresponding focusing NLS simulations). To complete the definition one needs to provide a suitable characterization of soliton positions x_i^0 within an N-soliton ensemble. As the individual solitons are generally not discernible in a dense gas (cf. Fig. 3 right) a more consistent term for x_i^0 would be "soliton spatial phases." A GHD inspired way to formally define x_i^0 in a dense soliton ensemble involves the concept of "asymptotic coordinates," see Refs. [78,86]. Within the IST framework the soliton spatial phases are determined by the phases of the so-called norming constants associated with solitonic spectrum, see Refs. [5,15,79]. (Note that different implementations of N-soliton solutions—standard IST vs Darboux transformation—involve different definitions of norming constants, cf. Ref. [87]. We get to this subtle point in Sec. IV A). In a dense N-soliton ensemble the soliton spatial phases x_i^0 are assumed to be uniformly distributed on some interval $I_s = I(x_0, L_s)$ where the "asymptotic space width" $L_s < L$ depends on N and on the spatial and spectral densities α and $\phi(\eta)$ [30]. We note that for critically dense soliton ensembles, termed soliton condensates, the interval I_s shrinks to a single point, $I_s \to \{x_0\}$ [30]. A detailed description of the IST-based numerical realization of dense N-soliton ensembles for the fNLS equation is presented in Sec. IV. We now turn to the spectral characterization of SG in terms of DOS. The definition (11) of the SG phenomenological DOS $f(\eta)$ holds for the dense homogeneous (equilibrium) SG. For a weakly nonhomogeneous SG, as described in Sec. II B 1, we assume scale separation, where the gas is considered to be at local equilibrium over the intermediate, mesoscopic, scale involving sufficiently large numbers of solitons, while appreciable (x, t)-variations of the DOS occur on a larger, macroscopic, Euler, scale. The generalization of Zakharov's kinetic equation to the case of a dense gas was derived in Ref. [19] (see Sec. III B below). It involves the same continuity Eq. (15) for the DOS but the approximate expression (14) for the tracer velocity is replaced by the exact integral *equation of state*:

$$s(\eta) = 4\eta^2 + \frac{1}{\eta}\int_\Gamma \ln\left|\frac{\eta+\mu}{\eta-\mu}\right| f(\mu)[s(\eta) - s(\mu)]d\mu, \quad (16)$$

where we have dropped for brevity the (x, t) dependence for $f(\eta)$ and $s(\eta)$. In simple terms, the integral Eq. (16) represents an extrapolation of the rarefied gas properties to a dense gas, realized by replacing $s_0(\eta) \to s(\eta)$ in the collision rate expression (13). This result is quite remarkable and highly nonintuitive since the very notion of the phase shift is a byproduct of the asymptotic representation of the exact two-soliton solution as a sum of two separate solitons at $t \to \pm\infty$ and as such is formally not applicable to a dense SG, in which solitons never separate. The KdV result (16) has led in Ref. [20] to the conjecture that the phenomenological interpretation of Eq. (16) as the "collision-rate ansatz" can be used as a general principle for the construction of the SG equation of states for other integrable models given the free soliton velocity $s_0(\eta)$ and the phase shift expression $\Delta(\eta,\mu) = \text{sgn}[(s_0(\eta) - s_0(\mu)]G(\eta,\mu)$, specific to each integrable system:

$$s(\eta) = s_0(\eta) + \int_\Gamma G(\eta,\mu)f(\mu)[s(\eta) - s(\mu)]d\mu. \tag{17}$$

We note that the collision-rate ansatz (17) was also pivotal to the (independent) formulation of GHD, see Sec. VII.

In the fNLS case [Eq. (6) with $\sigma = +1$] the solitonic spectrum $\{\lambda_j\}$ in the associated linear (Zakharov-Shabat) scattering problem is complex-valued (see Sec. IV A below) so that the DOS $f(\lambda)$ is generally supported on some compact Schwarz symmetric two-dimensional (2D) set $\Lambda \subset \mathbb{C}$ so it is sufficient to consider only the upper half plane part Λ^+ (here Schwarz symmetry means that if $\lambda \in \mathbb{C}$ is a point of the spectrum then so is the c.c. point λ^*). Then, using $s_0(\lambda) = -4\text{Re}\lambda$ for the free-soliton velocity and the expression (8) for the two-soliton scattering shift, the (conjectured) kinetic equation for the fNLS soliton gas assumes the form [20]

$$f_t + (fs)_x = 0,$$
$$s(\lambda;x,t) = -4\text{Re}\lambda + \frac{1}{\text{Im}\lambda}\iint_{\Lambda^+} \ln\left|\frac{\mu-\lambda^*}{\mu-\lambda}\right|[s(\lambda;x,t) - s(\mu;x,t)]f(\mu;x,t)d\xi d\zeta, \tag{18}$$

where $\mu = \xi + i\zeta$ and $\Lambda^+ \subset \mathbb{C}^+ \setminus i\mathbb{R}^+$. Equation (18) was rigorously derived in Ref. [24] (see also Ref. [29]) in the framework of the thermodynamic limit of finite-gap potentials, see Sec. III C confirming the formal construction proposed in Ref. [20]. The special case when all discrete spectrum points are located on the imaginary axis, $\Lambda^+ \subset i\mathbb{R}^+$, corresponds to nonpropagating multisoliton solutions called bound states [81]. This case requires a separate consideration since for the corresponding bound-state soliton gas $\text{Re}\lambda = 0$ the equation of state in (18) immediately yields $s(\lambda) = 0$, resulting in the stationary DOS, $f_t = 0$. Finally, the above construction of SG kinetic theory based on the collision-rate ansatz can be extended to the so-called anisotropic bidirectional SGs in integrable systems exhibiting different signs of phase shifts for overtaking and head-on collisions (a physically relevant example is the Kaup-Boussinesq system for shallow-water waves). The theory of bidirectional soliton gases was developed in Ref. [26].

C. Conserved quantities

One of the fundamental properties of integrable dynamics is the availability of an infinite set of conservation laws

$$\partial_t P_n + \partial_x Q_n = 0, \quad n = 1, 2, \dots, \tag{19}$$

where the P_n and Q_n are functions of the field variable u and its derivatives. For the KdV equation, of particular interest are the first three conserved densities:

$$P_1 = u, \quad P_2 = u^2, \quad P_3 = \frac{u_x^2}{2} - u^3, \tag{20}$$

typically associated with the "mass," "momentum" and "energy" conservation. Their counterparts for the fNLS equation have the form [81]

$$P_1 = |\psi|^2, \;\; P_2 = \text{Im}(\psi_x\psi^*), \;\; P_3 = |\psi|^4 - |\psi_x|^2. \tag{21}$$

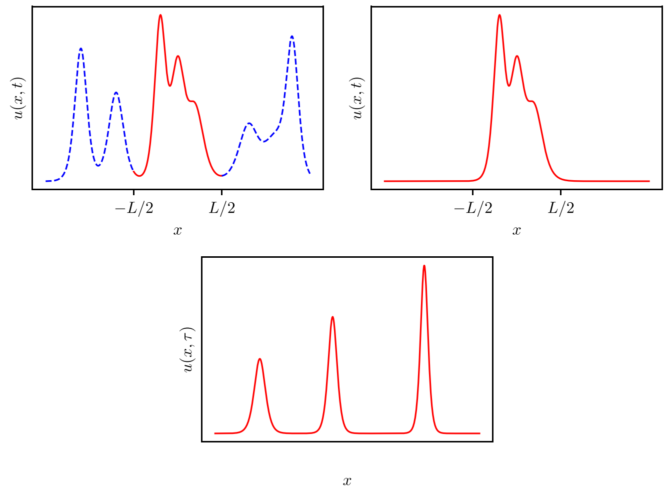

FIG. 4. Schematic illustrating the N-soliton approximation of the windowed portion $u_L(x)$ of a realization of KdV soliton gas at some $t = t^*$ (upper row) and its evolution into a soliton train at $t = \tau \gg t^*$ (lower plot).

For nonequilibrium soliton gas dynamics conservation Eq. (19) are replaced by their averaged analogs:

$$\partial_t\langle P_n[u]\rangle + \partial_x\langle Q_n[u]\rangle = 0, \quad n = 1, 2, \dots, \tag{22}$$

where $\langle\cdot\rangle$ denotes ensemble averaging, and the x, t variations in (19) occur on much larger scales than in (19).

In contrast with the discrete set of conservation laws (19) for the original equation, kinetic Eq. (15) possesses a continuum of conserved quantities. Indeed, (15) implies that for any $h(\eta) \neq 0$, $\int_\Gamma h(\eta)f(\eta;x,t)d\eta$ is a density of the conserved quantity with $\int_\Gamma h(\eta)f(\eta;x,t)s(\eta;x,t)d\eta$ being the corresponding flux density. For the KdV equation, the densities of the special "Kruskal" series (22) are given by [19,23]

$$\langle P_n[u]\rangle = C_n\int_\Gamma \eta^{2n-1}f(\eta)d\eta, \;\; n = 1, 2, \dots, \tag{23}$$

where the coefficients C_n depend on the normalization of the conserved densities. For the physical densities (20) we have

$$C_1 = 4, \quad C_2 = 16/3, \quad C_3 = 32/5. \tag{24}$$

Expressions (23) and (24) are readily obtained by considering a large portion of a homogeneous soliton gas: $u_L = \chi_{[0,L]}u(x,t)$ with $L \gg \alpha^{-1}$ at some arbitrary $t = t^*$, which is approximated by N-soliton ensemble defined in Sec. II B 2.[1] Assuming ergodicity, one can replace the ensemble average $\langle P_n\rangle$ by the spatial average $L^{-1}\int_{x_0}^{x_0+L} P_n[u_L]dx$ over a single realization. Since this spatial average is a conserved quantity it can be evaluated over the long-time asymptotic solution: $u_L \sim \sum_i u_s(x,t;\eta_i)$ as $t \to \infty$ (a soliton train), see Fig. 4 for the illustration of this concept.

[1]To avoid boundary effects one can assume that the transition to zero at the edges of the "window" $\chi_{[0,L]}$ is smooth but sufficiently rapid (e.g., exponential) so that such a windowed portion $u_L(x)$ of a soliton gas can be more faithfully approximated by the N-soliton solution for some $N \gg 1$, i.e., the continuous spectrum can be neglected.

A fundamental restriction imposed on the DOS $f(\eta)$ follows from non-negativity of the variance, $\mathcal{A} = \langle u^2\rangle - \langle u\rangle^2 \geqslant 0$, or equivalently, recalling (23) and (24),

$$\int_\Gamma \eta^3 f(\eta)d\eta - 3\left(\int_\Gamma \eta f(\eta)d\eta\right)^2 \geqslant 0. \quad (25)$$

Using the phenomenological DOS (11) the inequality (25) provides an upper bound on the admissible values of the spatial density α of a SG with a given spectral density $\phi(\eta)$, see Ref. [23].

For the fNLS equation the averaged conserved densities can also be expressed in terms of moments of the DOS as [29]

$$\langle P_n[\psi]\rangle = C_n \iint_{\Lambda^+} \mathrm{Im}(\lambda^n) f(\lambda) d\xi d\zeta, \quad n = 1, 2, \dots, \quad (26)$$

where $\lambda = \xi + i\zeta$ and the coefficients C_n for the physical conserved quantities (21) are

$$C_1 = 4, \quad C_2 = -4, \quad C_3 = 16/3. \quad (27)$$

The specific values (27) can be obtained by the above phenomenological windowing procedure applied to the fNLS gas [34].

III. SPECTRAL KINETIC THEORY OF SOLITON GAS

A. General framework

The phenomenological kinetic theory of soliton gas described in Sec. II B is essentially based on the interpretation of solitons as quasiparticles experiencing short-range pairwise interactions accompanied by the well-defined position shifts. As was already stressed, although this theoretical framework is justifiable in the case of rarefied gas, it is less satisfactory for a dense gas where solitons experience significant overlap and continual nonlinear interactions so that they could become indistinguishable as separate entities. This suggests that a more consistent theoretical approach involving the wave aspect of the soliton's "dual identity" is necessary. In this section we outline a general mathematical framework for the spectral theory of soliton gas based on the thermodynamic limit of nonlinear multiphase solutions of integrable equations. This approach has been first developed in Ref. [19] for KdV equation and more recently applied to the description of fNLS soliton and breather gases [24].

With the KdV equation as the simplest prototypical example in mind we consider the family of multiphase solutions of the form

$$u(x,t) = F_N(\theta_1, \dots, \theta_N), \quad \theta_j = k_j x - \omega_j t + \theta_j^0, \quad (28)$$

where k_j and ω_j, $j = 1, \dots, N$ are the wave numbers and frequencies (generally incommensurable), and the function F_N is 2π periodic with respect to each phase component $\theta_j \in [-\pi, \pi)$, θ_j^0 being initial phases. [In the context of the NLS Eq. (6) the representation (28) is valid for $|\psi|$]. We stress that the existence of multiphase quasiperiodic solutions (28) to a nonlinear dispersive equation is a unique property of integrable systems. Such solutions are typically expressed in terms of Riemann theta functions, see, e.g., Ref. [6], but we will not be using their specific form here.

It was discovered in 1970s that multiphase solutions to integrable equations have remarkable spectral properties defined within the (quasi-)periodic analog of IST called the finite-gap theory, see Refs. [5,6]. The fundamental result of the finite gap theory is that the IST spectrum $\mathcal{S}_N$ of the N-phase solution (28) lies in the union of $N+1$ disjoint bands $\gamma_j = [\lambda_{2j-1}, \lambda_{2j}]$, $j = 1, \dots, N+1$,

$$\lambda \in \mathcal{S}_N \equiv \cup_{i=1}^{N+1} \gamma_i, \quad \gamma_i \cap \gamma_j = \emptyset, \quad i \neq j, \quad (29)$$

separated by N finite gaps $c_j = (\lambda_{2j}, \lambda_{2j+1})$. The number N of spectral gaps is called the *genus*.

For the preliminary discussion of this section it is convenient to assume that the spectrum $\mathcal{S}_N$ is real-valued. This is the case for the (unidirectional) KdV equation and (bidirectional) defocusing NLS equation [Eq. (6) with $\sigma = -1$]. The case of complex band spectrum, $\mathcal{S}_N \subset \mathbb{C}$, arises for the fNLS equation, and this case will be considered separately in Sec. III C. We also note that one of the spectral bands could be semi-infinite (as is the case for the KdV equation), then $\gamma_{N+1} = [\lambda_{2N+1}, +\infty)$. Thus the spectrum of a finite-gap solution (also called finite-gap potential) is fully parametrized by the state vector $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_\mathcal{D})$, where $\mathcal{D} = 2N+1$ or $\mathcal{D} = 2N+2$ depending on the presence or absence of the semi-infinite band.

One of the important outcomes of the finite-gap theory are the *nonlinear dispersion relations* (NDRs) linking the physical parameters of the multiphase solution (28) such as the wave numbers, the frequencies and the mean with the components of the $\mathcal{D}$-dimensional spectral state vector $\boldsymbol{\lambda}$. In particular, for the N-component wave number and frequency vectors $\boldsymbol{k} = (k_1, \dots, k_N)$ and $\boldsymbol{\omega} = (\omega_1, \dots, \omega_N)$ in (28) the NDRs can be represented as

$$k_j = K_j(\boldsymbol{\lambda}), \quad \omega_j = \Omega_j(\boldsymbol{\lambda}), \quad j = 1, \dots, N, \quad (30)$$

where $K_j(\boldsymbol{\lambda})$, $\Omega_j(\boldsymbol{\lambda})$ are typically expressed in terms of complete hyperelliptic integrals, see, e.g., Refs. [6,24,65].

By manipulating the endpoints of spectral bands λ_j one can modify the waveform of the solution (28). In particular, by collapsing all spectral bands into double points, $\lambda_{2j-1}, \lambda_{2j} \to \lambda_*^j$, $j = 1, \dots, N$, the N-gap solution transforms into N-soliton solution with the discrete spectrum eigenvalues λ_*^j [5] (for the KdV equation $\lambda_*^j = -\eta_j^2$, see Sec. II A). This solitonic transition corresponds to the limit $k_j, \omega_j \to 0$, $(\omega_j/k_j) \to s_0(\lambda_*^j) = O(1)$, where $s_0(\lambda)$ is the velocity of a free soliton corresponding to the discrete spectral eigenvalue λ_*^j. (We note that any linear combination of wave numbers with integer coefficients is also a wave number, so by $\{k_j\}_{j=1}^N$ we always assume a particular, "fundamental," set of the wave numbers that vanish in the solitonic limit). Thus, finite-gap potentials represent periodic or quasiperiodic generalizations of multisoliton solutions. Importantly, finite-gap potentials, unlike N-soliton solutions, are nondecaying functions with nonzero mean, which makes them natural building blocks for the construction of equilibrium, spatially uniform, soliton gases. Another advantage of using finite-gap solutions for the soliton gas construction is the presence of a natural probability measure—the uniform measure on the N-dimensional phase torus $\mathbb{T}^N$ of (28), i.e., for each phase, $\theta_j^0 \in [-\pi, \pi)$ is assumed to be a random value uniformly distributed on the period. Assuming incommensurability of the wave numbers k_j and frequencies ω_j this measure gives rise to the ergodic random process with realizations defined by (28).

The dynamics of weakly nonuniform finite-gap potentials are described by the Whitham modulation theory [49,65,88], which prescribes slow evolution of the spectrum, $\lambda_j(x,t)$, on the spatiotemporal scales much larger than those associated with "rapid" variations of the wave (28) itself. The modulation system inherently includes wave conservation equations

$$\partial_t k_j + \partial_x \omega_j = 0, \quad j = 1, \dots, N, \tag{31}$$

where $k_j(\boldsymbol{\lambda})$ and $\omega_j(\boldsymbol{\lambda})$ are given by the NDRs (30).

We now define equilibrium soliton gas via the thermodynamic limit of finite-gap potentials [24]. Namely, we consider a sequence of finite-gap potentials (28) such that

$$N \to \infty : \quad k_1, \dots, k_N \to 0, \quad \sum_{j=1}^{N} k_j \to 2\pi\alpha = O(1), \tag{32}$$

with a similar behavior for the frequency components ω_j, so that $\omega_j / k_j = O(1)$. The limit (32) suggests the following asymptotic scaling for the fundamental wave numbers and frequencies as $N \to \infty$:

$$N \to \infty : \quad k_j \sim \omega_j \sim N^{-1}. \tag{33}$$

It can be shown quite generally that under the limit (32) the uniform distribution for the initial phases $\theta_j^0 \in [-\pi, \pi)$, $j = 1, \dots, N$ transforms to the Poisson distribution with the density parameter α on $\mathbb{R}$ for the "position phases" $l_j^0 \equiv \theta_j^0 / k_j$ [25], which corroborates the soliton position distribution in phenomenological model of rarefied SG in Sec. II B 1. We associate the limiting random process $\lim_{N\to\infty} F_N(\boldsymbol{\theta})$ satisfying (32) with SG assuming the existence of such a limit in some probabilistic sense. By construction this process is ergodic. As we shall see, the above definition is consistent with the phenomenological construction of SG in Sec. II B 2.

As we show in the next section, the thermodynamic limit (32) is achieved by imposing a special band-gap distribution (scaling) for the spectrum $\mathcal{S}_N$ for $N \gg 1$. Generally the spectral bands are required to be exponentially narrow compared with the gaps although the superexponential and subexponential scalings are also possible and these corresponds to the noninteracting (ideal) soliton gas and soliton condensate respectively. In all cases we call the corresponding limit as $N \to \infty$ of a function $F(\boldsymbol{\lambda})$ defined on $\mathcal{S}_N$ the thermodynamic limit of F.

The DOS $f(\lambda)$ is then defined via the thermodynamic limit of the partial sum

$$\frac{1}{2\pi} \sum_{j=1}^{M \leqslant N} k_j \to \int_{\lambda_{\min}}^{\lambda} f(\lambda') d\lambda', \tag{34}$$

where λ is a continuous spectral variable interpolating the discrete positions λ_j^* of the band centers; as a matter of fact $\int_{\lambda_{\min}}^{\lambda_{\max}} f(\lambda')d\lambda' = \alpha$, cf. (12).

Similarly, we have

$$\frac{1}{2\pi} \sum_{j=1}^{M \leqslant N} \omega_j \to \int_{\lambda_{\min}}^{\lambda} v(\lambda') d\lambda', \tag{35}$$

where $v(\lambda)$ is the spectral flux density; then $s(\lambda) = v(\lambda)/f(\lambda)$ has the meaning of the soliton gas transport velocity that can also be interpreted as an average tracer soliton velocity in the gas.

The next steps can be outlined as follows: Applying the thermodynamic limit (34) and (35) to the discrete NDRs (30) for finite-gap solutions one obtains the limiting, continuous NDRs for an equilibrium soliton gas which then yield the equation of state $s(\lambda) = \mathcal{F}[f(\lambda)]$ in the form of a linear integral Eq. (17). Furthermore, assuming $f \equiv f(\lambda; x, t)$, $s \equiv s(\lambda; x, t)$ and applying the thermodynamic limit to the modulation Eqs. (31), one obtains the continuity Eq. (15) for the DOS in a nonequilibrium gas. This procedure is detailed in the next section.

B. Korteweg–de Vries equation

We now present the results of the application of the above general spectral construction to the KdV Eq. (1) following Refs. [19,25] The key input ingredient of the theory are the discrete NDRs (30) for finite-gap potentials. The specific expressions for the KdV NDRs can be found elsewhere (see, e.g., Refs. [19,25,65]), here we only discuss their thermodynamic limit as $N \to \infty$.

First we recall that the N-soliton limit of an N-gap solution is achieved by collapsing all the finite bands γ_j in the spectral set $\mathcal{S}_N$ (29) into double points corresponding to the soliton discrete spectral values. It was proposed in Ref. [19] that the special infinite-soliton limit of the spectral N-gap KdV solutions, namely, the thermodynamic limit, provides spectral description the KdV soliton gas. The thermodynamic limit is achieved by assuming a special band-gap distribution (scaling) of the spectral set $\mathcal{S}_N$ for $N \gg 1$ on the fixed interval $[\lambda_1, \lambda_{2N+1}]$ (e.g., $[-1, 0]$). Specifically, we require the spectral bands γ_j to be exponentially narrow compared with the gaps c_j so that for $N \to \infty$ the spectral set $\mathcal{S}_N$ is asymptotically characterized by two continuous positive functions: the density $\phi(\eta)$ of the lattice points $\eta_j \in \Gamma \subset \mathbb{R}^+$ defining the band centers via $-\eta_j^2 = (\lambda_{2j} + \lambda_{2j-1})/2$, and the logarithmic bandwidth distribution $\tau(\eta)$ defined for $N \to \infty$ by

$$\eta_j - \eta_{j+1} \sim \frac{1}{N\phi(\eta_j)}, \quad \tau(\eta_j) \sim -\frac{1}{N} \ln(\lambda_{2j} - \lambda_{2j-1}), \tag{36}$$

with $\int_\Gamma \phi(\eta)d\eta = 1$. Note that the definition of $\phi(\eta)$ agrees with expression (10) for the density of soliton spectra in the phenomenological model of SG. Additionally, invoking the asymptotic behaviors (33) we introduce the interpolating functions $\kappa(\eta)$, $\nu(\eta)$ for the scaled wave numbers and frequencies

$$k_j \sim \frac{\kappa(\eta_j)}{N}, \quad \omega_j \sim \frac{\nu(\eta_j)}{N}. \tag{37}$$

Then the definitions (34) and (35) of the DOS and the spectral flux density imply

$$f(\eta) = \frac{1}{2\pi}\kappa(\eta)\phi(\eta), \quad v(\eta) = \frac{1}{2\pi}\nu(\eta)\phi(\eta), \tag{38}$$

where we have used a more convenient in the KdV context spectral variable η instead of $\lambda = -\eta^2$. Note that the expression for the DOS $f(\eta)$ in (38) is consistent with the phenomenological definition (11) with the important difference that now the spatial density of SG depends on the spectral parameter and is given by $\tilde{\alpha}(\eta) = \kappa(\eta)/2\pi$.

Now, considering the KdV finite-gap NDRs (30) subject to the thermodynamic scaling (36) and (37) and letting $N \to \infty$, yields the integral equations [19,25]

$$\begin{aligned}\int_\Gamma \ln\left|\frac{\mu+\eta}{\mu-\eta}\right| f(\mu)d\mu + f(\eta)\sigma(\eta) = \eta,\\ \int_\Gamma \ln\left|\frac{\mu+\eta}{\mu-\eta}\right| v(\mu)d\mu + v(\eta)\sigma(\eta) = 4\eta^3\end{aligned} \tag{39}$$

for all $\eta \in \Gamma$ (if Γ is a fixed, simply connected compact interval one can set $\Gamma = [0, 1]$ without loss of generality). Here the *spectral scaling function* $\sigma : \Gamma \to [0, \infty)$ is a continuous non-negative function that encodes the Lax spectrum of the soliton gas via $\sigma(\eta) = \tau(\eta)/\phi(\eta)$. Equations (39) are the KdV soliton-gas NDRs.

Eliminating $\sigma(\eta) > 0$ from the NDRs (39) yields the equation of state (16) for the KdV soliton gas. Next, for a nonhomogeneous soliton gas $f(\eta) \equiv f(\eta; x, t)$, $v(\eta) \equiv v(\eta; x, t)$, and the application of the thermodynamic limit to the modulation Eq. (31) yields the continuity Eq. (15) for the DOS. Indeed, (31) implies

$$\left(\frac{1}{2\pi}\sum_{j=1}^{M\leqslant N} k_j\right)_t + \left(\frac{1}{2\pi}\sum_{j=1}^{M\leqslant N} \omega_j\right)_x = 0 \tag{40}$$

for $M = 1, \dots, N$. Applying the thermodynamic limit (34) and (35) to (40) we obtain the kinetic equation $f_t + (fs)_x = 0$ as required, see Eq. (15). Assuming fixed spectral support Γ in the NDRs (39) it is not difficult to show that the evolution of $\sigma(\eta; x, t)$ in a nonhomogeneous SG satisfies the Riemann-type equation

$$\sigma_t + s(\eta; x, t)\sigma_x = 0, \tag{41}$$

which can be used instead of the continuity Eq. (15).

Summarizing, the SG spectral kinetic Eqs. (15) and (16) represent the thermodynamic limit of the KdV-Whitham modulation system [19]. We note that condition $\sigma > 0$ used in the derivation of the equation of state (16) implies the restriction $\eta^{-1}\int_\Gamma \ln|\frac{\mu+\eta}{\mu-\eta}| f(\mu)d\mu < 1$ on the admissible DOS $f(\eta)$, complementing the earlier formulated restriction (25). The limiting case $\sigma = 0$ corresponds to the special soliton gas termed *soliton condensate*, see Sec. III D below.

C. Focusing nonlinear Schrödinger equation

The spectral theory of SG for the fNLS equation was developed in Ref. [24]. It follows the same general framework of the thermodynamic limit of finite-gap potentials outlined in Sec. III A and resulting in the kinetic Eq. (18) for the dense gas of fundamental fNLS solitons. However, due to the fact that the finite-gap spectral set for the fNLS equation lies in the complex plane, $\lambda \in \mathbb{C}$, the spectral theory of fNLS soliton gas admits a much broader range of scenarios than the KdV theory. In particular, it covers the case of *breather gases*, including infinite random ensembles of interacting Akhmediev, Kuznetsov-Ma, and Peregrine breathers [66]. Another highly nontrivial object is the gas of bound state fNLS solitons (bound states are N-soliton solutions with all discrete spectral parameters λ_j, $j = 1, \dots, N$ having the same, possibly zero, real part [81]). The latter was shown in Ref. [47] to represent an accurate model for the nonlinear stage of the development of spontaneous (noise-induced) modulation instability, see Sec. VI B.

The SG theory for the fNLS equation is more technically involved than in the KdV case. Here we only present the NDRs for the fNLS soliton gas, a counterpart of the KdV NDRs (39):

$$\begin{aligned}\int_{\Gamma^+} \ln\left|\frac{\mu-\bar{\lambda}}{\mu-\lambda}\right| f(\mu)|d\mu| + \sigma(\lambda)f(\lambda) = \mathrm{Im}\lambda,\\ \int_{\Gamma^+} \ln\left|\frac{\mu-\bar{\lambda}}{\mu-\lambda}\right| v(\mu)|d\mu| + \sigma(\lambda)v(\lambda) = -4\mathrm{Im}\lambda\mathrm{Re}\lambda,\end{aligned} \tag{42}$$

where Γ^+ is the upper part of the 1D Schwarz-symmetric curve $\Gamma \subset \mathbb{C}$—the spectral support of the DOS $f(\lambda)$ (in the general 2D case the integration with respect to arc length of Γ^+ in (42) is replaced by the integration over a 2D compact domain $\Lambda^+ \subset \mathbb{C}^+$: $\int_{\Gamma^+} \cdots |d\mu| \to \iint_{\Lambda^+} \cdots d\xi d\zeta$, where $\mu = \xi + i\zeta$).

Eliminating the spectral scaling function $\sigma(\lambda)$ from the NDRs (42) we obtain the equation of state in the kinetic Eq. (18). The continuity equation in (18) is derived via the thermodynamic limit of the modulation Eq. (31), similar to the KdV case. See Ref. [24] for details.

D. Polychromatic soliton gases and soliton condensates

Integration of the spectral kinetic Eqs. (15) and (17) for a soliton gas in any generality represents a challenging mathematical problem. One can, however, consider some physically interesting particular cases that admit effective analytical treatment. The most obvious one is the case of spectrally polychromatic gases studied in Ref. [21]. The DOS of a polychromatic soliton gas represents a linear combination of the "monochromatic" components in the form of Dirac δ-functions centered at distinct spectral points $\zeta_j \in \Gamma$ (note that Γ can be real or complex domain, depending on the original dispersive equation)

$$f(\lambda; x, t) = \sum_{j=1}^{M} w_j(x, t)\delta(\lambda - \zeta_j), \tag{43}$$

where $w_j(x, t) > 0$ are the components' weights and $\{\zeta_j\}_{j=1}^M \subset \Gamma$, $(\zeta_j \neq \zeta_k \iff j \neq k)$. Substitution of (43) into the kinetic Eqs. (15) and (17) reduces it to a system of hyperbolic hydrodynamic conservation laws

$$(w_i)_t + (w_i s_i)_x = 0, \quad i = 1, \dots, M, \tag{44}$$

where the component densities $w_i(x,t)$ and the transport velocities $s_j(x,t) \equiv s(\zeta_j,x,t)$ are related algebraically:

$$s_j = s_{0j} + \sum_{m=1,m\neq j}^{M} G_{jm} w_m (s_j - s_m), \quad j = 1,2,\dots M. \tag{45}$$

Here we used the notation $s_{0j} \equiv s_0(\zeta_j)$, $G_{jm} \equiv G(\zeta_j,\zeta_m)$, $j \neq m$. One should also mention an important restriction $\sum_{m=1,m\neq j}^{M} G_{jm} w_m < 1$, a counterpart of the condition of positivity for the spectral scaling function σ in the thermodynamic limit construction.

We note that the δ-function representation (43) is a mathematical idealization, which has a formal sense in the context of the integral equation of state (17) but cannot be applied to the original dispersion relations where it appears in both the integral and the secular terms [cf. (39) for the KdV equation]. In a physically realistic description the δ functions in (43) should be replaced by some narrow distributions around the spectral points ζ_j, i.e., we first take the thermodynamic limit $N \to \infty$ and then allow the distributions to become sharply peaked, see Ref. [93].

For $M = 2$ system (45) can be solved to give explicit expressions for $s_{1,2}(w_1,w_2)$:

$$\begin{aligned} s_1 &= s_{01} + \frac{G_{12} w_2 (s_{01} - s_{02})}{1 - (G_{12} w_2 + G_{21} w_1)}, \\ s_2 &= s_{02} - \frac{G_{21} w_1 (s_{01} - s_{02})}{1 - (G_{12} w_2 + G_{21} w_1)}. \end{aligned} \tag{46}$$

As was shown in Ref. [20] (see also Ref. [27]) the two-component system (44) and (46) is equivalent to the so-called Chaplygin gas equations that occur in certain theories of cosmology (see, e.g., Ref. [89]), and to the Born-Infeld equations arising in nonlinear electromagnetic field theory [49,90].

It was shown in Ref. [21] that system (44) and (45) for any $M \in \mathbb{N}$ possesses M Riemann invariants and belongs to the special class of linearly degenerate, semi-Hamiltonian systems of hydrodynamic type [91]. Linear degeneracy of (44) and (45) implies the absence of wave breaking and shock formation for generic initial-value problems with smooth Cauchy data [92]. On the other hand, it implies that the solution to a Riemann (the evolution of an initial discontinuity) problem for polychromatic soliton gas will be given by a combination of differing constant states $w_i = \text{const}_j$, separated by *contact discontinuities* propagating with classical shock speeds found from the Rankine-Hugoniot conditions for the conservation laws (44) and (45). Such weak solutions were constructed in Refs. [20,22,25,26,93] for various soliton gases. More general solutions are available via the hodograph transform, see Refs. [21,27].

Another special class of soliton gases is presented by soliton condensates whose properties are dominated by the collective effect of soliton interactions while the individual soliton dynamics are completely suppressed. Soliton condensates were first introduced in Ref. [24] for the fNLS equation and then thoroughly studied in Ref. [30] for the KdV equation. Spectrally, a soliton condensate is realized by vanishing the spectral scaling function, $\sigma(\eta) \to 0$, in the soliton gas NDRs [Eq. (39) for KdV or (42) for fNLS]. For the KdV case the condensate NDRs are then given by [30]

$$\int_\Gamma \ln\left|\frac{\mu+\eta}{\mu-\eta}\right| f(\mu)d\mu = \eta, \quad \int_\Gamma \ln\left|\frac{\mu+\eta}{\mu-\eta}\right| v(\mu)d\mu = 4\eta^3. \tag{47}$$

For the simplest case $\Gamma = [0,q]$ these are solved by

$$f(\eta) = \frac{\eta}{\pi\sqrt{q^2-\eta^2}}, \quad v(\eta) = \frac{6\eta(2\eta^2-1)}{\pi\sqrt{q^2-\eta^2}}. \tag{48}$$

The counterpart fNLS solution of the NDRs (42) with $\lambda \in \Gamma^+ = [0, iq]$ and $\sigma = 0$ is given by [24]

$$f(\lambda) = \frac{-i\lambda}{\pi\sqrt{q^2+\lambda^2}}, \quad v(\lambda) = 0, \tag{49}$$

and describes the DOS $f(\lambda)$ in the nonpropagating, bound state ($s = v/f = 0$) soliton condensate. By choosing a different 1D support $\Gamma^+ \subset \mathbb{C}^+$ one can construct other types of fNLS soliton condensates. For example, if $\Gamma^+ = \{\xi + i\eta \mid \xi^2 + \eta^2 = 1, \ \eta > 0\}$ (a semicircle) then the corresponding condensate DOS $f(\lambda) = \frac{\text{Im}\lambda}{\pi}$ [24]. Such a "circular" soliton condensate propagates with the speed $s(\lambda) = -8\text{Re}\lambda$—twice the speed of a free fNLS soliton.

Concluding this section, we mention an important generalization of the spectral theory of KdV soliton condensates developed in Ref. [30] by assuming the spectral support Γ in (47) to be a union of $N+1$ finite disjoint intervals, termed "s-bands," $\Gamma = [0,\beta_1] \cup [\beta_2,\beta_3] \cup [\beta_{2j},\beta_{2j+1}]$, $j = 0,\dots N$, with $\beta_j = \beta_j(x,t)$. It was shown in Ref. [30] that the kinetic Eqs. (15) and (16) then imply that the endpoints β_j of the s-bands vary according to the genus N KdV-Whitham equations [65], providing the connection of nonequilibrium soliton gas with the fundamental objects of dispersive hydrodynamics such as rarefaction and dispersive shock waves [50]. The fNLS counterpart of the KdV theory of soliton condensates was recently developed in Ref. [94]. Finally we mention a very recent paper [34] where fNLS soliton condensates were used to explain important statistical features of integrable turbulence related to the formation of extreme events (rogue waves).

IV. INVERSE SCATTERING TRANSFORM APPROACHES TO SYNTHESIS AND ANALYSIS OF SOLITON GAS

As shown in the previous sections, the IST and finite-gap theory lay the foundations for the theory of soliton gas, demonstrating that soliton collisions are elastic and providing exact relations for the shifts in soliton positions, ultimately leading to the kinetic equation. Here, on the example of the fNLS equation, we discuss how the IST method allows one to observe the wave field of soliton gas in practice by generating such fields in numerical simulations or experiments from known soliton parameters. Also, we discuss the numerical techniques to solving the (opposite) direct-scattering problem and determining the complete set of soliton parameters—eigenvalues and norming constants—from numerically or experimentally observed wave fields. Combined, the solutions of these two problems form a complete recipe for the IST synthesis and analysis of soliton gas.

Note that the similar IST-based algorithms are also available for the KdV equation, see, e.g., Refs. [30,95] for the numerical synthesis of multisoliton solutions. Our choice of using the fNLS soliton gas as the main example is motivated by the rich phenomenology and its important physical applications supported by recent experiments, see Sec. V for details.

Also note that, in general, the IST can be formulated by using different types of boundary conditions, such as vanishing and constant [5,96]. Recently, a considerable progress has been made in the description of localized modulation instability by using the IST with constant boundary conditions [97–100]. In the present paper, we focus on the vanishing boundary conditions suitable for numerical simulations of multisoliton solutions.

In the case of a rarefied soliton gas, its wave field can be constructed as an arithmetic sum of wave fields of single solitons with eigenvalues and positions chosen in accordance with the desired DOS, see Sec. II B 1. The dynamical and statistical properties of such gases have been studied previously using the weak-interaction model, two-soliton interaction models and direct numerical simulations, see, e.g., Refs. [69,101–104].

Constructing a wave field for a dense soliton gas requires full consideration of the interaction of solitons. In this section, we describe an approach based on the numerical construction of exact dense N-soliton solutions containing a large number N of solitons, which is used in recent numerical and experimental studies [47,48,57,105]. Although multisoliton solutions are localized in space, for large N edge effects can be neglected and the central part of the wave field can be considered as a continuous section of soliton gas. By changing the soliton norming constants, it is possible to influence the distribution of solitons in the physical space, even though the exact mathematical link between the norming constants and the soliton spatial density (or, more generally, the DOS) is still missing.

Albeit explicit formulas for exact multisoliton solutions have been known for decades, see, e.g., Ref. [5], their practical application was impossible due to numerical errors in the form of extreme gradients that appeared already starting from $N \approx 10$ solitons. The main source of these errors is the roundoff during a large number of arithmetic operations with exponentially small and large numbers. A solution to this problem has been found only recently in Ref. [48] with a specific implementation of the dressing method combined with high-precision arithmetic computations, making it possible to successfully generate wave fields containing hundreds of solitons.

Concerning the direct-scattering procedure, there are several well established methods for the computation of soliton eigenvalues, see, e.g., the Fourier collocation and Boffetta–Osborne methods. In the present paper, we focus on a highly challenging problem of the accurate identification of soliton norming constants, which is hampered by several types of numerical instabilities and has been solved only very recently in Refs. [106,107].

In this section, we consider the fNLS equation in the form

$$i\psi_t + \tfrac{1}{2}\psi_{xx} + |\psi|^2\psi = 0, \tag{50}$$

following the studies [47,48,106,108,109] on the application of numerical IST and also other literature where the coefficients used in Eq. (50) are conventional.

A. Inverse scattering transform method formalism

The IST method is based on the correspondence between an integrable nonlinear PDE and a specific auxiliary system of two linear equations (Lax pair), which consists of a stationary eigenvalue problem and an evolutionary problem for the same auxiliary function. The considered PDE is then obtained from the Lax pair as a compatibility condition. Using this compatibility condition, one can prove the fundamental property of the auxiliary system that its eigenvalue spectrum does not change with the evolution of wave field [5].

For the fNLS Eq. (50), the Lax pair is known as the Zakharov–Shabat system [81] for a two-component vector wave function $\mathbf{\Phi}(x,\lambda) = (\phi_1, \phi_2)^{\mathrm{T}}$,

$$\mathbf{\Phi}_x = \begin{pmatrix} -i\lambda & \psi \\ -\psi^* & i\lambda \end{pmatrix}\mathbf{\Phi}, \tag{51a}$$

$$\mathbf{\Phi}_t = \begin{pmatrix} -i\lambda^2 + \frac{i}{2}|\psi|^2 & \lambda\psi + \frac{i}{2}\psi_x \\ -\lambda\psi^* + \frac{i}{2}\psi_x^* & i\lambda^2 - \frac{i}{2}|\psi|^2 \end{pmatrix}\mathbf{\Phi}, \tag{51b}$$

where the superscript T stands for the matrix transpose and $\lambda = \xi + i\eta$ is a complex-valued spectral parameter. The first Eq. (51a) is equivalent to the eigenvalue problem for λ written via the Lax operator $\widehat{\mathcal{L}}$ as

$$\widehat{\mathcal{L}}\mathbf{\Phi} = \lambda\mathbf{\Phi}, \quad \widehat{\mathcal{L}} = i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{\partial}{\partial x} - i\begin{pmatrix} 0 & \psi \\ \psi^* & 0 \end{pmatrix}. \tag{52}$$

One can check that the fNLS equation, i.e., Eq. (6) with $\sigma = 1$, can be obtained in the antidiagonal elements of the compatibility condition,

$$\mathbf{\Phi}_{xt} = \mathbf{\Phi}_{tx}. \tag{53}$$

Note that, for the KdV Eq. (1), the equivalent Lax operator represents the self-adjoint Schrödinger operator (2), for which the spectral theory is well developed in quantum mechanics, see, e.g., Refs. [110]. For the fNLS equation, the Lax operator is not self-adjoint, meaning that its eigenvalues can be located in the entire complex plane, although it is sufficient to consider only the upper half of it, $\eta = \mathrm{Im}\lambda \geqslant 0$. The latter follows from the fact that, for every solution $\mathbf{\Phi} = (\phi_1, \phi_2)^{\mathrm{T}}$ of the Zakharov–Shabat system, which corresponds to an eigenvalue λ, there exists a counterpart $\tilde{\mathbf{\Phi}} = (-\phi_2^*, \phi_1^*)^{\mathrm{T}}$ corresponding to the complex-conjugate eigenvalue λ^*. Despite these differences, there are many similarities in the spectral theory of the operator (52) and the Schrödinger operator, and we encourage the reader to keep in mind this analogy, according to which the wave field ψ of the fNLS equation is considered as a potential, and the vector function $\mathbf{\Phi}$ as a wave function. In what follows, we consider only a rapidly decaying potentials $\psi(x)$.

Similarly to quantum mechanics, the scattering problem (51a) for the wave function $\mathbf{\Phi}$ can be introduced with the following asymptotics at infinity (the so-called "right" scattering problem, in contrast with the "left" scattering problem, see, e.g., Refs. [111]),

$$\lim_{x\to-\infty}\left\{\mathbf{\Phi}-\begin{pmatrix}e^{-i\lambda x}\\0\end{pmatrix}\right\}=0, \tag{54}$$

$$\lim_{x\to+\infty}\left\{\mathbf{\Phi}-\begin{pmatrix}a(\lambda)e^{-i\lambda x}\\b(\lambda)e^{i\lambda x}\end{pmatrix}\right\}=0. \tag{55}$$

These asymptotics represent a two-component generalization of the "right" scattering problem for the Schrödinger operator. The scattering coefficients $a(\lambda)$ and $b(\lambda)$ have the meaning that a wave $(ae^{-i\lambda x},0)^{\mathrm{T}}$ comes from the right side of the potential $\psi(x)$ and then splits into the transmitted wave $(e^{-i\lambda x},0)^{\mathrm{T}}$ at $x\to-\infty$ and the reflected wave $(0,be^{i\lambda x})^{\mathrm{T}}$ at $x\to+\infty$. Hence, the quantity $r=b/a$ represents the so-called reflection coefficient. Note that the alternative choice of the asymptotics corresponding to the left-scattering problem is also common in the IST constructions, see, e.g., Ref. [111].

The eigenvalue spectrum of the scattering problem consists of the eigenvalues λ corresponding to bounded solutions $\mathbf{\Phi}$ of the Zakharov–Shabat system with asymptotics (54) and (55). Such solutions exist for real-valued spectral parameter, $\lambda=\xi\in\mathbb{R}$, and also for complex-valued λ, $\eta=\mathrm{Im}\lambda>0$, if and only if $a(\lambda)=0$. For rapidly decaying potentials $\psi(x)$, the latter part of the eigenvalue spectrum usually consists of a finite number of discrete points λ_n, $a(\lambda_n)=0$, $n=1,\dots,N$ (discrete spectrum), and the overall eigenvalue spectrum contains also the real line $\lambda=\xi\in\mathbb{R}$ (continuous spectrum), see Ref. [5]. The full set of the scattering data represents a combination of the discrete $\{\lambda_n,\rho_n\}$ and continuous $\{r\}$ spectra,

$$\{\lambda_n|a(\lambda_n)=0,\ \mathrm{Im}\lambda_n>0\},$$
$$\rho_n=\frac{b(\lambda_n)}{a'(\lambda_n)},\quad r(\xi)=\frac{b(\xi)}{a(\xi)}, \tag{56}$$

where $a'(\lambda)$ is complex derivative of $a(\lambda)$ with respect to λ, ρ_n are the so-called norming constants associated with the eigenvalues λ_n, and $r(\xi)$ is the reflection coefficient defined on the real line $\xi\in\mathbb{R}$. Most importantly, the time evolution of the scattering data (56) is trivial,

$$\forall\, n:\lambda_n=\mathrm{const},\quad \rho_n(t)=\rho_n(0)e^{2i\lambda_n^2 t},$$
$$r(\xi,t)=r(\xi,0)e^{2i\xi^2 t}, \tag{57}$$

and the wave field $\psi(x,t)$ can be recovered from it at any moment of time with the IST by solving the integral Gelfand-Levitan-Marchenko (GLM) equations [5]. However, in the general case, the latter procedure can only be done numerically, asymptotically at large time, or in the semiclassical approximation [112,113].

Note that the function $a(\lambda)$ is analytic in the upper half of the λ plane and has simple zeros at the eigenvalue points $a(\lambda_n)=0$ (we do not consider the degenerate case when an eigenvalue point represents a multiple zero), see, e.g., Refs. [5,96]. Meanwhile, the analyticity is not always the case for the function $b(\lambda)$. However, in numerical simulations or experiments, the wave field $\psi(x)$ is always confined to a finite region of space, i.e., it has compact support, and in this case the function $b(\lambda)$ is also analytic in the upper half of the λ plane [5,96]. This property of $a(\lambda)$ and $b(\lambda)$ is essential for algorithmic implementations of the direct-scattering transform discussed below.

In the physical space, the continuous spectrum with nonzero reflection coefficient $r(\xi)$ corresponds to nonlinear dispersive waves, while the discrete eigenvalues λ_n together with the norming constants ρ_n—to solitons. In particular, the eigenvalues $\lambda_n=\xi_n+i\eta_n$ contain information about the soliton amplitudes $A_n=2\eta_n$ and group velocities $V_n=-2\xi_n$, while the soliton norming constants—about their positions in space $x_n^{\mathrm{IST}}\in\mathbb{R}$ and complex phases $\theta_n^{\mathrm{IST}}\in[0,2\pi)$. In a weakly nonlinear case, the discrete spectrum disappears and the function $r(\xi)$ tends to a conventional Fourier spectrum of the wave field $\psi(x)$, so that the IST is often considered as a nonlinear analog to the Fourier transform.

In the (opposite) reflectionless case $r(\xi)=0$, the dispersive waves are absent and the IST procedure can be performed analytically by solving the GLM equations, leading to an exact N-soliton solution (N-SS) $\psi_{(N)}(x,t)$. There is also an alternative procedure for the construction of N-SS called the dressing method [5,114], also known as the Darboux transformation [115,116]. The dressing method allows one to add solitons to the resulting solution recursively by one at a time using a special algebraic construction [5,114,116]. The numerical implementation of this construction turns out to be much more stable and resource-efficient than solving the GLM equations, making it possible to build multisoliton wave fields containing large number of solitons [48].

The dressing procedure starts from the trivial potential of the fNLS equation, $\psi_{(0)}(x)=0$ for $x\in\mathbb{R}$, and the corresponding matrix solution of the Zakharov–Shabat system (51),

$$\mathbf{\Phi}^{(0)}(x,\lambda)=\begin{pmatrix}e^{-i\lambda x}&0\\0&e^{i\lambda x}\end{pmatrix}; \tag{58}$$

here we fix time, $t=0$, for definiteness. At the nth step of the recursive method, the n-soliton potential $\psi_{(n)}(x)$ is constructed via the $(n-1)$-soliton potential $\psi_{(n-1)}(x)$ and the corresponding matrix solution $\mathbf{\Phi}^{(n-1)}(x,\lambda)$ as

$$\psi_{(n)}(x)=\psi_{(n-1)}(x)+2i(\lambda_n-\lambda_n^*)\frac{q_{n1}^*q_{n2}}{|\mathbf{q_n}|^2}, \tag{59}$$

where the vector $\mathbf{q}_n=(q_{n1},q_{n2})^{\mathrm{T}}$ is determined by $\mathbf{\Phi}^{(n-1)}(x,\lambda)$ and the scattering data of the nth soliton $\{\lambda_n,C_n\}$,

$$\mathbf{q}_n(x)=[\mathbf{\Phi}^{(n-1)}(x,\lambda_n^*)]^*\cdot\begin{pmatrix}1\\C_n\end{pmatrix}. \tag{60}$$

Here C_n, $n=1,\dots,N$, are the soliton norming constants in the dressing method formalism, see the discussion below. The corresponding matrix solution $\mathbf{\Phi}^{(n)}(x,\lambda)$ of the Zakharov–Shabat system is calculated via the so-called dressing matrix $\boldsymbol{\sigma}^{(n)}(x,\lambda)$,

$$\mathbf{\Phi}^{(n)}(x,\lambda)=\boldsymbol{\sigma}^{(n)}(x,\lambda)\cdot\mathbf{\Phi}^{(n-1)}(x,\lambda), \tag{61}$$

$$\sigma_{ml}^{(n)}(x,\lambda)=\delta_{ml}+\frac{\lambda_n-\lambda_n^*}{\lambda-\lambda_n}\frac{q_{nm}^*q_{nl}}{|\mathbf{q_n}|^2}, \tag{62}$$

where $m,l=1,2$ and δ_{ml} is the Kronecker delta. The time dependency is recovered using the time-evolution of the norming constants,

$$C_n(t)=C_n(0)e^{-2i\lambda_n^2 t}, \tag{63}$$

and repeating the dressing procedure at each time t.

The norming constants C_n are related to the IST norming constants ρ_n as follows [106,117] (this equation is valid for pure multisoliton solutions only):

$$C_n(t) = \frac{1}{\rho_n(t)} \prod_{k=1}^{N} (\lambda_n - \lambda_k^*) \prod_{j\neq n}^{N} \frac{1}{\lambda_n - \lambda_j}, \tag{64}$$

and can be parametrized in terms of soliton positions x_n^{DM} and phases θ_n^{DM},

$$C_n = -\exp\left[2i\lambda_n x_n^{\mathrm{DM}} + i\theta_n^{\mathrm{DM}}\right]. \tag{65}$$

Note that the IST norming constants ρ_n have an alternative parametrization via IST positions x_n^{IST} and phases θ_n^{IST}, which coincide with the dressing method positions x_n^{DM} and phases θ_n^{DM} and also with the observed in the physical space positions and phases only for the one-soliton solution (7). In presence of other solitons or dispersive waves, all three types of positions and phases may differ significantly from each other; see, e.g., the discussion in Ref. [108] and the references wherein.

B. Inverse scattering transform synthesis of soliton-gas wave field

The discussed method for the numerical construction of soliton-gas wave field is based on the computation of N-SS for a large number of solitons N using the straightforward algorithmic implementation of the dressing method. The high-precision arithmetics is applied to accurately resolve operations with exponentially small and large numbers coming from the elements of vectors $\mathbf{q}_n$ in Eqs. (58)–(62). The required number of digits grows with N nontrivially and depends on specific choice of the soliton eigenvalues and norming constants but usually stays in the hundreds for $N \simeq 100$ and thousands for $N \simeq 1000$; see Refs. [47,48,108] for detail. Note that, while this inherent difficulty of the dressing method and other schemes based on the IST theory cannot be entirely avoided, the recently developed optimizations [118] can substantially reduce the necessary number of digits.

For the fNLS equation, soliton gas is characterized by the distribution of soliton eigenvalues (amplitudes and velocities) and soliton norming constants (positions and phases). Soliton eigenvalues are generally problem-specific and cannot be easily changed without modifying the context in which the soliton gas is studied. Soliton phases can usually be chosen as random values uniformly distributed over the interval $[0, 2\pi)$; in this case, evolution over time, see Eq. (63), preserves this distribution. In what follows, we focus on the study of dense soliton gases that are in equilibrium and have wave fields that are statistically homogeneous in space. This poses two problems: (i) how to achieve a high spatial density of solitons and (ii) how to construct multisoliton wave fields, which are statistically homogeneous over a wide region in the physical space for random soliton phases.

As has been observed empirically in Refs. [47,48], if soliton positions are distributed within the interval $x_n^{\mathrm{DM}} \in [-L_0/2, L_0/2]$ and L_0 approaches zero, then the characteristic size of the corresponding N-SS in the physical space shrinks to some finite nonzero limit and the soliton spatial density reaches its maximum value. However, for $L_0 = 0$ the N-SS becomes symmetric, $\psi(x) = \psi(-x)$. To avoid this artificial symmetry, one can use sufficiently small intervals $L_0 \simeq 1$, so that the symmetry is not observed and the characteristic size of the N-SS remains close to the size in the limiting case $L_0 = 0$.

In Ref. [48], a method has been developed for the construction of statistically homogeneous soliton gas wave fields, which starts from the computation of N-SS wave fields using rather arbitrary soliton positions from a small interval $x_n^{\mathrm{DM}} \in [-L_0/2, L_0/2]$, $L_0 \simeq 1$. Then, these wave fields are put into a sufficiently large box $x \in [-L_p/2, L_p/2]$ where they are small near the edges,

$$|\psi(\pm L_p)| \lesssim 10^{-16} \max|\psi(x)|,$$

so that one can treat this box as a periodic one and simulate the time evolution of the constructed solutions inside it using the direct numerical simulation of the fNLS equation. If soliton velocities are random, then after some time the wave fields spread over the box L_p and the system arrives to a statistically steady state, in which its basic statistical functions no longer depend on time. Then this state is used as a model of statistically homogeneous soliton gas of spatial density N/L_p in an infinite space; in Ref. [48], it has been confirmed that for large enough number of solitons N and box size L_p the results depend on them only in the combination N/L_p. Note that such "periodization" of solitons requires the periodic box L_p to be significantly larger than the characteristic size of the initial N-SS, decreasing the maximum soliton density that can be achieved with the described method.

In terms of the finite-gap theory, the periodic evolution in time replaces N-SS by N-band periodic solutions having exponentially narrow bands compared with the gaps, as the characteristic soliton width is much smaller than the box size L_p. This allows one to neglect the difference between the two types of solutions, similarly as it is done in Sec. III A, where, vice versa, the soliton gas is considered as a limit of finite-gap solutions.

Figure 5 illustrates the periodization method on the example of a single 128-SS generated from solitons having uniformly distributed positions $x_n^{\mathrm{DM}} \in [-L_0/2, L_0/2]$, $L_0 = 2$, and phases $\theta_n^{\mathrm{DM}} \in [0, 2\pi)$, equal amplitudes $A_n = \pi/3.2 \approx 1$ and Gaussian-distributed velocities with zero mean and standard deviation $V_0 = 2$, $V_n \sim \mathcal{N}(0, V_0^2)$. The initial 128-SS has characteristic size in the physical space $\delta X \simeq 280$, and on average the wave field is greater at the center than closer to the edges of the solution, see Fig. 5(a). Then, this solution is placed into the periodic box $x \in [-L_p/2, L_p/2]$, $L_p = 128\pi$, and the evolution is simulated until the final time $t = 200$, when the wave field in average becomes fairly uniform, see Fig. 5(b). As has been verified in Ref. [48], the soliton eigenvalues Λ_n calculated at the final simulation time with the Fourier collocation method [119] almost coincide with the eigenvalues λ_n of the initial 128-SS with the relative differences between the two $|\lambda_n - \Lambda_n|/|\lambda_n|$ of 10^{-9} order.

The described above periodization method can be applied only when the soliton velocities are distributed over some finite interval of values. In Ref. [47], it has been observed that for the special case of bound-state soliton gas, i.e., when all solitons have the same velocity (which one can set to zero for simplicity), a certain distribution of soliton eigenvalues (i.e., amplitudes) leads to a statistically homogeneous multisoliton wave fields in a wide region of the physical space

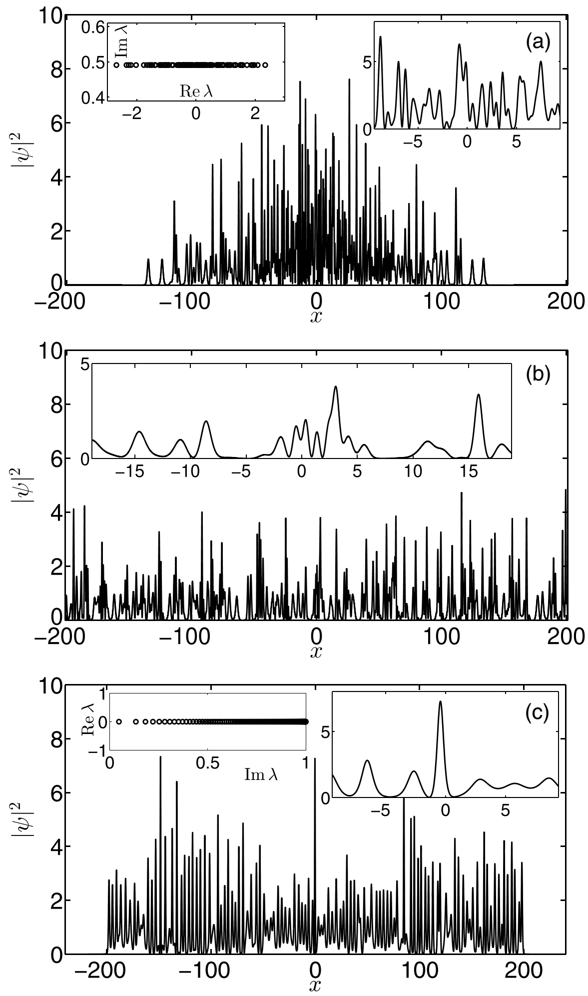

FIG. 5. (Adapted from Ref. [48] and [47]) (a) Wave field of 128-SS constructed from solitons having equal amplitudes $A_n = \pi/3.2 \approx 1$, Gaussian-distributed velocities with zero mean and standard deviation $V_0 = 2$, $V_n \sim \mathcal{N}(0, V_0^2)$, uniformly distributed positions $x_n^{\rm DM} \in [-L_0/2, L_0/2]$, $L_0 = 2$, and phases $\theta_n^{\rm DM} \in [0, 2\pi)$. (b) The same wave field after placing it into the periodic box $x \in [-L_p/2, L_p/2]$, $L_p = 128\pi$ and simulating the time evolution within the fNLS equation up to the final time $t = 200$. (c) Wave field of 128-SS constructed from solitons having amplitudes distributed according to the Bohr-Sommerfeld quantization rule for a rectangular box, see Eq. (69) below, zero velocities $V_n = 0$, uniformly distributed positions $x_n^{\rm DM} \in [-L_0/2, L_0/2]$, $L_0 = 2$, and phases $\theta_n^{\rm DM} \in [0, 2\pi)$. Right insets in panels (a) and (c) and the inset in panel (b) show a zoom of the wave fields. Left insets in panels (a) and (c) demonstrate soliton eigenvalues (note the swapped notations between the axes).

for sufficiently small soliton positions $|x_n^{\rm DM}| \lesssim 1$ and random soliton phases; see Fig. 5(c). The figure shows 128-SS constructed from solitons having zero velocity $V_n = 0$ and uniformly distributed positions and phases over the intervals $x_n^{\rm DM} \in [-L_0/2, L_0/2]$, $L_0 = 2$, and $\theta_n^{\rm DM} \in [0, 2\pi)$. The amplitudes are distributed according to the Bohr-Sommerfeld quantization rule, which is deduced from the solution of the direct-scattering problem for the rectangular box potential; see Sec. VI B for detail. The resulting wave field turns out to be statistically homogeneous $\langle|\psi(x)|^2\rangle \approx 1$ over more than 70% of its characteristic size in the physical space for random soliton phases [47]. Cutting out the remaining 30% at the edges where the wave field is not statistically homogeneous, one can use this 70% part as a model of statistically homogeneous bound-state soliton gas. As discussed in Sec. VI B, this soliton gas accurately models the long-time statistically stationary state of the noise-induced modulation instability of the plane-wave solution.

We believe that there are other distributions of soliton amplitudes leading to the statistically homogeneous multisoliton wave fields in a wide region of physical space for sufficiently small soliton positions $|x_n^{\rm DM}| \lesssim 1$ and random soliton phases. The general question of constructing multisoliton bound-state wave fields with a given profile $\langle|\psi(x)|^2\rangle = P(x)$ in the physical space and a given set of amplitudes A_n by using random soliton phases and a specific distribution of soliton positions represents a challenging problem for future studies.

C. Direct scattering transform analysis

In this section, we discuss the direct scattering transform (DST) analysis, which allows one to study the nonlinear composition of numerically or experimentally observed wave fields. Focusing only on the discrete spectrum (soliton eigenvalues and norming constants), we assume that the wave field in question is given in a simulation box $x \in [-L/2, L/2]$ and outside this box it equals zero. If the actual boundary conditions are different, then one can assume that the box L is large enough compared with the characteristic sizes of nonlinear structures, so that the difference in the boundary conditions and the resulting edge effects can be neglected. Note that in this formulation the scattering coefficients $a(\lambda)$ and $b(\lambda)$ are analytic functions in the upper half of the λ plane, that is essential for the algorithms discussed below.

In what follows, we describe the DST procedure presented in the recent study [109]. This procedure, based on the standard DST methods [119–121] supplemented by the latest studies [106,107,122] for the accurate calculation of the norming constants, contains several steps which are discussed below.

First, if there is a discontinuity of the wave field at $x = \pm L/2$, then it is smoothed using a smoothing window of the same size as the characteristic soliton width. It is assumed that the number of solitons inside the box L is large and that these discontinuities, together with their smoothing, do not introduce significant inaccuracies in the results.

Second, an approximate location of the soliton eigenvalues is found using the standard Fourier collocation method [119]. Being fast and fairly accurate, this method is based on the Fourier decomposition of the wave field, which artificially shifts the continuous spectrum eigenvalues to the upper half of the λ plane due to the implied periodization. Also, it does not distinguish between the eigenvalues of discrete and continuous spectra, leading to the problem of identifying low-amplitude solitons.

Third, to cope this this problem, the wave field is considered in two larger boxes $x \in [-3L/4, 3L/4]$ and $x \in [-L, L]$ by filling with zeros $\psi = 0$ the intervals where the wave field is not defined. Then, the Fourier collocation method is executed in both boxes and only the eigenvalues coinciding in both calculations are selected as belonging to the discrete spectrum. While the latter provides a good approximation of the soliton eigenvalues, i.e., zeros of the coefficient $a(\lambda)$, this approximation is still insufficient for the accurate calculation of the norming constants, which requires knowledge of roots $a(\lambda_n) = 0$ to hundreds of digits [106]. That is why the calculated eigenvalues are then used as seeding values for a high-accuracy root-finding procedure.

The fourth and the final step of the described DST procedure consists in application of the standard second-order Boffetta–Osborne method [120] on a fine interpolated grid using high-precision arithmetic operations, as suggested in Refs. [106,107]. The Boffetta–Osborne method is based on the calculation of the so-called extended 4×4 scattering matrix $\mathbf{S}$, which translates the solution of the Zakharov–Shabat system $\mathbf{\Phi}$ together with its derivative $\mathbf{\Phi}' = \partial\mathbf{\Phi}/\partial\lambda$ from $x = -L$ to $x = L$,

$$\begin{pmatrix} \mathbf{\Phi}(L) \\ \mathbf{\Phi}'(L) \end{pmatrix} = \underbrace{\begin{pmatrix} \Sigma & 0 \\ \Sigma' & \Sigma \end{pmatrix}}_{\mathbf{S}} \begin{pmatrix} \mathbf{\Phi}(-L) \\ \mathbf{\Phi}'(-L) \end{pmatrix}. \tag{66}$$

Here $\Sigma(\lambda)$ is 2×2 matrix, such that $\mathbf{\Phi}(L) = \Sigma(\lambda)\mathbf{\Phi}(-L)$, and the scattering coefficients are connected with the elements of matrix $\mathbf{S}$ as

$$a(\lambda) = S_{11}e^{2i\lambda L}, \quad b(\lambda) = S_{21},$$
$$a'(\lambda) = [S_{31} + iL(S_{11} + S_{33})]e^{2i\lambda L}. \tag{67}$$

Note that instead of the standard second-order Boffetta–Osborne method one can use the higher-order methods obtained with the Magnus expansion; see Refs. [107,122] for detail. A fine spatial grid and the high-precision arithmetic operations are necessary to (i) neglect the round-off errors when calculating the wave function $\mathbf{\Phi}$ of the Zakharov–Shabat system, (ii) avoid the anomalous errors in computation of the norming constants, and (iii) suppress the numerical instability of the wave scattering through a large potential, see Refs. [106,107] for detail. Also note that when avoiding the anomalous errors, one can supplement the DST procedure with the bidirectional algorithm and its improvements, see Ref. [123], to decrease the necessary number of digits in the high-precision operations.

The Boffetta–Osborne method allows one to find the scattering coefficients $a(\lambda)$ and $b(\lambda)$ for any value of λ by the direct numerical integration of the Zakharov–Shabat system on the interval $[-L/2, L/2]$ with boundary conditions (54). Note that $a(\lambda)$ and $b(\lambda)$ are analytic functions in the upper-half of the λ plane, as the potential $\psi(x)$ has a compact support. Then, with the help of the Newton method, one can find roots $a(\lambda_n) = 0$ with the necessary precision by using the eigenvalues obtained by the Fourier collocation method as seeding values. Finally, the norming constants are calculated according to their definition (56) using the extended scattering matrix $\mathbf{S}$, see Eq. (67), to find the derivative $a'(\lambda)$.

Figure 6 illustrates the performance of this DST procedure on the example of a periodic wave field that was grown from small statistically homogeneous in space noise within the fNLS equation, supplemented by a small linear pumping term, until the intensity, averaged over the simulation box, reached unity, $\overline{|\psi|^2} = 1$; see Ref. [109] for details. The solid blue and green lines in Figs. 6(a) and 6(b) show the amplitude $|\psi|$ and complex phase $\arg\psi$ of the grown up wave field, while the dots in Figs. 6(c) and 6(d) demonstrate the calculated soliton amplitudes A_n, velocities V_n, positions x_n^{DM} and phases θ_n^{DM}. Using these soliton parameters, one can construct the corresponding exact multisoliton solution as discussed in the previous section; it turns out that this solution approximates the original wave field very well, as illustrated by the dashed red and black lines in Figs. 6(a) and 6(b).

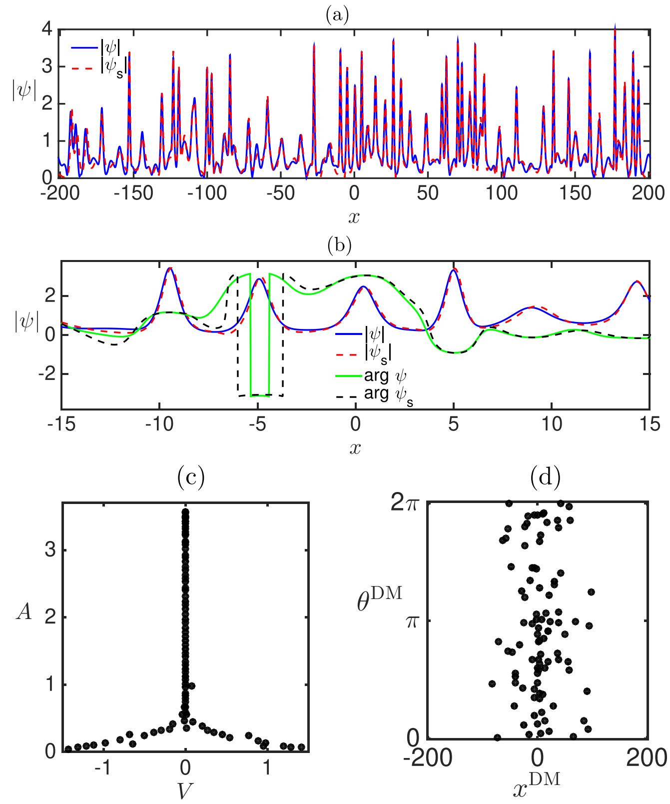

FIG. 6. (Adapted from Ref. [109].) Numerical DST analysis of a periodic wave field that was "grown" from small statistically homogeneous in space noise within the fNLS equation, supplemented by a small linear pumping term, until the intensity, averaged over the simulation box, reached unity, $\overline{|\psi|^2} = 1$; see Ref. [109] for details. Panel (a) shows the absolute values of the "grown up" wave field $|\psi|$ (solid blue) and the multisoliton solution $|\psi_s|$ (dashed red); ψ_s is constructed using the soliton parameters obtained in the DST procedure. Panel (b) represents zoom of panel (a), also demonstrating the complex phases of the grown up wave field (solid green) and the multisoliton solution (dashed black). The dots in panels (c) and (d) illustrate soliton amplitudes A_n, velocities V_n, positions x_n^{DM} and phases θ_n^{DM}.

Note that the average intensity of the multisoliton solution ψ_s in Fig. 6 equals 99% of that of the original grown up wave field ψ. Also, most of the solitons of this solution have zero velocities, forming a bound state. In Ref. [109], such a situation is observed if the initial noise amplitude and the pumping coefficient are small enough. If this is not the case, then the grown up wave fields with intensity of unity order $\overline{|\psi|^2} \approx 1$ also represent soliton-dominated states, which are not bound as these solitons have different velocities. Moreover, as shown in the paper, during the growth stage the solitonic part of the wave field becomes the dominant one very early when the average intensity is still small, $\overline{|\psi|^2} \simeq 0.1$, and the dispersion effects are leading in the dynamics. These observations indicate that the soliton gas model can be applicable even to weakly nonlinear cases, so that a soliton gas can be a very common object in nature.

Concluding this section we mention that a similar DST or IST based algorithms enable the numerical analysis and synthesis of breather gases, see Refs. [66,124]. Also, soliton gas can interact with a portion of dispersive radiation described by a nonzero reflection coefficient. In this case, one can use numerical IST algorithms directly solving the GLM equations or based on the Riemann-Hilbert problem to synthesize wave fields characterized by nonzero continuous and discrete spectra [125–127].

V. EXPERIMENTS

From the experimental point of view, a few attempts to generate and to observe soliton gases have been made in some optical fiber experiments performed at the end of the 1990s [35–37]. The soliton gas was generated by the synchronous injection of laser pulses inside a passive ring cavity. No direct observation but only averaged measurements of the Fourier power spectrum and of the second-order autocorrelation function characterizing the optical soliton gas have been reported in these pioneering experiments. Moreover, the dynamics of the ring resonator was so complex that many features ranging from purely temporal chaos to spatiotemporal chaos or turbulence were observed in this fiber system [37].

Analyzing ocean waves recordings, Costa *et al.* have reported the observation of random wave packets in shallow water waves in 2014 [38]. The wave packets have been analyzed using numerical tools of nonlinear spectral analysis [128] and interpreted as being composed of random solitons that might be associated with KdV soliton gas. One year later, large ensembles of interacting and colliding solitons have been observed in a laboratory environment [39]. The experimental system was a water cylinder deposited on a heated channel and levitating on its own generated vapor film owing to the Leidenfrost effect. Multiple soliton propagation was observed at the surface of the water cylinder and the Fourier analysis that was made in an attempt to characterize the multiple coherent structures revealed a "soliton turbulence-like spectrum." Note also that a striking transition between weak turbulence and solitonic regimes has been evinced in the hydrodynamic experiments reported in Ref. [129]. In these experiments, water waves have been generated by exciting horizontally a water container by using an oscillating table. The weak turbulence regime observed at low forcing and/or large depth was shown to abruptly evolve into a solitonic regime at larger forcing and/or small depth. Remarkably, these results establish a possible link between the field of integrable turbulence and the field of wave turbulence.

In the recent laboratory experiments reported in Ref. [41], Redor *et al.* have taken advantage of the process of fission of a sinusoidal wave train to generate a bidirectional shallow water soliton gas in a 34-m-long flume. The space-time observations revealed complex dynamics where large numbers of colliding solitons retained their profile adiabatically, although their amplitude was slowly decaying because of some unavoidable damping. The Fourier analysis of the observed nonlinear wave field has clearly revealed the interplay between multiple solitons and dispersive radiation. Further analysis making use of the periodic scattering transform have been implemented in Ref. [130] to discriminate linear wave motion states from integrable turbulence and soliton gas. Moreover the statistical properties of the soliton gas have been given in terms of probability density distribution, skewness, and kurtosis [130].

The experiments reported in Refs. [41,56,130] have been made in the presence of an unavoidable slow damping but it has been shown that a stationary state typified by the interplay among random bidirectional solitons can be achieved because of the continuous energy input by the wavemaker. In these shallow-water experiments, a route to integrable turbulence has been discovered through the disorganization of wave motion that is induced by the wave maker [130]. This route has been shown to depend on the nonlinearity of the waves but also on the amplitude amplification and reduction due to the wavemaker feedback on the wave field [130].

Using an approach fully based on the IST method while also relying on the concept of DOS, a soliton gas has been generated in hydrodynamic experiments performed in the deep-water regime where wave propagation is described at leading order by the 1D fNLS equation [57]. The experiment has been performed in a wave flume being 148 m long, 5 m wide, and 3 m deep. Unidirectional waves have been generated at one end of the tank with a computer assisted flap-type wave maker and the flume is equipped with an absorbing device strongly reducing wave reflection at the opposite end. In these experiments the space-time evolution of the generated wave packet is measured with 20 gauges uniformly distributed along the tank.

The experiment reported in Ref. [57] starts from the numerical generation or synthesis of a soliton gas by using the methodology described in Sec. IV. An ensemble of 128 solitons having spectral parameters being distributed in a rectangular region of the spectral IST plane has been numerically generated. The solitons have the modulus of their norming constants being equal to unity while their phases are randomly distributed between $-\pi$ and $+\pi$. In the experiment, the generated soliton gas has the form of a random wave field spreading over ≈1200 s, see Fig. 7. It represents a *dense* soliton gas in which solitons are not isolated and not well separated like in a rarefied gas.

In the experiments reported in Ref. [57], a large number of discrete eigenvalues were distributed with some density within a limited region of the complex plane. This justifies the introduction of a statistical description of the spectral (IST) data. This represents a key point for the analysis of the observed wave field in the framework of the SG theory. In Ref. [57], the DOS of a homogeneous soliton gas has been measured for the first time in experiments, which provides an essential first step towards experimental verification of the kinetic theory of nonequilibrium SGs. Nonlinear spectral analysis of the generated hydrodynamic soliton gas reveals that the DOS slowly changes under the influence of perturbative higher-order effects that break the integrability of the wave dynamics.

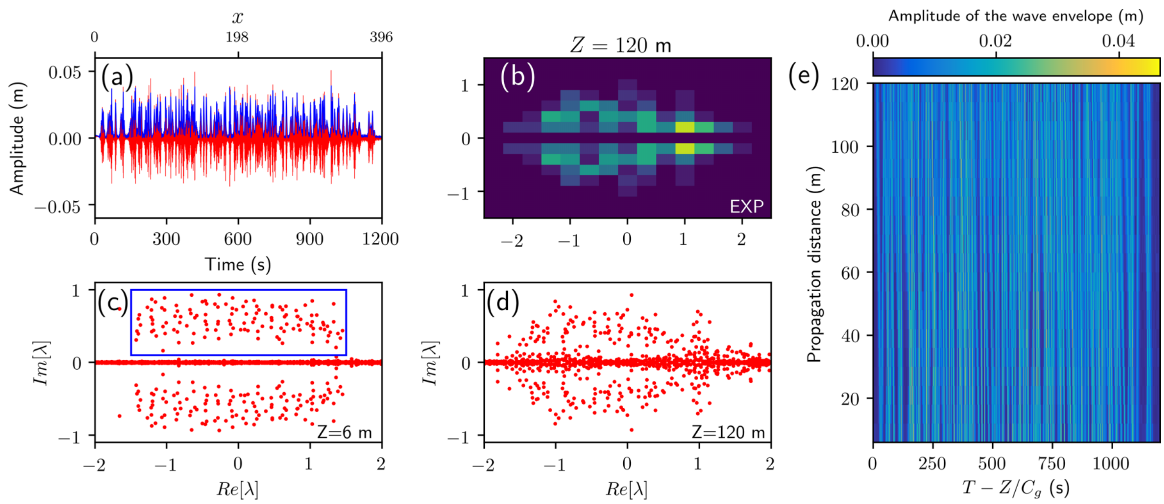

FIG. 7. Gas of 128 solitons propagating in a 140-m-long 1D water tank [57]. (a) Water elevation (red line) and modulus of the wave envelope measured at $Z = 6$ m, close to the wave maker. (b) DOS of the soliton gas measured at $Z = 120$ m. (c) Discrete IST spectrum measured at $Z = 6$ m. (d) Discrete IST spectrum measured at $Z = 120$ m. (e) Space-time evolution of modulus of the wave envelope recorded by the 20 gauges regularly spaced along the tank. Reproduced with permissions from Ref. [57]

Very recently an experiment on the interaction of monochromatic SGs modeled by the superposition of the δ-function DOS (see Sec. III D) was performed in a deep-water tank [131] confirming physical relevance of exact analytical solutions of the fNLSE SG kinetic equation obtained in Ref. [20]. Moreover, in recent optical fiber experiments, the effective velocity of a test soliton propagating in a homogeneous SG has been measured for the first time and compared quantitatively to the prediction of the SG theory [32]

VI. APPLICATIONS OF SOLITON GASES

Since the first paper of Zakharov [18], a peculiar interest has been ascribed to SG as a fundamental mathematical and physical concept. Importantly, it has been recently shown that SG theory can provide a powerful framework to describe theoretically the complex statistics underlying some well-known and fundamental nonlinear dispersive waves phenomena. It is indeed natural to expect that SG theory can be used to describe some specific regimes of integrable systems (integrable turbulence, see Sec. VI A) [7,12,54,55,70,132,133]. In particular, by using numerical simulations, it has been shown in 2019 that the long-term statistical properties of the so-called *spontaneous modulation instability* coincides with those of a specifically designed SG, see Sec. VI B and [47]. Very recently, a general relationship between the DOS of a SG and the kurtosis (fourth-order moment) of wave field has been derived, see Sec. VIII B. These results provide the first theoretical description of the long-term evolution of the noise-induced modulation instability and pave the way for the description of integrable turbulence by using the SG theory.

A. Integrable turbulence

A soliton gas is a peculiar case of a more general phenomenon named "integrable turbulence" by Zakharov [4]. Integrable turbulence represents the dynamical and statistical phenomena emerging during the propagation of nonlinear random waves in an integrable system. In *nonintegrable* systems, the propagation of nonlinear random waves can be described in the weakly nonlinear regime by the so-called wave turbulence theory [1,134]. In wave turbulence, the exchanges of energy among different scales are dominated by resonant interactions among Fourier components. In this framework, wave turbulence theory predicts, in particular, the out-of-equilibrium phenomena (Kolmogorov-Zakharov cascade) or thermodynamic equilibrium (Rayleigh-Jeans distribution) [1,134–137].

The physics of *integrable* wave systems is of profoundly different nature because of the infinite number of constants of motion and of the absence of resonances. In particular, nonlinear random integrable waves cannot reach the thermodynamical Rayleigh-Jeans equilibrium [135,138]. For this reason, Zakharov has introduced a new field of research, the *integrable turbulence* (IT), which is defined as the statistical description of integrable systems [4]. Since this seminal paper in 2009, integrable turbulence has received a growing interest both from the theoretical [4,7–9,47,70,75,103,132,139–144] and experimental [10,12,54,55,133,141,145] points of view.

In practice, integrable turbulence corresponds to the propagation of random waves in systems described by integrable equations such as the 1D NLS, the KdV or the Sine-Gordon equations. In this Sec. VI, we focus on recent results on the 1DNLS integrable turbulence. The one-dimensional focusing NLS equation provides a bridge between nonlinear optics and hydrodynamics [145,146]. The 1D focusing NLS equation describes at leading-order deep-water wave trains or optical fiber in anomalous dispersion regime and it plays a central role in the study of rogue waves [147–151]. The relevant approach to study nonlinear random waves is a statistical description, including probability density functions (PDFs) of wave amplitude ψ or of intensity $|\psi|^2$ and moments such as the kurtosis $\kappa_4 = \langle|\psi|^4\rangle/\langle|\psi|^2\rangle^2$. The last years, the statistical properties of integrable turbulence has been widely studied by using numerical simulation of the NLS equations. Preserving integrability in long-term simulations is a delicate and challenging task, but to the best of the knowledge, integrable turbulence is characterized by stationary statistical properties of the field for long time t. This existence of stationary statistical states in the long-time evolution of the wave system is the most fundamental known feature of integrable turbulence.

IT phenomena in the 1D NLS dynamics can be classified by considering the statistical properties of the initial conditions. Two classes of initial conditions have been extensively investigated : (i) the plane wave perturbed by a small random noise and, (ii) partially coherent waves.

The homogeneous solution of the 1D focusing NLS equation (the plane wave or *condensate*) is unstable in the presence of long-wave perturbation. When the perturbation is a random process, this dynamical mechanism, known as the "noise-induced" or *spontaneous modulation instability* (MI) [7,10,152], represents a prominent example of the IT phenomena. Surprisingly, the long-term statistical state is characterized by a Gaussian local statistics of the field ψ i.e., by a kurtosis $\kappa_4 = 2$ [7] while the other statistical properties such as the Fourier spectra or two-point correlations are not trivial [10]. These statistical quantities have been quantitatively measured in experiments but up to very recent studies, no theoretical description was available. In the Sec. VI B, we show that the soliton gas concept provides a powerful theoretical tool to predict quantitatively the statistical properties of the long term evolution of the spontaneous modulation instability.

Partially coherent waves provide a second kind of integrable turbulence. The initial condition is provided by the linear superposition of numerous Fourier modes and is characterized by a Gaussian statistics of the field [134,135]. Such initial conditions have been extensively investigated in numerical simulations of the defocusing and of the focusing NLS equation and in experiments [9,12,54,55,133,145,153]. The long-term evolution is characterized by non-Gaussian statistics. Note that in the focusing regime of NLS, the strongest deviation from Gaussianity is characterized by a kurtosis $\kappa_4 = 4$ [154] (corresponding to the numerous emergence of extreme event rogue waves).

The evolution of the statistical properties of partially coherent waves in the framework of 1D NLS integrable turbulence can be described by using a nonconventional wave turbulence theory approach [138,155]. This theoretical approach predicts the deviation from Gaussianity for a weak nonlinearity but is not valid in the high nonlinearity regime. In particular, up to now, the maximum value of the kurtosis $\kappa_4 = 4$ was not understood. In Sec VIII B, we summarize a very recent theoretical study based on SG theory which explains this maximum value of the kurtosis.

As mentioned above, *soliton gas is a peculiar case of integrable turbulence*. Indeed, in the framework of IST with zero boundary conditions, integrable turbulence can always be described by the combination of the discrete and of the continuous spectra. By focusing on the spontaneous MI, we show in the Sec. VI B that soliton gas theory provides a powerful framework to tackle the problem of the statistical description of integrable turbulence in pure solitonic cases.

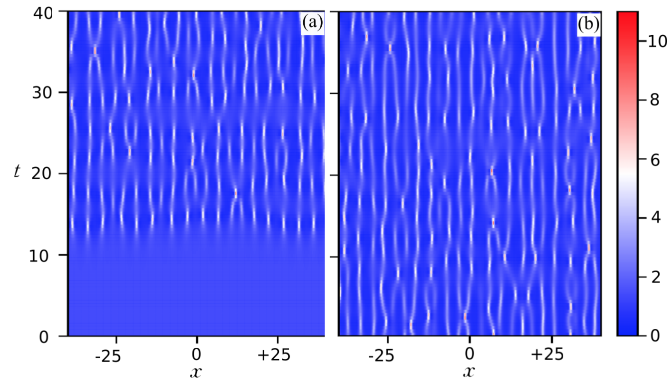

FIG. 8. Numerical simulations of the one-dimensional focusing NLS equation: Space-time diagrams of $|\psi(x,t)|^2$ (a) Noise-induced modulation instability of a plane wave (periodic boundary conditions). (b) Dynamics of the random-phase bound N-SS (only the central part of the N-SS having a total width $L_0 \simeq 400$ is plotted). Reproduced with permission from Ref. [47].

B. Spontaneous modulation instability

The MI appears in many physical systems, such as deep water waves [6], Bose-Einstein condensates [156] or nonlinear optical waves [157]. If the plane wave is perturbed by an initially small sinusoidal perturbation, the nonlinear stage of MI is described by homoclinic solutions of the 1D focusing NLS equation—the Akhmediev breathers [158–161]. As reminded above, in the case of random initial perturbation, single-point statistics evolves toward a stationary Gaussian distribution (and $\kappa_4 = \langle|\psi|^4\rangle/\langle|\psi|^2\rangle^2 = 2$) despite the presence of highly nonlinear breather-like structures [7,70,132]. The long-time (stationary) statistics is also typified by a quasiperiodic structure of the autocorrelation function $g^{(2)}$ of the wave field intensity [10].

In this section, we review numerical simulations that prove that the nonlinear stage of the spontaneous MI in the focusing regime of the Eq. (6) ($\sigma = +1$) can be quantitatively described by a specifically designed soliton gas [47].

Without loss of generality, we consider the plane-wave solution of Eq. (6)—the condensate—of unit amplitude $\psi_c(t, x) = \exp it$. In the classical formulation of the spontaneous MI problem, the initial condition reads [7,162]

$$\psi(t = 0, x) = 1 + \eta(x), \tag{68}$$

where η is a small noise, $\langle|\eta|^2\rangle \ll 1$, with zero average, $\langle\eta\rangle = 0$. The destabilization of the condensate with respect to long-wave perturbations was widely investigated, both numerically by using periodic boundary conditions in a box of large size [7,10,150,152], and experimentally [10,163,164]. The typical spatiotemporal dynamics of the spontaneous MI can be seen in Fig. 8(a).

To demonstrate that the statistical properties of the spontaneous MI coincide at long-time with those of SG, the first step used in Ref. [47] is to modify the boundary conditions. The idea is that if one fixes the time at which the nonlinear stage of MI is characterized [typically $t > 30$ in Fig. 8(a)], the plane wave with periodic boundary conditions can be replaced by a box with zero boundary conditions. The width of the box has to be sufficiently large to avoid any influence of the edges in the central part of the box at the considered time t.

By using the same idea, one expects that any homogeneous SG can be locally modeled by an N-SS (with zero boundary conditions). Moreover, in order to model the long-time dynamics of a stochastic field, it is natural to assume random norming constants phases because the phase rotations $-2i\lambda_n^2 t$ for large t introduce an effective randomization. Note that somehow, this random phases of the norming constant are similar to the so-called "random-phase approximation" (i.e., random phases of the Fourier components) in wave turbulence theory [1,134].

Finally, the last step is to determine the DOS of the SG underlying the dynamics of the field. Here, the answer is rather simple because the discrete spectrum of a real-valued rectangular box of unit amplitude and width L_0 is known. In the limit $L_0 \gg 1$, the discrete spectrum of the semiclassical Zakharov-Shabat scattering problem is given by the Bohr-Sommerfeld quantization rule, see, e.g., Refs. [5,81] and also Ref. [112]:

$$\lambda_n = i\beta_n = i\sqrt{1 - \left[\frac{\pi\left(n - \frac{1}{2}\right)}{L_0}\right]^2}, \quad n = 1, 2, \ldots, N, \tag{69}$$

where $N = \text{int}[L_0/\pi]$ (the density of the gas, i.e., the number of solitons per unit length is thus $1/\pi$). The continuum limit of Eq. (69) with $N \to \infty$, $\beta_n \to \beta$ gives the normalized distribution (density) $\phi(\beta)$ of the IST eigenvalues:

$$\phi(\beta) = \frac{1}{N}\frac{dn}{d\beta} = \frac{\beta}{\sqrt{1-\beta^2}}, \tag{70}$$

which is sometimes called the Weyl distribution. Finally, here, the DOS is simply [cf. (11)]

$$f(\beta) = \frac{1}{L_0}\frac{dn}{d\beta} = \frac{1}{\pi}\phi(\beta). \tag{71}$$

This is nothing but the 1D focusing NLS bound-state soliton condensate DOS (49) obtained in Sec. III D as the solution of the NDRs (42) in the limit $\sigma \to 0$ assuming the spectral support $\Gamma^+ = [0, i]$. In Ref. [47] a large number of realizations of N-SSs that fulfill the required eigenvalues distribution given by the Eq. (70) have been computed with random phase for the norming constants by using the procedure described in Sec. IV B. This realizations ensemble models a bound SG in the limit N large ($N = 128$ in [47]). As the N-SS are bound states ($\text{Re}\lambda_n = 0$), the expected dynamics of the wave field is identical to the dynamics observed in the nonlinear stage of the spontaneous MI. Indeed, the zero velocity of the solitons prevent any dilution of the gas during the evolution.

The Fig. 8 displays the comparison between two NLS equation simulations made with different initial conditions: Fig. 8(a) corresponds to the dynamics of the plane wave (initially perturbed with noise) while Fig. 8(b) corresponds to the dynamics of one realization of the N-SS. The features characterizing the spatiotemporal dynamics of the long-time evolution of the plane wave (typically for time $t > 20$) seems very similar to the one of the N-SS. As expected, the specifically designed N-SS apparently is a very good model of the nonlinear stage of the spontaneous MI.

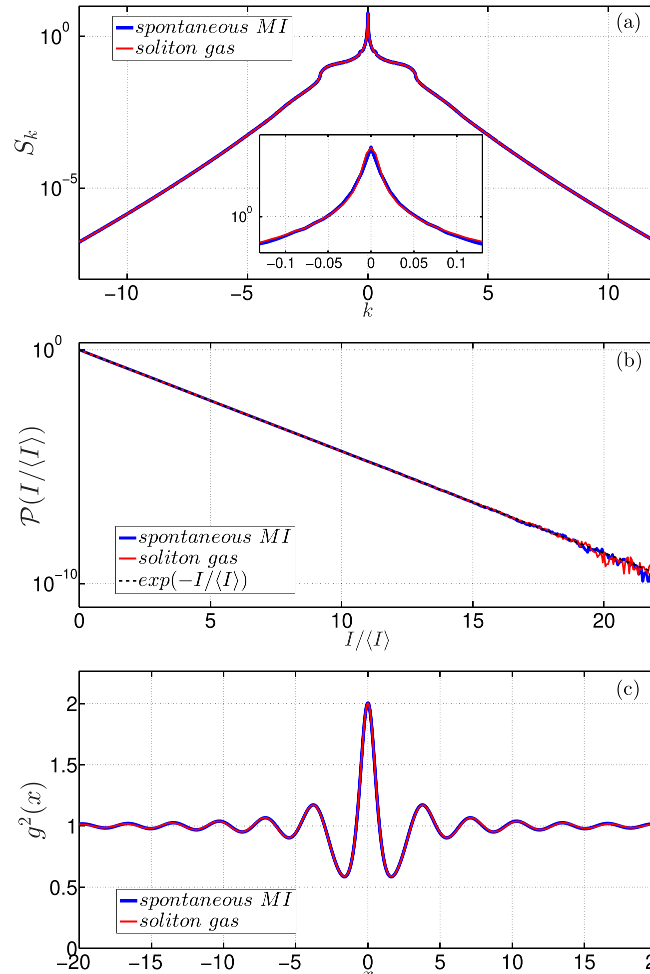

FIG. 9. Comparison of ensemble averaged statistical characteristics of the asymptotic state of the MI development and of random phase 128 SSs. (a) Wave action spectrum S_k. (b) The PDF $\mathcal{P}(I)$. (c) Autocorrelation function of intensity (second-order degree of coherence) $g^{(2)}(x)$. Reproduced with permission from Ref. [47].

More importantly, the statistical properties of SG coincide in a quantitative manner with those of the asymptotic stage of MI. For example, the long-term evolution of the noise-induced MI is characterized by stationary values the potential H_{nl} and kinetic H_l energy [7], $\langle H_l\rangle = 0.5$ and $\langle H_{nl}\rangle = -1$ where the total energy (Hamiltonian) H, which is one of the infinite constants of motion of the 1D-fNLS equation [5], reads

$$H = H_l + H_{nl}, \quad H_l = \frac{1}{2}\frac{1}{L}\int_{-L/2}^{L/2} |\psi_x|^2 dx,$$
$$H_{nl} = -\frac{1}{2}\frac{1}{L}\int_{-L/2}^{L/2} |\psi|^4 dx. \tag{72}$$

Figure 9 shows the comparison between three other statistical of both the long-time evolution of the spontaneous MI and of the ensemble of N-SS. Figure 9(a) displays the wave-action spectrum,

$$S_k \propto \langle|\psi_k|^2\rangle, \quad \psi_k = \frac{1}{L}\int_{-L/2}^{L/2} \psi e^{-ikx} dx. \tag{73}$$

Figure 9(b) displays the probability density function (PDF) $\mathcal{P}(I)$ of the field intensity $I = |\psi|^2$ which is known to follow the exponential distribution in the asymptotic statistics of the unstable condensate [7,10]. Finally, Fig. 9(c) displays the autocorrelation of the intensity $g^{(2)}(x)$:

$$g^{(2)}(x) = \frac{\langle I(y,t)I(y-x,t)\rangle}{\langle I(y,t)\rangle^2}, \quad (74)$$

which represents the second-order degree of coherence.

It is important to note that all these remarkable features of the spontaneous MI have been observed in optical fibers [10]. In other words, higher-order effects that break integrability do not play an important role before the nonlinear wave field reaches the stationary state. This validates the use of the (integrable) theory of SG in order to describe the fundamental phenomenon of spontaneous MI observed in realistic conditions.

Remarkably, all these statistical quantities computed in the asymptotic state of the MI coincide with excellent accuracy with those of the considered SG. In addition, further studies revealed that extreme amplitude waves emerging in the asymptotic state of the MI and the soliton gas have identical dynamical and statistical characteristics [105]. Note that this agreement is weakly depend on the exact eigenvalues chosen for the N-SS because the key ingredient are the statistical distribution of the eigenvalues and the use of random phases for the norming constants [in other words, similar statistical results have been obtained in the case of soliton eigenvalues randomly distributed according to the probability function (70) [47]].

What are the main conclusions of this numerical study? *First, the asymptotic state of the spontaneous MI can be modeled by a specific SG—the bound state soliton condensate*. This SG can be constructed with exact N-SSs of the 1D focusing NLS equation by using large values of N and the Weyl's distribution of IST eigenvalues coinciding with the one predicted for the box potential in the semiclassical limit [81]. Moreover, the long-term statistical state of MI corresponds to a full stochastization of the phases of the norming constants i.e., the solitons' phases. Finally, note that for other distributions of eigenvalues explored by the authors, the statistical properties of the SG do not coincide with those of the MI and/or are strongly nonhomogeneous in space [47].

These results open a promising direction in the theory of integrable turbulence by establishing a link between the MI and SG dynamics. It is important to note that this quantitative link is possible in the case of MI because for a "semiclassical" box, the contribution of the "nonsoliton" part of the field, i.e., of the continuous IST spectrum, decays exponentially with L_0 and so can be neglected [5]. As a consequence, this modeling of integrable turbulence by using SG can be *a priori* generalized to a broad class of IT problems when the (random) wave field is strongly nonlinear, so that the impact of the nonsolitonic content can be neglected in the asymptotic state ($t \to \infty$). For such a case, the general strategy to study the asymptotic state should be to build N-soliton solutions with the distribution $\phi(\lambda)$ of IST eigenvalues characterizing the field and *random phases* of the norming constants (see Sec. VIII B). We review in Sec. VIII B new results showing that this approach provides a framework allowing to compute theoretically the observed value of $\kappa_4 = 2$ for the long-term evolution in the spontaneous MI phenomenon [10] and $\kappa_4 = 4$ for the semiclassical limit of partially coherent waves [154].

VII. GENERALIZED HYDRODYNAMICS

A. The perspective of emergent hydrodynamics

As mentioned, it is very natural to understand the theory of soliton gases within a kinetic perspective, as the fundamental objects—the soliton DOS or phase-space density $f(\eta; x, t)$ introduced in Sec. II, its kinetic Eq. (15) and the equation of state Eq. (17) the effective velocity $s(\eta; x, t)$,—have a clear kinetic interpretation in terms of soliton propagation and scattering. This interpretation is mathematically accurate at low densities, but at high densities, although compelling, it remains nebulous. The kinetic viewpoint is in fact an *a-posteriori* interpretation: as reviewed in Sec. III, the soliton gas theory may be derived from an appropriate thermodynamic limit of finite-gap (quasiperiodic) solutions and their Whitham modulations.

Independently from the soliton gas theory, a framework for the emergent large-scale behaviors of quantum and classical many-body integrable systems out of equilibrium was more recently developed, dubbed "generalized hydrodynamics" (GHD) [76,77]. In this context, the problem is to determine the dynamics of many-body systems out of equilibrium, as done for instance in the quantum Lieb-Liniger gas [76], in the Heisenberg quantum spin chain [77], in the classical Toda model [165–167] and in the NLS [168] and sine-Gordon [169] as a classical field equations. One seeks, for instance, the full space-time profile of expectation values of local observables, from an initial state that present variations on large scales; or the full space-time profile of their correlation functions. As it was realized [170], it turns out that the main objects of GHD—the phase-space densities denoted in this context $\rho_{\rm p}(\eta; x, t)$, the kinetic equation referred to as the "GHD equation," and the equation of state for the effective velocity denoted $v^{\rm eff}(\eta; x, t)$—have exactly the same structure as in soliton gases. The spectral parameter η is identified with the quasimomentum of the thermodynamic Bethe ansatz (TBA), and, in quantum systems, quite surprisingly the two-body scattering shift is simply identified with the semiclassical shift of quantum wave packets, or the "kernel" of the TBA equations. The TBA [171–174] is a framework first developed at the beginning of the 1960s to construct the thermodynamics of Bethe-ansatz integrable systems.

However, by contrast to the theory of soliton gases, in GHD a different viewpoint is emphasized. Certainly, a kinetic perspective can be taken, as was done in one of the co-founding papers of GHD [77]: "Bethe quasiparticles" are the kinetic objects, and the effective velocity $v^{\rm eff}(\eta)$ had in fact been proposed earlier [175] as their emergent propagation velocity within finite-density states. However, in the quantum context it is more difficult to establish the validity of this perspective, even at low densities. Furthermore, a quantum modulation theory has not yet been developed. Instead, the currently prevalent viewpoint, emphasized in the other co-founding paper of GHD [76], is that of *the emergence of hydrodynamics at large space-time scales*. This physical idea implies that the structure of GHD is in fact that of *Euler equations*, instead of a kinetic theory, only generalized to infinitely many conservation laws.

Euler hydrodynamics is the idea that locally, within each "mesoscopic" region of space and time (sometimes referred to as fluid cell), the system's state looks as if it had relaxed. A mesoscopic space-time region covers a length that is large as compared with the microscopic scales (the interparticle scales and interaction distances) but small as compared with macroscopic scales (the length scales at which averages of local observables show variations); and a time that is likewise large compared with microscopic, and small compared with macroscopic, times. According to conventional physical wisdom, a state that has "relaxed" is space-time stationary and takes the Gibbs form. Thus, in Euler hydrodynamics, one assumes that at every point in space-time, the state looks like it is in Gibbs form. The Gibbs form arises from an entropy maximization principle, so these are "maximal entropy states," and we may therefore talk about "local entropy maximization." The local maximal entropy states depends on space-time, and upon imposing all the available local conservation laws, this gives the Euler equations for the system.

It has been worked out in the past 20 years (see the reviews [176,177]) that the so-called generalized Gibbs ensembles (GGEs), with density matrix

$$\text{GGE:} \quad \rho \propto e^{-\sum_i \beta_i Q_i}, \tag{75}$$

where $Q_i = \int dx q_i(x,t)$, with $dQ_i/dt = 0$, are extensive conserved quantities, correctly describe relaxation in many-body integrable models. In the infinite-volume limit, infinitely many conserved quantities Q_i must be considered (under some convergence condition). The emergent hydrodynamic perspective then simply states that the local MES are GGEs, so the local relaxation process is

$$\langle o(x,t)\rangle_{\text{initial state}} \to \langle o\rangle_{\text{GGE}(x,t)} \tag{76}$$

for "any" local observable $o(x,t)$. Here GGE(x,t) is described by "Lagrange parameters" $\beta_i(x,t)$ which depend on space-time. In the hydrodynamic approximation, the GGEs only depend slowly on space and time, and one then imposes the local conservation laws $\partial_t q_i(x,t) + \partial_x j_i(x,t) = 0$ for GGE-average local densities $q_i(x,t)$ and their currents $j_i(x,t)$, in order to obtain the long-wavelength, slow dynamics,

$$\frac{\partial}{\partial t}\langle q_i\rangle_{\text{GGE}(x,t)} + \frac{\partial}{\partial x}\langle j_i\rangle_{\text{GGE}(x,t)} = 0. \tag{77}$$

In principle, barring subtleties associated with hyperbolic systems of equations (see, for instance, Ref. [178]), these are enough equations to have a well-posed initial-value problem. The crucial ingredient in (77) is the "thermodynamic equations of state": the way the average currents $\langle j_i\rangle_{\text{GGE}}$ are related to the average densities $\langle q_i\rangle_{\text{GGE}}$. Once this is known explicitly, the hydrodynamic Eq. (77) is written explicitly.

Thus, the usual ideas of hydrodynamics are simply extended to the principle of "generalized thermalization" for many-body integrability; this is the hydrodynamic basis for GHD, justifying its name.

Crucially, *the perspective of emergent hydrodynamics applies equally well to soliton gases* [78]. For instance, KdV and NLS soliton gases can be seen as special examples within the frameworks of GHD and TBA. This has important consequences for soliton gases. First, this directly provides their thermodynamics (the free energy, the entropy, the temperature), and more generally, it puts the soliton gas theory in the general statistical mechanics framework; this was missing before the connection with GHD was made. Second, the large body of results in GHD, and indeed the statistical mechanics viewpoint on hydrodynamics that is at the basis of GHD, gives new formulas and insight which are not yet obtainable by more mathematically accurate methods from IST. These include, for instance, correlation functions in space and time (see the review [179]), numerically confirmed in the KdV soliton gas [78], and fluctuations of macroscopic currents [180,181]. Finally, GHD provides guidance for the generalization of soliton gas theory to, e.g., external forces and diffusive and dispersive corrections. Results coming from (and expected in the future to come from) connecting soliton gases with GHD are typically hard to obtain by other methods, yet are directly applicable to specific KdV and NLS soliton gas setups and questions that are currently of interest. We emphasize that, contrary to the historical theory of soliton gas, GHD provides crucial spatiotemporal information such as correlations, whose tests, we believe, will allow new understanding of experimental observations in the near future. We discuss such perspectives in Sec. VIII E and at the end of Sec. VIII C. But first, we overview the TBA framework at the basis of the thermodynamics of integrable systems, and the universal principles that explain why GHD and TBA are so widely applicable, with some recent predictions obtained in the KdV soliton gas.

B. The thermodynamic Bethe ansatz

In the hydrodynamic perspective, no kinetic theory is invoked. This perspective emphasizes not the kinetic interpretation of the equations, but rather their thermodynamic and hydrodynamic interpretations. But how does one deal with infinitely many conservation laws, and a large space of maximal entropy states? And how does a formulation that looks like a kinetic theory emerge?

This is thanks to the structure of the TBA. To describe it, take the integrable model of Bose particles interacting with a δ-function potential, the repulsive Lieb-Liniger gas (see, e.g., the review [182] where its GHD is explained),

$$H = -\sum_{n=1}^{N} \frac{1}{2}\frac{\partial^2}{\partial x_n^2} + \sum_{n<m=1}^{N} c\delta(x_n - x_m), \quad c > 0. \tag{78}$$

The fully symmetric N-particle Bethe ansatz eigenfunctions, parametrized by Bethe roots η_n's, take the form

$$\Psi(\{x\}) \propto \sum_{\mathcal{P}:\text{permutations}} \prod_{n<m} \text{sgn}(x_{\mathcal{P}(n)} - x_{\mathcal{P}(m)}) \times \exp\left[i\sum_n \eta_n x_{\mathcal{P}(n)} + \frac{i}{4}\sum_{n\neq m} \phi(\eta_n - \eta_m)\text{sgn}(x_{\mathcal{P}(n)} - x_{\mathcal{P}(m)})\right]. \tag{79}$$

The quantity $\phi(\eta_n - \eta_m) = 2\ \arctan\frac{\eta_n-\eta_m}{c}$ is the two-body quantum-scattering phase shift occurring when a particle of Bethe root η_n scatters with one of Bethe root η_m. Conserved quantities (including the Hamiltonian) take a simple form on these eigenfunctions:

$$Q_i\Psi(\{x\}) = \sum_{n=1}^{N} h_i(\eta_n)\Psi(\{x\}), \tag{80}$$

where the functions $h_i(\eta)$ are the "one-particle eigenvalues." These include the total number of particle Q_0 [with $q_0(x) = \sum_n \delta(x - x_n)$ and $h_0(\eta) = 1$], the momentum Q_1 [with $q_1(x) = \frac{1}{2}\sum_n -i\{\partial_{x_n}, \delta(x - x_n)\}$ and $h_1(\eta) = \eta$], and the energy $Q_2 = H$ [with $h_2(\eta) = \eta^2/2$]. In fact, local conserved charges—those admitting a local density $q_i(x,t)$—have $h_i(\eta) \propto \eta^i$ for all $i \in \mathbb{N}$. In a system of finite length L, the values of η_n are quantized as is usual in quantum mechanics; however, the quantization condition is nontrivial: these are the Bethe ansatz equations, involving $\phi(\eta)$ (see, for instance, Ref. [182]).

The TBA is based on the basic statistical mechanics principle of the equivalence of the microcanonical and macrocanonical ensembles, but generalized to all conserved charges, or equivalently all Bethe roots. Thus, the sum over eigenstates involved in a GGE concentrates on a *fixed distribution of Bethe roots* $\rho_{\rm p}(\eta)$, and one evaluates GGE averages of conserved densities by using this distribution, $\langle q_i\rangle_{\rm GGE} = \int d\eta\rho_{\rm p}(\eta)h_i(\eta)$, as follows from (80). The TBA gives an explicit map from β_i to $\rho_{\rm p}(\eta)$. This map is obtained by minimizing a free-energy functional that encodes the constraints on quasimomenta arising from the Bethe ansatz equations. The result may be written in the suggestive form:

$$\varepsilon(\eta) = \sum_i \beta_i h_i(\eta) - \int \frac{d\eta'}{2\pi}\varphi(\eta - \eta')\ln(1 + e^{-\varepsilon(\eta')}),$$
$$\rho_{\rm p}(\eta) = -2\pi\frac{\partial}{\partial\beta_0}\ln(1 + e^{-\varepsilon(\eta)}), \tag{81}$$

involving the *pseudoenergy* $\varepsilon(\eta)$, defined as the solution of the above nonlinear integral equation, and the *differential scattering phase*, defined by

$$\varphi(\eta) = \frac{d\phi(\eta)}{d\eta} = \frac{2c}{\eta^2 + c^2}. \tag{82}$$

In this sense, the phase-space density $\rho_{\rm p}(\eta;x,t)$ does not arise as a density for particle-like dynamical objects forming a gas, but rather as a way of characterizing all averages of local conserved densities in the x,t-dependent GGE that arises from the Euler hydrodynamic principle,

$$\langle q_i\rangle_{{\rm GGE}(x,t)} = \int d\eta\rho_{\rm p}(\eta;x,t)h_i(\eta). \tag{83}$$

Many-body integrable systems admit an infinite-dimensional space of conserved quantities Q_i, and the spectral parameter is just seen as a continuous parametrization of this space (interpreted as a particular choice of a "scattering basis," see, e.g., the discussion in Ref. [179]).

As mentioned above, the crucial ingredient is the relation between GGE averages of currents and densities. Historically, this was in fact the main stumbling block in developing the hydrodynamics of integrable systems.

Average currents in GGE were first evaluated [76] using the TBA and crossing symmetry of relativistic quantum field theory; they were later derived directly from the Bethe ansatz and other quantum integrability techniques and then from "self-conserved" currents using the symmetry of current-charge correlations; see the reviews [183,184]. The result is striking: it takes the form

$$\langle j_i\rangle_{{\rm GGE}(x,t)} = \int d\eta v^{\rm eff}(\eta;x,t)\rho_{\rm p}(\eta;x,t)h_i(\theta), \tag{84}$$

where the effective velocity, here obtained, we recall, via Bethe ansatz calculations, satisfies the classical-looking collision-rate ansatz (17), with $G(\eta,\eta') = -\varphi(\eta - \eta')$ and $s_0(\eta) = \eta$:

$$v^{\rm eff}(\eta) = \eta + \int d\mu\varphi(\eta - \mu)\rho_{\rm p}(\mu)[v^{\rm eff}(\mu) - v^{\rm eff}(\eta)]. \tag{85}$$

Its x, t dependence $v^{\rm eff}(\eta) \to v^{\rm eff}(\eta;x,t)$ comes from the (x,t)-dependent GGE $\rho_{\rm p}(\eta) \to \rho_{\rm p}(\eta;x,t)$. Note how the differential scattering phase $\varphi(\eta - \eta')$, Eq. (82), arises: this is exactly the semiclassical scattering shift of Bethe ansatz wave packets. One then obtains, from (77) and assuming some completeness of the space of functions $h_i(\theta)$, the GHD equation

$$\frac{\partial}{\partial t}\rho_{\rm p}(\eta;x,t) + \frac{\partial}{\partial x}[v^{\rm eff}(\eta;x,t)\rho_{\rm p}(\eta;x,t)] = 0. \tag{86}$$

This is, in this perspective, a Euler hydrodynamic equation, even though it looks like a kinetic Eq. (15).

We finally note that each value of the spectral parameter η corresponds to a hydrodynamic normal mode—a "sound mode" or the like—for the emergent Euler-scale equation, and $v^{\rm eff}(\eta;x,t)$ are the associated hydrodynamic velocities tangent to their characteristics. Riemann invariants can be explicitly constructed; indeed $\varepsilon(\eta;x,t)$, or any function of it, satisfies the diagonalized Euler-scale equation, $\partial_t\varepsilon(\eta;x,t) + v^{\rm eff}(\eta;x,t)\partial_x\varepsilon(\eta;x,t) = 0$, and so does the "cumulative density" or height field $\int_{-\infty}^{x} dx'\rho_{\rm p}(\eta;x',t)$ (cf. the counterpart Eq. (41) for the spectral scaling function $\sigma(\eta;x,t)$ in the SG theory); likewise, for linear perturbations on top of a homogeneous stationary background, $\rho_{\rm p}(\eta) + \delta\rho_{\rm p}(\eta;x,t)$, we have $\partial_t\delta\rho_{\rm p}(\eta;x,t) + v^{\rm eff}(\eta)\partial_x\delta\rho_{\rm p}(\eta;x,t) = 0$.

C. Universality and generalized hydrodynamics of the Korteweg–de Vries soliton gas

Note that, curiously, one obtains, using the above description, a reinterpretation of the Liouville equation of phase-space conservation in classical mechanics. Traditionally it is understood in kinetic theory as a "collisionless" Boltzmann equation. Now take, for instance, the Tonks-Girardeau limit $c \to \infty$, where the Lieb-Liniger model becomes a model of nonintegracting fermions, with $\varphi(\eta) = 0$. The resulting GHD equation is the Liouville equation. But here, it is seen as a *hydrodynamic equation*, for a continuum of sound modes emerging at large scales in this system of noninteracting particles! The same holds for any system of noninteracting particles, quantum or classical.

This latter observation leads us to emphasize an important concept: *the hydrodynamic perspective has the advantage that it is indifferent to the precise nature of the underlying many-body system*. It has a large amount of universality.

This universality arises at two levels. First, the general structure of hydrodynamics at the Euler scale is always the same, no matter the underlying many-body system, under fairly general conditions (local interactions, and perhaps microscopic reversibility). The important point in establishing the Euler-scale hydrodynamic theory of a given many-body system is to characterize its full manifold of maximal entropy states. One expects that the "extensive conserved quantities," widely studied in quantum many-body systems [185], span the tangent spaces to this manifold, and according to Euler hydrodynamics, their densities are the emergent dynamical degrees of freedom onto which the microscopic dynamics projects at large scales; at the linearized level, this phenomenon has been rigorously established in quantum spin chains [186] and lattices [187]. Once the space of extensive conserved quantities is understood, the hydrodynamic principles—local relaxation and the conservation laws—are completely general and do not require any strong dynamical assumptions such as chaos or any particular structures for the underlying microscopic theory.

Second, within the family of many-body integrable models, the description of Sec. VII B is also completely universal. The microscopic system may be quantum or classical, composed of continuous fields, particles, solitons, spins, etc.—the same structure emerges for its Euler-scale hydrodynamics. The model-dependent aspects are the phase space $\mathcal{S}$ of possible values of the spectral parameter η (it is $\mathbb{R}_{>0}$ in the KdV soliton gas, $\mathbb{R}$ in the repulsive LL model, $\mathbb{C}$ in the soliton gas of focusing NLS, etc.), and the basic dynamical quantities, including the two-body shift $G(\eta, \mu)$ [it is $-2c/[(\eta-\mu)^2+c^2]$ in the repulsive LL model, $\frac{1}{\eta}\ln|\frac{\eta+\mu}{\eta-\mu}|$ in the KdV soliton gas, etc.], as well as a "bare" velocity $s_0(\eta)$ entering as the source term in the equations of state, Eqs. (17), (16), and (85) (η in the LL model, $4\eta^2$ in the KdV soliton gas, etc.). Thus, GHD, as a theory for many-body integrable systems, includes soliton gases. The full equivalence between TBA and GHD quantities and those traditionally considered in soliton gases is given in Ref. [78], and the thermodynamics and GHD of soliton gases has been worked out for KdV [78] and focusing NLS [168]. One obtains, for instance for the KdV soliton gas, the thermodynamic entropy and free energy per unit space and spectral parameter as, respectively,

$$S(\eta) = f(\eta)\left(1 + \ln\frac{4\sigma(\eta)}{\pi}\right), \quad \mathcal{F}(\eta) = -\frac{\eta}{\sigma(\eta)}. \tag{87}$$

These results arise from the universality of the GHD and TBA frameworks, but have yet to be derived more accurately using the methods of the previous sections.

This universality of the hydrodynamic description of integrable models has its source in an important aspect of many-body integrability, that of *factorized, elastic scattering*. Factorized scattering for solitons was reviewed in Sec. II A, see Eq. (5). It is also made apparent in the LL Bethe ansatz wave function (79): the structure in the exponential implies that the phase of a full many-particle scattering is the sum of two-body scattering phases. If we put the LL model in a finite segment and let the particles expand in the vacuum, then η_n are the values of asymptotic momenta that will be seen at long times in this *time-of-flight "gedenkenexperiment"*. In general, for both quantum and classical models, the spectral space is nothing else than the set of possible objects that emerge at long times (solitons, particles, bound states, waves, etc.), and a basic dynamical analysis will give the bare velocities and two-body scattering shifts for such objects. It turns out that the TBA form of the thermodynamics then emerges quite generally solely from this scattering picture; in classical systems this was first observed [167] in the Toda model—thus the TBA does not require the Bethe ansatz! In a fluid, a mesoscopic cell can be "observed" by taking it out of the fluid and making a time-of-flight experiment on it, in order to determine the distribution of spectral parameters that characterize it. The manifold of GGEs is a manifold of distributions on the spectral space. In particular, the soliton gas is simply the case where we restrict the manifold of GGEs to be distributions of solitons only; and this restriction is stable under the Euler hydrodynamic evolution. This explains the general structure of GHD and why many of the results from GHD can immediately be applied to soliton gases.

In fact, the scattering picture has far-reaching ramifications. One of them is the geometric viewpoint on GHD, whereby the GHD equations are seen as arising from a change of coordinates—or a change of metric—from the free-particle Liouville equations [45,86]. The change of metric is state-dependent (much like in Einstein's gravity!), and represents the map to the freely propagating asymptotic coordinates. This leads to an integral-equation solution [86], a "solution by characteristics" akin to the hodograph transform.

VIII. OPEN PROBLEMS

Over the last few years, various fundamental questions inspired by the exciting theoretical and experimental challenges have emerged in the growing fields of SGs and of GHD. We summarize here some of the most important of these open problems.

A. Spectral theory and rigorous asymptotics

The spectral theory of soliton gas outlined in Sec. III is based on the thermodynamic limit of finite-gap potentials and their Whitham modulation equations. At the core of this theory is the special distribution (scaling) of finite-gap spectra ensuring appropriate balance of terms in the nonlinear dispersion relations. Can this thermodynamic spectral scaling be obtained as a long-time asymptotics in some class of initial-value problems for integrable equations? One possible scenario to be explored was proposed in Ref. [188] where one considers a chain of topological bifurcations of local invariant tori parametrized by slowly evolving finite-gap spectra that emerge in the zero-dispersion (semiclassical) limit of the fNLS equation. This scenario resembles the classical Landau-Hopf transition to turbulence (see, e.g., Ref. [189]) realized in the framework of an integrable dispersive system.

A related major open question is rigorous mathematical justification of the spectral kinetic theory. While the derivation of the SG kinetic equation via the thermodynamic limit of finite-gap modulation theory has been rigorously justified [29] the asymptotic validity of the kinetic equation in the framework of the original nonlinear dispersive PDE is yet to be established. It would be highly desirable to have a rigorous asymptotic derivation of the kinetic equation for KdV, NLS and other integrable models "from first principles." An important step in this direction has been recently made in Ref. [52] where it was shown that kinetic equation for soliton gas describes the leading order asymptotic behavior of a special class of "deterministic" soliton gases for the modified KdV equation constructed as an infinite-soliton limit of N-soliton solutions by invoking the theory of the so-called primitive potentials [75] (see also Ref. [74]). At the spectral level, the characterization of the gases studied in Ref. [52] coincides with that of soliton condensates [30] so the extension of the rigorous asymptotic theory to more general classes of inherently random soliton gases remains an outstanding problem. This would pose a number of challenging and interesting questions at the intersection of applied analysis and probability theory. Related to this, making a rigorous connection between the IST- or finite-gap-based spectral theory of soliton gas and the statistical mechanics foundations and results of GHD (generalized Gibbs ensembles, correlations, etc.) is a major open problem, see also Sec. VIII E. The first step in this direction was made very recently in Ref. [78] but much remains to be done from the point of view of rigorous analysis.

The existing numerical realizations of SGs are based on N-soliton solutions (see Sec. IV). At the same time the spectral kinetic theory described in Sec. III uses finite-gap multiperiodic solutions in the thermodynamic limit. Numerical realization of soliton gases via finite-gap potentials of large genus subject to the thermodynamic spectral scaling is of immediate relevance and interest.

Finally we mention that the spectral theory of soliton gas can be applied to any integrable dispersive PDE supporting finite-gap solutions associated with hyperelliptic Riemann surfaces. One can expect new interesting behaviors in integrable models qualitatively different from the already considered examples of the KdV and fNLS equations. These include the sine-Gordon equation (kink gas), the Camassa-Holm equation (peakon gas) and others. The theory of two-dimensional soliton gases (e.g., for the Kadomtsev-Petviashvili or Davey-Stewartson equations) is another completely uncharted territory yet to be explored.

B. Thermodynamics and Statistics

The statistical description of random waves in integrable systems represents a fundamental application of the SG theory. We have reviewed several important recent steps achieved in this challenging direction of research. Generalized hydrodynamics provides a framework to establish a thermodynamic description of soliton gases. However, up to now, there is no existing comparison between SGs experiments and GHD theoretical results. On the other hand, the possible correspondence between SGs and natural phenomena stimulates the study of statistical properties of SGs (for example, numerical simulations show that the so-called spontaneous modulation instability is with high accuracy by a specifically designed SG, see Sec. VI B).

Very recently, some of the authors of this review and their collaborators have derived a general formula for the kurtosis for a homogeneous SG [34]. Derived in the framework of the spectral SG theory, the kurtosis is expressed as a function of the DOS:

$$\kappa_4 = \frac{\langle|\psi|^4\rangle}{\langle|\psi|^2\rangle^2} = -\frac{\mathrm{Im}\left(\frac{2}{3}\overline{\lambda^3} + \frac{1}{4}\overline{\lambda^2 s(\lambda)}\right)}{\mathrm{Im}(\overline{\lambda})^2}, \tag{88}$$

where the averaging procedure is defined by $\overline{h(\lambda)} = \int h(\lambda) f(\lambda) d\lambda$ and $f(\lambda)$ is the DOS of the homogeneous SG (f does not depends on x and t). The use of Eq. (88) requires the knowledge of the DOS underlying the nonlinear waves under study. Note that the DOS is only known for some specific cases such as δ-correlated Gaussian noise for $\psi(t=0,x)$ [63] or the condensate with small additional noise (Weyl's bound state SG). The DOS can also be computed for slowly varying random semiclassical fields (the partially coherent waves).

Applying the Eq. (88) to the Weyl's bound state SG [$\lambda \in i\mathbb{R}^+$, $s(\lambda) \equiv 0$] corresponding to the long-term evolution of the spontaneous MI (see Sec. VI B), it is easy to show that $\kappa_4 = 2$. This corresponds to the value of κ_4 for the exponential distribution of $|\psi|^2$ empirically found in numerical simulations and experiments devoted to the spontaneous MI [7,10,54]. Note that this result is consistent with the virial theorem ($H_{nl} = 2H_l$) known in the context of zero boundary conditions in NLS [190]. Beyond the MI problem, it is possible to compute the DOS of any soliton gas generated by the propagation of a semiclassical field (if $H_{nl} \gg H_l$ initially). Using this approach, one can also show that the corresponding value of the kurtosis is $\kappa_4 = 4$ in the case of partially coherent waves. Remarkably, this corresponds to the largest value found recently in numerical simulation in the case of the strongly nonlinear regime of partially coherent waves [154].

These recent results pave the way to a general statistical description of nonlinear random waves naturally found in various physical systems. However, it is important to note that the evaluation of the kurtosis is only the first step toward a general statistical theory. Among the various questions, one finds the evaluation of the probability density functions (of the nonlinear wave field or its amplitude, for example) and of correlation functions [such as $g^{(2)}$, see Eq. (74)]. Note that the probability density function of intensity for the defocusing NLS and some correlations for the focusing NLS have been predicted in the framework of GHD [168,191].

The spectral power density (Fourier spectrum) $\langle|\widetilde{\psi}(k,t)\rangle|^2$ is a key measurable variable of turbulence, allowing, for example, the characterization of the Kolmogorov cascade. Moreover the spectrum can be easily and directly measured in optical experiments devoted to the observation of SGs. The analysis and the understanding of Fourier spectra of SGs thus represents an important direction of research. The natural framework of the SG theory is the IST and the relationship between the IST spectrum and the Fourier spectrum is highly nontrivial from the mathematical point of view.

It is important to emphasize again that the SG theory provides a promising framework to describe and understand the statistics of wave systems close to integrability. The spontaneous modulation instability in the focusing regime of the one-dimensional focusing NLS equation is the first example of physical phenomena quantitatively described by a SG (see Sec. VI B) and Ref. [47]). One natural question is the possible link between natural phenomena and breather gases. In particular, as the Akhmediev breather is an exact solution associated with the sinusoidal perturbation of a plane wave, one might expect that the spontaneous modulation instability can also be described by a breather gas.

The general description of integrable turbulence (random waves in integrable systems) is still an open question. Any random waves in integrable systems can be decomposed into radiative modes and solitons, the former being associated with the continuous spectrum and the latter being associated with the discrete spectrum in the framework of the IST (see Sec. II A). SGs thus correspond to the peculiar case of integrable turbulence having no continuous spectrum. The study of nonlinear random waves phenomena by using a SG description is based on the conjecture that continuous spectrum can be neglected in the strongly nonlinear regime. One can naturally ask: what happens, for example, with partially coherent wave for weaker nonlinearity?

The general description of integrable turbulence thus requires the development of a statistical theory involving both discrete and continuous spectrum. In principle, the general case can be described in the framework of the finite gap theory (see Sec. III). On the other hand, by taking into account the nonresonant interactions, a nonstandard wave kinetic theory (developed in the basis of Fourier components) describes the statistical behavior and the Fourier spectrum of integrable turbulence [135,138,155]. One of the fundamental and interesting open questions is the IST formulation in the weakly nonlinear regime when the wave system is dominated by radiation components (continuous spectrum). Investigations of this question may build a bridge between finite gap theory and wave turbulence theory.

C. Experimental challenges

Experiments devoted to the study of SGs can be classified by the wave generation techniques and by the data analysis types. While solitons can simply be generated one by one in diluted SGs, the experimental realization of dense SG is highly nontrivial. One possible approach is the use of dynamical phenomenon such as the soliton fission [41] in which the DOS is not controlled. Another strategy has been recently demonstrated in order to achieve a controlled generation of dense SG [57]: by using the numerical procedure described in Sec. IV B, N-soliton solutions with random parameters are computed and then used to build the experimental SG.

It is important to note that, up to now, the procedure based on N solitons allows for the generation of *homogeneous* dense SG having an arbitrary DOS $f(\lambda)$. The generation of a *nonhomogeneous* dense SG with a space-dependent DOS $f(\lambda, x)$ is an open problem. In the context of the focusing NLSE, this extremely challenging task will require a deep theoretical understanding of the precise link between the positions of the solitons and the amplitudes of the norming constants in the N-soliton solution with $N \gg 1$. Solving this problem would be a fundamental milestone in the study of SGs. Indeed, the most intriguing and complex phenomena are expected to emerge in the context of nonhomogeneous SG whose nonequilibrium, macroscopic dynamics are described by the nontrivial continuity Eq. (18). The experimental test of this continuity equation requires the generation of nontrivial space-dependent initial DOS $f(\lambda; x, 0)$. The first example of the experimental verification of the continuity equation for the collision of spectrally “monochromatic” SGs realized in deep water tank was reported recently in Ref. [33].

On the other hand, the study of nonhomogeneous SG will also require the development of new tools for the data analysis. The measurement of a space-dependent DOS is not trivial; one will have first to define the local DOS of a measurable field. One of the difficulties is the scale separation: in the theory, the DOS evolves spatially very slowly and the number of solitons in one fluid cell dx tends to infinity. In experiments, the number of solitons is limited and thus, the measurement of the local DOS is a complex and challenging task.

In Sec. V we have reported several experiments devoted to the study of NLS SG. Some studies have also been reported with KdV-type soliton gas in shallow water experiments [41]. Extensive experimental studies of KdV SG (in hydrodynamics or electric lines for example) are interesting future direction of research. In particular, one of the challenges is the implementation in the context of the physical systems described by KdV of the approach used in NLS study: compute numerically large N-soliton solutions of KdV and launch them as initial condition in an experiment.

The experimental test of the GHD is another open exciting challenge. This includes, for example, the measurement of space-time correlation in SGs, the measurement of GGEs, etc.

D. Breakdown of integrability

In the “real-world” experiments, integrable equations such as the 1DNLS or KdV, only describe the systems at leading order. This means that in any experiments, at long time (or long propagation distance), high-order effects break integrability and play a role in the dynamics and in the statistics of the wave field. Integrability can be broken by linear effects (losses or high-order dispersion for example) or nonlinear (stimulated Raman scattering in optical fiber for example). These effects induce nonelastic collisions of solitons (for example, two interacting solitons do not recover their initial amplitudes and velocities over large time). The study of the influence of higher-order effects on SG is of fundamental and practical importance. This includes also the influence of external forces on solitons (induced for example by some potential).

In various systems, the high-order effects can be considered as small perturbations of the integrable system. As a consequence, IST spectra can be seen as slowly varying quantities that evolve adiabatically. The IST perturbation theory of nearly integrable systems is well elaborated for simple wave field patterns, such as single and two-soliton pulses [192]; meanwhile, the collective multisoliton dynamics under the influence of weak external forces now is treated only with numerical simulations [109,193]. Building a theory of SG including perturbative effects is an open and fundamental problem. GHD is a promising framework to investigate perturbative effects (see Sec. VIII E).

E. Lessons from generalized hydrodynamics: Correlations, external forces, diffusion, integrability breaking

Communities working on soliton gases, and on quantum and classical many-body systems and statistical mechanics, have been mostly disconnected until recently. Certainly, making a better connection between the ideas that have arisen in both communities would be fruitful.

For instance, the metric transform from the Liouville equation to the GHD equation [86] is nothing else but a generalization of the transformation used extensively in addressing the hard rod gas [194]. In this transformation, each quasiparticle is given a precise location, and occupies a certain momentum-dependent space that, if taken away, reduces the quasiparticles' dynamics to that of free particles. Can something like this be achieved in KdV or NLS soliton gases?

Furthermore, the hydrodynamic viewpoint on GHD has been extremely powerful. It has allowed for the extension of known structures of hydrodynamics to the realm of integrability. Taking and developing the full hydrodynamic perspective in soliton gases should lead to interesting new result, and this is still at its infancy. Here we briefly mention four directions: correlation functions, the inclusion of external forcing, the diffusive and higher-order corrections, and the inclusion of small integrability-breaking effects via Boltzmann-like equations. In fact, many more exact expressions potentially applicable to integrable turbulence have been obtained using quantum methods, we mention for instance the probability density function of the modulus of the NLS field [191].

Correlation functions in space-time are natural objects to be studied by hydrodynamics. The basic idea is that the propagation of hydrodynamic modes gives the leading large-scale correlations between local observables. Technically, one studies the linearized Euler equation for small variations $\delta\langle q_i\rangle$ on top of a homogeneous, stationary state. This gives the following form for the Fourier transform of connected correlation functions $S_{ij}(k,t)=\int dx e^{ikx}\langle q_i(x,t)q_j(0,0)\rangle^{\rm c}$ in that state:

$$S_{ij}(k,t)\sim(\exp[ikt\mathsf{A}]\mathsf{C})_{ij},\quad \mathsf{A}_{ij}=\frac{\partial\langle j_i\rangle}{\partial\langle q_j\rangle},$$
$$\mathsf{C}_{ij}=-\frac{\partial\langle q_i\rangle}{\partial\beta_j}\quad(k\to0,\ t\to\infty,\ kt\ \text{fixed}). \tag{89}$$

The flux Jacobian A and static covariance C can be written in terms of TBA quantities, giving rather explicitly (see the review [179])

$$S_{ij}(k,t)\sim\int d\eta\rho_{\rm p}(\eta)f_{\rm stat}(\varepsilon(\eta))\frac{\partial\varepsilon(\eta)}{\partial\beta_i}\frac{\partial\varepsilon(\eta)}{\partial\beta_j}e^{iktv^{\rm eff}(\eta)}, \tag{90}$$

where $f_{\rm stat}(\varepsilon)$ encodes the *statistics* of the fundamental particles (the asymptotic objects), e.g., $f_{\rm stat}(\varepsilon)=1/(1+e^{-\varepsilon})$ in the LL model, and $f_{\rm stat}(\varepsilon)=1$ in the KdV soliton gas [78]. This formula was verified numerically in various integrable models; see the review [179] and more recent results in the KdV soliton gas [78] and the Toda model [195]. For instance, in the KdV soliton gas one obtains the following correlation function of the KdV field at large time separations:

$$\langle u(\xi t,t)u(0,0)\rangle-\langle u(0,0)\rangle^2\overset{t\to\infty}{\sim}t^{-1}\sum_{\eta:s(\eta)=\xi}\frac{16\sigma(\eta)^2f(\eta)^3}{|s'(\eta)|}. \tag{91}$$

We emphasize that this formula arises from the general hydrodynamic response theory as applied to GHD, and still needs to be understood from the IST perspective.

One can go much further and obtain two-point correlation functions not just of conserved densities, but also of currents, and in fact of arbitrary observables, by the use of hydrodynamic projections [186], as well as, quite surprisingly, two-point correlation functions in nonstationary backgrounds. Going beyond, based on similar ideas, the Euler-scale large-deviation theory of integrated currents and other extensive quantities (ballistic fluctuation theory) and nonlinear-response functions have been obtained. See the review [179]. More generally, the ballistic macroscopic fluctuation theory [181], which has in particular been applied to GHD, gives a complete framework where many-point correlation functions and Euler-scale large-deviation theory can be evaluated, predicting novel long-range spatial correlations in moving fluids [180]. All these results apply, in principle, to soliton gases as well. For instance, one obtains, in the NLS soliton gas, predictions for "dynamical free energies," such as $\mathcal{G}_4(\ell)$ defined via the asymptotic

$$\ln\left\langle\exp\left[\ell\int_0^T dt(|\psi(0,t)|^4-2|\psi_x(0,t)|^2)\right]\right\rangle\overset{T\to\infty}{\sim}T\mathcal{G}_4(\ell).$$

This quantity is in principle available by combining the aforementioned ballistic fluctuation theory with the soliton-gas thermodynamics of Ref. [168]. But, in this context, numerical verifications and a full theoretical underpinning are still very much lacking.

Generalized external forces may be written as external fields coupled to conserved densities. These change the Hamiltonian to $H+V$ where $V=\sum_i\int dx V_i(x)q_i(x)$. Although generically V breaks the integrability of H, with $V_i(x)$ slowly varying in space, Euler hydrodynamic equations with generalized force terms remain valid for all original conservation laws—indeed, for conventional gases, Euler equations can be written within external force fields, even when such fields break momentum conservation. Within GHD, the corresponding force terms have been obtained [196], with (86) modified to

$$\frac{\partial}{\partial t}\rho_{\rm p}(\eta;x,t)+\frac{\partial}{\partial x}[v^{\rm eff}(\eta;,x,t)\rho_{\rm p}(\eta;x,t)]$$
$$+\frac{\partial}{\partial\eta}[a^{\rm eff}(\eta;,x,t)\rho_{\rm p}(\eta;x,t)]=0. \tag{92}$$

Quite surprisingly, the effective acceleration $a^{\rm eff}(\eta;,x,t)$ satisfies a "collision-rate ansatz" as (85) but with the bare velocity η replaced by the bare acceleration $a(\eta;x)=-\sum_i V_i'(x)h_i(\eta)$. It is this GHD equation, for the LL model and with a simple external force field, was verified experimentally in cold atomic gases restrained to one dimension of space [197–199], see the review [182]. This is the simplest situation of externally changing parameters: the more general situation was worked out [200], including time dependence, and varying the coupling strength $c \to c(x,t)$ in (78), something which is crucial for comparison with some experiments. External force fields and slowly varying couplings also naturally occur in many situations where soliton gases emerge. The theory from GHD is in principle fully applicable to soliton gases; however, again up to now, the application to soliton gases and the IST perspective on such GHD results are still completely missing.

Hydrodynamics is a derivative expansion, and as such, one may wonder about the higher-derivative corrections. At second derivative, this is the *diffusive correction*, such as the viscosity term in Navier-Stokes equations. Again, an exact expression of the diffusive matrix—or diffusive operator on spectral space—has been evaluated in GHD with convincing comparisons against numerical results, see the review [179]. The form obtained is

$$\frac{\partial}{\partial t}\rho_{\rm p}(\eta;x,t)+\frac{\partial}{\partial x}[v^{\rm eff}(\eta;,x,t)\rho_{\rm p}(\eta;x,t)] = \frac{1}{2}\frac{\partial}{\partial x}\left(\int d\eta'\mathcal{D}_{\eta,\eta'}[\rho_{\rm p}(\cdot;x,t)]\frac{\partial}{\partial x}\rho_{\rm p}(\eta';x,t)\right). \quad (93)$$

The diffusion kernel $\mathcal{D}_{\eta,\eta'}[\rho_{\rm p}]$ is evaluated from the Kubo formula involving space-time integrated current two-point functions, using form factor methods of quantum integrability [184,201]. The general formula, applicable to quantum and classical models alike, is conjectured by comparison with the diffusion kernel obtained in the 1980s for the classical hard-rod gas [202]. Again, the general formula involves the statistical factor $f(\varepsilon)$. The combination of diffusion with external forces has also been evaluated [203]. The third-order, *dispersive* correction was proposed recently [204], although much work is still needed to fully establish it.

Is there diffusion in soliton gases? If so, is it correctly described by the GHD formula? Furthermore, can we evaluate the exact third-order dispersion term? A natural conjecture concerns the condensate limit; in the GHD of quantum integrable models, the condensate limit had been studied earlier, and is known as zero-entropy GHD [205]. The connection between soliton-gas condensate limit and zero-entropy GHD was partially made in Ref. [30]. Do dispersive terms of GHD (soliton gases) reproduce, in the zero-entropy (condensate) limit, dispersive terms of the fundamental dynamical equations (e.g., the KdV equation)?

Finally, the effects of small perturbations that break integrability has been studied. The development is still in its infancy, with various approaches and different physical situations proposed, see the review [206]. The perspective taken in GHD is different from that taken in soliton gases, and it would be fruitful to make a better connection. One important point that has been emphasized [207] generalizes the viewpoint discussed above, whereby the Liouville equation—the kinetic equation for free particles—is seen as a Euler-scale hydrodynamic equation. It is possible to modify the Euler-scale hydrodynamic equation to account for terms that break the conservation laws on which it is based. There are general Kubo-like formulas this modification, and when applied to GHD, these give terms that can be written, at least in quantum models, in a form-factor expansion. Specialized to the GHD of free particles, these terms are nothing else but Boltzmann collision terms from the Boltzmann equation; form factors of interacting integrable models generalize Boltzmann collision terms. Is there a parallel notion of form factors that can be used to evaluate Boltzmann collision terms in soliton gases? Thus, again, we obtain a different viewpoint: the Boltzmann equation, a kinetic equation, is re-interpreted as a hydrodynamic equation, with terms that break the infinitely many conservation laws admitted by free particles. This reinterpretation has, potentially, far-reaching consequences, which still need to be addressed.

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