Amplification of the anomalous scaling in the Kazantsev-Kraichnan model with finite-time correlations and spatial parity violation

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By using the field theoretic renormalization group technique together with the operator product expansion, simultaneous influence of the spatial parity violation and finite-time correlations of an electrically conductive turbulent environment on the inertial-range scaling behavior of correlation functions of a passively advected weak magnetic field is investigated within the corresponding generalized Kazantsev-Kraichnan model in the second order of the perturbation theory (in the two-loop approximation). The explicit dependence of the anomalous dimensions of the leading composite operators on the fixed point value of the parameter that controls the presence of finite-time correlations of the turbulent field as well as on the parameter that drives the amount of the spatial parity violation (helicity) in the system is found even in the case with the presence of the large-scale anisotropy. In accordance with the Kolmogorov's local isotropy restoration hypothesis, it is shown that, regardless of the amount of the spatial parity violation, the scaling properties of the model are always driven by the anomalous dimensions of the composite operators near the isotropic shell. The asymptotic (inertial-range) scaling form of all single-time two-point correlation functions of arbitrary order of the passively advected magnetic field is found. The explicit dependence of the corresponding scaling exponents on the helicity parameter as well as on the parameter that controls the finite-time velocity correlations is determined. It is shown that, regardless of the amount of the finite-time correlations of the given Gaussian turbulent environment, the presence of the spatial parity violation always leads to more negative values of the scaling exponents, i.e., to the more pronounced anomalous scaling of the magnetic correlation functions. At the same time, it is shown that the stronger the violation of spatial parity, the larger the anomalous behavior of magnetic correlations.

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I. INTRODUCTION

One of the fundamental characteristics of fully developed turbulent systems is undoubtedly the existence of the anomalous scaling [1–12], i.e., the existence of deviations from the simple inertial-range scaling behavior of various correlation functions predicted by the famous classical phenomenological Kolmogorov-Obukhov (KO) theory [13]. Let us recall that, in the framework of the KO theory, it is postulated that the statistical properties of arbitrary fully developed turbulent system deep inside the inertial interval defined by inequalities $l \ll r \ll L$ are independent of the so-called integral scale L, i.e., of a typical scale at which the energy is pumped into the system in order to maintain the steady state (the first Kolmogorov hypothesis), as well as of the so-called dissipation scale l, i.e., of a typical scale at which the energy begins to dissipate intensively (the second Kolmogorov hypothesis). When one supposes the validity of these two Kolmogorov hypotheses, then inertial-range scaling exponents of various correlation functions are unambiguously given and can be found by using simple dimensional analysis. Note also that one of the consequences of the full validity of the two Kolmogorov hypotheses would be the inevitable restoration of the full isotropy of the system in the statistical sense in the inertial range.

However, it is well known that numerical simulations and theoretical analyses as well as real experiments show that, in contradiction with the first Kolmogorov hypothesis, turbulent systems have the ability to remember at least some general properties of their origin (the form and properties of the energy pumping at large scales into the system). This ability is given by the preservation of the explicit dependence of the correlation functions of various random quantities on the integral scale even deep inside the inertial interval. This dependence of the correlation functions on the integral scale L in the inertial range is manifested explicitly in the change of their scaling properties, i.e., in the presence of the aforementioned anomalous scaling (see, e.g. Refs. [1–12,14–18] as well as references cited therein).

From a physical point of view, the presence of the anomalous scaling in turbulent systems is given by the existence of strong developed fluctuations of the dissipative rate, i.e., by the intermittency [1–4,6,11]. Geometrically it means that the turbulent flows behave like fractals [more precisely, like multifractals (see, e.g., Ref. [3])] in the sense that not whole volume of a given turbulent environment is filled by vortices (such a situation in fact corresponds to the pure dimensional scaling in the KO theory) but there always exist dynamically changing places with pure laminar flows at all scales of the inertial range.

In this connection it should also be mentioned the well-known fact that the anomalous scaling is even more strongly pronounced (more visible) in the behavior of various correlation functions of passively advected scalar or vector quantities by turbulent environments than in the behavior of turbulent velocity fields themselves (see, e.g., Refs. [4,6,9] and references cited therein). The central role in these investigations has been played by the well-known Kraichnan model [19] and the Kazantsev-Kraichnan model [20] of the passively advected scalar and vector fields by simple turbulent environments described by δ-correlated in time random velocity fields with the Gaussian spatial statistics. Namely, in the framework of the Kraichnan model of the passive scalar advection, a systematic theoretical analysis of anomalous scaling in turbulent systems was performed for the first time using the so-called zero-mode technique (see Ref. [6] as well as references cited therein).

The very existence of the anomalous scaling in turbulent systems also has nontrivial impact on their inertial-range properties. For instance, it can lead to the natural persistence of various symmetry breaking, generated by the form of the energy pumping into the system, even deep inside the inertial interval. In this respect, one of the most effective theoretical techniques for the systematic investigation of the properties of the anomalous scaling in turbulent systems, especially when symmetry breaking of various types is present, is the field theoretic renormalization group (RG) technique. In the framework of this theoretical approach, the anomalous inertial-range scaling behavior of various correlation functions is described through the existence of the so-called dangerous operators with the negative critical exponents in the operator product expansion (OPE) [21–23]. Many interesting fundamental facts about the properties of the anomalous scaling of various random quantities in the Gaussian as well as non-Gaussian (driven by the stochastic Navier-Stokes equation) turbulent environments were obtained using the field theoretic RG approach during the last 25 years (see, e.g., Refs. [9,24–42] as well as references cited therein).

The symmetry breaking in various turbulent systems, which is recently intensively studied since it plays important role in many geophysical as well as astrophysical turbulent processes (see, e.g., Refs. [43–54] as well as references cited therein), is the spatial parity violation (helicity). Therefore, from a fundamental point of view, it is undoubtedly important to understand also the role of helicity in fully developed turbulent systems. In this respect, using the field theoretic RG technique, it was shown in Ref. [55] that the spatial parity violation of turbulent environments can have a nontrivial impact on diffusion processes of scalar as well as vector quantities in such turbulent systems. At the same time, as was shown in Refs. [56,57], it seems that the presence of the helicity has no impact on the inertial-range scaling behavior of the correlation functions of passively advected scalar fields by turbulent environments, regardless whether the turbulent velocity field is Gaussian or non-Gausian, i.e., driven by the corresponding stochastic Navier-Stokes equation. On the other hand, as was shown in Refs. [58,59] in the framework of the helical Kazantsev-Kraichnan model, it seems that the presence of the spatial parity violation can lead to significant changes in the scaling properties of the correlation functions of the passive magnetic field in the kinematic magnetohydrodynamic (MHD) turbulence. Namely, as calculations have shown, the anomalous scaling of the passive magnetic field becomes more pronounced in the helical turbulent environment than in the absence of the spatial parity violation.

In Refs. [58,59], the influence of the helicity on the anomalous scaling of the passive vector (magnetic) field was investigated in the framework of the so-called "rapid-change" Kazantsev-Kraichnan model of the kinematic MHD turbulence with δ-time correlations of the velocity field. The aim of the present study is to go beyond this restriction and to investigate in detail the simultaneous influence of the spatial parity violation and of the finite-time correlations of the Gaussian turbulent velocity field (in the framework of the corresponding generalized Kazantsev-Kraichnan model) on the inertial-range scaling properties of the correlation functions of the passively advected magnetic field. As will be shown, the presence of the helicity amplifies the anomalous scaling of the magnetic field correlations even when the presence of the finite-time velocity correlations of an electrically conductive turbulent environment is assumed. This behavior of the magnetic correlation functions is again significantly different in comparison with the behavior of the correlation functions of the passively advected scalar field in the framework of the corresponding generalized Kraichnan model [56], where the presence of the helicity has no impact on the anomalous scaling even when the finite-time correlations of the turbulent velocity field is considered.

The paper is organized as follows. In Sec. II the generalized helical Kazantsev-Kraichnan model of the kinematic MHD turbulence is described. In Sec. III the field theoretic formulation of the model is given and the basic facts of its ultraviolet (UV) renormalization are discussed. The explicit dependence of the critical dimensions of the leading composite operators on the helicity parameter is found and discussed in Sec. IV. The inertial-range scaling behavior of the single-time two-point correlation functions of the magnetic field under the simultaneous influence of the spatial parity violation and of the finite-time velocity correlations is investigated in Sec. V. Obtained results are briefly reviewed and discussed in Sec. VI.

II. THE KAZANTSEV-KRAICHNAN MODEL WITH THE SPATIAL PARITY VIOLATION AND FINITE-TIME CORRELATIONS OF THE VELOCITY FIELD

As was already mentioned in Introduction, our aim is to investigate in detail the influence of the presence of the spatial parity violation on the scaling properties of correlation functions of the fluctuating part of the magnetic field $\mathbf{b} \equiv \mathbf{b}(t, \mathbf{x})$ in the kinematic MHD turbulence driven by the Gaussian turbulent velocity field with finite-time correlations. Such a stochastic system is described by the generalized Kazantsev-Kraichnan model given by the following stochastic equation:

$$\partial_t \mathbf{b} = \nu_0 \Delta \mathbf{b} - (\mathbf{v} \cdot \partial)\mathbf{b} + (\mathbf{b} \cdot \partial)\mathbf{v} + \mathbf{f}^{\mathbf{b}}, \qquad (1)$$

where $\partial_t \equiv \partial/\partial t$, $\partial_i \equiv \partial/\partial x_i$, $\Delta \equiv \partial^2$ is the Laplace operator, $\nu_0 = c^2/(4\pi\sigma_0)$ is the magnetic diffusivity, c is the speed of light, σ_0 is the conductivity, $\mathbf{v} \equiv \mathbf{v}(t, \mathbf{x})$ is the fluctuating part of the velocity field of the electrically conductive turbulent environment, and $\mathbf{f}^{\mathbf{b}} = \mathbf{f}^{\mathbf{b}}(t, \mathbf{x})$ is a random noise (the source of fluctuations of the magnetic field). Since, in what follows, we will use the field theoretic RG technique for the analysis of the model, the subscript 0 will always denote bare parameters of the unrenormalized theory. In addition, we will also suppose that the studied turbulent system is incompressible, i.e., we postulate that both fields $\mathbf{v}$ and $\mathbf{b}$ are solenoidal (divergent-free) vector fields ($\partial \cdot \mathbf{v} = \partial \cdot \mathbf{b} = 0$).

Because of the incompressibility assumption, the random noise $\mathbf{f^b}$ is also transverse and maintains the steady state of the studied dissipative system (it represents the source of the magnetic energy pumping). It is supposed that its statistics is Gaussian with zero mean and the correlation function

$$D^b_{ij}(t_1, \mathbf{x_1}; t_2, \mathbf{x_2}) \equiv \langle f^b_i(t_1, \mathbf{x_1}) f^b_j(t_2, \mathbf{x_2})\rangle = \delta(t_1 - t_2) C_{ij}(\mathbf{r}/L), \tag{2}$$

where $\mathbf{r} = \mathbf{x_1} - \mathbf{x_2}$, L represents an integral scale related to the corresponding stirring, and C_{ij} are some functions that are finite in the limit $L \to \infty$. We will not specify their explicit form here since it is not important in what follows (they will not enter into any calculation). The only condition that must be satisfied by the functions C_{ij} is that they must decrease rapidly for $|\mathbf{r}| \gg L$. Note, however, that the large-scale anisotropy can be introduced into the system through the correlator (2). For instance, if the system is placed in a constant large-scale (macroscopic) magnetic field $\mathbf{B}$ (the source of the uniaxial large-scale anisotropy), then one possible way to define the random noise $\mathbf{f^b}$ is $(\mathbf{B} \cdot \partial)\mathbf{v}$ with the explicit uniaxial anisotropic properties (see, e.g., Ref. [26] for more details).

In the framework of the Kazantsev-Kraichnan model of the kinematic MHD turbulence the turbulent velocity field $\mathbf{v}(t, \mathbf{x})$ also has the Gaussian statistics. Moreover, in the case when the presence of finite-time correlations of the velocity field is supposed, then the correlation function of the velocity field is taken in the following form [25,60,61]:

$$D^v_{ij}(t, \mathbf{x}; t', \mathbf{x}') \equiv \langle v_i(t, \mathbf{x}) v_j(t', \mathbf{x}')\rangle = g_0 \nu_0^3 \int \frac{d\omega\, d\mathbf{k}}{(2\pi)^{d+1}} \frac{k^{4-d-2\varepsilon-\eta} R_{ij}(\mathbf{k}) e^{i[\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}') - \omega(t-t')]}}{\omega^2 + (u_0 \nu_0 k^{2-\eta})^2}, \tag{3}$$

where d denotes the spatial dimension of the system, $\mathbf{k}$ is the momentum (wave number vector), $k = |\mathbf{k}|$, $g_0 > 0$ is the coupling constant of the model, the parameter u_0 represents the ratio of the turnover time of the vector field and the velocity correlation time, $R_{ij}(\mathbf{k})$ is a transverse projector (due to the incompressibility) explicit form of which will be discussed below, and the exponents ε and η are RG expansion parameters (see, e.g., Ref. [25] for details).

The energy spectrum $E(k)$ of the system is controlled by the coupling constant g_0 and by the exponent ε in Eq. (3) through the relation $E(k) \simeq (g_0 \nu_0^2/u_0) k^{1-2\varepsilon}$. On the other hand, the finite-time correlations of the velocity field are described by the parameter u_0 and by the exponent η through the dispersion relation $\omega \simeq u_0 \nu_0 k^{2-\eta}$ between the frequency ω and the momentum k [25,62–65]. Note that the physical values of the parameters ε and η are $\varepsilon = \eta = 4/3$ since the value $\varepsilon = 4/3$ leads to the Kolmogorov "two-thirds law" for the spatial statistics of the velocity field (or to the "five-thirds law" for the energy spectrum) and $\eta = 4/3$ corresponds to the Kolmogorov frequency.

Using the dimensional analysis one can also find that the coupling constant g_0 and the parameter u_0 are related to the characteristic ultraviolet (UV) momentum scale Λ ($\Lambda \sim 1/l$, where l is the characteristic inner length) by the relations

$$g_0 \simeq \Lambda^{2\varepsilon+\eta}, \quad u_0 \simeq \Lambda^{\eta}. \tag{4}$$

The geometric properties of the velocity fluctuations in Eq. (3) are described by the form of the transverse projector $R_{ij}(\mathbf{k})$. In the case without the presence of any symmetry breaking, the transverse projector $R_{ij}(\mathbf{k})$ is reduced to the ordinary isotropic transverse projector $P_{ij}(\mathbf{k}) = \delta_{ij} - k_i k_j/k^2$. On the other hand, our aim is to investigate the influence of the spatial parity violation (helicity) on the statistical properties of the magnetic correlations. The simplest way to introduce the spatial parity violation into the studied system is to extend the ordinary transverse projector $P_{ij}(\mathbf{k})$ by a part proportional to the tensor $\epsilon_{ijl} k_l/|k|$, where ϵ_{ijl} is Levi-Civita's completely antisymmetric tensor of rank 3. The presence of such a tensor in the transverse projector $R_{ij}(\mathbf{k})$ in the velocity correlator (3) explicitly violates the spatial parity of the turbulent system. Thus, in what follows, the transverse projector $R_{ij}(\mathbf{k})$ in Eq. (3) is taken in the following form:

$$R_{ij}(\mathbf{k}) = \delta_{ij} - k_i k_j/k^2 + i\rho\epsilon_{ijl} k_l/|k|, \tag{5}$$

where the real parameter $0 \leqslant |\rho| \leqslant 1$ describes the amount of the helicity in the studied turbulent environment. The value $\rho = 0$ means the absence of the spatial parity violation in the system and, on the other hand, the maximal spatial parity violation is obtained for $|\rho| = 1$.

For completeness, let us also note that the Gaussian statistics of the velocity field described by the correlator (3) has two important special limit cases (see, e.g., Refs. [25,61]). The first of them is obtained in the limit $u_0 \to \infty$ together with the assumption that the ratio $g'_0 \equiv g_0/u_0^2$ remains constant. This represents the so-called rapid-change model limit, in the framework of which the correlator (3) has the following form:

$$D^v_{ij}(t, \mathbf{x}; t', \mathbf{x}') = \delta(t - t') g'_0 \nu_0 \int \frac{d\mathbf{k}}{(2\pi)^d} R_{ij}(\mathbf{k}) k^{-d-2\varepsilon+\eta} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}, \tag{6}$$

with the δ-time correlations. Thus, in this case, the studied stochastic model is reduced into the genuine Kazantsev-Kraichnan model [20] with the presence of the spatial parity violation [58].

The second nontrivial limit of the studied model is obtained when $u_0 \to 0$ and, at the same time, the ratio $g''_0 \equiv g_0/u_0$ is held constant. This is the so-called quenched (time-independent or frozen) velocity field limit with the velocity correlator in the form

$$D^v_{ij}(t, \mathbf{x}; t', \mathbf{x}') = \frac{g''_0 \nu_0^2}{2} \int \frac{d\mathbf{k}}{(2\pi)^d} R_{ij}(\mathbf{k}) k^{2-d-2\varepsilon} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}. \tag{7}$$

In this case, the model is similar to models of random walks in a random environment with long-range correlations (see, e.g., Refs. [66,67]).

Note also that the necessary infrared (IR) regularization of the integral in Eq. (3), as well as in Eqs. (6) and (7), is realized by the cutoff from below $k = k_{\min} \equiv 1/L$, where L represents the integral turbulent scale. It is, in general, different from the stirring scale L introduced in Eq. (2), but, in what follows, this difference is unimportant.

Finally, it is worth mentioning that the stochastic model given in Eqs. (1)–(3) represents a simplification of real MHD turbulence problem at least in two points. First of all, the properties of the velocity field in the stochastic model (1)–(3) are given by the correlator (3), i.e., all nonlinearities in the statistics of the velocity field are neglected. The second significant simplification is given by the fact that the influence of the magnetic field on the conductive turbulent environment is also completely neglected. Therefore, the random magnetic field **b** in the studied model behaves like a passive vector admixture.

III. FIELD THEORETIC FORMULATION OF THE MODEL AND ITS UV RENORMALIZATION

Since the field theoretic formulation of the model as well as its two-loop RG analysis with the determination of all possible stable scaling regimes were already given in detail in Ref. [68], it is not necessary to repeat all the technical details of such an analysis. Instead, here we give only the basic facts of the RG analysis as well as the results with the explicit presence of the spatial parity violation that will be important in what follows.

A. Field theoretic formulation of the model

Using the well-known formalism [69], the stochastic model defined by Eqs. (1)–(3) can be rewritten into the field theoretic model with the action functional (see, e.g., Ref. [61])

$$S(\Phi) = -\frac{1}{2}\int dt_1\, d^d\mathbf{x_1}\, dt_2\, d^d\mathbf{x_2}\ \left\{v_i(t_1,\mathbf{x_1})\left[D^v_{ij}(t_1,\mathbf{x_1};t_2,\mathbf{x_2})\right]^{-1} v_j(t_2,\mathbf{x_2}) - b'_i(t_1,\mathbf{x_1}) D^b_{ij} t_1,\mathbf{x_1};t_2,\mathbf{x_2}) b'_j(t_2,\mathbf{x_2})\right\} + \int dt\, d^d\mathbf{x}\ \mathbf{b}'\cdot[-\partial_t\mathbf{b} + \nu_0\Delta\mathbf{b} - (\mathbf{v}\cdot\partial)\mathbf{b} + (\mathbf{b}\cdot\partial)\mathbf{v}], \quad (8)$$

where $\mathbf{b}' = \mathbf{b}'(t,\mathbf{x})$ is a solenoidal auxiliary field, $\Phi = \{\mathbf{v},\mathbf{b},\mathbf{b}'\}$, D^b_{ij} and D^v_{ij} are correlators (2) and (3), respectively, and the required summation over all dummy indices is assumed. The second term in the first line together with the second line in the action functional (8) represent the De Dominicis-Janssen action at fixed **v** of the studied stochastic problem and the first term in the first line represents the Gaussian averaging over the velocity field **v**.

The field theoretic model defined by the action functional (8) can be investigated using the perturbation theory that can be realized through the standard diagrammatic technique with two necessary propagators and one interaction vertex, graphical representations of which are shown explicitly in Fig. 1. The analytical form of the propagators is (in the frequency-momentum representation)

$$\langle b_i b'_j\rangle_0 = \frac{P_{ij}(\mathbf{k})}{-i\omega + \nu_0 k^2}, \quad \langle b'_i b_j\rangle_0 = \langle b_i b'_j\rangle_0^* \quad (9)$$

and

$$\langle v_i v_j\rangle_0 = \frac{g_0 \nu_0^3 k^{4-d-2\varepsilon-\eta} R_{ij}(\mathbf{k})}{\omega^2 + (u_0\nu_0 k^{2-\eta})^2}. \quad (10)$$

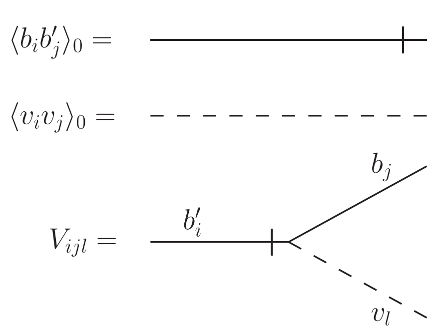

FIG. 1. Graphical representation of the used propagators and interaction vertex of the model.

On the other hand, the only interaction (triple) vertex of the model, which follows from the action (8), is given as

$$b'_i[-v_j\partial_j b_i + b_j\partial_j v_i] = b'_i V_{ijl} b_j v_l, \quad (11)$$

with the following analytic form of the vertex factor V_{ijl} (again in the frequency-momentum representation):

$$V_{ijl} = i(k_l\delta_{ij} - k_j\delta_{il}). \quad (12)$$

Note that the momentum **k** is flowing into the vertex via the auxiliary field $\mathbf{b}'$.

In what follows, we will use the field theoretic functional formulation (8) of the stochastic problem (1)–(3) for the investigation of some of its statistical properties. In this respect, the main advantage of the functional formulation is the possibility to apply the well-defined field theoretic means, such as the RG technique and the OPE expansion, to analyze the problem. Here the statistical averages of random quantities in the stochastic problem are replaced with the corresponding functional averages with weight $\exp S(\Phi)$ (see, e.g., Ref. [23]).

B. Basic facts of the two-loop RG analysis of the model

The standard canonical dimensional analysis [40,61,68] shows that the field theoretic model described by the action functional (8) (i) belongs to the so-called two-scale models [23], in the framework of which the total canonical dimension of any quantity is given as a linear combination of its momentum dimension and frequency dimension, (ii) is multiplicatively renormalizable (the only superficially UV divergent function of the model is the 1-irreducible Green's function $\langle b'_i b_j\rangle_{1-ir}$ and the corresponding divergences can be removed multiplicatively by the counterterm of the form $b'_i\Delta b_j$), and (iii) is logarithmic for $\varepsilon = \eta = 0$ (the coupling constants g_0 and u_0 of the model are dimensionless at this point). Therefore, in the so-called minimal subtraction (MS) scheme [22], which is always used in what follows, all UV divergences in the correlation functions have, in general, the form of poles in parameters ε and η as well as in their linear combinations.

Using the two-loop RG results obtained in Refs. [40,61,68], one can conclude that, depending on the value of parameters ε and η, the model exhibits five IR stable fixed points that drive all possible asymptotic inertial-range scaling regimes of the model. Two of them are related to the rapid-change limit of the model [the velocity correlations given by Eq. (6)]: one trivial with zero fixed point value of the coupling constant g' and the second one nontrivial with $g'_* > 0$ (the asterisk index "*" will always denote the fixed point value of arbitrary quantity), and two of them are related to the frozen limit of the model [the velocity correlations given by Eq. (7)]: again, one trivial with zero fixed point value of the coupling constant g'' and the second one nontrivial with $g''_* > 0$.

Finally, the most general scaling regime with the presence of the finite-time correlations of the turbulent velocity field is described by the fixed point with arbitrary finite fixed point value of the parameter u and with the fixed point value of the coupling constant g as the explicit function of u_* as well as of the parameter of the spatial parity violation ρ. In the two-loop approximation it can be written in the following integral form:

$$g_* \frac{S_d}{(2\pi)^d} = \frac{2du_*(1+u_*)}{d-1}\varepsilon - \frac{4d^2u_*(1+u_*)}{(d-1)^3}\frac{S_{d-1}}{S_d}\varepsilon^2 \int_0^1 dx(1-x^2)^{\frac{d-1}{2}} \times \left[\frac{(d+u_*)x}{(1+u_*)} \frac{\arctan\left(\frac{u_*-x+1}{\sqrt{u_*^2+2u_*-x^2+1}}\right) - \arctan\left(\frac{u_*+x+1}{\sqrt{u_*^2+2u_*-x^2+1}}\right)}{\sqrt{u_*^2+2u_*-x^2+1}} + \rho^2\delta_{3d}\frac{(d-2)(d-5)}{d-1} \frac{\pi - \arctan\left(\frac{u_*-x+1}{\sqrt{u_*^2+2u_*-x^2+1}}\right) - \arctan\left(\frac{u_*+x+1}{\sqrt{u_*^2+2u_*-x^2+1}}\right)}{\sqrt{u_*^2+2u_*-x^2+1}} \right], \tag{13}$$

where S_d denotes the surface area of the d-dimensional unit sphere

$$S_d \equiv \frac{2\pi^{d/2}}{\Gamma(d/2)}, \tag{14}$$

$\Gamma(x)$ is the Euler's gamma function, and the Konecker symbol δ_{3d} is used to show explicitly that the helical term has meaning only for $d = 3$. This fixed point is realized for $\varepsilon = \eta$ and is always stable for the physically most interesting case $\varepsilon = \eta = 4/3$, regardless of the value of the helicity parameter ρ.

The existence of stable fixed point means that, for given values of the model parameters, various correlation functions of the model exhibit IR scaling behavior with corresponding critical dimensions. From a phenomenological point of view, the most interesting are the multiplicatively renormalizable equal-time two-point quantities $G(r)$ (see, e.g., Refs. [40,61] for the general scaling analysis of such quantities).

In what follows, we will be interested in the scaling behavior of the equal-time two-point correlation functions of the magnetic field

$$B_{N-m,m}(r) \equiv \langle b_r^{N-m}(t,\mathbf{x}) b_r^m(t,\mathbf{x}')\rangle, \quad r = |\mathbf{x}-\mathbf{x}'|, \tag{15}$$

where b_r denotes the component of the magnetic field directed along the vector $\mathbf{r} = \mathbf{x} - \mathbf{x}'$ (see, e.g., Refs. [26,61]). The IR scaling behavior of these correlation functions (driven by the corresponding IR stable fixed point) can be written as follows (see Ref. [40] for details):

$$B_{N-m,m}(r) \simeq \nu_0^{-N/2}(r/l)^{-\gamma^*_{N-m}-\gamma^*_m} R_{N,m}(r/L), \tag{16}$$

where γ^*_{N-m} and γ^*_m are the anomalous dimensions of the composite operators b_r^{N-m} and b_r^m, taken at the corresponding fixed point values g_* and u_*, and the scaling functions $R_{N,m}(r/L)$ remain unknown in the framework of the standard RG analysis.

On the other hand, the inertial range ($r/L \to 0$) asymptotic behavior of the scaling functions $R_{N,m}(r/L)$ can be studied using the OPE technique [22], in the framework of which the scaling functions can be written as follows:

$$R_{N,m}(r/L) = \sum_i C_{F_i}(r/L)(r/L)^{\Delta_{F_i}}, \quad r/L \to 0, \tag{17}$$

where the sum runs through all possible renormalized composite operators F_i allowed by the symmetry of the problem with their critical dimensions Δ_{F_i}. $C_{F_i}(r/L)$ are the corresponding coefficient functions, which are regular in r/L. It is clear that the nontrivial contribution to the inertial-range behavior of the scaling functions [and therefore also to the inertial-range behavior of the correlation functions (15)] is given by operators with negative critical dimensions, which give singular contributions to the OPE (17) in the limit $r/L \to 0$ (see, e.g., Ref. [26] for details). At the same time, if more than one such "dangerous" operators exist, then the leading contribution to the expansion (17) is given by the composite operators with the smallest critical dimensions This behavior is commonly known as the anomalous scaling and is typical for fully developed turbulent systems.

Dimensional analysis shows (see, e.g., Ref. [40]) that, in the studied model even with the presence of the spatial parity violation, the most singular contributions in the OPE are given by the operators constructed solely from the magnetic field $\mathbf{b}(x)$ in the following form (see, e.g., Refs. [26,34,36]):

$$F_{N,p} = (\mathbf{n}\cdot\mathbf{b})^p(\mathbf{b}\cdot\mathbf{b})^l, \quad N = 2l + p. \tag{18}$$

Note that the from of the operators (18) is also suitable for investigation of the model with the presence of the uniaxial anisotropy represented by the unit vector $\mathbf{n}$. It can be defined, e.g., as $\mathbf{n} = \mathbf{B}/|\mathbf{B}|$, if the studied system is placed in a constant large-scale magnetic field $\mathbf{B}$ (see Sec. II).

Since our aim is to investigate the influence of the spatial parity violation on the scaling properties of the single-time two-point correlation functions (15) of the magnetic field, it is necessary first to determine the dependence of the critical exponents of the composite operators (18) on the helicity parameter ρ. This dependence is determined and discussed in the next section.

IV. INFLUENCE OF HELICITY ON THE CRITICAL DIMENSIONS OF THE LEADING COMPOSITE OPERATORS

Thus, our first aim is to determine the dependence of the critical dimensions $\Delta_{N,p}$ of the composite operators (18) on the helicity parameter ρ. For this purpose it is necessary to perform the corresponding UV renormalization procedure of these operators, a general description of which can be found elsewhere (see, e.g., Refs. [26,34,36,61]). Therefore, in what follows, we will discuss only some of its basic results.

One of the most important feature of the renormalization procedure of the system of composite operators (18) is the fact that not only the operators with different values of N do not mix during the renormalization but also the corresponding matrices of renormalization constants $Z_{[N,p][N,p']}$ for all values of N are triangular. As a result, the anomalous dimensions $\gamma_{N,p}$ of the basic composite operators (18) are determined directly by the diagonal elements of the matrix $Z_{[N,p][N,p']}$, i.e., by the elements $Z_{N,p} \equiv Z_{[N,p][N,p]}$, by the following relation:

$$\gamma_{N,p} = \mu\partial_\mu \ln Z_{N,p}. \tag{19}$$

At the same time, using the general relation between the critical dimensions $\Delta_{N,p}$ and the anomalous dimensions $\gamma_{N,p}$ [26,34,36,40,61], one can come to the conclusion that, in our case,

$$\Delta_{N,p} = \gamma^*_{N,p}, \tag{20}$$

i.e., the critical dimensions of the composite operators (18) are equal to their anomalous dimensions taken at the corresponding fixed point of the studied scaling regime (see the previous section).

At the two-loop level of approximation, the anomalous dimensions $\gamma_{N,p}$ are given by the corresponding UV renormalization analysis of the one- and two-loop Feynman diagrams shown in Fig. 2. After the corresponding calculations, the anomalous dimensions $\gamma_{N,p}$ taken at the fixed point can be

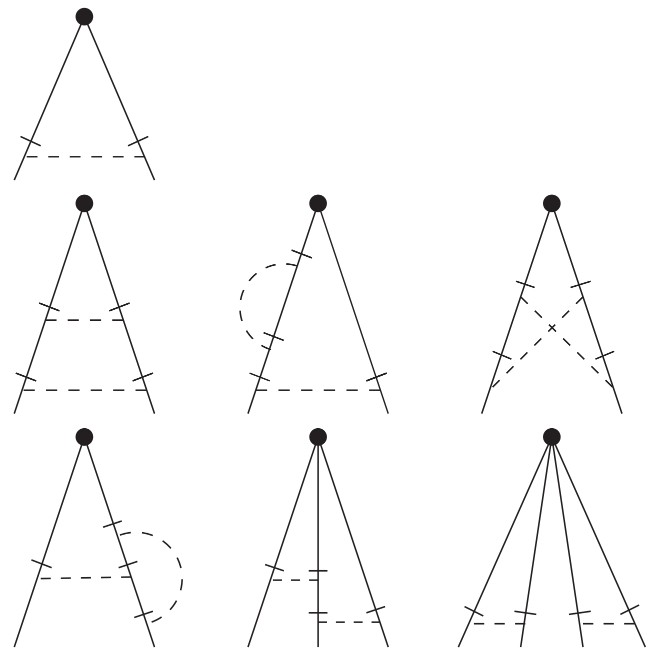

FIG. 2. Relevant one- and two-loop Feynamn diagrams for the UV renormalization of the composite operators (18). The black circle in each diagram represents vertex related to the composite operator $F_{N,p}$ (see, e.g., Ref. [40] for details), and the graphical representation of all other Feynman rules is given in Fig. 1.

written in the following form with the explicit dependence on the helicity parameter ρ:

$$\gamma^*_{N,p} = \gamma^{*(1)}_{N,p}\varepsilon + \left[\gamma^{*(2)}_{N,p} + \gamma^{*(2)\rho}_{N,p}\right]\varepsilon^2 + O(\varepsilon^3), \tag{21}$$

where the explicit expression for the one-loop contribution

$$\gamma^{*(1)}_{N,p} = \frac{2N(N-1) - (N-p)(d+N+p-2)(d+1)}{2(d+2)(d-1)} \tag{22}$$

is known for a long time (see, e.g., Ref. [61]) and the nonhelical two-loop contribution $\gamma^{*(2)}_{N,p}$ to the total anomalous dimensions $\gamma^*_{N,p}$ was calculated recently in Ref. [40]. Since its explicit expression is enormous we will not present it here (it can be found in Ref. [40]). On the other hand, the main aim of this paper is to determine the two-loop helical contribution $\gamma^{*(2)\rho}_{N,p}$, which is present only in the system with the breaking of the spatial parity. Our calculations show that this contribution has the following explicit form:

$$\gamma^{*(2)\rho}_{N,p} = \frac{\rho^2\delta_{3d}(d-2)d[(d+1)k_1 - 2k_2]}{(d+2)(d-1)^3(1+u_*)}\frac{\Gamma\left(\frac{d}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{d-1}{2}\right)} \times \int_0^1 dx(1-x^2)^{\frac{d-1}{2}}\left\{\frac{(d-5)(1+u_*)K_1}{d-1} + \frac{A_1K_1 + A_2K_2 + A_3K_3}{u_*(1-u_*)[1+u_*^2+2u_*(2x^2-1)]}\right\}, \tag{23}$$

where

$$k_1 = (N-p)(d+N+p-2), \tag{24}$$

$$k_2 = N(N-1), \tag{25}$$

$$A_1 = [(u_*+1)^2(u_*+2) - 4u_*x^2] \times [4u_*x^2 + (u_*-1)^2], \tag{26}$$

$$A_2 = 2(u_*-1)(u_*+1)^2[4u_*x^2 + (u_*-1)^2] \times [u_*(u_*+4x^2-1)+1], \tag{27}$$

$$A_3 = -2u_*^2\{u_*[u_*^2 + (3u_*+8)x^2 + u_* - 4x^4 - 1] + x^2 - 1\}, \tag{28}$$

and

$$K_1 = \frac{1}{\sqrt{u_*^2+2u_*-x^2+1}} \times \left[\pi - \arctan\left(\frac{u_*-x+1}{\sqrt{u_*^2+2u_*-x^2+1}}\right) - \arctan\left(\frac{u_*+x+1}{\sqrt{u_*^2+2u_*-x^2+1}}\right)\right], \tag{29}$$

$$K_2 = \frac{\pi - \arctan\left(\frac{1-x}{\sqrt{1-x^2}}\right) - \arctan\left(\frac{x+1}{\sqrt{1-x^2}}\right)}{\sqrt{1-x^2}[1+u_*^2+2u_*(2x^2-1)]}, \tag{30}$$

$$K_3 = \frac{Y_1+Y_2}{\sqrt{2u_*-x^2+2}}, \tag{31}$$

where

$$Y_1 = \pi - \arctan\left(\frac{2-x}{\sqrt{2u_*-x^2+2}}\right) - \arctan\left(\frac{x+2}{\sqrt{2u_*-x^2+2}}\right), \tag{32}$$

$$Y_2 = \pi - \arctan\left(\frac{u_*-x+1}{\sqrt{2u_*-x^2+2}}\right) - \arctan\left(\frac{u_*+x+1}{\sqrt{2u_*-x^2+2}}\right). \tag{33}$$

Before we analyze the behavior of the anomalous dimensions $\gamma^*_{N,p}$, which directly define the critical dimensions of the composite operators (18) through the relation (20), let us note that in the rapid-change limit ($u_* \to \infty$), i.e., in the framework of the Kazantsev-Kraichnan model with the presence of the helicity at $d = 3$, one comes to the expression for $\gamma^{*(2)}_{N,p}$ found and present in Ref. [58].

To proceed with the investigation of the simultaneous influence of the finite-time correlations of the velocity field and of the spatial parity violation of the conductive turbulent environment on the anomalous scaling in the framework of the studied model with the presence of the large-scale anisotropy, it is necessary first to identify which of various anisotropic contributions $\gamma^*_{N,p}$ with a given value of N and various values of p is the smallest and therefore plays the leading role for the determination of the scaling properties of various phenomenologically interesting quantities, e.g., of the single-time two-point correlation functions of the magnetic field shown in Eq. (15). In this respect, the validity of various hierarchy relations among anomalous dimensions $\gamma^*_{N,p}$ can help, and it is known that the following relations are valid in the framework of the nonhelical Kazantsev-Kraichnan model with the finite-time correlations of the velocity field [40]:

$$\gamma^*_{N,p} < \gamma^*_{N,p'}, \quad p < p', \tag{34}$$

$$\gamma^*_{N,0} < \gamma^*_{N',0}, \quad N > N', \tag{35}$$

$$\gamma^*_{N,1} < \gamma^*_{N',1}, \quad N > N', \tag{36}$$

where relation (35) holds for even values of N and N' and relation (36) is valid for odd values of N and N', respectively. Direct calculations show that, at least at the studied two-loop level of approximation, all these relations remain valid when the spatial parity violation of the turbulent environment is considered. It means that, at least up to the two-loop approximation, the scaling behavior of various statistical quantities deep inside the inertial interval is driven by the anomalous dimensions $\gamma^*_{N,0}$ (for even values of N) and $\gamma^*_{N,1}$ (for odd values of N), respectively, even in the helical Kazantsev-Kraichnan model with finite-time correlations of the velocity field. Note that this behavior is in accordance with the Kolmogorov's local isotropy restoration hypothesis.

The explicit simultaneous dependence of the anomalous dimensions $\gamma^*_{N,0}$ for $N = 2, 4, 6$, and 8 and $\gamma^*_{N,1}$ for $N = 3, 5, 7$, and 9 on the parameter u_*, which controls the presence of finite-time correlations of the turbulent velocity field, and on the absolute value of the parameter ρ, which controls the amount of the spatial parity violation, is shown in Figs. 3–10. As follows from all these figures, regardless of the value of u_* (i.e., regardless of the strength of the finite-time correlations of the velocity field), the presence of the spatial parity violation in the conductive turbulent environment always decreases the fixed point values of the leading anomalous dimensions $\gamma^*_{N,0}$ and $\gamma^*_{N,1}$, respectively. Therefore, one can also expect that the anomalous scaling of various correlation functions of the fluctuating part of the magnetic field will be more pronounced in the helical environments than in the system without the presence of spatial parity violation. At the same time, the most anomalous behavior of the model is expected in the systems with the maximal spatial parity violation. The fact that such behavior really takes place is demonstrated in the next section.

V. SIMULTANEOUS INFLUENCE OF HELICITY AND FINITE-TIME VELOCITY FIELD CORRELATIONS ON THE SCALING BEHAVIOR OF MAGNETIC FIELD CORRELATION FUNCTIONS

The existence of strict hierarchy relations (34)–(36), which are valid in the one-loop as well as in the two-loop approximation of the studied model, leads to the definite prediction of the asymptotic inertial-range behavior of the correlation functions (15) in the presence of the large-scale uniaxial anisotropy. Its final form depends on the values of N and m [26,34,36],

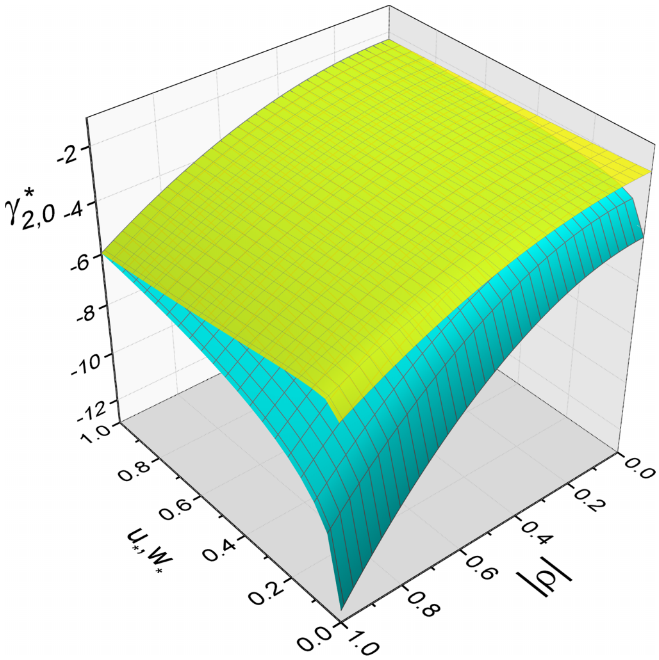

FIG. 3. The explicit dependence of the total two-loop anomalous dimensions $\gamma^*_{2,0}$ on the helicity parameter ρ as well as on the parameter $u_* \in [0, 1]$ (the lower surface) and $w_* \equiv 1/u_* \in [0, 1]$ (the upper surface) for $d = 3$ and $\varepsilon = 4/3$.

namely,

$$B_{N-m,m}(r) \sim r^{\gamma^*_{N,0}-\gamma^*_{N-m,0}-\gamma^*_{m,0}}, \qquad (37)$$

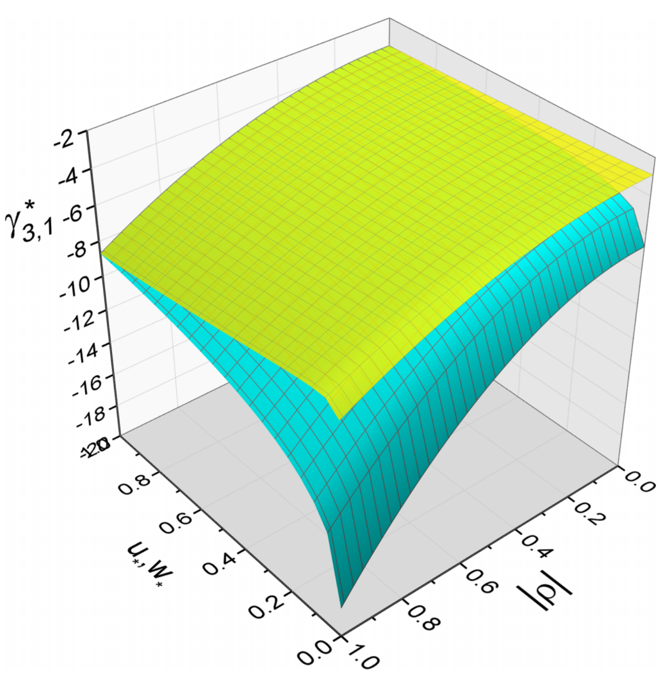

FIG. 4. Explicit dependence of the total two-loop anomalous dimensions $\gamma^*_{3,1}$ on the helicity parameter ρ as well as on the parameter $u_* \in [0, 1]$ (the lower surface) and $w_* \equiv 1/u_* \in [0, 1]$ (the upper surface) for $d = 3$ and $\varepsilon = 4/3$.

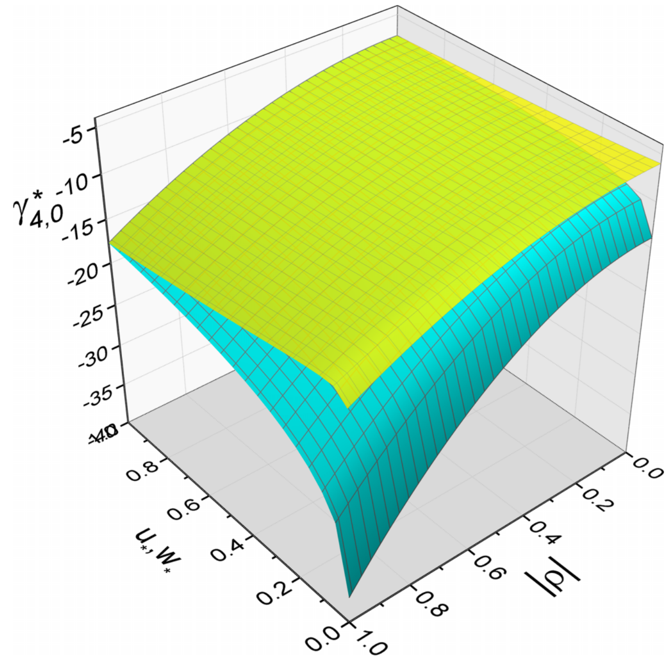

FIG. 5. Explicit dependence of the total two-loop anomalous dimensions $\gamma^*_{4,0}$ on the helicity parameter ρ as well as on the parameter $u_* \in [0, 1]$ (the lower surface) and $w_* \equiv 1/u_* \in [0, 1]$ (the upper surface) for $d = 3$ and $\varepsilon = 4/3$.

for even values of N and m,

$$B_{N-m,m}(r) \sim r^{\gamma^*_{N,0}-\gamma^*_{N-m,1}-\gamma^*_{m,1}}, \qquad (38)$$

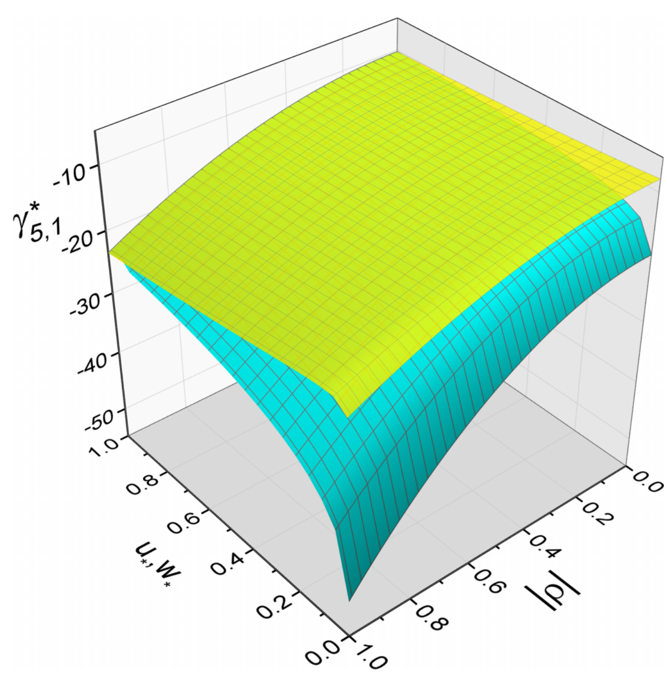

FIG. 6. Explicit dependence of the total two-loop anomalous dimensions $\gamma^*_{5,1}$ on the helicity parameter ρ as well as on the parameter $u_* \in [0, 1]$ (the lower surface) and $w_* \equiv 1/u_* \in [0, 1]$ (the upper surface) for $d = 3$ and $\varepsilon = 4/3$.

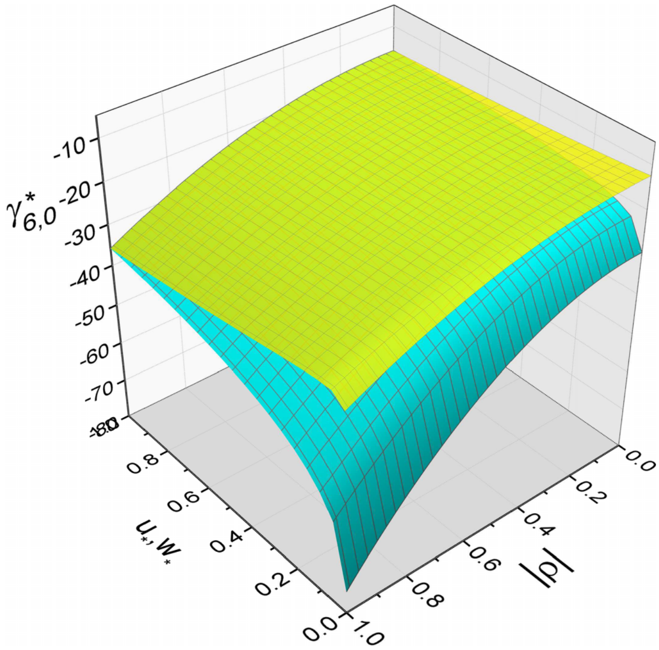

FIG. 7. Explicit dependence of the total two-loop anomalous dimensions $\gamma^*_{6,0}$ on the helicity parameter ρ as well as on the parameter $u_* \in [0, 1]$ (the lower surface) and $w_* \equiv 1/u_* \in [0, 1]$ (the upper surface) for $d = 3$ and $\varepsilon = 4/3$.

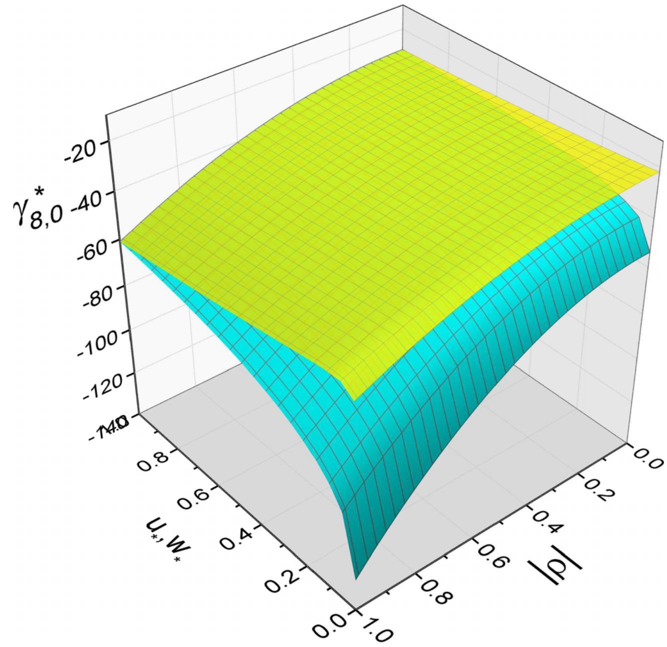

FIG. 9. Explicit dependence of the total two-loop anomalous dimensions $\gamma^*_{8,0}$ on the helicity parameter ρ as well as on the parameter $u_* \in [0, 1]$ (the lower surface) and $w_* \equiv 1/u_* \in [0, 1]$ (the upper surface) for $d = 3$ and $\varepsilon = 4/3$.

for even value of N and odd value of m, and

$$B_{N-m,m}(r) \sim r^{\gamma^*_{N,1}-\gamma^*_{N-m,0}-\gamma^*_{m,1}}, \tag{39}$$

for odd values of N and m. The fourth possibility with odd value of N and even value of m is in fact contained in the last case.

Now, using the explicit expressions (21)–(33) for the anomalous dimensions taken at the general fixed point (13)

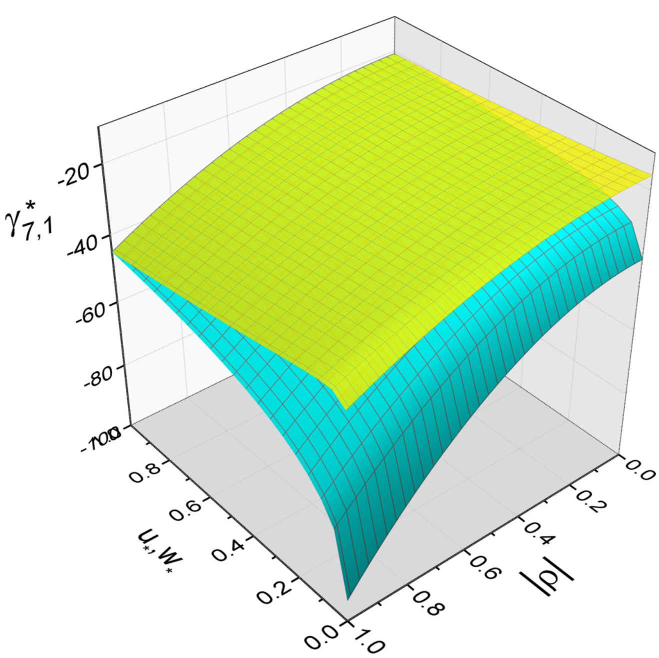

FIG. 8. Explicit dependence of the total two-loop anomalous dimensions $\gamma^*_{7,1}$ on the helicity parameter ρ as well as on the parameter $u_* \in [0, 1]$ (the lower surface) and $w_* \equiv 1/u_* \in [0, 1]$ (the upper surface) for $d = 3$ and $\varepsilon = 4/3$.

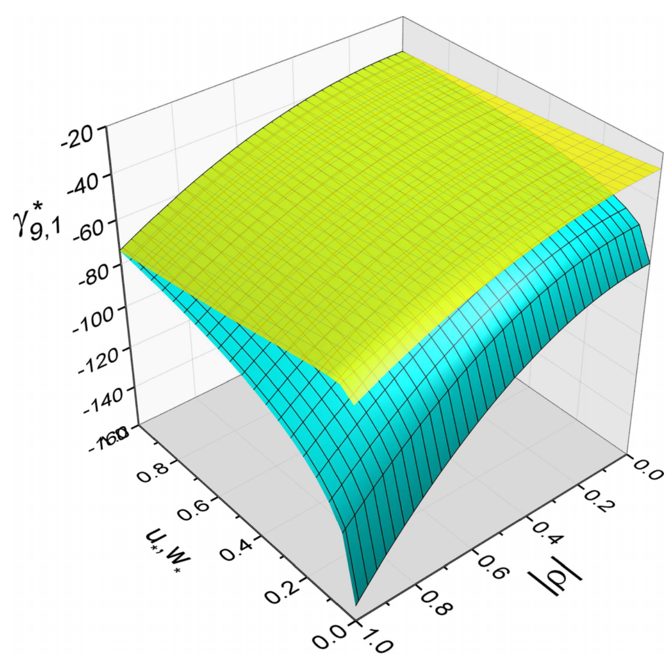

FIG. 10. Explicit dependence of the total two-loop anomalous dimensions $\gamma^*_{9,1}$ on the helicity parameter ρ as well as on the parameter $u_* \in [0, 1]$ (the lower surface) and $w_* \equiv 1/u_* \in [0, 1]$ (the upper surface) for $d = 3$ and $\varepsilon = 4/3$.

of the model, the final scaling asymptotic behavior of the correlation functions $B_{N-m,m}(r)$ in the two-loop approximation can be written in the following general form:

$$B_{N-m,m}(r) \sim r^{\zeta_{N,m}} = r^{\zeta^{(1)}_{N,m}\varepsilon + [\zeta^{(2)}_{N,m} + \zeta^{(2)\rho}_{N,m}]\varepsilon^2}, \tag{40}$$

where the well-known one-loop result $\zeta^{(1)}_{N,m}$ has the form [26,36]

$$\zeta^{(1)}_{N,m} = -\frac{m(N-m)}{d+2}, \tag{41}$$

when N and m are simultaneously even or odd and

$$\zeta^{(1)}_{N,m} = -\frac{m(N-m)+d+1}{d+2}, \tag{42}$$

for even values of N and odd values of m. At the same time, the nonhelical two-loop corrections $\zeta^{(2)}_{N,m}$ to the exponents $\zeta_{N,m}$ in Eq. (40) can be found in Ref. [41]. On the other hand, the determination of the two-loop helical corrections $\zeta^{(2)\rho}_{N,m}$ to the exponents $\zeta_{N,m}$ in Eq. (40) represent one of the main results of this paper and have the following explicit form:

$$\zeta^{(2)\rho}_{N,m} = \frac{\rho^2 \delta_{3d}(d-2)dD_1}{(d+2)(d-1)^2(1+u_*)} \frac{\Gamma\left(\frac{d}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{d-1}{2}\right)} \times \int_0^1 dx (1-x^2)^{\frac{d-1}{2}} \left\{ \frac{(d-5)(1+u_*)K_1}{d-1} + \frac{A_1K_1 + A_2K_2 + A_3K_3}{u_*(1-u_*)[1+u_*^2+2u_*(2x^2-1)]} \right\}, \tag{43}$$

where the functions A_i and K_i for $i = 1, 2, 3$ are given in Eqs. (26)–(31) and

$$D_1 = 2m(N-m), \tag{44}$$

when both N and m are simultaneously even or odd and

$$D_1 = 2[m(N-m)+d+1], \tag{45}$$

for even N and odd m.

As follows from the explicit two-loop expressions for the exponents $\zeta_{N,m}$, the presence of the two-loop corrections leads to their explicit dependence on the spatial parity violation of the turbulent environment through the parameter ρ. Note once more that this nontrivial fact is completely invisible at the one-loop level of approximation sice the corresponding coefficients $\zeta^{(1)}_{N,m}$ in Eqs. (41) and (42) are independent of ρ. Moreover, the same is true for the finite-time correlations of the turbulent velocity field. Again, the scaling exponents of the single-time two-point correlation functions of the passively advected magnetic field are independent of the parameter u_* at the one-loop level of approximation. Thus, at least the two-loop approximation (studied in the present paper) is needed to be able to investigate the simultaneous influence of the finite-time correlations and the spatial parity violation of the turbulent velocity field on the scaling properties of the correlations of the studied magnetic field.

At the same time, note that the existence of a nontrivial dependence of the scaling exponents $\zeta_{N,m}$ of the correlation functions of the magnetic field (15) on the helicity parameter ρ in the two-loop approximation in the framework of the studied generalized Kazantsev-Kraichnan model also demonstrates a fundamental difference between passive advection processes of scalar and vector fields in turbulent environments. This conclusion follows from the fact that, in contrast to the magnetic field advection studied in this paper, the two-loop field theoretic RG calculations show that the scaling properties of single-time two-point correlation or structure functions of a passively advected scalar field are independent of the presence of the spatial parity violation of the turbulent environment even when the presence of finite-time correlations of the turbulent velocity field is assumed, i.e., in the framework of the generalized Kraichnan model [56]. Moreover, as was shown in Ref. [57], it seems that the scaling properties of various correlation or structure functions of the passively advected scalar field are independent of the spatial parity violation of the turbulent environment not only when the Gaussian statistic of the velocity field is assumed but even in the case when the turbulent velocity field is driven by the stochastic Navier-Stokes equation.

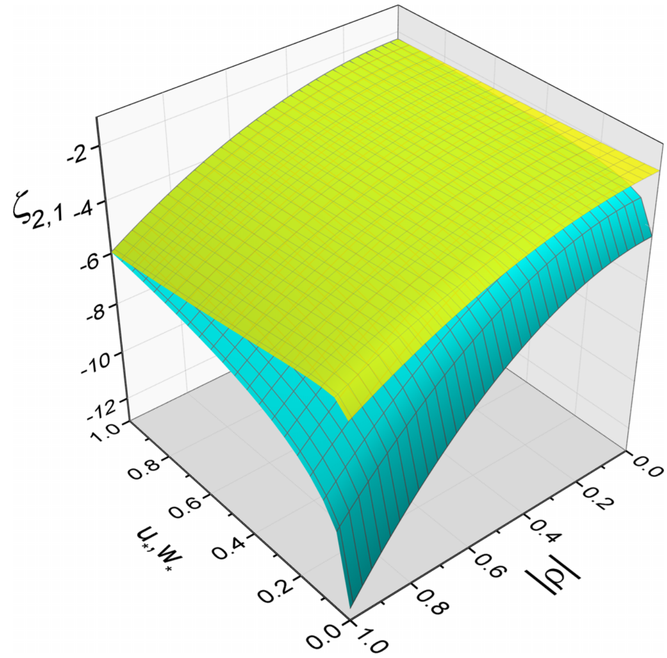

FIG. 11. Dependence of the scaling exponents $\zeta_{2,1}$ on the absolute value of the helicity parameter ρ as well as on the parameter $u_* \in [0, 1]$ (the lower surface) and $w_* \equiv 1/u_* \in [0, 1]$ (the upper surface) for $d = 3$ and $\varepsilon = 4/3$.

The explicit dependence of all scaling exponents $\zeta_{N,m}$ for $N = 2, \ldots, 7$ of the single-time two-point correlation functions of the passively advected magnetic field given in Eq. (15) on the absolute value of the helicity parameter ρ and on the parameter u_*, which controls the presence of the finite-time velocity correlations, in the studied two-loop approximation is shown in Figs. 11–22 for the spatial dimension $d = 3$ (the only spatial dimension for which the helical contribution can be considered) and for the physically most important value of the exponent ε, i.e., for $\varepsilon = 4/3$. As follows from all these figures, regardless of the value of the parameter u_*, the presence of helicity always leads to more negative values of the scaling exponents $\zeta_{N,m}$, i.e., to the amplification of

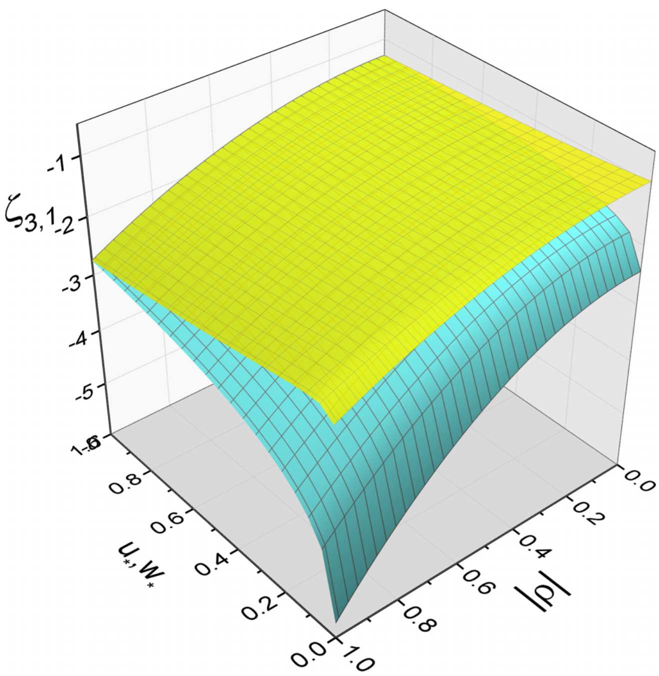

FIG. 12. Dependence of the scaling exponents $\zeta_{3,1}$ on the absolute value of the helicity parameter ρ as well as on the parameter $u_* \in [0, 1]$ (the lower surface) and $w_* \equiv 1/u_* \in [0, 1]$ (the upper surface) for $d = 3$ and $\varepsilon = 4/3$.

the anomalous scaling of the studied correlation functions of the magnetic field. At the same time, regardless of the absolute value of the helicity parameter ρ, the scaling exponents $\zeta_{N,m}$ always decrease when the parameter $u_* \in [0, 1]$

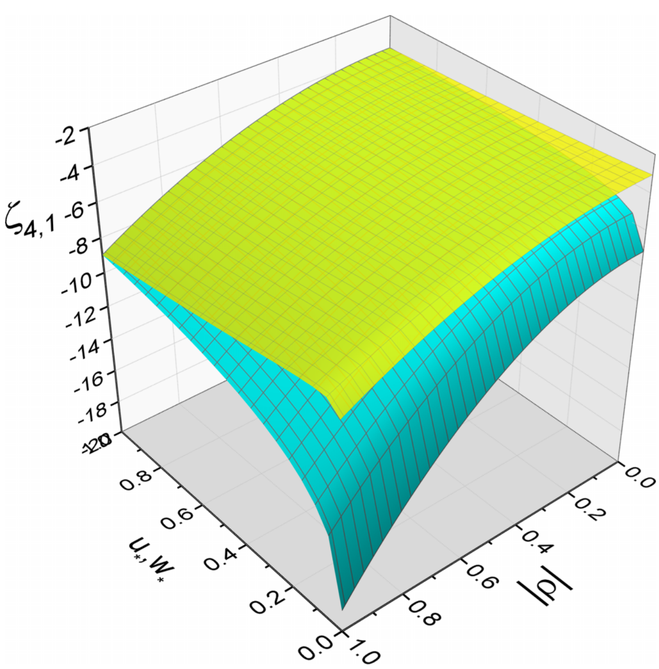

FIG. 13. Dependence of the scaling exponents $\zeta_{4,1}$ on the absolute value of the helicity parameter ρ as well as on the parameter $u_* \in [0, 1]$ (the lower surface) and $w_* \equiv 1/u_* \in [0, 1]$ (the upper surface) for $d = 3$ and $\varepsilon = 4/3$.

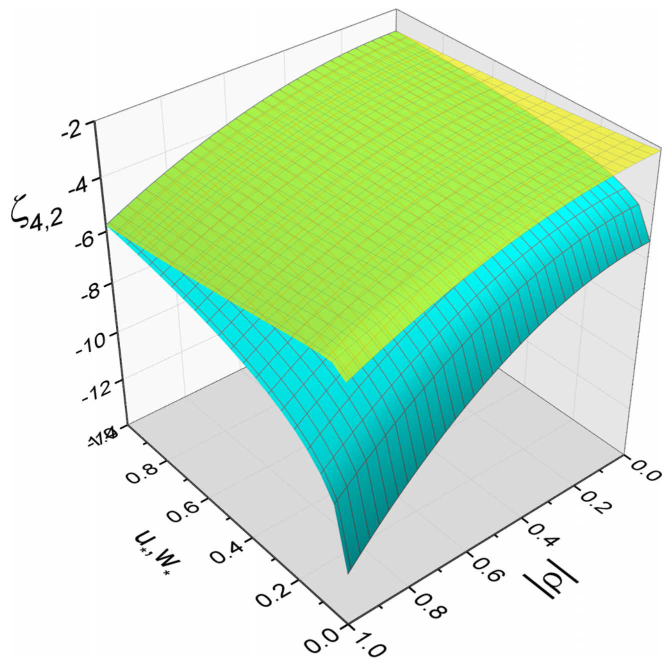

FIG. 14. Dependence of the scaling exponents $\zeta_{4,2}$ on the absolute value of the helicity parameter ρ as well as on the parameter $u_* \in [0, 1]$ (the lower surface) and $w_* \equiv 1/u_* \in [0, 1]$ (the upper surface) for $d = 3$ and $\varepsilon = 4/3$.

decreases. On the other hand, a similar behavior of the scaling exponents $\zeta_{N,m}$ is observed when the parameter $w_* = 1/u_* \in [0, 1]$ increases but only for small enough values of the parameter ρ, i.e., for $|\rho| \ll 1$. When the absolute value

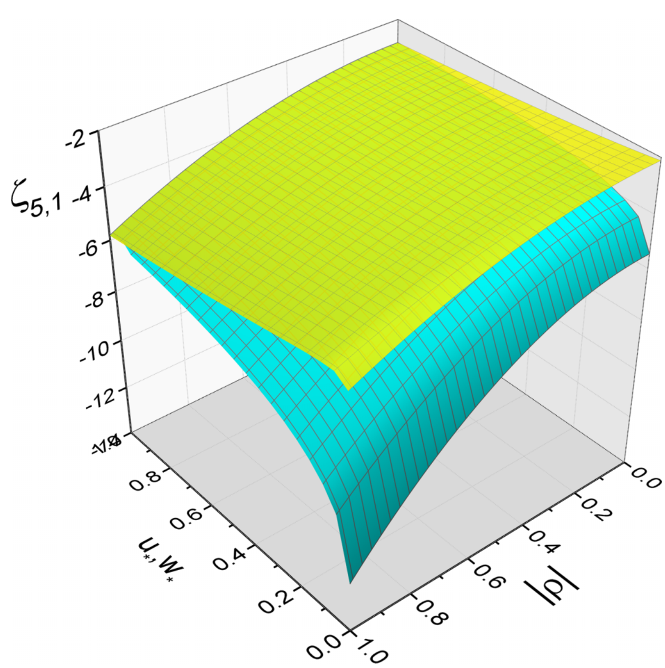

FIG. 15. Dependence of the scaling exponents $\zeta_{5,1}$ on the absolute value of the helicity parameter ρ as well as on the parameter $u_* \in [0, 1]$ (the lower surface) and $w_* \equiv 1/u_* \in [0, 1]$ (the upper surface) for $d = 3$ and $\varepsilon = 4/3$.

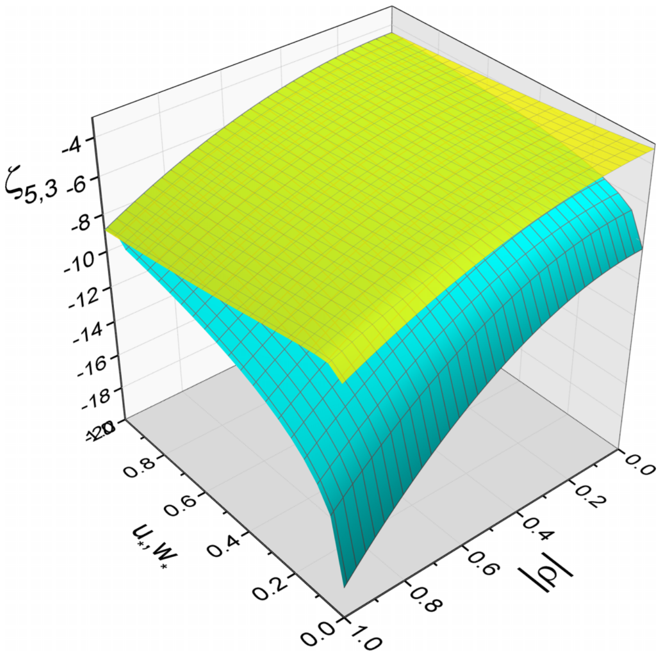

FIG. 16. Dependence of the scaling exponents $\zeta_{5,3}$ on the absolute value of the helicity parameter ρ as well as on the parameter $u_* \in [0, 1]$ (the lower surface) and $w_* \equiv 1/u_* \in [0, 1]$ (the upper surface) for $d = 3$ and $\varepsilon = 4/3$.

of the helicity parameter ρ is large enough, the anomalous scaling of the studied correlation functions of the magnetic field becomes more pronounced for the rapid-change limit of the model ($w_* = 0$) than for the model with the presence of small but nonzero time-correlated turbulent velocity

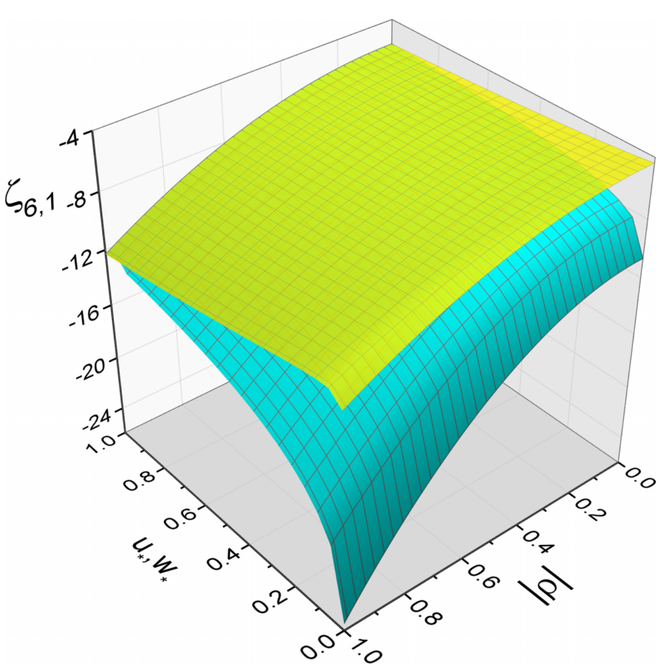

FIG. 17. Dependence of the scaling exponents $\zeta_{6,1}$ on the absolute value of the helicity parameter ρ as well as on the parameter $u_* \in [0, 1]$ (the lower surface) and $w_* \equiv 1/u_* \in [0, 1]$ (the upper surface) for $d = 3$ and $\varepsilon = 4/3$.

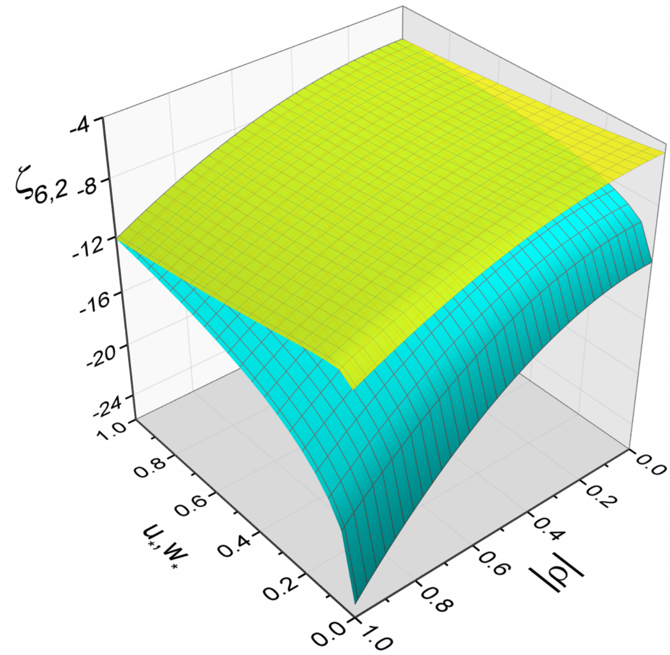

FIG. 18. Dependence of the scaling exponents $\zeta_{6,2}$ on the absolute value of the helicity parameter ρ as well as on the parameter $u_* \in [0, 1]$ (the lower surface) and $w_* \equiv 1/u_* \in [0, 1]$ (the upper surface) for $d = 3$ and $\varepsilon = 4/3$.

field. However, this behavior is valid only for very small values of the parameter $w_* \ll 1$ (see Figs. 11–22) and in fact may just be an artifact of the studied two-loop approximation. Therefore, despite this fact, a general conclusion that follows from our analysis is that the electrically conduc-

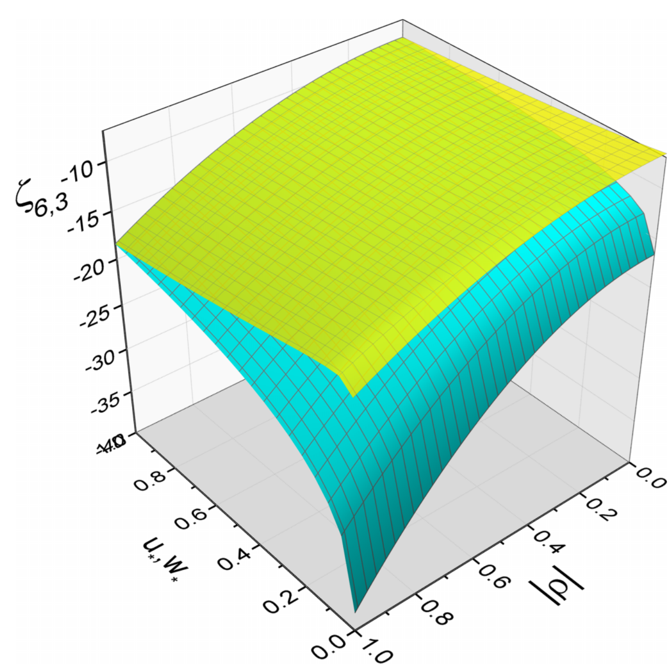

FIG. 19. Dependence of the scaling exponents $\zeta_{6,3}$ on the absolute value of the helicity parameter ρ as well as on the parameter $u_* \in [0, 1]$ (the lower surface) and $w_* \equiv 1/u_* \in [0, 1]$ (the upper surface) for $d = 3$ and $\varepsilon = 4/3$.

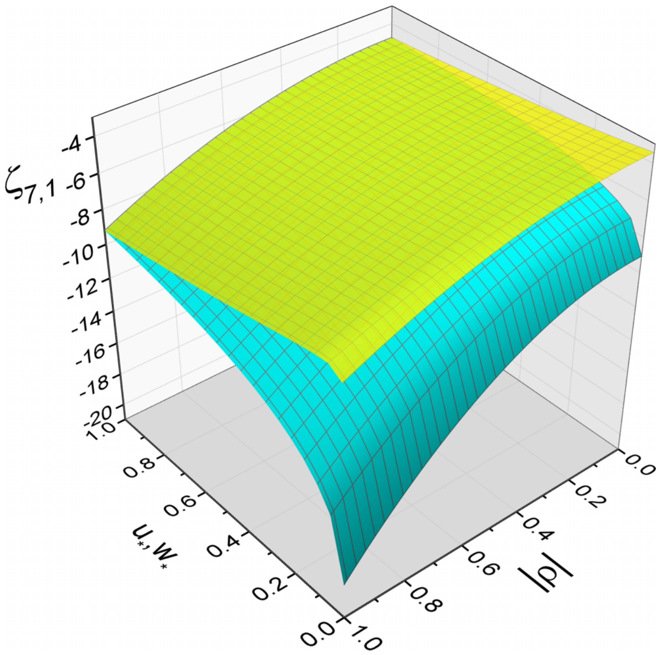

FIG. 20. Dependence of the scaling exponents $\zeta_{7,1}$ on the absolute value of the helicity parameter ρ as well as on the parameter $u_* \in [0, 1]$ (the lower surface) and $w_* \equiv 1/u_* \in [0, 1]$ (the upper surface) for $d = 3$ and $\varepsilon = 4/3$.

tive turbulent environment with the spatial parity violation leads to significant amplification of the anomalous scaling behavior of the correlation functions of the magnetic field even when this turbulent environment is time correlated, i.e.,

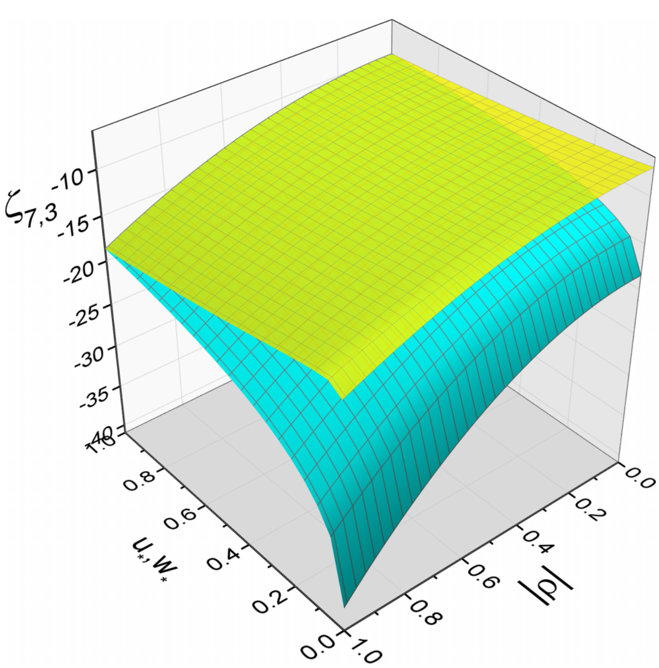

FIG. 21. Dependence of the scaling exponents $\zeta_{7,3}$ on the absolute value of the helicity parameter ρ as well as on the parameter $u_* \in [0, 1]$ (the lower surface) and $w_* \equiv 1/u_* \in [0, 1]$ (the upper surface) for $d = 3$ and $\varepsilon = 4/3$.

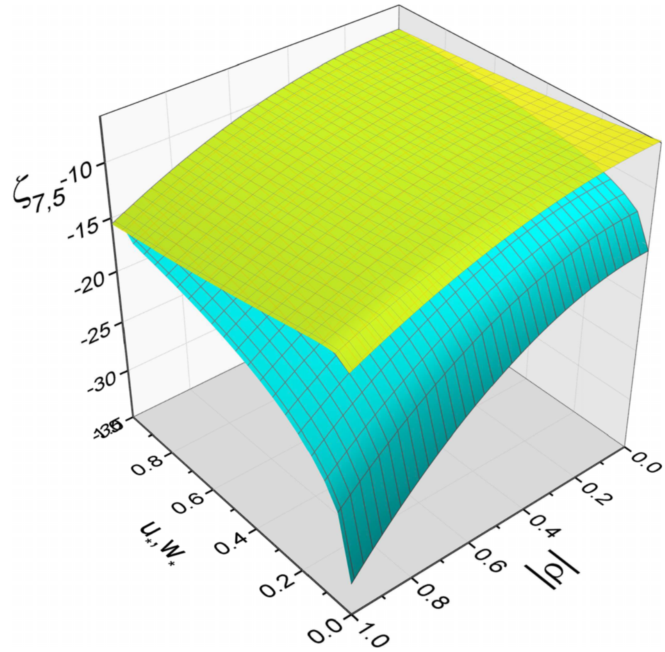

FIG. 22. Dependence of the scaling exponents $\zeta_{7,5}$ on the absolute value of the helicity parameter ρ as well as on the parameter $u_* \in [0, 1]$ (the lower surface) and $w_* \equiv 1/u_* \in [0, 1]$ (the upper surface) for $d = 3$ and $\varepsilon = 4/3$.

with the presence of finite-time correlations of the velocity field.

VI. CONCLUSION

To conclude, let us summarize briefly the main results obtained in the present study. In this paper, the simultaneous influence of the spatial parity violation and the finite-time correlations of the electrically conductive turbulent environment on the inertial-range scaling properties of the correlation functions of the weak magnetic field is investigated in the framework of the kinematic MHD turbulence described by the corresponding generalized Kazantsev-Kraichnan model. This influence is investigated using the field theoretic RG technique together with the OPE in the two-loop approximation. The two-loop anomalous dimensions of the leading composite operators in the OPE, which drive the scaling behavior of the correlation functions of passively advected magnetic field, are calculated as the explicit functions of the parameters that control the amount of the spatial parity violation as well as of the form of the finite-time correlations in the studied turbulent system. It is shown that, in accordance with the Kolmogorov's hypothesis about the local isotropy restoration, the anomalous dimensions of the relevant composite operators of the model always obey the well-defined anisotropy hierarchies with the most negative values of the anomalous dimensions given by the composite operators from the vicinity of the isotropic shell, regardless of the amount of the spatial parity violation in the system. It means that, namely, the anomalous dimensions of the composite operators near the isotropy shell will drive the asymptotic inertial-range scaling behavior of the correlation functions of the passive magnetic field. In this respect, the properties of the two-loop anomalous dimensions $\gamma^*_{N,0}$ (for even values of N) and $\gamma^*_{N,1}$ (for odd values of N) of the composite operators (18) are investigated in detail for various values of N up to $N = 9$. Their explicit dependence on the parameters that control the presence of the spatial parity violation and of the finite-time correlations in the studied turbulent system is shown in Figs. 3–10. The obtained results show that, regardless of the form of the finite-time correlations of the velocity field in the conductive turbulent environment, the presence of the spatial parity violation always leads to the more negative values of the leading anomalous dimensions $\gamma^*_{N,0}$ and $\gamma^*_{N,1}$. Therefore, it is also natural to expect that the anomalous scaling of various correlation functions of the fluctuating part of the magnetic field have to be more pronounced in the helical environments than in the system without the presence of the spatial parity violation with the most anomalous behavior of the model in the system with the maximal spatial parity violation.

This expectation is confirmed in Sec. V, where the asymptotic inertial-range behavior of the single-time two-point correlation functions of the magnetic field is investigated in detail. The explicit dependence of the corresponding two-loop scaling exponents $\zeta_{N,m}$ [see Eq. (40)] on the parameter that control the presence of the spatial parity violation in the system is found and discussed in detail for all correlation functions up to $N = 7$ (see Figs. 11–22). As follows from all these figures, the scaling properties of the correlation functions of passively advected weak magnetic field in the studied Gaussian turbulent environment strongly depend on the amount of the spatial parity violation in the system, regardless of the form of the finite-time correlations of the velocity field.

Although the results of the present paper are obtained in the framework of a Gaussian model of the kinematic MHD turbulence, nevertheless we suppose that a similar behavior of the correlation functions of the magnetic field in the helical turbulent environment will also be valid in the framework of the real kinematic MHD turbulence, i.e., in the case when the weak magnetic field is advected by the turbulent velocity field described by the stochastic Navier-Stokes equation. This problem is, however, much more complicated from the mathematical point of view and will be studied elsewhere.

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[1] A. S. Monin and A. M. Yaglom, *Statistical Fluid Mechanics*, Vol. 2 (MIT Press, Cambridge, MA, 1975).
[2] W. D. McComb, *The Physics of Fluid Turbulence* (Clarendon Press, Oxford, 1990).
[3] U. Frisch, *Turbulence: The Legacy of A. N. Kolmogorov* (Cambridge University Press, Cambridge, 1995).
[4] K. R. Sreenivasan and R. A. Antonia, Annu. Rev. Fluid Mech. **29**, 435 (1997).
[5] L. Ts. Adzhemyan, N. V. Antonov, and A. N. Vasil'ev, *The Field Theoretic Renormalization Group in Fully Developed Turbulence* (Gordon & Breach, London, 1999).
[6] G. Falkovich, K. Gawędzki, and M. Vergassola, Rev. Mod. Phys. **73**, 913 (2001).
[7] A. Yoshizawa, S.-I. Itoh, and K. Itoh, *Plasma and Fluid Turbulence: Theory and Modelling* (Institute of Physics, Bristol and Philadelphia, 2003).
[8] D. Biskamp, *Magnetohydrodynamic Turbulence* (Cambridge University Press, Cambridge, 2003).
[9] N. V. Antonov, J. Phys. A **39**, 7825 (2006).
[10] Y. Zhou, Phys. Rep. **488**, 1 (2010).
[11] P. A. Davidson, *Turbulence* (Oxford University Press, Oxford, 2015).
[12] Y. Zhou, Phys. Rep. **935**, 1 (2021).
[13] A. N. Kolmogorov, Dokl. Akad. Nauk SSSR **30**, 301 (1941) [reprinted in Proc. R. Soc. London A **434**, 9 (1991)]; **31**, 538 (1941); **32**, 16 (1941) [reprinted in **434**, 15 (1991)].
[14] R. A. Antonia, B. R. Satyaprakash, and A. K. F. Hussain, J. Fluid Mech. **119**, 55 (1982).
[15] F. Anselmet, Y. Gagne, E. Hopfinger, and R. A. Antonia, J. Fluid Mech. **140**, 63 (1984).
[16] C. Meneveau and K. R. Sreenivasan, Phys. Rev. A **41**, 2246 (1990).
[17] V. R. Kuznetsov and V. A. Sabel'nikov, *Turbulence and Combustion* (Hemisphere Publications, New York, 1990).
[18] M. S. Borgas, Phys. Fluids A **4**, 2055 (1992).
[19] R. H. Kraichnan, Phys. Fluids **11**, 945 (1968).
[20] A. P. Kazantsev, Zh. Eksp. Teor. Fiz. **53**, 1806 (1968) [Sov. Phys. JETP **26**, 1031 (1968)].
[21] D. J. Amit, *Field Theory, Renormalization Group, and Critical Phenomena* (McGraw-Hill, New York, 1978).
[22] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena* (Clarendon Press, Oxford, 1989).
[23] A. N. Vasil'ev, *Quantum-Field Renormalization Group in the Theory of Critical Phenomena and Stochastic Dynamics* (Chapman & Hall/CRC, Boca Raton, FL, 2004).
[24] L. T. Adzhemyan, N. V. Antonov, and A. N. Vasil'ev, Phys. Rev. E **58**, 1823 (1998).
[25] N. V. Antonov, Phys. Rev. E **60**, 6691 (1999).
[26] N. V. Antonov, A. Lanotte, and A. Mazzino, Phys. Rev. E **61**, 6586 (2000).
[27] L. T. Adzhemyan, N. V. Antonov, V. A. Barinov, Yu. S. Kabrits, and A. N. Vasil'ev, Phys. Rev. E **63**, 025303(R) (2001); **64**, 056306 (2001).
[28] L. T. Adzhemyan, and N. V. Antonov, Phys. Rev. E **58**, 7381 (1998); N. V. Antonov and J. Honkonen, *ibid.* **63**, 036302 (2001).
[29] L. T. Adzhemyan, N. V. Antonov, and A. V. Runov, Phys. Rev. E **64**, 046310 (2001).
[30] L. T. Adzhemyan, N. V. Antonov, J. Honkonen, and T. L. Kim, Phys. Rev. E **71**, 016303 (2005).
[31] E. Jurčišinová, M. Jurčišin, and R. Remecký, Phys. Rev. E **80**, 046302 (2009).
[32] E. Jurčišinová, M. Jurčišin, and R. Remecký, J. Phys. A: Math. Theor. **42**, 275501 (2009).

[33] M. Hnatič, M. Jurčišin, A. Mazzino, and S. Šprinc, Acta Phys. Slov. **52**, 559 (2002); S. V. Novikov, J. Phys. A: Math. Gen. **39**, 8133 (2006); E. Jurčišinová, M. Jurčišin, R. Remecký, and M. Scholtz, Phys. Part. Nuclei Lett. **5**, 219 (2008); N. V. Antonov and N. M. Gulitskiy, Theor. Math Phys. **176**, 851 (2013); L. T. Adzhemyan, N. V. Antonov, P. B. Gol'din, and M. V. Kompaniets, J. Phys. A: Math. Theor. **46**, 135002 (2013); N. V. Antonov and N. M. Gulitskiy, Phys. Rev. E **91**, 013002 (2015); **92**, 043018 (2015); N. V. Antonov and M. M. Kostenko, *ibid.* **92**, 053013 (2015); N. V. Antonov and N. M. Gulitskiy, EPJ Web Conf. **108**, 02008 (2016).

[34] M. Hnatich, J. Honkonen, M. Jurčišin, A. Mazzino, and S. Šprinc, Phys. Rev. E **71**, 066312 (2005).

[35] E. Jurčišinová, M. Jurčišin, and R. Remecký, Phys. Rev. E **84**, 046311 (2011).

[36] E. Jurčišinová and M. Jurčišin, J. Phys. A: Math. Theor. **45**, 485501 (2012).

[37] N. V. Antonov and N. M. Gulitskiy, Phys. Rev. E **85**, 065301(R) (2012); Lecture Notes Comput. Sci. **7125**, 128 (2012).

[38] E. Jurčišinová and M. Jurčišin, Phys. Rev. E **88**, 011004(R) (2013).

[39] N. V. Antonov, N. M. Gulitskiy, M. M. Kostenko, and A. V. Malyshev, Phys. Rev. E **97**, 033101 (2018).

[40] E. Jurčišinová, M. Jurčišin, and M. Menkyna, Eur. Phys. J. B **91**, 313 (2018).

[41] E. Jurčišinová, M. Jurčišin, M. Menkyna, and R. Remecký, Phys. Rev. E **104**, 015101 (2021).

[42] E. Jurčišinová, M. Jurčišin, and R. Remecký, Phys. Rev. E **107**, 025106 (2023).

[43] P. D. Mininni, D. O. Gómez, and S. M. Mahajan, Astrophys. J. **619**, 1019 (2005).

[44] P. D. Mininni and A. Pouquet, Phys. Fluids **22**, 035106 (2010).

[45] R. Stepanov, P. Frick, and I. Mizeva, Astrophys. J. **798**, L35 (2015).

[46] O. G. Chkhetiani and E. B. Gledzer, Physica A **486**, 416 (2017).

[47] A. Brandenburg, T. Kahniashvili, S. Mandal, A. Roper Pol, A. G. Tevzadze, and T. Vachaspati, Phys. Rev. Fluids **4**, 024608 (2019).

[48] D. Dierkes, M. Oberlack, and A. Cheviakov, Phys. Fluids **32**, 053604 (2020).

[49] B. Deußen, D. Dierkes, and M. Oberlack, Phys. Fluids **32**, 065109 (2020).

[50] J.-Z. Zhu, Phys. Plasmas **28**, 032302 (2021).

[51] A. Pouquet and N. Yokoi, Philos. Trans. R. Soc. A **380**, 20210087 (2022).

[52] V. V. Titov, J. Appl. Mech. Tech. Phys. **63**, 1079 (2022).

[53] A. Armua, A. Berera, and J. Calderón-Figueroa, Phys. Rev. E **107**, 055206 (2023).

[54] *Helicities in Geophysics, Astrophysics, and Beyond*, edited by K. Kuzanyan, N. Yokoi, M. K. Georgoulis, and R. Stepanov (John Wiley & Sons, New York, 2024).

[55] E. Jurčišinová, M. Jurčišin, R. Remecký, and P. Zalom, Phys. Rev. E **87**, 043010 (2013).

[56] O. G. Chkhetiani, M. Hnatich, E. Jurčišinová, M. Jurčišin, A. Mazzino, and M. Repašan, Phys. Rev. E **74**, 036310 (2006); J. Phys. A: Math. Gen. **39**, 7913 (2006); Czech. J. Phys. **56**, 827 (2006).

[57] A. V. Gladyshev, E. Jurčišinová, M. Jurčišin, and R. Remecký, Phys. Part. Nuclei **41**, 1023 (2010).

[58] E. Jurčišinová and M. Jurčišin, Phys. Rev. E **91**, 063009 (2015).

[59] E. Jurčišinová, M. Jurčišin, and M. Menkyna, Phys. Rev. E **95**, 053210 (2017).

[60] L. T. Adzhemyan, N. V. Antonov, and J. Honkonen, Phys. Rev. E **66**, 036313 (2002).

[61] N. V. Antonov, M. Hnatich, J. Honkonen, and M. Jurčišin, Phys. Rev. E **68**, 046306 (2003).

[62] K. Gawędzki and A. Kupiainen, Phys. Rev. Lett. **75**, 3834 (1995); D. Bernard, K. Gawędzki, and A. Kupiainen, Phys. Rev. E **54**, 2564 (1996); M. Chertkov, G. Falkovich, I. Kolokolov, and V. Lebedev, *ibid.* **52**, 4924 (1995); M. Chertkov and G. Falkovich, Phys. Rev. Lett. **76**, 2706 (1996).

[63] Q. Zhang and J. Glimm, Commun. Math. Phys. **146**, 217 (1992).

[64] M. Chertkov and G. Falkovich, and V. Lebedev, Phys. Rev. Lett. **76**, 3707 (1996).

[65] G. Eyink, Phys. Rev. E **54**, 1497 (1996).

[66] J. P. Bouchaud, A. Comtet, A. Georges, and P. Le Doussal, J. Phys. (Paris) **48**, 1445 (1987); **49**, 369 (1988); J. P. Bouchaud and A. Georges, Phys. Rep. **195**, 127 (1990).

[67] J. Honkonen and E. Karjalainen, J. Phys. A **21**, 4217 (1988); J. Honkonen, Y. M. Pis'mak, and A. N. Vasil'ev, *ibid.* **21**, L835 (1988); J. Honkonen and Yu. M. Pis'mak, *ibid.* **22**, L899 (1989).

[68] E. Jurčišinová and M. Jurčišin, Phys. Part. Nuclei **44**, 360 (2013).

[69] P. C. Martin, E. D. Siggia, and H. A. Rose, Phys. Rev. A **8**, 423 (1973); C. De Dominicis, J. Phys. (Paris), Colloq. **37**, C1-247 (1976); H. K. Janssen, Z. Phys. B **23**, 377 (1976); R. Bausch, H. K. Janssen, and H. Wagner, *ibid.* **24**, 113 (1976).