Fiedler value: The cumulated dynamical contribution value of all edges in a complex network

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Fiedler value, as the minimal real part of (or the minimal) nonzero Laplacian eigenvalue, garners significant attention as a metric for evaluating network topology and its dynamics. In this paper, we address the quantification relation between Fiedler value and each edge in a directed complex network, considering undirected networks as a special case. We propose an approach to measure the dynamical contribution value of each edge. Interestingly, these contribution values can be both positive and negative, which are determined by the left and right Fiedler vectors. Further, we show that the cumulated dynamical contribution value of all edges is exactly the Fiedler value. This provides a promising angle on the Fiedler value in terms of dynamics and network structure. Therefore, the percentage of contribution of each edge to the Fiedler value is quantified. Numerical results reveal that network dynamics is significantly influenced by a small fraction of edges, say, one single directed edge contributes to over 90% of the Fiedler value in the Cat Cerebral Cortex network.

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I. INTRODUCTION

Complex dynamical networks, comprised of a collection of dynamical nodes and edges, display fascinating dynamics, such as synchronization and diffusion [1–12]. The Fiedler value [13,14], which measures synchronizability [1] and diffusion rate [2], indicates the ability of convergence of all nodes to a consistent state. A greater Fiedler value means better dynamical performance.

In an undirected network, several methods have been proposed to improve the Fiedler value by modifying the topology, including adding or removing edges [15,16], rewiring edges [17,18], and other ways [19–21]. Researchers have also designed optimal synchronized networks with a fixed number of nodes and edges [22,23]. Additionally, significant attention has been given to the study of superdiffusion, which is a phenomenon where the Fiedler value of a multiplex network exceeds that of each individual layer [24–27].

Different from undirected networks, edges may have negative effects on Fiedler value in directed networks. Researchers have primarily concentrated on studying the effect of adding reverse edges to some specific directed networks. For example, whether the diffusion rate (or consensus rate) of a directed chain or a stem is inhibited by an additional reverse edge [28–31], what is the necessary and sufficient condition of an interfering reverse edge in a directed acyclic network [14]. In the leader-follower interaction topology, the addition of edges to a strongly connected follower graph can be qualitatively analyzed using the right Fiedler vector [32,33].

However, in a general directed network, how to identify the positive or negative effect of edges and the quantifiable relation between Fiedler value and each edge is still a challenge. How much does each edge contribute to the Fiedler value? Which edges have a larger contribution? How do the position and direction of edges affect the Fiedler value? This paper systematically addresses these fundamental questions.

In Ref. [12], based on the sensitivity analysis of Fiedler value, we have put forward an importance index of cycle (edge) for an undirected network, which relies on the right Fiedler vector. In this paper, we extend the method to measure the dynamical contribution values of directed edges in a network. The left and right Fiedler vectors jointly determine the specific contribution values of all directed edges, with the directionality of the edge requiring the involvement of the left Fiedler vector. Further, we show that the Fiedler value is exactly equal to the cumulated dynamical contribution value of all edges. The undirected network is regarded as a special directed network, ultimately resulting in a unified conclusion. As a result, the quantitative relation between each edge and network dynamics is established. The contribution of edges to the dynamics in different networks is quantitatively analyzed by numerical examples. Interestingly, the"Pareto principle" is observed in the dynamical contribution of edges, with only a small portion of edges having a significant impact on the network.

II. DYNAMICAL CONTRIBUTION VALUE

Consider a directed connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, 2, ..., N\}$ is the set of nodes and $\mathcal{E} = \{e_{ij} = (i, j) \subseteq \mathcal{V} \times \mathcal{V}\}$ the set of edges with initial weight one. The network dynamics is described as

$$\dot{\xi}_i(t) = f(\xi_i(t)) - \sum_{j=1}^{N} L_{ij}\xi_j(t), i = 1, 2, ..., N,$$

where $\xi_i(t) \in \mathcal{R}$ is the state of the ith node, and $f \in \mathcal{C}[\mathcal{R}, \mathcal{R}]$ the smooth nonlinear node dynamics. The network dynamics is diffusion if $f = 0$. The Laplacian matrix is defined as

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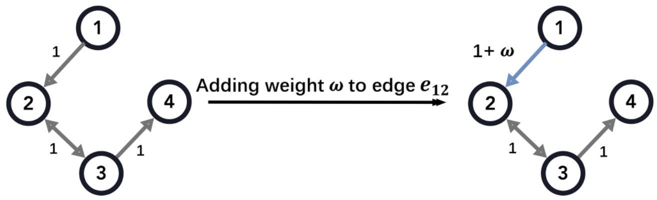


FIG. 1. Adding weight ω to e_{12} in a sample network with initial weight of one.

$L = D - A$, where $A_{ji} = 1$ if and only if there is a directed edge e_{ij} from node i to node j, otherwise $A_{ji} = 0$. D is a diagonal matrix with elements representing the sum of the corresponding rows of A. Provided with a network that contains a directed spanning tree, the real part of nonzero Laplacian eigenvalues are positive. Connectivity of a network refers to the Fiedler value, the minimal real part of the nonzero Laplacian eigenvalue, denoted as $Re[\lambda_2(L)]$. If $\lambda_2(L)$ is a complex value, there exists a conjugate eigenvalue $\lambda_3(L)$ that satisfies $Re[\lambda_2(L)] = Re[\lambda_3(L)]$. Without loss of generality, we define the eigenvalue with the nonnegative imaginary part as $\lambda_2(L)$. The corresponding normalized left and right eigenvectors are referred to as the left and right Fiedler vectors, respectively.

First, we introduce the contribution value of each edge in a directed network. The graph $\mathcal{G}$ after the addition of weight ω to edge e_{ij} is denoted as $\mathcal{G}^{\omega}_{e_{ij}}$. Consequently, the corresponding Laplacian matrix is modified to $L^{\omega}_{e_{ij}} = L(\mathcal{G}) + \omega L_{e_{ij}}$. For example, the Laplacian matrix after introducing weight ω to edge e_{12} in Fig. 1 is

$$L^{\omega}_{e_{12}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} + \omega \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

It is clear that the larger the absolute value of $I^d_{e_{ij}} = \frac{Re[\lambda_2(L^{\omega}_{e_{ij}})] - Re[\lambda_2(L)]}{\omega}$ $(\omega \ll 1)$, the more sensitive edge e_{ij} is, and the greater the positive or negative contribution of the edge to the Fiedler value. Defined as the dynamical contribution value of edge e_{ij}, $I^d_{e_{ij}}$ represents the initial slope of the Fiedler value with the edge weight variations. It provides insights into the dynamical contribution value of each edge.

Undirected graphs can be classified as a special case of directed graphs, characterized by bidirectional edges. The contribution value of an undirected edge e'_{ij} is defined as the sum of the contribution value of its corresponding directed edges e_{ij} and e_{ji}, denoted as $I^u_{e'_{ij}} = I^d_{e_{ij}} + I^d_{e_{ji}}$. Both e'_{ij} and e'_{ji} refer to the same edge. We denote $\mathcal{E}'$ as the set of all undirected edges in an undirected network.

III. MAIN RESULTS

In this section, we derive a concise expression of the contribution value and establish the relation between the edges and the Fiedler value.

A. Directed networks

The perturbation theory [34] is used to calculate $I^d_{e_{ij}}$. With $\omega \ll 1$, $\lambda_2(L^{\omega}_{e_{ij}})$ and the corresponding right eigenvector $\mathbf{x}_2(L^{\omega}_{e_{ij}})$ are expressed as

$$\lambda_2(L^{\omega}_{e_{ij}}) = \lambda_2(L) + k_1\omega + O(\omega^2), \tag{1}$$

$$\mathbf{x}_2(L^{\omega}_{e_{ij}}) = \mathbf{x}_2(L) + \mathbf{t}_1\omega + O(\omega^2), \tag{2}$$

where k_1 and $\mathbf{t}_1$ represent some constant and constant vector, respectively. Due to that, $L^{\omega}_{e_{ij}} = L + \omega L_{e_{ij}}$ and

$$L^{\omega}_{e_{ij}}\mathbf{x}_2(L^{\omega}_{e_{ij}}) = \lambda_2(L^{\omega}_{e_{ij}})\mathbf{x}_2(L^{\omega}_{e_{ij}}), \tag{3}$$

and substituting Eq. (1) and Eq. (2) into Eq. (3), one gets

$$L\mathbf{t_1} + L_{e_{ij}}\mathbf{x}_2(L) = \lambda_2(L)\mathbf{t_1} + k_1\mathbf{x}_2(L). \tag{4}$$

Left multiply Eq. (4) by left eigenvector $\mathbf{y}_2^T(L)$, where $\mathbf{y}_2^T(L)L = \lambda_2(L)\mathbf{y}_2^T(L)$, one has

$$k_1 = \frac{\mathbf{y}_2^T(L)L_{e_{ij}}\mathbf{x}_2(L)}{\mathbf{y}_2^T(L)\mathbf{x}_2(L)}. \tag{5}$$

Setting $\mathbf{y}_2^T(L)\mathbf{x}_2(L) = 1$ and ignoring $O(\omega^2)$, one easily gets

$$\begin{aligned} Re\big[\lambda_2(L^{\omega}_{e_{ij}})\big] &= Re[\lambda_2(L)] + Re\big[\mathbf{y}_2^T(L)L_{e_{ij}}\mathbf{x}_2(L)\big]\omega \\ &= Re[\lambda_2(L)] + Re[y_j(x_j - x_i)]\omega, \end{aligned}$$

where x_p and y_p are the pth component of $\mathbf{x}_2(L)$ and $\mathbf{y}_2(L)$, respectively. This implies that the first-order derivative of the Fiedler value can be obtained by the left and right eigenvctors of $\lambda_2(L)$.

As a result, the dynamical contribution value of directed edge e_{ij} is $I^d_{e_{ij}} = Re[y_j(x_j - x_i)]$. The right Fiedler vector provides the information of the starting and ending nodes, whereas the left Fiedler vector only provides the information of the end node. Consequently, the combination of the left and right Fiedler vectors provides comprehensive information about the edge's position and direction.

Further, for any vectors $\mathbf{u}$ and $\mathbf{v}$, due to the characteristics of Laplacian matrix, one obtains

$$\begin{aligned} \mathbf{u}^T L\mathbf{v} &= \sum_{i,j=1}^{N} L_{ji}u_j v_i \\ &= \sum_{i,j=1}^{N} (D_{ji} - A_{ji})u_j v_i \\ &= \sum_{j}^{N} D_{jj}u_j v_j - \sum_{(i,j)\in\mathcal{E}} u_j v_i \\ &= \sum_{(i,j)\in\mathcal{E}} u_j(v_j - v_i). \end{aligned}$$

When $\mathbf{u}, \mathbf{v}$ are the right Fiedler vector $\mathbf{x}_2(L)$ and the left Fiedler vector $\mathbf{y}_2(L)$ correspondingly, with $\mathbf{y}_2^T(L)\mathbf{x}_2(L) = 1$, one gets

$$\mathbf{y}_2^T(L)L\mathbf{x}_2(L) = \lambda_2(L).$$

Hence, one obtains

$$Re[\lambda_2(L)] = \sum_{(i,j)\in\mathcal{E}} Re[y_j(x_j - x_i)] = \sum_{(i,j)\in\mathcal{E}} I^d_{e_{ij}}. \quad (6)$$

Equation (6) illustrates not only the individual dynamical contribution value of each edge, but also the quantitative relation between each edge and Fiedler value. Interestingly, the Fiedler value represents the cumulated contribution of all edges in a directed network.

B. Undirected networks

In the case of undirected networks, the Laplacian matrix is symmetric. Consequently, the left and right eigenvectors of the same eigenvalue are equal and both composed of real values. Then $I^u_{e'_{ij}} = I^d_{e_{ij}} + I^d_{e_{ji}} = x_j(x_j - x_i) + x_i(x_i - x_j) = (x_i - x_j)^2$, where x_p is the pth component of Fiedler vector $\mathbf{x}_2(L)$, which is exactly the unit eigenvector corresponding to $\lambda_2(L)$. This is consistent with the importance index of undirected edges [12]. Further, one also obtains

$$\begin{aligned}\lambda_2(L) &= \mathbf{x}_2^T(L)L\mathbf{x}_2(L)\\ &= \sum_{(i,j)\in\mathcal{E}'} (x_i - x_j)^2 = \sum_{(i,j)\in\mathcal{E}'} I^u_{e'_{ij}}. \end{aligned} \quad (7)$$

Hence, the quantitative relation between the Fiedler value and each edge is unified, regardless of whether the network is directed or undirected.

IV. NUMERICAL RESULTS

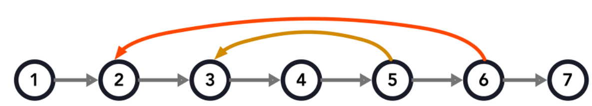
To verify the effectiveness of our method, some sample networks and two experimental networks are shown.

A. Sample networks

References [28,29] have highlighted that the reverse edges can inhibit the diffusion rate in a directed chain network. Besides detecting the inhibition of the reverse edges, our method quantifies the degree of inhibition. Figure 2 reveals that the contribution values of the forward edges $e_{12}, e_{23}, \ldots, e_{67}$ are positive, and their contribution decreases in turn, while the reverse edges e_{62} and e_{53} contribute to the diffusion rate negatively. The contribution value becomes more negative as the reverse range increases. The sensitivity simulation verifies that if the weight of a reverse edge is increased, there is a corresponding decrease in the Fiedler value. Even if the dynamical contribution value works with small ω, simulations demonstrate that the edge ranking is effective with relatively large ω. Consequently, the quantitative analysis method clearly depicts how each edge affects network dynamics.

In a general directed network, the contribution values of edges are shown in Fig. 3. By using the left and right Fiedler vectors, the positive and negative effects of all edges are identified and quantified. The cumulated dynamical contribution value of all edges is $Re(\lambda_2) = 1.4811$.

Different from directed edges, all edges in undirected networks have nonnegative effects $[I^u_{e'_{ij}} = (x_i - x_j)^2]$. The result is intuitively illustrated in two sample networks.



Edge	e_{12}	e_{23}	e_{34}	e_{45}	e_{56}	e_{67}	e_{53}	e_{62}
$I^d_{e_{ij}}$	0.0577	0.0477	0.0302	0.0302	0.011	0	-0.0333	-0.0525

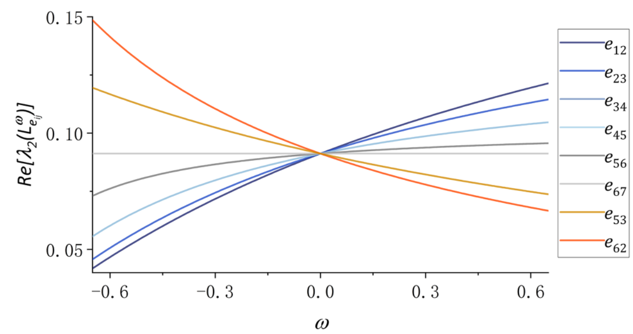
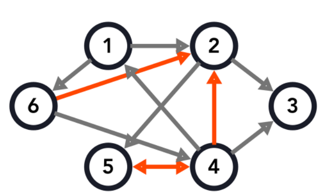


FIG. 2. A directed chain network with seven nodes and two reverse edges. The table provides the dynamical contribution value $I^d_{e_{ij}}$ of each edge. The Fiedler value $Re(\lambda_2) = 0.0912 = \sum_{(i,j)\in\mathcal{E}} I^d_{e_{ij}}$. The graph depicts the Fiedler value versus ω.

Figure 4 shows two representative networks. In Fig. 4(a), the contribution values of the bridge edge e'_{17} and any other edge are 0.1982 and 0.0060, respectively. The total sum of contribution values amounts to $\lambda_2 = 0.2582$. Notably, the contribution rate of the bridge edge is 76.8%. Fiedler value of a fully connected network with N nodes is N. However, when one node is hanged, as seen in Fig. 4(b), the Fiedler value decreases to one. It suggests that even small local changes significantly impact the network dynamics. It is found that the most important edge is e'_{17}, whose contribution value is 0.8333, followed by the edges connected to node one with contribution values 0.0060 s, while the remaining edges have contribution values of 0 s. Although there are many edges in this network, only a few really contribute to the network dynamics. This is the reason why the Fiedler value of this network remains relatively small, even with a fully connected portion.

The above intuitive examples show that in networks with severely uneven distribution of degrees, the distribution of edge contribution values follows the "Pareto principle." This principle is also observed in networks with relatively even distribution of degrees, such as random networks, as shown



Edge	$I^d_{e_{ij}}$
e_{41}	0.6428
e_{16}	0.4795
e_{64}	0.2886
e_{12}	0.1342
e_{25}	0.1253
e_{23}	0
e_{43}	0

Edge	$I^d_{e_{ij}}$
e_{45}	-0.0291
e_{62}	-0.0291
e_{42}	-0.0656
e_{54}	-0.0656

FIG. 3. A directed network comprising six nodes and 11 edges. The gray (red) edges indicate the nonnegative (negative) effects on the Fiedler value. The quantified contribution values are presented in tables.

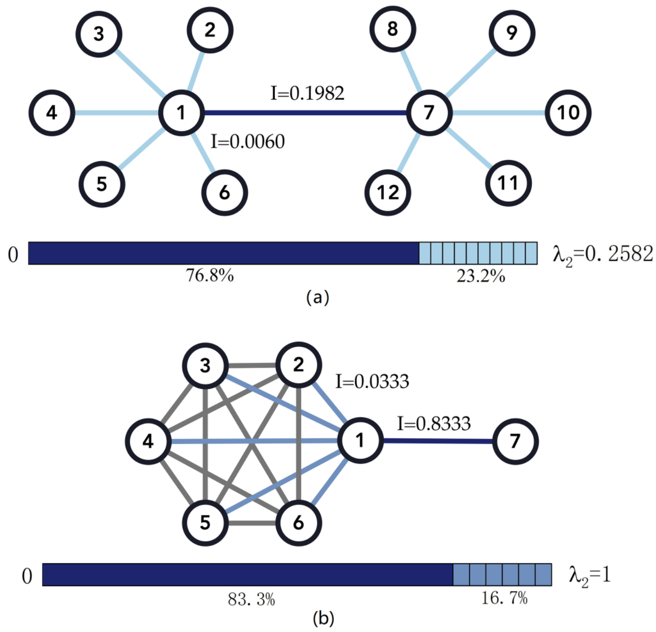


FIG. 4. (a) A bistar network with 12 nodes and a bridge edge. (b) A fully connected network with an additional hanging node. The contribution value $I^{u}_{e'_{ij}}$ of each edge is marked next to the corresponding edge, in which edges with the same color signify equal contribution value. The contribution rate of each edge is then presented below the network, which is the ratio of the contribution value to the Fiedler value.

in Fig. 5. This phenomenon is further supported by the subsequent experimental networks [35].

B. Experimental networks

In Fig. 6, it is observed that the contribution rate of directed edge $e_{27,25}$ is 94.3%. If edge $e_{27,25}$ does not exist, the network is a leader-follower type with only one source node 25. The edge $e_{27,25}$ represents the feedback provided by the followers to the leader. The Fiedler value of the original network in Fig. 6 is 0.9852. When this edge is deleted, the Fiedler value becomes 0.0492, accompanied by a noticeable decrease in the diffusion rate of the network in Fig. 7. The second

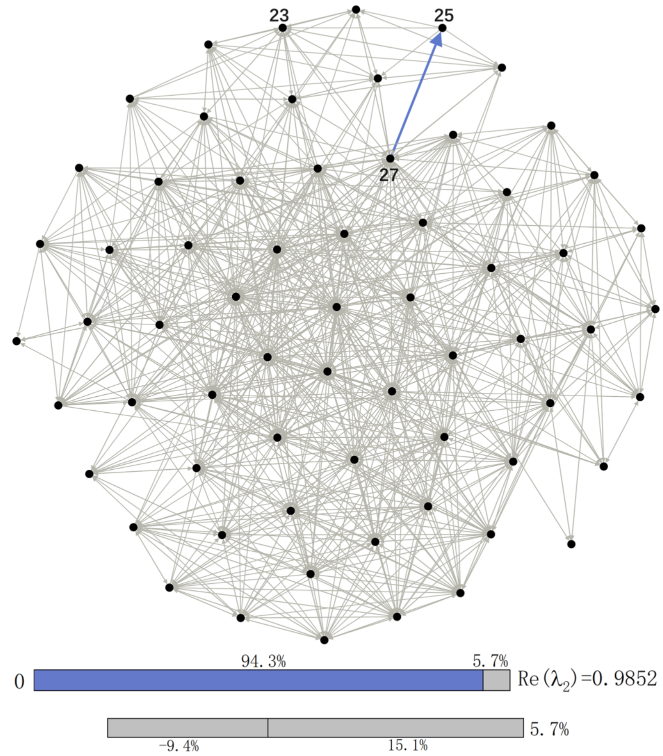


FIG. 6. A directed Cat Cerebral Cortex network with 65 nodes and 1139 edges. The contribution rate of edge $e_{27,25}$ is 94.3%, while the total contribution rates of all the gray edges are 5.7%, composed of 15.1% and −9.4%.

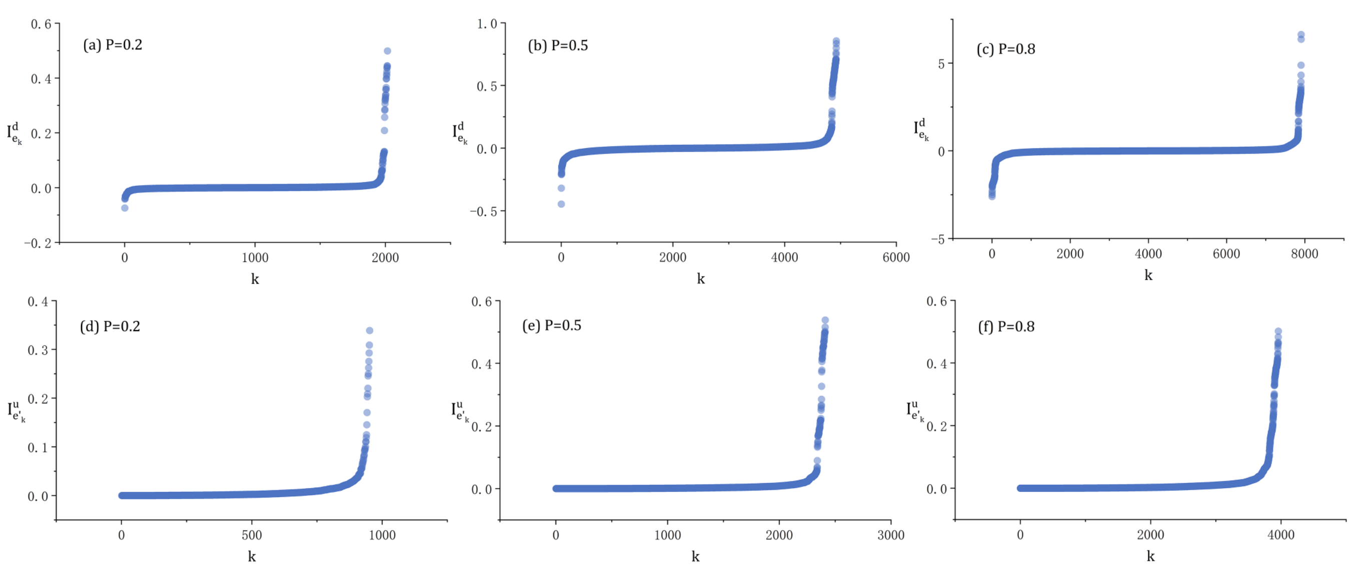


FIG. 5. Distribution of dynamical contribution values for all edges across varying probabilities p in random networks with 100 nodes, where each dot represents the dynamical contribution value of edge e_k numbered in ascending order [Directed networks: (a), (b), (c); Undirected networks: (d), (e), (f)].

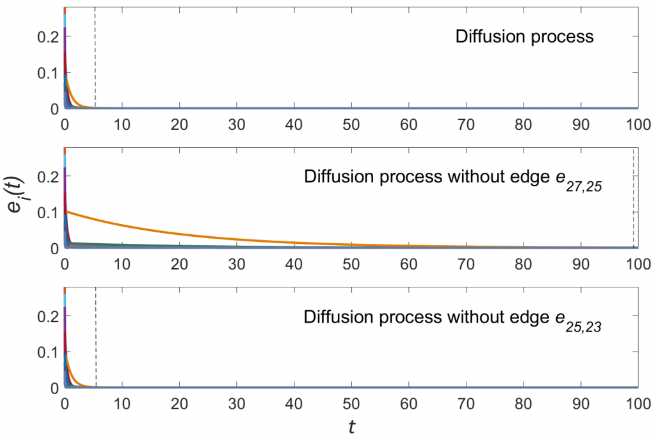


FIG. 7. The diffusion error $e_i(t) = |\xi_i(t) - s(t)|$ versus time t in the original Cat Cerebral Cortex network, in the network with the removal of edge $e_{27,25}$ and $e_{25,23}$, respectively, where $\dot{\xi}_i(t) = -\sum_{j=1}^{N} L_{ij}\xi_j(t)$ and $s(t) = \frac{\sum_{i=1}^{65}\xi_i(t)}{65}$. The error state is obtained by averaging the initial values within the range of $[-1, 1]$ after 20 simulations. Each colored curve represents the error state of a node. The orange curve specifically corresponds to node 25, which is the end of edge $e_{27,25}$. When all error states are equal to zero, it signifies that the nodes have reached a steady state.

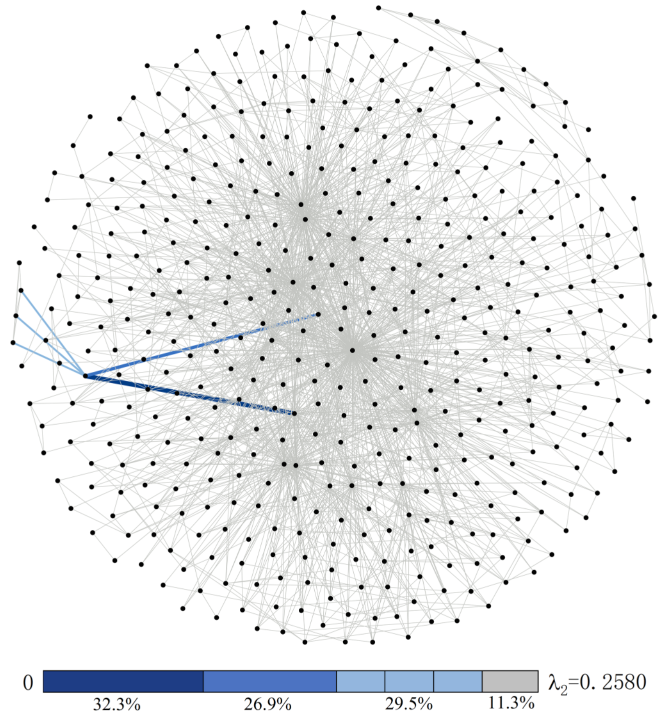


FIG. 8. An undirected C. elegans metabolic network with 453 nodes and 2025 edges. The blue edges contribute totally 88.7%, while the gray edges make a total contribution of 11.3%.

important edge $e_{25,23}$, which contributes only 0.5%, has little impact on the diffusion rate. Undoubtedly, edge $e_{27,25}$ is of significant importance (the contribution rate is 94.3%), while the remaining edges of the network make a humble dynamical contribution (the total contribution rates of all the remaining edges are 5.7%, composed of 15.0% positive contribution rates and 9.4% negative contribution rates). Figure 8 depicts the C. elegans metabolic network, which was employed to observe the distribution of important triangles in Ref. [12]. It is seen that only five edges in blue, in total 2025 edges, account for an 88.7% contribution rate to the Fiedler value. Notably, these edges coincide with the ones comprising the vital triangles of Ref. [12].

V. DISCUSSION AND CONCLUSION

In this section, we further conclude the relation between edges and the Fiedler value in weighted networks. The weighted Laplacian matrix is denoted as W, where W_{ji} is the weight of the edge e_{ij}. In a weighted directed network, the contribution value is expressed as $I_{e_{ij}}^{wd} = W_{ji}Re[y_j(x_j - x_i)]$, where x_p and y_p are the pth component of left Fiedler vector $\mathbf{x}_2(W)$ and right Fiedler vector $\mathbf{y}_2(W)$, respectively. Further, one obtains $Re[\lambda_2(W)] = \sum_{(i,j)\in\mathcal{E}} I_{e_{ij}}^{wd}$. In a weighted undirected network, the contribution value is expressed as $I_{e'_{ij}}^{wu} = I_{e_{ij}}^{wd} + I_{e_{ji}}^{wd} = W_{ij}(x_i - x_j)^2$, where x_p is the pth component of Fiedler vector $\mathbf{x}_2(W)$. Further, one gets $\lambda_2(W) = \sum_{(i,j)\in\mathcal{E}'} I_{e'_{ij}}^{wu}$. Hence, a comprehensive set of methods for edge dynamics has been developed, regardless of the directionality and weight of the edge.

In conclusion, this paper has effectively quantified the relation between edges and Fiedler value in a network, providing valuable insights into network dynamics. The dynamical contribution value of each edge has been measured using only two Fiedler vectors. The right Fiedler vector provides the information of the starting and ending nodes, while the left Fiedler vector only provides the information of the end node. Interestingly, the edge dynamical contribution value can predict the result of a period of weight variation. It is critical not only for the diffusion analysis but also for the assessment of edge vulnerabilities.

Further, the network dynamics index, the Fiedler value, has been illustrated from an interesting lens: the cumulated contribution value of all edges. The summation formula, from an algebraic perspective, has been endowed with dynamical significance. What's more, numerical examples have verified the effectiveness of the results and revealed that only a few edges make a significant contribution to the network dynamics. Especially in the Cat Cerebral Cortex network with 2025 edges, one single directed edge has been found to contribute over 90% of the Fiedler value, greatly influencing the diffusion rate of the network. This "Pareto principle" of edge dynamical contribution provides a theoretical basis for edge-based pinning control and helps to promote further research. These findings would have potential implications in the fields of physics, biology, chemistry, and engineering.

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