



Subsystem eigenstate thermalization hypothesis for translation invariant systems

Zhiqiang Huang (黄志强) ^{1,*} and Xiao-Kan Guo (郭肖侃) ^{2,†}

¹*State Key Laboratory of Magnetic Resonance and Atomic and Molecular Physics,
Innovation Academy for Precision Measurement Science and Technology,
Chinese Academy of Sciences, Wuhan 430071, China*

²*Department of Applied Mathematics, Yancheng Institute of Technology, Jiangsu 224051, China*



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The eigenstate thermalization hypothesis for translation invariant quantum spin systems has been proved recently by using random matrices. In this paper, we study the subsystem version of the eigenstate thermalization hypothesis for translation invariant quantum systems without referring to random matrices. We first find a relation between the quantum variance and the Belavkin-Staszewski relative entropy. Then, by showing the small upper bounds on the quantum variance and the Belavkin-Staszewski relative entropy, we prove the subsystem eigenstate thermalization hypothesis for translation invariant quantum systems with an algebraic speed of convergence in an elementary way. The proof holds for most of the translation invariant quantum lattice models with exponential or algebraic decays of correlations.

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I. INTRODUCTION

The equilibration and the thermalization of an isolated quantum system are fundamental for understanding the emergence of quantum statistical mechanics from unitary quantum mechanics. By thermalization, it means that either an isolated quantum system would evolve into a thermal state or the observables would attain their values in a statistical ensemble, after a unitary quantum evolution of the isolated quantum system for a period of time that is long enough. Since a unitary quantum evolution preserves the pure state, it is not easy to understand how the statistical mixture emerges if the initial state of an isolated quantum system is a pure state. Numerous approaches have been proposed to understand various aspects of this problem; cf. the reviews [1,2].

The eigenstate thermalization hypothesis (ETH) [3,4], that the expectation values of quantum observables in an energy eigenstate should approximately coincide with the thermal expectation values, provides a possible mechanism for the thermalization of an isolated quantum system. Although the ETH has more and more numerical and experimental evidence in specific closed quantum models and systems, its physical origin and mathematical description are not yet completely understood. In the original proposal by Deutsch and Srednicki [5–7], a random perturbation is added to a closed quantum system, and the ETH holds if the perturbed system becomes chaotic. By modeling the random perturbations as random matrices, the ETH for deterministic observables with the Hamiltonians sampled from the Wigner random matrix ensemble without further unitary symmetry is mathematically proved in a recent work [8]. This scenario, however, is not universal. For one thing, if further unitary symmetries are present,

the conserved quantities would obstruct the thermalization to Gibbs states and the original ETH would fail. More recently, in [9], the ETH for translation invariant spin systems is proved using the same method from random matrices, thereby generalizing its validity to various translation invariant lattice spin models.

In many studies of the “weak” ETH, for example, Refs. [10–14], one does not presume the random energy perturbations or simply the random Hamiltonians, but tries to derive the statistical properties solely from quantum properties. From this perspective, the quantum entanglement inside a closed quantum system, together with its dynamics under the global unitary evolution, should play a crucial role for thermalization, which has indeed been experimentally observed in [15]. To quantify the entanglement in a closed quantum system, we need to work at the level of subsystems of the total system to compute the entanglement entropies and alike. This observation leads to the *subsystem* ETH [16,17], which hypothesizes the convergence of the subsystem density matrices to the thermal Gibbs density matrix. In fact, the trace distance between two density matrices is bounded by the relative entropy between two density matrices. Since the entanglement entropies and relative entropies are calculable in many conformal field theories (CFTs), the subsystem ETH and its violation have been tested in many CFTs [17–23]. Notice that the conformal symmetry forms an infinite-dimensional group, so the infinite number of conserved Korteweg-de Vries (KdV) charges make the generalized Gibbs states the proper equilibrated states for CFTs [24,25]. It is then natural to ask for a quantum system and model with a smaller symmetry group such that the subsystem ETH still holds.

For translation invariant quantum lattice systems, we already know that the strong ETH [9], the weak ETH [11,12], and the canonical typicality [26] are true. In addition, a version of the generalized ETH, i.e., thermalization to the generalized Gibbs ensemble, for translation invariant quasifree

*hzq@wipm.ac.cn

†kankuohsiao@whu.edu.cn

fermionic integrable models is also proved in [27]. We therefore see that the translation invariant quantum lattice systems are a good test for checking various versions of ETH. In this paper, we make an effort to prove the subsystem ETH for translation invariant systems without referring to random matrices.

We will work in the setting of a translation invariant quantum lattice system in the sense of [12]. Unlike the considerations by Iyoda *et al.* [12], we find a formal relation between the quantum variance and the Belavkin-Staszewski relative entropy in an average sense, thereby establishing a connection of the scaling analysis on the variance given in [12] and the subsystem ETH formulated as the relative-entropic bounds on the trace distance between the subsystem state and the canonical thermal state. In fact, we are able to prove the following form of subsystem ETH:

$$\|\rho_{\text{sub}} - \rho_A^c\| \sim \mathcal{O}(N_A^{1/2}/N^{1/2}), \quad (1)$$

$$\|\sigma_{\text{sub}}\| \sim \mathcal{O}(N_A^{1/2}/N^{1/2}), \quad (2)$$

where ρ_{sub} is the state of a subsystem A , σ_{sub} is a traceless (or “off-diagonal”) matrix of a subsystem, and ρ_A^c is the reduced density matrix of the canonical thermal state, for translation invariant quantum lattice systems. Notice that in our results (1) and (2) the errors decay algebraically as $\mathcal{O}(N_A^{1/2}/N^{1/2})$, with N_A the degrees of freedom (or number of lattice sites) in the subsystem A and N the total degrees of freedom. This decaying behavior is weaker than the exponential decays as usually expected in ETH, but corroborates the algebraic decay of error terms in the random-matrix proof of ETH for translation invariant systems [9].

We begin in Sec. II with some preliminary results about ETH, subsystem ETH, and, in particular, the setting of a translation invariant quantum lattice system from [12]. In Sec. III, we introduce the main technical input, i.e., the formal relation between the quantum variance and the Belavkin-Staszewski relative entropy in an average sense. Using this relation, we analyze the scaling of both the variance and the Belavkin-Staszewski relative entropy and prove the subsystem ETH in Sec. IV. In Sec. V, we discuss the role of correlation decay in our proof. In the final Sec. VI, we conclude this paper and discuss some related issues.

II. PRELIMINARIES

In this section, we recollect the basics of ETH and subsystem ETH, and the weak ETH with eigenstate typicality in the sense of [12].

A. ETH and subsystem ETH

Consider an isolated or closed quantum system B with Hamiltonian h . This Hamiltonian h could include a random perturbation h_{pert} . Suppose h has eigenvectors $|E_i\rangle$, $i = 1, 2, \dots, N$, with energy eigenvalues E_i , i.e., $h|E_i\rangle = E_i|E_i\rangle$. For a few-body observable A , the local ETH can be formulated in terms of the expectation values of A in the energy eigenstates as

$$\langle E_i|A|E_j\rangle = \mathcal{A}(E)\delta_{ij} + e^{-S(E)/2}f(E, \omega)R_{ij}, \quad (3)$$

where $E = \frac{1}{2}(E_i + E_j)$, $\omega = E_i - E_j$, and $e^{S(E)} = E \sum_i \delta(E - E_i)$ is the density of states of the system B . The $\mathcal{A}(E)$ and $f(E, \omega)$ are smooth functions, while the fluctuation factor R_{ij} is of the order of 1. In particular, the thermalization requires that $\mathcal{A}(E)$ should be approximately the thermal average of A in the canonical ensemble, $\mathcal{A} = \langle A \rangle_c + \mathcal{O}(N^{-1}) + \mathcal{O}(e^{-S/2})$, in the large- N limit.

This local form (3) of ETH can be derived based on Berry’s chaotic conjecture [7]. If we sample the Hamiltonian h from a random matrix ensemble, the form of inequality for ETH,

$$|\langle E_i|A|E_j\rangle - \langle A \rangle_{\text{mc}}(E)\delta_{ij}| \leq \mathcal{O}(e^{-S/2}), \quad (4)$$

where $\langle \cdot \rangle_{\text{mc}}$ denotes the thermal average in the microcanonical ensemble, can be proved mathematically in several cases, including the translation invariant systems, by using properties of random matrices [8,9].

Both (3) and (4) are local conditions, as the ETH is assumed for each energy eigenstate. Therefore, in analogy to the canonical typicality of a subsystem B_1 ,¹ we can envision the *subsystem* ETH,

$$\|\rho_i^{B_1} - \rho^c(E_i)\| \sim \mathcal{O}(e^{-S/2}), \quad (5)$$

$$\|\rho_{ij}^{B_1}\| \sim \mathcal{O}(e^{-S/2}), \quad i \neq j, \quad (6)$$

where $\rho_i^{B_1} = \text{Tr}_{\bar{B}_1} |E_i\rangle \langle E_i|$ is the reduced density matrix of the subsystem B_1 , ρ^c is a universal density matrix that could be the thermal canonical one, and $\rho_{ij}^{B_1} = \text{Tr}_{\bar{B}_1} |E_i\rangle \langle E_j|$. The norm here refers to the trace distance, or Schatten 1-norm, $\|\rho_1 - \rho_2\| = \frac{1}{2} \text{Tr} \sqrt{(\rho_1 - \rho_2)^2}$. The subsystem ETH as given by (5) and (6) is, in fact, stronger than the local ETH as in (3), due to the following inequality [17]:

$$|\langle A \rangle - \langle A \rangle_c| \leq \sqrt{\|\rho - \rho^c\| \text{Tr}[(\rho + \rho^c)A^2]}, \quad (7)$$

where $\langle A \rangle = \text{Tr}(\rho A)$ and $\langle A \rangle_c = \text{Tr}(\rho^c A)$.

What is important in the following is that the trace distance in (5) can be bounded by the relative entropy between two density matrices,

$$\|\rho^{B_1} - \rho^c(E_i)\|^2 \leq 2S(\rho^{B_1} \|\rho^c), \quad (8)$$

where $S(\rho_1 \|\rho_2) = \text{tr}(\rho_1 \ln \rho_1) - \text{tr}(\rho_1 \ln \rho_2)$ is the (Umegaki) quantum relative entropy. This inequality (8) is the so-called quantum Pinsker inequality in quantum information theory [28].

B. Weak ETH with eigenstate typicality

In proving the weak ETH for translation invariant quantum lattice systems [12], the quantum uncertainty of measuring an observable plays an important role. Conventionally, the uncertainties, either classical or quantum, can be quantified by the variance [29]. For instance, given a quantum state ρ , the quantum uncertainty of measuring an observable A in the state ρ can be quantified by the variance

$$V(\rho, A) = \text{Tr}(\rho A A^\dagger) - |\text{Tr}(\rho A)|^2 = \text{Tr}[\rho(A - \langle A \rangle)(A - \langle A \rangle)^\dagger]. \quad (9)$$

¹We emphasize that throughout this paper, B without indices denotes the total system and B_i and likewise denote the subsystems.

Let $\rho_B = \sum_j p_j \Pi_B^j$ be the state of the total system B expanded in the orthonormal basis $\{\Pi_B^j\}$ of rank-1 projectors. Then, in terms of these projectors, one can particularly define the following quantity, which is called a *fluctuation* in [12]:

$$\Delta(\rho, A) = \sum_j p_j |\text{Tr}[\Pi_B^j A]|^2 - |\text{Tr}[\rho A]|^2. \quad (10)$$

We have

$$\Delta(\rho, A) \leq \sum_{j,k} p_j \text{Tr}(\Pi_B^j A \Pi_B^k A^\dagger) - |\text{Tr}[\rho A]|^2 = V(\rho, A) \quad (11)$$

because the additional off-diagonal terms are positive, i.e., $\text{Tr}(\Pi_B^j A \Pi_B^k A^\dagger) = |\langle j|A|k \rangle|^2 \geq 0$. This $\Delta(\rho, A)$ is related to the following (in)distinguishability measure of quantum states:

$$d(\Pi_B^j, \rho; A) = |\text{Tr}[(\Pi_B^j - \rho)A]|^2. \quad (12)$$

Indeed, $\Delta(\rho, A)$ can be considered as the quantification of the probabilistic typicality or concentration with respect to the measure (12),

$$\Delta(\rho, A) = \int d(\Pi_B^j, \rho; A) p_d, \quad (13)$$

where the probability distribution p_d is obtained from the p_j 's through a Radon-Nikodym derivative. By the Chebyshev inequality, we have that

$$P_\rho(|\text{Tr}(\Pi_B^j A) - \text{Tr}(\rho A)| \geq \epsilon) \leq \frac{\Delta(\rho, A)^2}{\epsilon^2}, \quad \forall \epsilon \in \mathbb{R}^+. \quad (14)$$

Therefore, when $\Delta(\rho, A)$ is very small, the expectation of the projectively measured observable would concentrate on the expectation of observable calculated with respect to the state ρ . In other words, the indistinguishability of the measurement outcomes induces a description by a mixed state.

In ETH, one considers the local energy eigenstates. So, let $\sigma_{B_1}^j = \text{Tr}_{B_1} \Pi_B^j$ be the reduced, projection or state on the subsystem B_1 . Then we should consider

$$d(\sigma_{B_1}^j, \rho; A^{B_1}) = |\text{Tr}[(\sigma_{B_1}^j - \rho_{B_1})A^{B_1}]|^2 \leq \|\sigma_{B_1}^j - \rho_{B_1}\|_1^2 \|A^{B_1}\|_\infty^2, \quad (15)$$

where $\|\cdot\|_k$ is the Schatten k -norm. Next, let ρ^{mc} be the density matrix for the microcanonical ensemble. According to Eqs. (10), (11), and (14), if

$$\Delta(\rho^{\text{mc}}, A_{B_1}) \sim \mathcal{O}(N^{-\alpha}), \quad (16)$$

with $0 < \alpha < 1$, i.e., the expectations of a local observable A^{B_1} with respect to the results of local measurements concentrate the expectation of A^{B_1} with respect to ρ^{mc} , then we know that each pure state $\sigma_{B_1}^j$ cannot be distinguished from the microcanonical ρ^{mc} in the large- N limit. This is the *weak ETH with eigenstate typicality* [12]. Furthermore, by using the equivalence of the ensembles, one also has a similar weak ETH on the concentration of $\sigma_{B_1}^j$ to the canonical-ensemble density matrix ρ_c .

In the proofs of the weak ETH with eigenstate typicality for quantum lattice systems [12], the translation invariance in

the following sense is crucial. Let us partition the lattice of system B into $\mathcal{C} = |B|/|B_1|$ blocks with the same size, where $|B|$ means the number of lattice points in B . These \mathcal{C} blocks are identical copies of B_1 . Let us also define the translational copies A^{B_i} of A^{B_1} defined on B_i obtained by the translations from block to block. Then the translation invariance means

$$\text{Tr}[\Pi_B^j A^{B_i}] = \text{Tr}[\Pi_B^j A^{B_1}]. \quad (17)$$

We can introduce the average observable,

$$A^B = \frac{1}{\mathcal{C}} \sum_i A^{B_i}. \quad (18)$$

Then the translation invariance (17) gives $\Delta(\rho, A^{B_1}) = \Delta(\rho, A^B)$. Therefore, the weak ETH can be proved by bounding $\Delta(\rho, A^B)$. Since the translation invariance of the Hamiltonian does not guarantee the translation invariance of the energy eigenstate, (17) is not unconditionally true for any energy eigenstate and any measurement. If we only rely on the average observable (18), then for the translationally invariant state ρ , we can also consider

$$\begin{aligned} d(\Pi_B^j, \rho; A^B) &= \left| \frac{1}{\mathcal{C}} \sum_k \text{Tr}[(\sigma_{B_k}^j - \rho_{B_k})A^{B_k}] \right|^2 \\ &= \left| \text{Tr} \left[\left(\frac{1}{\mathcal{C}} \sum_k \sigma_{B_1}^{j,k} - \rho_{B_1} \right) A^{B_1} \right] \right|^2 \\ &= d\left(\frac{1}{\mathcal{C}} \sum_k \sigma_{B_1}^{j,k}, \rho_{B_1}; A^{B_1} \right), \end{aligned} \quad (19)$$

where $\sigma_{B_1}^{j,k}$ is the translational copies of $\sigma_{B_1}^j$. Equation (19) actually converts the average observable and the average local state into each other.

III. RELATING VARIANCE TO RELATIVE ENTROPY

Equations (9) and (12) depend on the measured observable A . In order to quantify the quantum uncertainty in a way that depends only on the quantum measurements but not on the measured observables, the following entropic uncertainty used in the entropic uncertainty relation [30] serves the purpose:

$$H_\Pi(\rho) = \sum_i p_i \mathcal{S}(\rho_i || \rho), \quad (20)$$

where $\rho_i = \Pi^i \rho \Pi^i / p_i$, with $p_i = \text{Tr}(\Pi^i \rho)$ and $\{\Pi^i\}$ being the (not necessarily rank-1) measurement operators.

In view of the frequent usages of the maps between the total system B and its subsystems B_i in the proofs in [12], we consider the Belavkin-Staszewski (BS) relative entropy [31,32],

$$\hat{\mathcal{S}}(\sigma || \rho) = \text{Tr}\{\sigma \ln [\mathcal{J}_\sigma^{1/2}(\rho^{-1})]\} \quad (21)$$

$$= \text{Tr}[\sigma \ln(\rho^{-1} \sigma)] \quad (22)$$

$$= \text{Tr}\{\rho \mathcal{J}_\rho^{-1/2}(\sigma) \ln [\mathcal{J}_\rho^{-1/2}(\sigma)]\}, \quad (23)$$

where $\mathcal{J}_\rho^\alpha(\cdot) := \rho^\alpha(\cdot)\rho^\alpha$ is a rescaling map. Notice that in the above definitions of BS entropy, there is the inverse ρ^{-1} , which requires that the density matrix should be strictly

positive; this requirement is naturally fulfilled in our considerations, as the density matrices at the position of ρ in the above formulas are the canonical ensemble ρ^c or the subsystem states. Now that different ρ_i are orthogonal to each other by definition, the entropic uncertainty can be generalized by using (21) as

$$\sum_i p_i \hat{S}(\rho_i || \rho) = \text{Tr} \left\{ \rho \ln \left[\sum_i \mathcal{J}_{\rho_i}^{1/2}(\rho^{-1}) \right] \right\}. \quad (24)$$

When the Hilbert-Schmidt norm $\|X - I\|_2 \leq 1$, the power series of the matrix logarithm,

$$\ln(X) = (X - I) - \frac{1}{2}(X - I)^2 + \dots, \quad (25)$$

converges. Using it, we can obtain the following first-order relations:

$$\begin{aligned} \sum_i \text{Tr} \rho [\mathcal{J}_{\rho_i}^{1/2}(\rho^{-1}) - 1] &= \sum_i p_i \{ \text{Tr} \rho_i [\mathcal{J}_{\rho_i}^{1/2}(\rho^{-1})] - 1 \} \\ &= \sum_i p_i \left(\text{Tr} \{ \rho [\mathcal{J}_{\rho_i}^{-1/2}(\rho_i)]^2 \} - 1 \right) = \sum_i p_i V(\rho, O_i), \end{aligned} \quad (26)$$

where, in the first line, we have used $\text{Tr} \rho_i = 1$, the second line follows from (23), and

$$O_i := \mathcal{J}_{\rho_i}^{-1/2}(\rho_i), \quad (27)$$

with $\langle O_i \rangle = 1$. This O_i plays the role of observable in quantum variance and is defined by ρ_i in a one-to-one manner. Although an observable O can be mathematically related to a particular density matrix ρ , the physical meaning of such an O is possibly unclear. Therefore, we do not interpret this O_i in (27) and merely take it as an intermediate technical step. Formally, Eq. (26) establishes a link between the variance of O_i in the states ρ and the entropic uncertainty in the first-order sense. Since the quantum relative entropy encodes the closeness between two density matrices, the $V(\rho, O_i)$ is again a quantity measuring the (in)distinguishability between state ρ_i and ρ .

The relation (26) is suitable for studying localized states on subsystems. Let $\rho_B = \sum_j p_j \Pi_B^j$, as before. For the pure state Π_B^j , its reduced density matrix on a subsystem, say B_1 , $\sigma_{B_1}^j = \text{Tr}_{\bar{B}_1} \Pi_B^j$, is no longer pure in general, so that it can be arbitrary subsystem states of B_1 . The reduced density matrix of ρ_B on B_1 is $\rho_{B_1} = \sum_i p_i \sigma_{B_1}^i$. In this setting, we can consider the formal observable

$$O_i^{B_1} = \mathcal{J}_{\rho_{B_1}}^{-1/2}(\sigma_{B_1}^i). \quad (28)$$

Again, we have $\langle O_i^{B_1} \rangle = 1$. Similar to (26), we also have

$$\begin{aligned} V(\rho_B, O_i^{B_1}) &= \text{Tr} \rho_B \{ [\mathcal{J}_{\rho_{B_1}}^{-1/2}(\sigma_{B_1}^i)]^2 - 1 \} \\ &= \text{Tr} \{ \sigma_{B_1}^i [\mathcal{J}_{\sigma_{B_1}^i}^{1/2}(\rho_{B_1}^{-1}) - 1] \} \end{aligned} \quad (29)$$

as the first-order expansions of $\hat{S}(\sigma_{B_1}^i || \rho_{B_1})$. In (29), we have used the property that $\text{Tr} \rho_B = 1$ as a normalized density matrix. Equation (29) relates the indistinguishability of localized states and the measurement uncertainty (of $O_i^{B_1}$) in ρ , in an average sense.

Recall that the BS relative entropy and the quantum relative entropy satisfy $\hat{S}(\sigma || \rho) \geq S(\sigma || \rho)$ [32]; thereby,

$$\hat{S}(\sigma_{B_1}^i || \rho_{B_1}) \geq \frac{1}{2} \|\sigma_{B_1}^i - \rho_{B_1}\|_1^2, \quad (30)$$

where the Schatten-1 norm $\|\cdot\|_1$ is just the trace distance introduced above. On the other hand, the variance $V(\rho_B, O_i^{B_1})$ before the series expansion is, by definition, a Schatten-2 norm,

$$\begin{aligned} V(\rho_B, O_i^{B_1}) &= \text{Tr} [(\sigma_{B_1}^i - \rho_{B_1}) \rho_{B_1}^{-1} (\sigma_{B_1}^i - \rho_{B_1})] \\ &= \|(\sigma_{B_1}^i - \rho_{B_1}) \rho_{B_1}^{-1/2}\|_2^2. \end{aligned} \quad (31)$$

By Hölder's inequality, we have

$$\begin{aligned} \|(\sigma_{B_1}^i - \rho_{B_1})\|_1^2 &= \|(\sigma_{B_1}^i - \rho_{B_1}) \rho_{B_1}^{-1/2} \rho_{B_1}^{1/2}\|_1^2 \\ &\leq \|(\sigma_{B_1}^i - \rho_{B_1}) \rho_{B_1}^{-1/2}\|_2^2 \|\rho_{B_1}^{1/2}\|_2^2 \\ &= \|(\sigma_{B_1}^i - \rho_{B_1}) \rho_{B_1}^{-1/2}\|_2^2 = V(\rho_B, O_i^{B_1}). \end{aligned} \quad (32)$$

Similarly, we can consider the ‘‘off-diagonal’’ observable,

$$O_{ij}^{B_1} = \mathcal{J}_{\rho_{B_1}}^{-1/2}(\sigma_{B_1}^{ij}), \quad i \neq j, \quad (33)$$

with $\sigma_{B_1}^{ij} = \text{Tr}_{\bar{B}_1} \Pi_B^{ij}$ an off-diagonal reduced density matrix. Now we have $\langle O_{ij}^{B_1} \rangle = 0$. Again, we have

$$\begin{aligned} \|\sigma_{B_1}^{ij}\|_1^2 &= \|\sigma_{B_1}^{ij} \rho_{B_1}^{-1/2} \rho_{B_1}^{1/2}\|_1^2 \\ &\leq \|\sigma_{B_1}^{ij} \rho_{B_1}^{-1/2}\|_2^2 \|\rho_{B_1}^{1/2}\|_2^2 \\ &= \|\sigma_{B_1}^{ij} \rho_{B_1}^{-1/2}\|_2^2 = V(\rho_B, O_{ij}^{B_1}), \end{aligned} \quad (34)$$

where the second line follows from Hölder's inequality and the third line holds by definition.

Similar to the definition (22), we can also rewrite the variance (29) as

$$\text{Tr} \{ [\mathcal{J}_{\rho_B}^{1/2} \circ \mathcal{J}_{\rho_{B_1}}^{-1/2}(\sigma_{B_1}^i)] [\rho_B^{-1} \mathcal{J}_{\rho_B}^{1/2} \circ \mathcal{J}_{\rho_{B_1}}^{-1/2}(\sigma_{B_1}^i) - 1] \}, \quad (35)$$

which is the first-order expansion of $\hat{S}[\mathcal{J}_{\rho_B}^{1/2} \circ \mathcal{J}_{\rho_{B_1}}^{-1/2}(\sigma_{B_1}^i) || \rho_B]$. In this form (35), we find that the map

$$\mathcal{R}_{\rho}^{B_k \rightarrow B} = \mathcal{J}_{\rho_B}^{1/2} \circ \mathcal{J}_{\rho_{B_k}}^{-1/2} \quad (36)$$

is just the Petz recovery map of the completely positive trace-preserving (CPTP) map $\mathcal{N}_{B \rightarrow B_k} = \text{Tr}_{\bar{B}_k}$ with respect to the reference state ρ_B ; cf. [33]. In this way, we can rewrite, by using (22) and (23),

$$\begin{aligned} \hat{S}(\sigma_{B_1}^i || \rho_{B_1}) &= \text{Tr} \{ \rho_{B_1} \mathcal{J}_{\rho_{B_1}}^{-1/2}(\sigma_{B_1}^i) \ln [\mathcal{J}_{\rho_{B_1}}^{-1/2}(\sigma_{B_1}^i)] \} \\ &= \text{Tr} \{ \mathcal{R}_{\rho}^{B_1 \rightarrow B}(\sigma_{B_1}^i) \ln [\rho_B^{-1} \mathcal{R}_{\rho}^{B_1 \rightarrow B}(\sigma_{B_1}^i)] \} \\ &= \hat{S}[\mathcal{R}_{\rho}^{B_1 \rightarrow B}(\sigma_{B_1}^i) || \rho_B]. \end{aligned} \quad (37)$$

The final expression pulls the subsystem BS entropy to the global one, which would be easier to make bounds.

A thing we should keep in mind is that the relations derived in this section are mainly mathematical relations with their physical meanings uninterpreted. The punch line is that we can approach the subsystem ETH (5) and (6) by bounding either $V(\rho_B, O_i^{B_1})$ or $\hat{S}[\mathcal{R}_{\rho}^{B_1 \rightarrow B}(\sigma_{B_1}^i) || \rho_B]$, and $V(\rho_B, O_{ij}^{B_1})$ based on (30), (32), (34), and (37).

IV. SUBSYSTEM ETH FOR TRANSLATION INVARIANT SYSTEMS

Now we can turn to the proof of the subsystem ETH. The strategy is to derive general bounds on the trace distance and show that they are small in the large- N limit.

We consider the macroscopic observable that is composed solely of local operators as in [13] or the translation invariant quantum lattice systems as in the last paragraph of Sec. II B of [12]. As in (18), we define the average *formal* observable,

$$O_i^B = \frac{1}{\mathcal{C}} \sum_k O_i^{B_k} = \frac{1}{\mathcal{C}} \sum_k \mathcal{J}_{\rho_{B_k}}^{-1/2}(\sigma_{B_k}^{i,1}), \quad (38)$$

where $\sigma_{B_k}^{i,1}$ are the translational copies of $\sigma_{B_1}^i$. It can also be obtained by translating state Π_B^i and then taking the partial trace. Here, we assume an equipartition of the lattice into subsystems with the same size, so that

$$\mathcal{C} = N/N_A, \quad (39)$$

if the number of sites in B_1 is N_A . We still have $\langle O_i^B \rangle = 1$. The quantum variance

$$V(\rho, O_i^B) = \text{Tr} \left\{ \left[\frac{1}{\mathcal{C}} \sum_k \mathcal{R}_{\rho}^{B_k \rightarrow B}(\sigma_{B_k}^{i,1}) \right] \left[\rho_B^{-1} \frac{1}{\mathcal{C}} \sum_l \mathcal{R}_{\rho}^{B_l \rightarrow B}(\sigma_{B_l}^{i,1}) - 1 \right] \right\}, \quad (40)$$

as given by (35), is the first-order expansion of the BS relative entropy,

$$\hat{S} \left[\frac{1}{\mathcal{C}} \sum_k \mathcal{R}_{\rho}^{B_k \rightarrow B}(\sigma_{B_k}^{i,1}) \middle| \middle| \rho_B \right]. \quad (41)$$

Since the Petz recovery map $\mathcal{R}_{\rho}^{B_k \rightarrow B}$ is also CPTP, we see that $\frac{1}{\mathcal{C}} \sum_k \mathcal{R}_{\rho}^{B_k \rightarrow B}(\sigma_{B_k}^{i,1})$ is also a legitimate density matrix. For example, consider that there is no correlation between the blocks B_1, \dots, B_C , i.e., $\rho_B = \rho_{B_1} \otimes \dots \otimes \rho_{B_C}$. Then,

$$\mathcal{R}_{\rho}^{B_k \rightarrow B}(\sigma_{B_k}^{i,1}) = \rho_{B_1} \otimes \dots \otimes \sigma_{B_k}^{i,1} \otimes \dots \otimes \rho_{B_C}.$$

By the joint convexity of relative entropy, it is easy to show that

$$\begin{aligned} \hat{S} \left[\frac{1}{\mathcal{C}} \sum_k \mathcal{R}_{\rho}^{B_k \rightarrow B}(\sigma_{B_k}^{i,1}) \middle| \middle| \rho_B \right] &\leq \frac{1}{\mathcal{C}} \sum_k \hat{S}[\mathcal{R}_{\rho}^{B_k \rightarrow B}(\sigma_{B_k}^{i,1}) \middle| \middle| \rho_B] \\ &= \frac{1}{\mathcal{C}} \sum_k \hat{S}(\sigma_{B_k}^{i,1} \middle| \middle| \rho_{B_k}) = \hat{S}(\sigma_{B_1}^{i,1} \middle| \middle| \rho_{B_1}), \end{aligned} \quad (42)$$

the last expression of which is just the local (in)distinguishability. In (42), we supposed that the state ρ_B is translation invariant; this requirement is naturally fulfilled by the canonical ensemble. As we can see from (42), if the $\hat{S}(\sigma_{B_k}^{i,1} \middle| \middle| \rho_{B_k})$ are small for all blocks B_i , then $\hat{S}[\frac{1}{\mathcal{C}} \sum_k \mathcal{R}_{\rho}^{B_k \rightarrow B}(\sigma_{B_k}^{i,1}) \middle| \middle| \rho_B]$ must be small, but the converse is not true.

To prove the subsystem ETH, we need to show that

$$\hat{S} \left[\frac{1}{\mathcal{C}} \sum_k \mathcal{R}_{\rho}^{B_k \rightarrow B}(\sigma_{B_k}^{i,1}) \middle| \middle| \rho_B^c \right] \sim \mathcal{O}(N_A/N) \quad (43)$$

$$\text{or } V(\rho^c, O_i^B) \sim \mathcal{O}(N_A/N). \quad (44)$$

First, the quantum variance can be rewritten as

$$\begin{aligned} V(\rho, O_i^B) &= \frac{1}{\mathcal{C}^2} \sum_k V(\rho, O_i^{B_k}) \\ &+ \frac{1}{\mathcal{C}^2} \sum_{k \neq l} \text{Tr} [O_i^{B_k} \otimes O_i^{B_l} (\rho_{B_k B_l} - \rho_{B_k} \otimes \rho_{B_l})]. \end{aligned} \quad (45)$$

The first term in (45) is the local variance, in which the terms

$$V(\rho, O_i^{B_k}) = \|(\sigma_{B_k}^{i,1} - \rho_{B_k}) \rho_{B_k}^{-1/2}\|_2^2 = V(\rho, O_i^{B_1}) \quad (46)$$

will not grow with \mathcal{C} . So we have

$$\frac{1}{\mathcal{C}^2} \sum_k V(\rho, O_i^{B_k}) = V(\rho, O_i^{B_1}) \times \mathcal{C}^{-1}. \quad (47)$$

The second term in (45) depends on the correlations between B_k and B_l . Suppose that the correlations of the canonical thermal state decay algebraically, i.e.,

$$\|\rho_{B_k B_l}^c - \rho_{B_k}^c \otimes \rho_{B_l}^c\| \leq d(B_k, B_l)^{-\gamma}, \quad \gamma \geq D_L, \quad (48)$$

where D_L is spatial dimension of the lattice and $d(A, B)$ is the shortest lattice path length between two regions A and B . The γ characterizes the decay of the correlations, which is related to the specific model. Then the second term of (45) is less than or equal to

$$\frac{O_{\max}^2}{\mathcal{C}^2} \sum_k \sum_{d=1}^{\infty} n_d d^{-\gamma} = d_{\text{eff}}^{-\gamma} \times \frac{O_{\max}^2}{\mathcal{C}}, \quad (49)$$

where $O_{\max} = \|O_i^{B_1}\|_{\infty}$, and n_d is the number of blocks that are of distance d from B_k . For lattices with spatial dimension D_L , we have, in general, $n_d \propto d^{D_L-1}$. The d_{eff} is the effective distance given by $\sum_{d=1}^{\infty} n_d d^{-\gamma}$, while the \sum_k in (49) gives \mathcal{C} . Combining the above bounds, we see that (44) holds. Due to the translation invariance of ρ^c , the variance for different blocks should give the same result. Therefore, for many O_i^B or equivalent σ^i , there should be

$$V(\rho, O_i^B) \sim V(\rho, O_i^{B_k}) = V(\rho, O_i^{B_1}). \quad (50)$$

This is an analog of the relation $\Delta(\rho, A^{B_1}) = \Delta(\rho, A^B)$ below (18) since the $O_i^{B_k}$ are also the translational copies of $O_i^{B_1}$ according to (38). With (50), we can rewrite (45) as

$$\begin{aligned} V(\rho, O_i^B) &\sim \frac{1}{\mathcal{C}(\mathcal{C}-1)} \sum_{k \neq l} \\ &\times \text{Tr} [O_i^{B_k} \otimes O_i^{B_l} (\rho_{B_k B_l} - \rho_{B_k} \otimes \rho_{B_l})]. \end{aligned} \quad (51)$$

It can provide a slightly tighter bound.

Second, we study the bounds on the BS relative entropy (41). To this end, define the m th moment of the (expanded logarithm) operator $O_i^B - 1$,

$$M^{(m)} = \text{Tr} [\rho_B^c (O_i^B - 1)^m], \quad (52)$$

which is the higher-moment generalization of (29). Then, by the power series of the matrix logarithm, we have

$$\begin{aligned} & \hat{S} \left[\frac{1}{\mathcal{C}} \sum_k \mathcal{R}_\rho^{B_k \rightarrow B}(\sigma_{B_k}^{i,1}) \middle| \middle| \rho_B^c \right] \\ &= \frac{1}{\mathcal{C}} \sum_k \text{Tr} \left[\rho_B^c O^{B_k} \ln \left(\frac{1}{\mathcal{C}} \sum_l O^{B_l} \right) \right] \\ &= V(\rho^c, O_i^B) + \dots + \frac{(-1)^n}{n-1} (M^{(n)} + M^{(n-1)}) + \dots \\ &= \frac{1}{2} V(\rho^c, O_i^B) + \sum_{n=3}^{\infty} \frac{(-1)^n}{(n-1)n} M^{(n)}. \end{aligned} \quad (53)$$

The first term $V(\rho^c, O_i^B)$ has been bounded as in (49). The other terms in (53) depend on the multipartite correlations, and the higher moments $M^{(m)}$ in them can be bounded in the same way as in [13],

$$M^{(m)} \leq \frac{1}{\mathcal{C}^m} \mathcal{O}(\mathcal{C}^{m/2}) \sim \mathcal{O}(\mathcal{C}^{-m/2}), \quad (54)$$

where we omit those parts that do not increase with \mathcal{C} . This $\mathcal{O}(\mathcal{C}^{-m/2})$ behavior decays faster than $\mathcal{O}(\mathcal{C}^{-1})$, so the BS entropy should be mainly bound by the behavior of the first variance term. Thus, we obtain the overall bounds (43) on the BS relative entropy.

Third, if we replace the canonical thermal state ρ^c by another local state σ_{B_i} in the above formulas, we will find that the scaling analysis still holds. In other words, for two local states, we have

$$\|\sigma_{B_k}^i - \sigma_{B_l}^i\| \sim \mathcal{O}(\mathcal{C}^{-1/2}). \quad (55)$$

This means the concentration of states of different subsystems to a certain common equilibrium state. However, this (55) is not the off-diagonal subsystem ETH (6). In fact, (6) holds in the following sense: From the inequality (34) and the fact that the bound on variance in the first step of the proof does not depend on the specific forms of measurements, one obtains

$$\|\sigma^{ij}\|_1 \sim \mathcal{O}(\mathcal{C}^{-1/2}). \quad (56)$$

We have therefore successfully proved the subsystem ETH by showing the bounds or decaying behaviors (43) and (44). We remark that this decay behavior $\mathcal{O}(N_A^{1/2}/N^{1/2})$ is qualitatively consistent with the observations made in [16] that the subsystem must be small compared to the total system size. This is simply because for larger N_A , the faster the decaying speed, and hence the remaining bound should be the smaller N_A .

In the previous proof, we mainly considered the case where ρ_B is a Gibbs state. But in fact, our proof mainly uses the strict positivity of ρ_B and the correlation decay (48). Therefore, as long as these two properties are satisfied, other states can also be used, such as microcanonical ensembles or certain evolutionary steady states. Of course, when other states are selected, the bounds of O_{\max} will also be affected, thus affecting the tightness of the bound.

Compared to the proofs given in (the Appendix) of [12], we have changed the (in)distinguishability measure (12) to the variance or BS relative entropy. We can apply such a replacement back to the proofs of the weak ETH with eigenstate

typicality as in [12] to see what happens. Similar to Eqs. (15) and (32), the (in)distinguishability measure in (19) satisfies

$$|\text{Tr}[(\Pi^i - \rho_B)A^B]|^2 \leq V \left[\rho, \frac{1}{\mathcal{C}} \sum_k \mathcal{J}_{\rho_{B_1}}^{-1/2}(\sigma_{B_1}^{i,k}) \right] \|A^{B_1}\|_{\infty}^2. \quad (57)$$

By replacing $d(\Pi_B^j, \rho; A)$ with the variance, we see that the probabilistic typicality (13) becomes

$$\langle V_{\text{dg}} \rangle := \sum_i p_i V \left[\rho, \frac{1}{\mathcal{C}} \sum_k \mathcal{J}_{\rho_{B_1}}^{-1/2}(\sigma_{B_1}^{i,k}) \right]. \quad (58)$$

Similarly, we can consider the off-diagonal probabilistic typicality,

$$\langle V_{\text{off}} \rangle := \sum_{i \neq j} p_i V \left[\rho, \frac{1}{\mathcal{C}} \sum_k \mathcal{J}_{\rho_{B_1}}^{-1/2}(\sigma_{B_1}^{ij,k}) \right]. \quad (59)$$

Similar to Eqs. (15) and (32), the off-diagonal measure also satisfies

$$|\text{Tr}[\Pi_B^{ij}A^B]|^2 \leq V \left[\rho, \frac{1}{\mathcal{C}} \sum_k \mathcal{J}_{\rho_{B_1}}^{-1/2}(\sigma_{B_1}^{ij,k}) \right] \|A^{B_1}\|_{\infty}^2, \quad i \neq j. \quad (60)$$

Let $\rho_{B_1} = \sum_{\alpha} p'_{\alpha} \Pi_{B_1}^{\alpha}$ be the state of the subsystem B_1 expanded in the orthonormal basis $\{\Pi_{B_1}^{\alpha}\}$ of rank-1 projectors. With these projectors and Eqs. (31) and (34), one can rewrite Eqs. (58) and (59) as

$$\begin{aligned} \langle V_{\text{dg}} \rangle + \langle V_{\text{off}} \rangle &= \sum_{i,\alpha,\beta} p_i p'_{\alpha}^{-1} \left| \text{Tr} \left[\left(\frac{1}{\mathcal{C}} \sum_k \sigma_{B_1}^{i,k} - \rho_{B_1} \right) \Pi_{B_1}^{\alpha\beta} \right] \right|^2 \\ &+ \sum_{i \neq j, \alpha, \beta} p_i p'_{\alpha}^{-1} \left| \text{Tr} \left[\left(\frac{1}{\mathcal{C}} \sum_k \sigma_{B_1}^{ij,k} \right) \Pi_{B_1}^{\alpha\beta} \right] \right|^2. \end{aligned} \quad (61)$$

Notice that the transformation (19) also applies to off-diagonal terms,

$$\left| \text{Tr} \left[\left(\frac{1}{\mathcal{C}} \sum_k \sigma_{B_1}^{ij,k} \right) A^{B_1} \right] \right|^2 = \left| \text{Tr} \left[\Pi^{ij} \left(\frac{1}{\mathcal{C}} \sum_k A^{B_k} \right) \right] \right|^2, \quad i \neq j. \quad (62)$$

Using Eqs. (19) and (62), we can convert the average local state back to the average observable. Then, according to the form of variance in formula (11), we have

$$\langle V_{\text{dg}} \rangle + \langle V_{\text{off}} \rangle = \sum_{\beta, \alpha} p'_{\beta} V \left[\rho, \frac{1}{\mathcal{C}} \sum_k \mathcal{J}_{\rho_{B_k}}^{-1/2}(\Pi_{B_k}^{\alpha\beta}) \right], \quad (63)$$

where $\Pi_{B_k}^{\alpha\beta}$ is the translational copies of $\Pi_{B_1}^{\alpha\beta}$. In (63), we have used the property that

$$\mathcal{J}_{\rho_{B_k}}^{-1/2}(\Pi_{B_k}^{\alpha\beta}) = (p'_{\alpha} p'_{\beta})^{-1/2} \Pi_{B_k}^{\alpha\beta}. \quad (64)$$

Since we assume that state ρ is translation invariant, therefore $\Pi_{B_k}^{\alpha}$ is still the diagonal basis of ρ_{B_k} . The right-hand side of inequality (63) can be bounded like inequality (45). It should be pointed out that due to the orthogonal relationship between operators,

$$\begin{aligned} \mathcal{J}_{\rho_{B_k}}^{-1/2}(\Pi_{B_k}^{\alpha}) [\mathcal{J}_{\rho_{B_k}}^{-1/2}(\Pi_{B_k}^{\alpha\beta})]^{\dagger} &= 0, \quad \beta \neq \alpha, \\ \mathcal{J}_{\rho_{B_k}}^{-1/2}(\Pi_{B_k}^{\alpha\gamma}) [\mathcal{J}_{\rho_{B_k}}^{-1/2}(\Pi_{B_k}^{\alpha\beta})]^{\dagger} &= 0, \quad \beta \neq \gamma, \end{aligned} \quad (65)$$

the corresponding local variance term satisfies

$$\begin{aligned} & p'_\alpha V[\rho, \mathcal{J}_{\rho_{B_k}}^{-1/2}(\Pi_{B_k}^\alpha)] + \sum_{\beta, \beta \neq \alpha} p'_\beta V[\rho, \mathcal{J}_{\rho_{B_k}}^{-1/2}(\Pi_{B_k}^{\alpha\beta})] \\ &= p'_\alpha V\left[\rho, p'_\alpha^{-1} \left(\Pi_{B_k}^\alpha + \sum_{\beta, \beta \neq \alpha} \Pi_{B_k}^{\alpha\beta} \right)\right]. \end{aligned} \quad (66)$$

The other terms are very small as long as the correlations decay fast enough. Combining Eqs. (58), (59), and (63), we have the Chebyshev-type inequality,

$$\begin{aligned} & P_\rho \left\{ \sum_j V\left[\rho, \frac{1}{C} \sum_k \mathcal{J}_{\rho_{B_1}}^{-1/2}(\sigma_{B_1}^{ij,k})\right] \geq \epsilon^2 \right\} \\ & \leq \frac{1}{\epsilon^2} \left\{ \sum_{\beta, \alpha} p'_\beta V\left[\rho, \frac{1}{C} \sum_k \mathcal{J}_{\rho_{B_k}}^{-1/2}(\Pi_{B_k}^{\alpha\beta})\right] \right\}, \end{aligned} \quad (67)$$

$$\begin{aligned} \langle V_{\text{off}} \rangle + \langle V_{\text{dg}} \rangle &= \sum_{i,j} p_i V\left[\rho, \frac{1}{C} \sum_k \mathcal{J}_{\rho_{B_1}}^{-1/2}(\sigma_{B_1}^{ij,k})\right] \\ &= \sum_{i \neq j, \alpha, \beta} p_i p_\alpha \left| \text{Tr} \left[\mathcal{J}_{\rho_{B_1}}^{-1/2} \left(\frac{1}{C} \sum_k \sigma_{B_1}^{ij,k} \right) \sigma_{B_1}^{\alpha\beta} \right] \right|^2 + \sum_{i, \alpha, \beta} p_i p_\alpha \left| \text{Tr} \left[\mathcal{J}_{\rho_{B_1}}^{-1/2} \left(\frac{1}{C} \sum_k \sigma_{B_1}^{i,k} - \rho_{B_1} \right) \sigma_{B_1}^{\alpha\beta} \right] \right|^2 \\ &= \sum_{i, j, \alpha \neq \beta} p_i p_\alpha \left| \text{Tr} \left[\frac{1}{C} \sum_k \sigma_{B_k}^{ij} \mathcal{J}_{\rho_{B_k}}^{-1/2}(\sigma_{B_k}^{\alpha\beta,1}) \right] \right|^2 + \sum_{i, j, \alpha} p_i p_\alpha \left| \text{Tr} \left[\frac{1}{C} \sum_k \sigma_{B_k}^{ij} \mathcal{J}_{\rho_{B_k}}^{-1/2}(\sigma_{B_k}^{\alpha,1} - \rho_{B_k}) \right] \right|^2 \\ &= \sum_{\alpha, \beta} p_\alpha V\left[\rho, \frac{1}{C} \sum_k \mathcal{J}_{\rho_{B_k}}^{-1/2}(\sigma_{B_k}^{\alpha\beta,1})\right]. \end{aligned} \quad (68)$$

This equation establishes the connection between V_{dg} , V_{off} , and $V(\rho, O_i^B)$, $V(\rho, O_{i \neq j}^B)$.

Now we briefly discuss the equivalence between the microcanonical and canonical ensembles. To this end, we consider a microcanonical energy shell $(E - \delta, E]$ with width δ with the index set

$$\mathcal{M}_{E,\delta} = \{i | E_i \in (E - \delta, E]\}. \quad (69)$$

The (in)distinguishability of the microcanonical and canonical ensembles can be bounded with

$$\begin{aligned} \|\rho_{B_1}^{\text{mc}} - \rho_{B_1}^c\|_1 &\leq \sum_{i \in \mathcal{M}_{E,\delta}} \frac{1}{D} \left\| \frac{1}{C} \sum_k \sigma_{B_1}^{i,k} - \rho_{B_1}^c \right\|_1 \\ &\leq \sum_{i \in \mathcal{M}_{E,\delta}} \frac{1}{D} \left\{ V\left[\rho^c, \frac{1}{C} \sum_k \mathcal{J}_{\rho_{B_1}^c}^{-1/2}(\sigma_{B_1}^{i,k})\right] \right\}^{1/2}, \end{aligned} \quad (70)$$

where we have used the joint convexity of Schatten norm and (32). In the large- N limit, we have, from (67), that the right-hand side of (70) is very small, so we can conclude the equivalence between the microcanonical and canonical ensembles in this case.

V. THE BOUND FROM THE CLUSTERING OF CORRELATIONS

It seems that the Hamiltonian of the system does not make an appearance in the above proof, but, in fact, the Hamiltonian

for $\epsilon > 0$. When the right-hand side of Eq. (67) is very small, we can conclude that the measurement results concentrate on the results predicted by ρ . It is similar to the weak ETH with eigenstate typicality, but it includes both diagonal and off-diagonal ETH and does not depend on specific measurements.

The local observable in V_{dg} and V_{off} only measures the state of B_1 , but it is determined by the average state of each block. On the contrary, the observable (38) will measure the state of each block, but is only determined by the state of B_1 . They look very different, but they are deeply connected, as we will show below. In Eq. (61), we use the variation form (31) and the spectral decomposition of ρ_{B_1} . If we use the variation form (35) and the spectral decomposition of ρ_B instead, we get

is important in the condition (48) of correlations leading to (49). We see that as long as the correlations decay fast enough, i.e., $\gamma \geq D_L$, the scaling (49) and hence the above proof of the subsystem ETH holds. For models with exponentially decaying correlations, the above conditions can be easily satisfied.

We remark that the behavior of (49) holds not only for short-range interactions, but also for some types of long-range interactions. To see this, let us recall that the mutual information of the Gibbs state has some general bounds; in particular, for long-range interactions of the form $1/d^{\eta+D_L}$, $\eta > 0$, we have, for high temperatures, the following bound on mutual information between two regions A and C :

$$I(A : C) \leq \beta \min(N_A, N_C) \frac{C_\beta}{d(A, C)^\eta}, \quad (71)$$

where C_β is a function of the inverse temperature β independent of the system size which can be found in [34]. The mutual information can be related to the relative entropy through $I(A : C) = S(\rho_{AC} \| \rho_A \otimes \rho_C)$, whence

$$\|\rho_{B_k B_l}^c - \rho_{B_k}^c \otimes \rho_{B_l}^c\| \leq \sqrt{2I(B_k : B_l)} \leq \frac{N_A^{1/2} \sqrt{2\beta C_\beta}}{d(B_k, B_l)^{\eta/2}}, \quad (72)$$

where we have assumed $N_A < N_C$ and used (30). Since $n_d \propto d^{D_L-1}$, we obtain

$$\lim_{C \rightarrow \infty} \sum_{d=1}^{C^{1/D_L}} d^{D_L-1} d^{-\eta/2} \times \left(\sum_k \frac{1}{C^2} \right) = O(C^{-\eta/(2D_L)}). \quad (73)$$

When $\eta > 0$, it is possible that the estimate (49) still holds. We see that for one-dimensional systems ($D_L = 1$), we require $\eta = 2$ to conform to the estimate (49). Compared to the numerical results reported in [35], this value is within the range of validity of strong ETH, i.e., $\eta + D_L \geq 0.6$, although with a slower speed of convergence.

VI. CONCLUSION AND DISCUSSION

We have studied the subsystem ETH for translation invariant quantum systems. We develop upon the setting for translation invariant systems given in [12] by relating the quantum variance to the BS relative entropy. Surprisingly, with this technical input, we are able to prove the subsystem ETH for translation invariant systems using the similar scaling analysis as in [12]. The proof given above is elementary, without referring to the advanced techniques from random matrix theory. Since the subsystem is stronger than the local ETH, our results corroborate the previous results for local ETH for translation invariant systems [9,11,12].

We have remarked that our results apply to some long-range interacting systems. Compared with the recent numerical test for one-dimensional translation invariant systems [35], the constraint on the interaction parameter here is less

stringent, but can be applied to other dimensions. However, adding an external driving field will make the system nonequilibrium [36], even when the system is translation invariant. Another point is that our results only restrict the decaying of error terms algebraically. The exponential decays of errors is a quite strong result, which might not be universal in view of the examples from large- c CFTs with $\mathcal{O}(c^0)$ decay [20,21,23].

In the analysis of (53), the higher moments are relevant. The higher-moment versions of ETH can be related to many interesting structures, such as the out-of-time-ordered correlation functions indicating quantum chaos [37]. This could be a possible approach to relating the chaotic conjecture and the present analysis without referring to random matrices. Moreover, it is also interesting to study the eigenstate fluctuation theorems [12,38] at the subsystem level, which might be a suitable situation for thermalized open quantum systems. These aspects are left to future investigations.

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