Critical Casimir effect in a disordered O(2)-symmetric model

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The critical Casimir effect appears when critical fluctuations of an order parameter interact with classical boundaries. We investigate this effect in the setting of a Landau-Ginzburg model with continuous symmetry in the presence of quenched disorder. The quenched free energy is written as an asymptotic series of moments of the model's partition function. Our main result is that, in the presence of a strong disorder, Goldstone modes of the system contribute either with either an attractive or a repulsive force. This result was obtained using the distributional zeta-function method without relying on any particular ansatz in the functional space of the moments of the partition function.

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I. INTRODUCTION

Quantum fields are mathematical objects that allow a general description of the physical world. In the axiomatic approach they are operator-valued generalized functions acting over test function spaces [1,2]. Such a description leads the local energy of quantum fields to attain negative values [3]. In the presence of boundaries, negative local energies generate attractive forces. This result, known in the literature as the Casimir effect [4,5], manifests itself for all types of fundamental fields, scalar, fermionic, and vector [6–9].

In an Euclidean functional integral description, due to the randomness properties of quantum fields, they need to be integrated over the functional space [10]. From such a functional or classical probabilistic point of view, it is known that if the mean of a nonzero random variable vanishes, their variance differs from zero. This fact alone suffices to give rise to Casimir forces. The physical reason behind the Casimir effect can be traced to the presence of massless excitations and the change of the thermodynamic equilibrium of the vacuum (state with zero number occupation) due to the presence of boundaries that change the fluctuating spectrum of the theory [11].

Holding the physical interpretation of the Casimir forces, one can expect that a similar effect happens for critical systems with infinite correlation lengths in the presence of boundaries. Such a situation was discussed in fluids first by Fisher and de Gennes [12]. As a matter of fact, thermal fluctuations can induce Casimir-like long-ranged forces in any correlated medium, with a critical system being an example. In such a situation the massless excitations are not associated with photons but with some other quasiparticles, *e.g.*, phonons or Goldstone bosons. Such an effect is referred to as the critical or the thermodynamic Casimir effect. So far the critical Casimir effect has enjoyed only a few reviews, *e.g.,* those of Refs. [13–17].

The quantum Nyquist theorem [18] allows one to identify regimes where thermal fluctuations dominate over those of quantum origin, with the possibility of systems becoming critical. Such situations are the subject of statistical field theory. When a system reaches the critical regime, correlations become long ranged and critical Casimir forces appear. Besides thermal fluctuations, disorder fluctuations can also drive a system to criticality [19]. A prototype model featuring disorder fluctuations is a binary fluid in a porous medium [20], whose critical behavior can be studied as a continuous field in the presence of a random field. When the binary-fluid correlation length is smaller than the porous radius, one has a system with finite-size effects in the presence of a surface field. When the binary-fluid correlation length is much larger than that of the porous radius, the random porous can exert a random field effect. In the latter case, introduction of boundaries gives rise to the critical Casimir effect [15].

A similar scenario, but with a discrete symmetry, was studied in Ref. [21]. The main result of that study was that a change in the sign of the Casimir force can happen depending on the ratio of the inverse of the correlation length and the disorder strength. This result is analogous to the situation of the electromagnetic Casimir effect which can change sign depending on the ratio between the permeability and the

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dielectric constant [22]; disorder fluctuations lead to a Casimir force that is attractive or repulsive. We would like to point that there are some papers in the literature discussing the critical Casimir effect with the disorder at the surface; see, e.g., Refs. [23,24].

The purpose of this work is to revisit the Casimir effect in disordered systems with a continuous symmetry. More specifically, we consider continuous fields that model order parameters possessing a continuous symmetry in scenarios where the disorder fluctuations dominate over the thermal fluctuations. Examples of systems realizing such a scenario include a binary fluid in the presence of an external random field in the critical regime, superfluids, and liquid crystals. In such a situation, when the criticality is reached, one has to take into account the soft modes (Goldstone bosons) due to the symmetry breaking [25,26]. Another difference, and perhaps a more technical one, is that in the approach that we adopt here we *do not choose any ansatz for the functional space* in the series of the quenched free energy. This procedure will be clarified along the following sections. Our primary aim in this work is to answer whether the soft modes associated with the Goldstone boson favor or suppress the Casimir force and whether they affect the sign of the force. The result that we obtain for such a question is that the soft modes do not affect the change of the sign of the force. However an interesting effect due to the disorder arises. In the regime of strong disorder, where we only have the Casimir effect due the presence of the soft mode, the Goldstone mode contribution may change from attractive to repulsive. In other words, the presence of disorder may change the sign of the "universal amplitude" due to the Goldstone modes.

The paper is organized as follows. In the firsts two sections we introduce the two main mathematical tools utilized in the paper. Section II presents the spectral zeta-function regularization method and how one can use it to obtain the Casimir energy of a system, while in Sec. III we introduce the distributional zeta-function method to evaluate the quenched free energy, revisiting the critical Casimir force due to the disorder. In Sec. IV we present our main results and calculations, and Sec. V contains our main conclusions alongside future perspectives.

II. CASIMIR ENERGY AND SPECTRAL ZETA-FUNCTION REGULARIZATION

In quantum field theory, Casimir force can be computed by analyzing either the local energy density [27–33] or the total energy [34–36] of the quantized fields. In this section we study the Casimir energy of a statistical field theory model describing a Gaussian scalar field $\phi(x_1,\dots,x_d)$ in a slab geometry with one compactified dimension, $\Omega_L \equiv \mathbb{R}^{d-1}\times[0,L]$. For simplicity, we assume Dirichlet boundary conditions:

$$\phi(x_1,\dots,x_{d-1},0)=\phi(x_1,\dots,x_{d-1},L)=0. \tag{1}$$

We start discussing the scalar field satisfying Dirichlet boundary conditions inside a box with sides L_1, L_2,...,L_d. The partition function of the theory is

$$Z=\int_{\Omega}[\mathrm{d}\phi]\; e^{-\frac{1}{2}\int \mathrm{d}^d x\; \phi(x)(-\Delta+m_0^2)\phi(x)}, \tag{2}$$

where Ω in the integral specifies the space of fields satisfying the boundary conditions, $[\mathrm{d}\phi]\equiv\prod_{x\in\Omega}\mathrm{d}\phi(x)$ is a formal measure over the space of functions Ω, $-\Delta$ is the Laplace operator, and m_0^2 the bare mass of the free field. Since the action is quadratic in the fields, the functional integral can be evaluated, yielding

$$Z=\left[\det\left(-\Delta+m_0^2\right)_{\Omega}\right]^{-\frac{1}{2}}, \tag{3}$$

where we omitted a normalization factor due to the total volume of the functional space, and the symbol Ω indicates the boundary condition under which the determinant must be computed. Using the fact that a positive semidefinite self-adjoint operator satisfies an eigenvalue equation, we can write such a determinant as

$$\det\left(-\Delta+m_0^2\right)_{\Omega}=\prod_{i=1}^{\infty}\lambda_i, \tag{4}$$

with the set of all λ_i being the spectrum defined by the operator and its boundary condition. Equation (4) is formally divergent and requires regularization. We use the spectral zeta-function regularization method [37–41]. The zeta-function regularization procedure is a special case of analytic regularization. The use of the latter regularization in Casimir effect was discussed in Refs. [42,43]. References [44–47] compare results for the Casimir energy obtained with an analytic regularization procedure and the traditional regularization using cutoff.

To give a meaning to Eq. (4), one starts defining the spectral zeta function, $\zeta_D(s)$, first for $\mathrm{Re}(s) > d/2$ as

$$\zeta_D(s)\equiv\sum_{i=1}^{\infty}\frac{1}{\lambda_i^s}, \tag{5}$$

where D specifies the differential operator under consideration. Second, extend it analytically to a maximal domain. Observe that zero belongs to its domain. Formally, from Eq. (5),

$$\frac{\mathrm{d}}{\mathrm{d}s}\zeta_D(s)|_{s=0}=-\sum_{i=1}^{\infty}\ln\lambda_i. \tag{6}$$

One can combine Eqs. (4) and (6) to write the partition function in Eq. (3) as

$$Z=\exp\left(-\frac{1}{2}\sum_{i=1}^{\infty}\ln\lambda_i\right)=\exp\left(\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\zeta_D(s)\Big|_{s=0}\right). \tag{7}$$

To proceed with the calculations, we must construct the appropriate $\zeta_D(s)$. It can be constructed by using the appropriate spectral measure in the Riemann-Stieltjes integral. All the information about the domain Ω_L and the boundary conditions are taken into account by the spectral measure. So in the continuous limit, one obtains $\zeta_D(s)$ as

$$\zeta_D(s)=\frac{A_{d-1}}{(2\pi)^{d-1}}\int \mathrm{d}^{d-1}p\sum_{n=1}^{\infty}\left[p^2+m_0^2+\left(\frac{\pi n}{L}\right)^2\right]^{-s}, \tag{8}$$

where $p^2 = p_1^2 + \cdots p_{d-1}^2$, and A_{d-1} is the area of the hypersurface in $d-1$ dimensions,

$$A_{d-1} \equiv \prod_{i=1}^{d-1} \lim_{L_i \to \infty} L_i, \tag{9}$$

where such a limit must be understood as $L_i \gg L$, $\forall\, i = 1, \ldots, d-1$. From here on one could proceed with the exact calculations of Ref. [38]; see also Ref. [48]. In the following we introduce the calculation method we will use later in Sec. IV. Such a method will reproduce the result in the literature via direct calculations. To proceed, let us set

$$\mathrm{d}^{d-1}p = \frac{2\pi^{\frac{d-1}{2}}}{\Gamma\left(\frac{d-1}{2}\right)} p^{d-2}\mathrm{d}p, \tag{10}$$

and the Mellin representation of a^{-s},

$$a^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{d}t\; t^{s-1} e^{-ta}, \tag{11}$$

to rewrite Eq. (8) as

$$\zeta_D(s) = \frac{2A_{d-1}\pi^{\frac{d-1}{2}}}{(2\pi)^{d-1}\Gamma\left(\frac{d-1}{2}\right)\Gamma(s)} \left(\frac{L^2}{\pi}\right)^s \times \int_0^\infty \mathrm{d}t\; t^{s-1} \sum_{n=1}^{\infty} e^{-tn^2\pi} \times \int_0^\infty \mathrm{d}p\; p^{d-2} \exp\left[\frac{-tL^2}{\pi}\left(p^2 + m_0^2\right)\right]. \tag{12}$$

The integration over the continuum modes can be readily performed. Additionally, we set $m_0^2 = 0$, because that is the case where the Casimir force appears (infinite correlation length), and rename $\zeta_D(s) \to \zeta_G(s)$, where G stands for Goldstone. Performing the integral, one obtains for $\zeta_G(s)$,

$$\zeta_G(s) = C_d(L, s) \int_0^\infty \mathrm{d}t\; t^{s-\frac{1}{2}(d+1)} \psi(t), \tag{13}$$

where we have defined the following quantities:

$$C_d(L, s) \equiv \frac{A_{d-1}}{(2L)^{d-1}\Gamma(s)} \left(\frac{L^2}{\pi}\right)^s, \tag{14}$$

$$\psi(t) \equiv \sum_{n=1}^{\infty} e^{-tn^2\pi}. \tag{15}$$

As one can see, the contribution of $\psi(t)$ is rapidly decreasing as $t \to \infty$. However, depending on the values of s and d, there are singularities at $t \to 0$ that need to be taken care of. As discussed in Ref. [38], the singularity can be removed assuming the system is confined to a large, but finite, box, which entails an infrared cutoff in the p integrals above. Instead of introducing an explicit infrared cutoff, we extract the finite part of the integral by using the following relations of $\psi(t)$ and the weight 1/2 modular form $\Theta(t)$ [49]:

$$\psi(t) = \tfrac{1}{2}[\Theta(t) - 1], \tag{16}$$

where

$$\Theta(t) \equiv \sum_{n\in\mathbb{Z}} e^{-tn^2\pi} \quad \text{and} \quad \Theta(1/t) = \sqrt{t}\,\Theta(t). \tag{17}$$

Combining the relation between $\psi(t)$ and $\Theta(t)$ together with the modular property of $\Theta(t)$ we can write

$$\psi(1/t) = t^{1/2}\psi(t) + \tfrac{1}{2}t^{1/2} - \tfrac{1}{2}. \tag{18}$$

Now we can carry out the analytic continuation of Eq. (13) with the change of variables $t \to 1/t$ and using Eq. (18), which leads to

$$\zeta_G(s) = \frac{C_d(L, s)}{2}\left[2I_{1,d}^G(s) + I_{2,d}^G(s) - I_{3,d}^G(s)\right], \tag{19}$$

with $I_{1,d}^G, \ldots$ being the integrals,

$$I_{1,d}^G(s) = \int_0^\infty \mathrm{d}t\; t^{\frac{d}{2}-s-1}\psi(t), \tag{20}$$

$$I_{2,d}^G(s) = \int_0^\infty \mathrm{d}t\; t^{\frac{d}{2}-s-1}, \quad \text{and,} \tag{21}$$

$$I_{3,d}^G(s) = \int_0^\infty \mathrm{d}t\; t^{\frac{d}{2}-s-\frac{3}{2}}. \tag{22}$$

The integral $I_{1,d}(s)$ is convergent for any values of s and d, whereas $I_{2,d}(s)$ diverges for $\mathrm{Re}(2s) < d$ and $I_d^{(3)}(s)$ diverges for $\mathrm{Re}(2s) < d-1$. As can be checked in Eq. (14), we have that $C_d(L, s) \to 0$ as $s \to 0$, which implies

$$\left.\frac{\mathrm{d}\zeta_G(s)}{\mathrm{d}s}\right|_{s=0} = \frac{1}{2}\left.\frac{\mathrm{d}C_d(L, s)}{\mathrm{d}s}\right|_{s=0} \times \left[2I_{1,d}^G(0) + I_{2,d}^G(0) - I_{3,d}^G(0)\right]. \tag{23}$$

The integral $I_{1,d}^G(0)$ is finite, positive definite, and does not depend on the distance of the plates L; it depends only on the dimension d and can be performed analytically. On the other hand, the divergent integrals $I_{2,d}^G(0)$ and $I_{3,d}^G(0)$ do not depend on the distance between the plates and can be dropped, considering that we have a large box, which implies a large, but finite, wavelength, as argued in Ref. [38] and mentioned above. Divergences would not appear if $m_0 \neq 0$. After some simplifications one can obtain that

$$\left.\frac{\mathrm{d}\zeta_G(s)}{\mathrm{d}s}\right|_{s=0} = \frac{A_{d-1}}{(2L)^{d-1}} I_{1,d}^G(0) = \frac{A_{d-1}}{(2L)^{d-1}} \frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{1}{n^d} = \frac{A_{d-1}}{(2L)^{d-1}} \frac{\zeta(d)}{2\pi}. \tag{24}$$

Using that $F = E - TS$ and the fact that $T = 0$ in our case, one concludes that

$$Z = e^{-F} = e^{-E} \Rightarrow E = -\frac{1}{2}\left.\frac{\mathrm{d}\zeta_G(s)}{\mathrm{d}s}\right|_{s=0}. \tag{25}$$

Now we can define the energy density and find that

$$\frac{E}{A_{d-1}} \equiv \epsilon_d(L) = -\frac{1}{2(2L)^{d-1}} \frac{\zeta(d)}{2\pi}, \tag{26}$$

which has, evidently, the correct sign and power law with L.

For $d = 3$, Eq. (26) results in

$$\epsilon_3(L) = -\frac{\zeta(3)}{16\pi L^2}, \tag{27}$$

which is the "universal" amplitude of the Goldstone modes [25]. The reason for the quotation marks will become clear at the end of this work. The Casimir force per unit of area

(Casimir pressure) can be calculated as the negative of the derivative with respect to L of Eq. (26).

In the next section we briefly review the technique that will be used to take into account the disorder, the distributional zeta-function method.

III. DISTRIBUTIONAL ZETA-FUNCTION METHOD

This section aims to review the distributional zeta-function method [50,51], the method we used to obtain the disorder-averaged free energy for a system described by statistical field theory or Euclidean quantum field theory. To exemplify the method, we use it to derive the Casimir force for a general configuration of the field multiplets, without using a saddle point approximation.

The partition function of the model for one disorder realization in the presence of an external source $j(x)$ is given by

$$Z(j,h)=\int[\mathrm{d}\phi]\,\exp\left[-S(\phi,h)+\int \mathrm{d}^d x\, j(x)\phi(x)\right], \tag{28}$$

where the action functional in the presence of additive (linearly coupled) disorder is

$$S(\phi,h)=S(\phi)+\int \mathrm{d}^d x\, h(x)\phi(x). \tag{29}$$

Here, $S(\phi)$ is the pure system action, and $h(x)$ is a quenched random field.

In a general situation, one can model a disordered medium by a real random field $h(x)$ in $\mathbb{R}^d$ with $\mathbb{E}[h(x)]=0$ and covariance $\mathbb{E}[h(x)h(y)]$, where $\mathbb{E}[\cdots]$ specifies the mean over the ensemble of realizations of the disorder. Some works have studied the case of a disorder modeled by a complex random field; see Refs. [52,53] and Sec. IV. As in the pure system case, one can define the system's free energy for one disorder realization $W(j,h)=\ln Z(j,h)$, the generating functional of connected correlation functions for one disorder realization. From $W(j,h)$ one can obtain the quenched free energy by performing the average over the ensemble of all disorder realizations:

$$\mathbb{E}\big[W(j,h)\big]=\int [\mathrm{d}h]P(h)\ln Z(j,h), \tag{30}$$

where $[dh]=\prod_{x\in\mathbb{R}^d} dh(x)$ is a formal functional measure, and $[dh]P(h)$ is the probability distribution of the disorder field.

For a general disorder probability distribution, the distributional zeta function, $\Phi(s)$, is defined as

$$\Phi(s)=\int[\mathrm{d}h]P(h)\frac{1}{Z(j,h)^s}. \tag{31}$$

For $s\in\mathbb{C}$, this function is defined in the region where the above integral converges. One defines the complex exponential $n^{-s}=\exp(-s\log n)$ for $\log n\in\mathbb{R}$. As proved in Refs. [50,51], $\Phi(s)$ is defined for $\mathrm{Re}(s)\geqslant 0$. Therefore the integral is defined in the half-complex plane, and an analytic continuation is unnecessary. We have that

$$\mathbb{E}[W(j,h)]=-\left.\frac{\mathrm{d}\Phi(s)}{\mathrm{d}s}\right|_{s=0^+},\quad \mathrm{Re}(s)\geqslant 0. \tag{32}$$

Using the Euler's integral representation for the γ function, we get

$$\Phi(s)=\frac{1}{\Gamma(s)}\int[\mathrm{d}h]P(h)\int_0^\infty dt\, t^{s-1}e^{-Z(j,h)t}. \tag{33}$$

The next step consists in expanding the exponential in the integral in a power series. The series expansion has a uniform convergence for each h in the domain $t\in[0,a]$, where a is a dimensionless arbitrary constant. We then split the integral into two pieces, one that is uniformly convergent in the interval $t\in[0,a]$ for finite a, and one that becomes small for $a\to\infty$. The contribution from the first piece then becomes a sum over all integer moments of the partition function, $\mathbb{E}[Z^k(j,h)]=\mathbb{E}\,[(Z(j,h))^k]$, while the second vanishes exponentially for large a. Explicitly, the average free energy can be represented by the following series of the moments of the partition function:

$$\mathbb{E}[W(j,h)]=\sum_{k=1}^\infty\frac{(-1)^{k+1}a^k}{kk!}\,\mathbb{E}\,[Z^k(j,h)] - \ln(a)-\gamma+R(a,j), \tag{34}$$

where γ is the Euler-Mascheroni constant, and $R(a,j)$ is given by

$$R(a,j)=-\int[\mathrm{d}h]P(h)\int_a^\infty\frac{\mathrm{d}t}{t}\,e^{-Z(j,h)t}. \tag{35}$$

For large a, $|R(a,j)|$ is small; therefore the dominant contribution to the average free energy is given by the moments of the partition function of the model.

For concreteness, we assume a Gaussian form for the probability distribution of the disorder field $[dh]\,P(h)$:

$$P(h)=p_0\,\exp\left[-\frac{1}{2\rho^2}\int\mathrm{d}^d x\, h^2(x)\right], \tag{36}$$

where ρ is a positive parameter and p_0 is a normalization constant. In this case we have a δ-correlated disorder:

$$\mathbb{E}[h(x)h(y)]=\rho^2\,\delta^d(x-y). \tag{37}$$

After integrating the disorder, one obtains that each moment of the partition function $\mathbb{E}\,[Z^k(j,h)]$ can be written as

$$\mathbb{E}\,[Z^k(j,h)]=\int\prod_{i=1}^k[\mathrm{d}\phi_i^k]\,e^{-S_{\text{eff}}(\phi_i^k,j_i^k)}, \tag{38}$$

where $S_{\text{eff}}(\phi_i^k,j_i^k)$ is obtained integrating over the disorder field, a standard procedure in the literature [54,55]. In the above equation the superscript k in ϕ_i^k identifies the term of the series expansion given by Eq. (34), the subscript i is the component of the kth multiplet, and $\prod_{i=1}^k[d\phi_i^k]$ represents a product of formal functional measures. Also, from now on we set $j_i^k(x)=0\ \forall\ i$ and suppress its appearance as an argument of the quantities of interest.

To proceed, we use a Ginzburg-Landau model with $\lambda\phi^4$ interaction. After performing the disorder average, one obtains the effective action:

$$S_{\mathrm{eff}}(\phi_i^k) = \int \mathrm{d}^d x \sum_{i=1}^{k}\left[\frac{1}{2}\phi_i^k(x)(-\Delta + m_0^2)\phi_i^k(x) - \frac{\rho^2}{2}\sum_{i,j=1}^{k}\phi_i^k(x)\phi_j^k(x) + \frac{\lambda}{4}\sum_{i=1}^{k}(\phi_i^k(x))^4\right]. \quad (39)$$

The ϕ^4 term is necessary to stabilize a ground state of the system, since the disorder average introduces a negative contribution, quadratic in the fields. For simplicity, we assume *in this section* the ansatz $\phi_i^k(x) = \phi_j^k(x)$ for the function space, in which case the effective action becomes

$$S_{\mathrm{eff}}(\phi_i^k) = \int \mathrm{d}^d x \sum_{i=1}^{k}\left[\frac{1}{2}\phi_i^k(x)(-\Delta + m_0^2 - k\rho^2)\phi_i^k(x) + \frac{\lambda}{4}\sum_{i=1}^{k}(\phi_i^k(x))^4\right]. \quad (40)$$

Such a simplified ansatz has been studied in several works using this method [21,56–61] and leads to consistent results. Very recently [62], we have shown that one can avoid such a simplification and work with the full set of arbitrary field configurations $\{\phi_i^k(x)\}$. For now, to explain the zeta-distributional method to compute the Casimir energy, we proceed with the simplified ansatz.

One sees in Eq. (40) that there exists a combination of m_0^2, k, and ρ for which $m_0^2 - k\rho^2 < 0$, signaling the spontaneous breaking of the discrete symmetry $\phi_i^k \to -\phi_j^k$. As usual, one can move from the "false" vacuum to the "true" vacuum by an appropriate shift of the fields and identify the mass in the Gaussian contribution to the action:

$$m_\rho^2 \equiv 2(k\rho^2 - m_0^2) > 0. \quad (41)$$

To discuss the Casimir energy, it is enough to consider the Gaussian contribution. This is so because, as shown by several studies within quantum-field-theory scenarios [63–66], radiative corrections are always subleading compared to the free-field contribution. Since the critical Casimir effect studied here is formally identical to the quantum scalar case, the scenario is the same. Therefore, we drop the non-Gaussian terms in the action. Now, compacting one dimension and assuming Dirichlet boundary conditions, one can recast the mean over the kth moment, Eq. (38), as

$$\mathbb{E}\,[Z^k(h)] = \left[\det(-\Delta + m_\rho^2)_{\Omega_L}\right]^{-\frac{k}{2}}. \quad (42)$$

From now on, we consider the situation $m_\rho^2 > 0$. Using the spectral zeta-function regularization, Sec. II, we can write the functional determinant as

$$\mathbb{E}\,[Z^k(h)] = \exp\left[\frac{k}{2}\frac{\mathrm{d}}{\mathrm{d}s}\zeta_\rho(s)|_{s=0}\right]. \quad (43)$$

The $\zeta_\rho(s)$ can be constructed as

$$\zeta_\rho(s) = \frac{A_{d-1}}{(2\pi)^{d-1}}\int \mathrm{d}^{d-1}p \sum_{n=1}^{\infty}\left[p^2 + m_\rho^2 + \left(\frac{\pi n}{L}\right)^2\right]^{-s}. \quad (44)$$

Following the same steps as those between Eqs. (8) and (23), but for a nonzero mass, we obtain

$$\left.\frac{\mathrm{d}\zeta_\rho(s)}{\mathrm{d}s}\right|_{s=0} = \frac{1}{2}\left.\frac{\mathrm{d}C_d(L,s)}{\mathrm{d}s}\right|_{s=0} \times \left[2I_{1,d}^\rho(0) + I_{2,d}^\rho(0) - I_{3,d}^\rho(0)\right], \quad (45)$$

with

$$I_{1,d}^\rho(s) = \int_0^\infty \mathrm{d}t\; t^{\frac{d}{2}-s-1} e^{\frac{-L^2 m_\rho^2}{\pi t}}\psi(t), \quad (46)$$

$$I_{2,d}^\rho(s) = \int_0^\infty \mathrm{d}t\; t^{\frac{d}{2}-s-1} e^{\frac{-L^2 m_\rho^2}{\pi t}}, \quad (47)$$

$$I_{3,d}^\rho(s) = \int_0^\infty \mathrm{d}t\; t^{\frac{d}{2}-s-\frac{3}{2}} e^{\frac{-tL^2 m_\rho^2}{\pi}}. \quad (48)$$

Since now we have a nonzero mass, all integrals are convergent. Some care must be taken to define the energy of the system. First of all, we recall that at zero temperature the quenched free energy can be written as

$$F_q(L) = E_q(L) = -\mathbb{E}[W(j,h)] = \sum_{k=1}^{\infty}\frac{(-1)^k a^k}{kk!}\,\mathbb{E}\,[(Z(j,h))^k]. \quad (49)$$

Using the previous results and exponentiating the a^k, we obtain the Casimir energy in the presence of quenched disorder. From now on we call such a quantity *quenched Casimir* energy,

$$E_q(L) = \sum_{k=k_c}^{\infty}\frac{(-1)^k}{kk!}\exp\left[k\ln a + \frac{k}{2}\frac{\mathrm{d}}{\mathrm{d}s}\zeta_\rho(s)\Big|_{s=0}\right], \quad (50)$$

with k_c defined as

$$k_c \equiv \left\lfloor \frac{m_0^2}{\rho^2}\right\rfloor, \quad (51)$$

where $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

Analyzing the behavior of the integrals, Eqs. (46)–(48), it is immediate to see that for each $k > k_c$ the exponential damping makes their contributions subleading. So the main contribution in the expression for the Casimir energy will be

$$E_q(L) = \frac{(-1)^{k_c}}{k_c k_c!}\exp\left[k_c\ln a + \frac{k_c}{2}\frac{\mathrm{d}}{\mathrm{d}s}\zeta_\rho(s)\Big|_{s=0}\right]. \quad (52)$$

Clearly, from the last equation we can see the connection between a and the thermodynamic limit: since $\zeta_\rho(s)$ is an extensive quantity, a must be chosen to maximize the exponential. Therefore the Casimir force is given by

$$f_d(L) \equiv -\frac{\partial E_q(L)}{\partial L} = \frac{(-1)^{k_c+1}}{2k_c!}\frac{\partial}{\partial L}\frac{\mathrm{d}}{\mathrm{d}s}\zeta_\rho(s)|_{s=0}. \quad (53)$$

With the results obtained up to now, we have that

$$f_d(L) = \frac{A_{d-1}}{2^{d+1}}\frac{(-1)^{k_c+1}}{k_c!} \times \left\{-\frac{1}{L^d}\left[2I_{1,d}^\rho(0) + I_{2,d}^\rho(0) - I_{3,d}^\rho(0)\right] + \frac{L^{1-d}}{d-1}\frac{\partial}{\partial L}\left[2I_{1,d}^\rho(0) + I_{2,d}^\rho(0) - I_{3,d}^\rho(0)\right]\right\}. \quad (54)$$

The derivative of $I^{\rho}_{i,d}$ deserves a closer look. All of those integrals have an exponential which depends on L^2, and thanks to the exponential and the $\psi(t)$ term, their derivatives with respect to $L/2$ do not change their convergence properties. In a power series expansion in $L/2$, the contribution of the second term of Eq. (54) has a global contribution proportional to $-L^{2-d}$, which ensures that such a contribution is the leading one in powers of $L/2$. Now, defining the quenched Casimir pressure as the quenched Casimir force per unit area ($d-1$ volume), we can write

$$p_d(L) = \frac{(-1)^{k_c}}{2^{d+1}k_c!L^d}\left[\frac{L^2}{d-1}B_d(0) + D_d(0)\right], \tag{55}$$

where $B_d(0)$ and $D_d(0)$ are defined by

$$B_d(0) \equiv -\frac{1}{L}\frac{\partial}{\partial L}\left[2I^{\rho}_{1,d}(0) + I^{\rho}_{2,d}(0) - I^{\rho}_{3,d}(0)\right], \tag{56}$$

$$D_d(0) \equiv 2I^{\rho}_{1,d}(0) + I^{\rho}_{2,d}(0) - I^{\rho}_{3,d}(0), \tag{57}$$

which are positive constants. Clearly, for $m^2_{\rho} = 0$ the $B_d(0)$ vanishes and the well-known behavior is recovered. The most interesting feature of Eqs. (53) and (55) is the fact that the factor of $(-1)^{k_c}$ can change the force or pressure from repulsive to attractive depending on the values of m^2_0 and ρ^2. In the next section we further explore such a feature. Alongside considering the breaking of a continuous symmetry breaking, which creates soft modes in the system, we also do not make any ansatz over the function space.

IV. INTERPLAY BETWEEN SOFT AND CRITICAL MODES

In order to verify and go beyond the results of Ref. [21], we now consider a system with a continuous symmetry $U(1) \cong O(2)$. Another difference will be in the function space that we obtain after taking the average of the logarithm of the partition function. To start, let us consider the action

$$S(\phi, \phi^*) = \frac{1}{2}\int d^dx\big[\phi^*(x)\big(-\Delta + m^2_0\big)\phi(x) + \lambda V(\phi, \phi^*) + h^*(x)\phi(x) + h(x)\phi^*(x)\big]; \tag{58}$$

as before, m^2_0 is the bare mass, λ is a strictly positive constant, and $V(\phi, \phi^*)$ is a polynomial in the field variables. Here we would like to point out that in the case of interacting field theories confined in compact domains, is necessary to introduce surface counterterms [67–71]. The main difference here is that $h(x)$ is now a complex random field [52,53,72], with a probability distribution $P(h, h^*)$. Again, to simplify the problem, we consider a Gaussian distribution

$$P(h, h^*) \equiv p_0 e^{-\frac{1}{\rho^2}\int d^dx|h(x)|^2}. \tag{59}$$

The kth moment in the series, Eq. (38), with $j(x) = 0$, generalizes to

$$\mathbb{E}[Z^k(h)] = \int \prod_{i,j=1}^{k}\big[d\phi^k_i\big]\big[d\phi^{k*}_j\big]\, e^{-S_{\text{eff}}(\phi^k_i, \phi^{k*}_j)}, \tag{60}$$

with

$$S_{\text{eff}}\big(\phi^k_i, \phi^{k*}_j\big) = \sum_{i,j}\big[S_0\big(\phi^k_i, \phi^{k*}_j\big) + \lambda S_I\big(\phi^k_i, \phi^{k*}_j\big)\big]. \tag{61}$$

Here, $S_0(\phi^k_i, \phi^{k*}_j)$ is the quadratic action:

$$S_0\big(\phi^k_i, \phi^{k*}_j\big) = \frac{1}{2}\int d^dx\ \phi^{k*}_i(x)\big(G^0_{ij} - \rho^2\big)\phi^k_j(x), \tag{62}$$

in which, for later convenience, we defined

$$G^0_{ij} \equiv \big(-\Delta + m^2_0\big)\delta_{ij}, \tag{63}$$

and $S_I(\phi_i, \phi^*_j)$ is the interaction action corresponding to $V(\phi, \phi^*)$. The propagator corresponding to $S_0(\phi_i, \phi^*_j)$ is not diagonal in the (i, j) space. Such a nagging feature has been previously dealt with in different ways in the literature. For example, one can work with a nondiagonal propagator, as in some of the minimal supersymmetric standard model extensions [73,74], or one can use a Hubbard-Stratonovich identity as in the Bose-Hubbard model [75]. Still another way is to use the ansatz $\phi^k_i = \phi^k_j$, as discussed in the last section. Although such an ansatz leads to consistent results, it is an unnecessary simplification, as one can use the spectral theorem of linear algebra to formally diagonalize the propagator [62]. This diagonalization is a new development in the distributional zeta-function method, introduced in Ref. [62] in a different context.

The diagonalization proceeds as follows. We define the matrix of the $k \times k$ propagator as

$$G \equiv \begin{bmatrix} G^0_{11} - \rho^2 & -\rho^2 & \cdots & -\rho^2 \\ -\rho^2 & G^0_{22} - \rho^2 & \cdots & -\rho^2 \\ \vdots & \cdots & \ddots & \vdots \\ -\rho^2 & -\rho^2 & \cdots & G^0_{kk} - \rho^2 \end{bmatrix}_{k\times k}, \tag{64}$$

where G^0_{ij} was defined in Eq. (63). Since G is a symmetric matrix, it can be diagonalized by an orthogonal matrix S whose columns are the eigenvectors of G:

$$D = \langle S, GS\rangle, \tag{65}$$

where $\langle , \rangle$ denotes the natural inner product in (i, j) space, and D is the (diagonal) matrix of eigenvalues of G. Using the vector $\Phi(x)$ as the vector which has components $\phi_i(x)$, we can rewrite the sum of the quadratic actions as

$$\sum_{i,j=1}^{k} S_0(\phi_i, \phi^*_j) = \frac{1}{2}\int d^dx\ \langle\Phi(x), G\Phi^*(x)\rangle = \frac{1}{2}\int d^dx\ \langle\tilde{\Phi}(x), D\tilde{\Phi}^*(x)\rangle, \tag{66}$$

where $\tilde{\Phi}(x) = S\Phi(x)$ and

$$D = \begin{bmatrix} G^0_{11} - k\rho^2 & 0 & \cdots & 0 \\ 0 & G^0_{22} & \cdots & 0 \\ \vdots & \cdots & \ddots & \vdots \\ 0 & \cdots & & G^0_{kk} \end{bmatrix}_{k\times k}. \tag{67}$$

The matrix S can be calculated exactly; due to the degeneracy of the spectrum, there are many matrices that diagonalize G. The components of $\tilde{\Phi}(x)$ will be given by a linear combination of the $\phi_i(x)$ determined by S. Let $\varphi_i(x)$ denote the components of $\tilde{\Phi}(x)$ by $\varphi_i(x)$. Using the component notation, one can write the diagonal form of the quadratic action in Eq. (66) as

$$\sum_{i,j=1}^{k} S_0(\phi_i,\phi_j^*) = \frac{1}{2}\int \mathrm{d}^d x\ \varphi^*(x)\big(-\Delta + m_0^2 - k\rho^2\big)\varphi(x) + \frac{1}{2}\sum_{a=1}^{k-1}\int \mathrm{d}^d x\ \varphi_a^*(x)\big(-\Delta + m_0^2\big)\varphi_a(x), \tag{68}$$

where, to simplify the notation henceforth, we defined $\varphi_1(x) \equiv \varphi(x)$ and also changed the dummy index in the second line. Since S is an orthogonal matrix, one has that

$$\prod_{i,j=1}^{k}[\mathrm{d}\phi_i][\mathrm{d}\phi_j^*] = [\mathrm{d}\varphi][\mathrm{d}\varphi^*]\prod_{a,b=1}^{k-1}[\mathrm{d}\varphi_a][\mathrm{d}\varphi_b^*]. \tag{69}$$

Therefore, inserting Eqs. (68) and (69) into Eq. (60), we obtain

$$\mathbb{E}[Z^k(h)] = \int [\mathrm{d}\varphi][\mathrm{d}\varphi^*]\prod_{a,b=1}^{k-1}[\mathrm{d}\varphi_a][\mathrm{d}\varphi_b^*] \times e^{-S_\rho(\varphi,\varphi^*)-\sum_a S_O(\varphi_a,\varphi_a^*)-\lambda S_I(\varphi_a,\varphi_a^*)}, \tag{70}$$

where $S_\rho(\varphi,\varphi^*)$ is the action carrying the information on the strength ρ of the disorder,

$$S_\rho(\varphi,\varphi^*) = \frac{1}{2}\int \mathrm{d}^d x\, \varphi^*(x)\big(-\Delta + m_0^2 - k\rho^2\big)\varphi(x), \tag{71}$$

and $S_O(\varphi_a,\varphi_a^*)$ is a $O(k-1)$-symmetric action, independent of the disorder, given by

$$S_O(\varphi_a,\varphi_a^*) = \frac{1}{2}\int \mathrm{d}^d x\, \varphi_a^*(x)\big(-\Delta + m_0^2\big)\varphi_a(x). \tag{72}$$

The action $S_I(\varphi_a,\varphi_a^*)$ will not be needed in our study of the Casimir effect, but its presence with a $\lambda > 0$ is required to guarantee the action boundness. Its explicit expression is readily obtained by replacing Φ in the original action by $\tilde{\Phi} = S\Phi$.

We proceed recalling that each moment of the partition function contributes to the total quenched free energy, Eq. (34). To obtain the Casimir energy, we make one of the dimensions compact, $\mathbb{R}^d \to \mathbb{R}^{d-1}\times[0,L]$, and impose some boundary conditions. As can be seen in Eq. (71), there is a combination of k, m_0^2, and ρ for which the effective mass $m_0^2 - k\rho^2$ becomes negative, indicating the symmetry breaking $U(1)\to\mathbb{Z}_2$, giving rise to a Goldstone (soft) mode. Of course, the Casimir force is present even for those terms in the sum with a positive effective mass, as the condition for its presence is that the correlation length becomes of the order of the system's compactified size L. That is, the total energy receives contributions from symmetry-preserving and symmetry-breaking terms. Our interest in this work is to study the interplay between the contributions to the energy of the symmetry-breaking soft mode and the critical mode, both induced by the disorder. Therefore we neglect the symmetry-preserving modes. We assess this interplay by first performing a shift in the field $\varphi(x)$ to expose the symmetry breaking and then neglect all non-Gaussian terms, and finally, take the large L limit.

We perform the symmetry-breaking field shift for the situation with $m_0^2 - k\rho^2 < 0$ in Eq. (71). In the Cartesian representation of the fields $\varphi(x)$ and $\varphi^*(x)$ we have that

$$\varphi(x) = \frac{1}{\sqrt{2}}[\psi_1(x) + i\psi_2(x)], \tag{73}$$

$$\varphi^*(x) = \frac{1}{\sqrt{2}}[\psi_1(x) - i\psi_2(x)]. \tag{74}$$

The minima of the action lie on the circle

$$\psi_1^2 + \psi_2^2 = \frac{2\big(k\rho^2 - m_0^2\big)}{\lambda} \equiv v^2. \tag{75}$$

Defining the shifted fields $\chi = \psi_1 - v$ and $\psi = \psi_2$, the Gaussian part of the action becomes

$$S_\rho(\chi,\psi) = \frac{1}{2}\int \mathrm{d}^d x\big[\chi(x)\big(-\Delta + m_\rho^2\big)\chi(x) + \psi(x)(-\Delta)\psi(x)\big], \tag{76}$$

where we defined $m_\rho^2 = 2(k\rho^2 - m_0^2)$. In the new variables, after dropping all non-Gaussian terms, Eq. (70) assumes the following enlightening form:

$$\mathbb{E}[Z^k(h)] = Z_\rho Z_G [Z_O]^{k-1}, \tag{77}$$

where

$$Z_\rho = \int [\mathrm{d}\chi]\, e^{-\frac{1}{2}\int \mathrm{d}^d x\, \chi(x)(-\Delta + m_\rho^2)\chi(x)}, \tag{78}$$

$$Z_G = \int [\mathrm{d}\psi]\, e^{-\frac{1}{2}\int \mathrm{d}^d x\, \psi(x)(-\Delta)\psi(x)}, \tag{79}$$

$$Z_O = \int [\mathrm{d}\varphi][\mathrm{d}\varphi^*]\, e^{-\frac{1}{2}\int \mathrm{d}^d x\, \varphi^*(x)(-\Delta + m_0^2)\varphi(x)}, \tag{80}$$

where the partition functions are respectively the contributions of the disorder, the Goldstone mode, and an $O(k-1)$ symmetric model.

Now, we take a slab geometry with one compactified dimension, $\Omega_L = \mathbb{R}^{d-1}\times[0,L]$, and impose Dirichlet boundary conditions to all fields:

$$A_\alpha(x_1,\dots,x_{d-1},0) = A_\alpha(x_1,\dots,x_{d-1},L) = 0, \tag{81}$$

with $\alpha = \{\rho, G, O\}$ and $\{A_\rho, A_G, A_O\} = \{\chi,\psi,\varphi\}$ respectively. Using the result in Eq. (3) for each of the partition functions in Eqs. (78), (79), and (80), we obtain for the kth moment of the partition function, Eq. (77), the following expression:

$$\mathbb{E}[Z^k(h)] = \big[\det\big(-\Delta + m_\rho^2\big)_{\Omega_L}\big]^{-\frac{1}{2}}[\det(-\Delta)_{\Omega_L}]^{-\frac{1}{2}} \times \big[\det\big(-\Delta + m_0^2\big)_{\Omega_L}\big]^{-\frac{k-1}{2}}. \tag{82}$$

The last term contributes neither to the critical nor to the soft Goldstone modes. As such, it can be dropped by redefining the energy.

The relevant contributions to the Casimir energy can be regularized using the spectral zeta regularization,

$$\mathbb{E}[Z^k(h)] = \exp\left\{\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}[\zeta_\rho(s) + \zeta_G(s)]\bigg|_{s=0}\right\}. \tag{83}$$

By the same arguments used to obtain Eq. (52) in the previous section, one concludes that the main contribution to the total

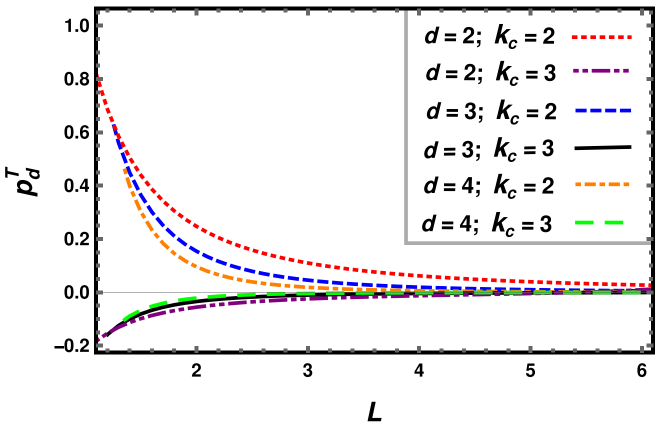

FIG. 1. Plot of the quenched Casimir pressure, Eq. (91), for dimensions 2,3, and 4, and $k_c = 2$, and 3.

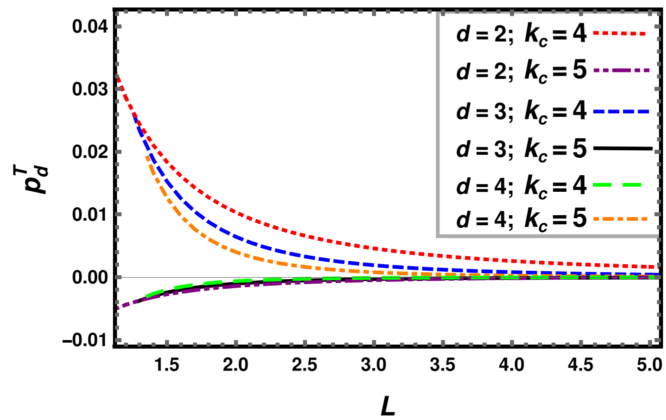

FIG. 2. Plot of the quenched Casimir pressure, Eq. (91), for dimensions 2,3, and 4, and $k_c = 4$, and 5.

quenched Casimir energy is given by

$$E_c^T = \frac{(-1)^{k_c}}{k_c k_c!} \exp\left\{ k_c \ln a + \frac{1}{2}\frac{d}{ds}[\zeta_\rho(s) + \zeta_G(s)]\bigg|_{s=0}\right\}. \tag{84}$$

We define the following zeta function,

$$\zeta_\alpha(s) = \frac{A_{d-1}}{(2\pi)^{d-1}} \int d^{d-1}p \sum_{n=1} \left[p^2 + m_\alpha^2 + \left(\frac{\pi n}{L}\right)^2 \right]^{-s}, \tag{85}$$

with $\alpha = \{\rho, G\}$ and $m_G^2 = 0$. Using the same definitions and arguments in Sec. II, one can rewrite $\zeta_\alpha(s)$ as

$$\zeta_\alpha(s) = C_d(L,s) \int_0^\infty dt\ t^{s-\frac{1}{2}(d+1)} e^{\frac{-tL^2}{\pi} m_\alpha^2} \psi(t). \tag{86}$$

Following the same steps taken between Eqs. (13) and (19), it is straightforward to obtain that

$$\zeta_\alpha(s) = C_d(L,s)\left[2I_{1,d}^\alpha(s) + I_{2,d}^\alpha(s) - I_{3,d}^\alpha(s)\right], \tag{87}$$

where

$$I_{1,d}^\alpha(s) = \int_0^\infty dt\ t^{\frac{d}{2}-s-1} e^{\frac{-L^2}{\pi t} m_\alpha^2} \psi(t), \tag{88}$$

$$I_{2,d}^\alpha(s) = \int_0^\infty dt\ t^{\frac{d}{2}-s-1} e^{\frac{-L^2}{\pi t} m_\alpha^2}, \tag{89}$$

$$I_{3,d}^\alpha(s) = \int_0^\infty dt\ t^{\frac{d}{2}-s-\frac{3}{2}} e^{\frac{-tL^2}{\pi} m_\alpha^2}. \tag{90}$$

One obtains the quenched Casimir force analogously to Eq. (54). Such a force receives contributions from the spectral zeta functions of soft and critical modes. In the case of $\alpha = G$ we have the same situation of Sec. II for $m_0 = 0$, *i.e.*, the contribution of the soft modes to the Casimir force is given by Eq. (24). For $\alpha = \rho$, we have the calculation of Sec. III, and the corresponding contribution is given by Eq. (45). Putting this all together, we obtain for the total quenched Casimir pressure of the system the following expression:

$$p_d^T(L) = \frac{(-1)^{k_c}}{k_c k_c! 2^{d-1} L^d}\left[\frac{L^2}{d-1} B_d(0) + D_d(0) + \frac{\zeta(d)}{2\pi}\right]. \tag{91}$$

Such a result can be plotted as function of L for different dimensions and values of k_c. Figures 1 and 2 a display $p_d^T(L)$ for dimensions 2, 3, and 4 for different values of k_c. Note the different scales in the axes of the two figures.

This result has some interesting features. First of all, if we ignore the Goldstone mode contributions, the resulting equation differs from Eq. (55) by a multiplicative factor, $4/k_c$. This factor comes from the exact diagonalization of the quadratic actions; when one uses the ansatz $\phi_i^k(x) = \phi_j^k(x)\ \forall\, i,\ j$, as used in Sec. III, the multiplicative factor does not appear. Of course, such a difference is irrelevant to gathering qualitative understanding. However, the qualitative similarity between the results holds only when one can neglect the contribution from the partition function Z_O, Eq. (80). This is the case whenever the corresponding action does not reach criticality, a situation that can occur due to nonzero temperature or finite-size effects. Another feature of Eq. (91) is that the critical and the soft mode effects are noncompetitive, they are of the same sign. Still another interesting feature is that when $k_c\rho \gg m_0^2$, one can neglect the contribution of Z_ρ, Eq. (78), to the Casimir energy; in practice, one can set $B_d(0) = D_d(0) = 0$ in Eq. (91). This is interesting because then only soft modes contribute but with a factor proportional to $(-1)^{k_c}$, which means that a change of sign may occur. In other words, there is a universal constant due to the soft modes, given by $\zeta(3)/16\pi$, with an overall sign that can be either negative (as usual) or positive, depending on the value of k_c.

V. CONCLUSIONS

In this work we analyzed the interplay in the Casimir energy between the soft modes from the breaking of a continuous symmetry, and the critical modes, due to a disorder linearly coupled to a complex scalar field. We found that both modes always have a cooperative effect, making the quenched Casimir pressure stronger. More interesting, we have seen that in the scenario of a strong disordered system, the Goldstone mode contribution to the pressure can be either positive or negative, depending on the ratio between the strength of disorder and mass parameter. This fact can be relevant in stability analyses of systems at nanoscale, where those effects are expected to be larger than 1 atm [16].

From a technical point of view, in this work we made use of a significant improvement on the application of the zeta distributional method regarding the functional space of fields.

The functional space is ansatz-free in the sense that we have not made any special choices of the fields in the nondiagonal effective action resulting from the disorder averaging. Moreover, we have made use of the spectral theorem of linear algebra to formally diagonalize the effective action in the full functional space. These features seem to be applicable to any Gaussian theory, bosonic or fermionic.

Further topics in the critical Casimir effect in disordered systems which deserve attention include analyses on how boundary shape and temperature and/or finite-size effects may affect the procedure that we described here. In addition, it would be interesting to extend the "ansatz-free" approach to interacting field theories, both for additive and multiplicative disorder. These subjects are under investigation by the authors.

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