Inclusive curvaturelike framework for describing dissipation: Metriplectic 4-bracket dynamics

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An inclusive framework for joined Hamiltonian and dissipative dynamical systems that are thermodynamically consistent, i.e., preserve energy and produce entropy, is given. The dissipative dynamics of the framework is based on the *metriplectic 4-bracket*, a quantity like the Poisson bracket defined on phase space functions, but unlike the Poisson bracket has four slots with symmetries and properties motivated by Riemannian curvature. Metriplectic 4-bracket dynamics is generated using two generators, the Hamiltonian and the entropy, with the entropy being a Casimir of the Hamiltonian part of the system. The formalism includes known previous binary bracket theories for dissipation or relaxation as special cases. Rich geometrical significance of the formalism and methods for constructing metriplectic 4-brackets are explored. Many examples of both finite and infinite dimensions are given.

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I. INTRODUCTION

A. Description

Various proposals have been suggested for categorizing or placing into a general formalism dissipative effects added to Lagrangian or Hamiltonian dynamical systems. An early example is the formalism of Rayleigh [1] who proposed a generalization of Lagrangian mechanics, by adding a term, the Rayleigh dissipation function, to Lagrange's equations of motion. However, here we follow on the early 1980s formalisms based on adding to generalizations of the Poisson bracket (see, e.g., [2,3]) a bilinear bracket which, akin to the Poisson bracket, is defined on phase space functions. These bracket formalisms were proposed in [4–9] for describing dynamics with dissipation in finite-dimensional systems, fluid mechanics, plasma models, and kinetic theories. In this paper we present an encompassing geometric formulation in terms of a quantity called the metriplectic 4-bracket, which like the Poisson bracket is defined on phase space functions, but has properties motivated by the Riemann curvature tensor, which subsumes ideas from the above publications as well as the double bracket formalism of [10–14].

A variety of metriplectic 4-bracket examples of both finite and infinite dimension will be described here, the former by reduction or mechanical modeling whereas the latter comes from fluid mechanics, plasma dynamics, and kinetic theory. However, we note, there are gradientlike systems for describing dissipative dynamics in other areas, such as the Cahn-Hilliard equation [15], which describes phase separation for binary fluids, the gradient structure of porous medium dynamics [16], and even the Ricci flows [15,17] instrumental in the proof of the Poincaré conjecture on S^3 [18]. We have found that the metriplectic 4-bracket formalism encompasses the cases we have examined, but additional examples along these lines are reserved for future publications. Indeed, the metriplectic 4-bracket formalism is most inclusive.

Our starting point is to consider finite-dimensional systems with a phase space $\mathcal{Z}$ being an n-dimensional manifold, on which a multibracket of the form $(f, g, h, \dots)$ is defined on smooth real-valued functions $f, g, h, \dots \in C^\infty(\mathcal{Z})$, so that we have

$$(\,\cdot,\,\cdot,\,\cdot,\,\cdot\,\dots) : C^\infty(\mathcal{Z}) \times C^\infty(\mathcal{Z}) \times C^\infty(\mathcal{Z}) \times \cdots \to C^\infty(\mathcal{Z}).$$

Examples of multilinear brackets of this form include Albeggiani's Poisson bracket of the nth order (see [19], p. 337), the well-known and oft cited Nambu bracket [20] (which was predated by Albeggiani), and the Lie algebra generalization of Nambu given in [21]. An n-bracket of this type generates nondissipative dynamics upon specification of $n-1$ "Hamiltonians" as follows:

$$\dot{o} = (o, H_1, H_2, \dots, H_{n-1}),$$

where $H_1, H_2, \dots, H_{n-1}$ are the Hamiltonians and the observable o is any dynamical variable. Usual Poisson brackets, canonical or noncanonical, correspond to the case of a bilinear bracket with a single Hamiltonian.

For infinite-dimensional systems, i.e., field theories, multibrackets are defined on functionals, maps of real-valued functions defined on some function space, that contain the dynamical variables defined on $\mathcal{D}$, the continuum label space of the field theory. Here, usually out of necessity, since the main systems of interest are nonlinear partial or partial integro-differential equations, we operate on a formal level and assume whatever is necessary for our operations to exist.

While the finite- and infinite-dimensional multibrackets described above generate nondissipative dynamics, the purpose of metriplectic dynamics, as developed in [7–9,22–26], is to place the first and second laws of thermodynamics

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into a dynamical systems setting using both a noncanonical (degenerate) Poisson bracket and a symmetric bilinear dissipative bracket, which together produce a combination of a Hamiltonian system with a gradient system associated with a degenerate metriclike tensor, providing a Lyapunov function by construction. Consequently, the two main functions of thermodynamics, an energy or Hamiltonian H and an entropy S, play important roles in the dynamics. It was for this reason that a parent 3-bracket $(f, g; h)$ was given in [7] that reduces to the dissipative bilinear bracket of metriplectic dynamics, where the H dependence was designed as a projector in order to maintain energy conservation. Additional projector examples were given in [9] and subsequent work and a general dissipative multibracket that can preserve additional variables were provided in [24].

Given that dissipation is governed by a kind of gradient system with a kind of metric tensor, the idea that a curvature tensor like object could be associated with dissipation was put forth in [9]. So, given a curvaturelike tensor, with two important functions associated with the dynamics, *viz.*, H and S, with the former conserved and the latter produced, we are led to the idea of a 4-bracket of the form $(f, k; g, h)$ with symmetries and properties consistent with those possessed by a fully contravariant curvature 4-tensor. It turns out that this is a general point of view that encompasses a wide variety of dissipative dynamics formalisms, including previous bilinear brackets for dissipation. We note that the formalism called GENERIC (rooted in [27] but developed in [28] and subsequent works) does not fit directly into the framework given here. This is because GENERIC (often used improperly to mean the prior metriplectic dynamics) is not bilinear and lacks the requisite symmetry in its binary dissipative bracket, a symmetry that would be induced, as we will see, by an underlying metriplectic 4-bracket. However, a procedure is given for turning GENERIC brackets into metriplectic brackets and thereby fitting them into the present theory.

B. Overview

The paper is organized as follows. Section II is about finite-dimensional systems. Here, in Secs. II A and II B, we review Poisson bracket and metriplectic dynamics, respectively, and establish notation that is to be used. Section II C contains the main introduced formalism of the paper, dissipative dynamics generated by a metriplectic 4-bracket. The basic notion is described in Sec. II C 1, where the *minimal metriplectic* properties of the 4-bracket are introduced, and in Sec. II C 2 it is shown how a metriplectic 4-bracket with these properties generates dissipative dynamics consistent with the laws of thermodynamics. Here we see how the metriplectic 2-bracket of [7,9] emerges from the formalism. In Sec. II C 3 we describe a geometric setting for manifolds with both Poissonian and Riemannian structure, which we call *Lie-metriplectic* manifolds. Section II C 4 shows that there is, in a sense that we define, a unique torsion-free metriplectic 4-bracket.

Section II D describes paths for constructing metriplectic 4-brackets. In Sec. II D 1 we see how they emerge from manifolds with Riemannian structure, affine or Levi-Civita, showing there is a large class of possibilities. In Sec. II D 2 the Kulkarni-Nomizu product is adapted for out purposes—it provides a way for building in the requisite symmetries given two symmetric bivector fields of one's choosing. In Sec. II D 3 we describe Lie algebra-based metriplectic 4-brackets, a formalism akin to the Lie-Poisson manifold construction, with a special pure Lie algebra case based on the Cartan-Killing metric. Section II D 4 uses a Poisson bracket-induced connection (see [29]) to describe a class of metriplectic 4-brackets with rich geometry.

In Sec. II E we show how the metriplectic 4-bracket formalism subsumes previous binary brackets. In Sec. II E 1 we see how it reduces to the Kaufman-Morrison bracket, while in Sec. II E 2 we explore how it relates to the double bracket formalism. Finally, in this subsection, in Sec. II E 3, we examine how GENERIC can be linearized and symmetrized, and then emerge naturally from the metriplectic 4-bracket formalism.

Section II F contains a collection of finite-dimensional examples, beginning in Sec. II F 1 with the ubiquitous free rigid body followed in Sec. II F 2 by the Kida vortex, another Lie algebra-based example. We conclude this section with Sec. II F 3, where other examples are mentioned.

Section III describes the leap from finite to infinite dimensions. In Sec. III A we review noncanonical Hamiltonian field theory and various metriplectic and double bracket field theories, and present a general form for the field-theoretic metriplectic 4-bracket. Section III B is the infinite-dimensional version of Sec. II E, where we show how metriplectic 4-bracket field theory subsumes previous theories.

Infinite-dimensional examples are given in Secs. III C and III D. Here we see how efficient it can be to construct metriplectic 4-bracket field theories. In Secs. III C 1, III C 2, and III C 3 various one-, two-, and three-dimensional fluid and plasmalike theories are developed, including one where the fluid helicity plays the role of entropy. In Secs. III D 1 and III D 2 we are concerned with kineticlike theories. The former gives a generalization of the Landau collision operator that has proven useful for computing equilibria, while the latter concerns finding the metriplectic 4-bracket from which the Boltzmann bracket of [27] emerges.

Finally, in Sec. IV we conclude. Here we briefly summarize some main points of the paper and discuss the usefulness of the metriplectic 4-bracket formalism, We discuss how it can be used to develop "honest" models, and we speculate about its usefulness for structure-preserving computation.

II. FINITE-DIMENSIONAL METRIPLECTIC DYNAMICAL SYSTEMS AND THE METRIPLECTIC 4-BRACKET

We consider dynamical systems with a real phase space manifold $\mathcal{Z}$ of dimension N. In a coordinate patch we denote a point of $\mathcal{Z}$ by $z = (z^1, z^2, \dots, z^N)$ with the usual tensorial notation. For example, given a vector field $\mathbf{Z} \in \mathfrak{X}(\mathcal{Z})$, where $\mathfrak{X}(\mathcal{Z})$ denotes differentiable vector fields on our phase space manifold $\mathcal{Z}$, we have the dynamical system, a set of autonomous first-order differential equations, given by

$$\dot{z}^i = Z^i(z), \quad i = 1, 2, \dots, N, \tag{1}$$

with, as usual, $\cdot$ denoting time differentiation. Such dynamics will be generated by bracket operations

$$C^\infty(\mathcal{Z}) \times C^\infty(\mathcal{Z}) \times \cdots \mapsto C^\infty(\mathcal{Z}) \tag{2}$$

defined on smooth functions $C^\infty(\mathcal{Z})$. It is conventional to call $C^\infty(\mathcal{Z})$ the space of 0-forms, $\Lambda^0(\mathcal{Z})$, and we will use these expressions interchangeably. Here all quantities are assumed to be real-valued, although we note that extensions from $\mathbb{R}$ to $\mathbb{C}$ are possible and remain to be fully explored.

A. Poisson dynamics

A phase space with Poisson manifold structure uses noncanonical Poisson brackets to generate flows. (See [30,31] for important seminal work and [3,32] for a physicist's perspective.) Such a binary operation,

$$\{\cdot,\cdot\} : C^\infty(\mathcal{Z}) \times C^\infty(\mathcal{Z}) \to C^\infty(\mathcal{Z}), \tag{3}$$

in addition to being bilinear, satisfies the following for all $f, g, h \in C^\infty(\mathcal{Z})$:

$$\text{antisymmetry:} \quad \{f, g\} = -\{g, f\} \tag{4}$$

$$\text{Jacobi identity:} \quad \{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0, \tag{5}$$

which provide a Lie algebra realization on $C^\infty(\mathcal{Z})$ (see, e.g., [32] chap. 14), and

$$\text{derivation:} \quad \{fg, h\} = f\{g, h\} + \{f, h\}g, \tag{6}$$

where fg denotes pointwise multiplication of functions in $C^\infty(\mathcal{Z})$. The Leibniz derivation property of (6) ensures that $\mathbf{Z}_f := \{\cdot, f\} \in \mathfrak{X}(\mathcal{Z})$ is a kind of Hamiltonian vector field.

For a Hamiltonian $H \in C^\infty(\mathcal{Z})$ the equations of motion in a coordinate patch take the following form in tensorial notation:

$$\dot{z}^i = \{z^i, H\} = J^{ij}\frac{\partial H}{\partial z^j}, \quad i, j = 1, 2, \dots, N, \tag{7}$$

where

$$\{f, g\} = \frac{\partial f}{\partial z^i} J^{ij} \frac{\partial g}{\partial z^j} \tag{8}$$

and repeated index notation is used in the second equality of (7) and in (8). We will call the bivector field (J^{ij}) the Poisson tensor. We can write (8) in two ways, namely,

$$\{f, g\} = J(\mathbf{d}f, \mathbf{d}g) = \langle \mathbf{d}f, J\mathbf{d}g\rangle,$$

where for $f, g \in \Lambda^0(\mathcal{Z})$, the exterior derivative gives $\mathbf{d}f, \mathbf{d}g \in \Lambda^1(\mathcal{Z})$, the space of 1-forms, and in the second equality we have the duality pairing $\langle\ ,\ \rangle$ between one-forms and vectors, with J considered as a bundle map $J : T^*\mathcal{Z} \to T\mathcal{Z}$ satisfying $J^* = -J$.

On symplectic manifolds, a special case of Poisson manifolds, $N = 2M$, and we may choose coordinates such that the Poisson tensor has the canonical form

$$J_c = \begin{pmatrix} O_M & I_M \\ -I_M & O_M \end{pmatrix}. \tag{9}$$

The choice of these coordinates reflects the usual splitting of z into the canonical coordinates (q, p).

Unlike the nondegenerate canonical Poisson bracket with Poisson tensor (9), where $\{f, C\} = 0\ \forall f \ \Leftrightarrow f = \text{constant}$ (a real number), on Poisson manifolds $\{f, C\} = 0\ \forall f$ is satisfied by nontrivial Casimir invariants. Generally speaking, the level sets of these quantities foliate the Poisson manifold and dynamics is confined to the leaf tagged by an initial condition for *any* Hamiltonian function. Such degenerate brackets, with Poisson tensor fields $(J^{ij}) \neq (J_c^{ij})$, were called noncanonical Poisson brackets in [33,34]. Casimir invariants play a special role as candidates for entropy functions in both the metriplectic formalism of Sec. II B and the curvature 4-bracket dynamics of Sec. II C 1.

A Lie Poisson bracket is a special kind of noncanonical Poisson bracket that is associated with a Lie algebra $\mathfrak{g}$. The natural phase space is actually the dual $\mathfrak{g}^*$. For $f, g \in C^\infty(\mathfrak{g}^*)$, $z \in \mathfrak{g}^*$, and $\mathbf{d}f \in \mathfrak{g}$, the bracket has the form

$$\begin{aligned}\{f, g\} &= \langle z, [\mathbf{d}f, \mathbf{d}g]\rangle \\ &= \frac{\partial f}{\partial z^i} c^{ij}{}_k z^k \frac{\partial g}{\partial z^j},\end{aligned} \tag{10}$$

where $[\ ,\ ]$ is the Lie bracket of $\mathfrak{g}$, $i, j, k = 1, 2, \dots, \dim(\mathfrak{g})$, z^i are coordinates for $\mathfrak{g}^*$, and $c^{ij}{}_k$ are the structure constants of $\mathfrak{g}$.

Just as these Lie-Poisson brackets are special Poisson brackets associated with Lie algebras, we will find in Sec. II D 3 that there are special metriplectic 4-brackets associated with Lie algebras.

B. Metriplectic dynamics

As noted above, metriplectic dynamics emerged in the 1980s from [4] and the adjacent papers [6,7], and [8], with the full set of axioms first appearing in [7,8]. The name metriplectic was introduced in [9] with the three basic axioms of [7,8] (given below) restated and several examples provided. The review here is adapted from the more recent publication [24]. As we will see, metriplectic dynamics nicely places the first and second laws of thermodynamics into a dynamical systems setting.

As above, a *metriplectic system* consists of a phase space manifold $\mathcal{Z}$, a Poisson bundle map $J : T^*\mathcal{Z} \to T\mathcal{Z}$, a bundle map $G : T^*\mathcal{Z} \to T\mathcal{Z}$, and two functions $H, S \in C^\infty(\mathcal{Z})$ with H being the Hamiltonian (energy) and S being the entropy. The dynamics will be defined in terms of binary brackets on functions $f, g, h \in C^\infty(\mathcal{Z})$, which we assume have the following properties:

(i) $(f, g) := \langle \mathbf{d}f, G\mathbf{d}g\rangle$ is a positive semidefinite symmetric bracket, i.e., $(\cdot,\cdot)$ is bilinear and symmetric, so $G^* = G$, and $(f, f) \geqslant 0\ \forall f \in C^\infty(\mathcal{Z})$; in coordinates the symmetric bracket has the form

$$(f, g) = \frac{\partial f}{\partial z^i} G^{ij} \frac{\partial g}{\partial z^j}, \tag{11}$$

and $G^{ij} = G^{ji},\ \forall\, i, j = 1, 2, \dots N$.

(ii) $\{S, f\} = 0$ and $(H, f) = 0\ \forall f \in C^\infty(\mathcal{Z}) \Longleftrightarrow J\mathbf{d}S = G\mathbf{d}H = 0$; in coordinates this expresses the null space conditions

$$J^{ij}\frac{\partial S}{\partial z^j} \equiv 0 \quad \text{and} \quad G^{ij}\frac{\partial H}{\partial z^j} \equiv 0. \tag{12}$$

The *metriplectic dynamics* of any observable (dynamical variable) o is given in terms of the two brackets and a generator $\mathcal{F} := H - \mathcal{T}S$ by

$$\begin{aligned}\dot{o} &= \{o, \mathcal{F}\} - (o, \mathcal{F}) \\ &= \{o, H - \mathcal{T}S\} - (o, H - \mathcal{T}S) \\ &= \{o, H\} + \mathcal{T}(o, S), \quad \forall\, o \in C^\infty(\mathcal{Z}),\end{aligned} \tag{13}$$

where $\mathcal{F}$ can be interpreted as a Helmholtz free energy and $\mathcal{T}$ interpreted as a global constant temperature [26]. For convenience, without loss of generality, we will henceforth set $\mathcal{T} = 1$.

In terms of coordinates in tensorial notation we have the ordinary differential equations

$$\dot{z}^i = J^{ij}\frac{\partial H}{\partial z^j} + G^{ij}\frac{\partial S}{\partial z^j}. \tag{14}$$

Geometrically, the vector field $\mathbf{Z}_H := J\mathbf{d}H \in \mathfrak{X}(\mathcal{Z})$ expresses the Hamiltonian part of these equations, while $\mathbf{Y}_S := G\mathbf{d}S \in \mathfrak{X}(\mathcal{Z})$ gives the dissipative part of the full metriplectic dynamics of (13) or (14).

The name metriplectic, as first given in [9], was chosen because these systems blend dissipative and Hamiltonian dynamics. The dissipative part, being generated by the symmetric bracket, is a degenerate gradient flow determined by a metriclike tensor G^{ij} accounting for the "metri" of metriplectic. We will see in this paper that there can also be an actual Riemannian metric, say, g. To distinguish the two, we will refer to the tensor G^{ij} as the G metric (even though it is degenerate). Because dynamics in a Poisson manifold is symplectic on a Casimir leaf, this motivated the "plectic" of metriplectic.

The definition of metriplectic systems was designed to have three immediate and important consequences:

(i) *Energy conservation: First law*:

$$\dot{H} = \{H, H\} + (H, S) \equiv 0. \tag{15}$$

(ii) *Entropy production: Second law*:

$$\begin{aligned}\dot{S} &= \{S, H\} + (S, S) \\ &= (S, S) \geqslant 0,\end{aligned} \tag{16}$$

where the second equality follows because the entropy is a Casimir. Here, in line with thermodynamics, we have entropy production; however, reversing the sign of the entropy gives a decreasing quantity as is typical for Lyapunov functions.

(iii) *Maximum entropy principle yields equilibria*: Suppose that a point z_* has any neighborhood U such that for every point $z \in U\backslash\{z_*\}$ such that $H(z) = H(z^*)$, $S(z) < S(z_*)$. Then, by the second law, z_* is necessarily an equilibrium of the metriplectic dynamics. This is akin to the free energy extremization of thermodynamics, as noted in [8,9], where it was suggested that one can build in degeneracies associated with Hamiltonian "dynamical constraints." For example, a good collision operator should conserve mass and momentum, in addition to energy (see also [24,35]). We will see that similar degeneracies can be naturally built into our metriplectic 4-bracket.

Proving conventional nondegenerate gradient flows achieve equilibrium states has a large literature dating to [36,37]. Some results for the degenerate flows generated by the metric 4-brackets of Sec. II C of the present paper are apparent, but the nature of the level sets of H and S complicates matters, with multiple possible basins of attraction and so on. Consequently, we leave this for future publication (some results will be included in [38]).

Although not treated in detail here, conservation of other invariants in addition to the Hamiltonian may be of interest. Suppose that $I \in C^\infty(\mathcal{Z})$ is a quantity conserved by the Hamiltonian part of the metriplectic dynamics, i.e., $\{I, H\} = 0$. Then, on an integral curve of the metriplectic dynamics, we have

$$\dot{I} = \{I, H\} + (I, S) = (I, S). \tag{17}$$

Thus, as pointed out in [9], this immediately implies that a function that is simultaneously conserved by both the full metriplectic dynamics and its Hamiltonian part is necessarily conserved by the dissipative part. Physically, it may be desirable for general metriplectic systems to conserve dynamical constraints, i.e., conserved quantities of its Hamiltonian part; the examples given in, e.g., [7–9,26] satisfy this condition, and a method based on multilinear brackets was given in [24].

C. The metriplectic 4-bracket and dynamics

1. The metriplectic 4-bracket

To motivate our metriplectic 4-bracket, we begin by supposing our phase space manifold $\mathcal{Z}$ is a Riemannian manifold with a curvature tensor R,

$$R : \mathfrak{X}(\mathcal{Z}) \times \mathfrak{X}(\mathcal{Z}) \times \mathfrak{X}(\mathcal{Z}) \to \mathfrak{X}(\mathcal{Z}), \tag{18}$$

i.e, for $\mathbf{X}, \mathbf{Y}, \mathbf{Z} \in \mathfrak{X}(\mathcal{Z})$, $R(\mathbf{X}, \mathbf{Y})\mathbf{Z} \in \mathfrak{X}(\mathcal{Z})$. As usual, we may write the Riemann curvature tensor in coordinate form as $R^i_{\ jkl}$, with $i, j, k, l = 1, 2, \dots, \dim(\mathcal{Z})$. Given a metric

$$g : \mathfrak{X}(\mathcal{Z}) \times \mathfrak{X}(\mathcal{Z}) \to \Lambda^0(\mathcal{Z}), \tag{19}$$

which has the usual covariant tensor expression g_{ij}, we can construct the totally covariant tensor

$$R : \mathfrak{X}(\mathcal{Z}) \times \mathfrak{X}(\mathcal{Z}) \times \mathfrak{X}(\mathcal{Z}) \times \mathfrak{X}(\mathcal{Z}) \to \Lambda^0(\mathcal{Z}) \tag{20}$$

defined by

$$R(\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathbf{W}) = g(\mathbf{R}(\mathbf{X}, \mathbf{Y})\mathbf{Z}, \mathbf{W}), \tag{21}$$

or in index form $R_{ijkl} = g_{im}R^m_{\ jkl}$.

The totally covariant Riemann tensor possess the following algebraic symmetries:

$$R_{ijkl} = -R_{jikl}, \tag{22}$$

$$R_{ijkl} = -R_{ijlk}, \tag{23}$$

$$R_{ijkl} = R_{klij}, \tag{24}$$

as well as the cyclic or as it is sometimes called the algebraic or first Bianchi identity,

$$R_{ijkl} + R_{iklj} + R_{iljk} = 0. \tag{25}$$

The differential or second Bianchi identity will not play a role in this first paper on 4-brackets.

The symbol R in (18) and (20) is used in different senses. In the remainder of this paper we will use R in other senses as well, ones not even necessarily related to Riemannian curvature. Similarly the symbol g in (19) and (21), the usual metric tensor, is also used for the cometric (inverse of metric tensor) and as an arbitrary function, $g \in \Lambda^0$. We do this to avoid the proliferation of symbols and trust the usage will be clear from context.

Given the above background, we follow a path analogous to that of Sec. II A to motivate our metriplectic bracket. Suppose we are given a fully contravariant tensor

$$R : \Lambda^1(\mathcal{Z}) \times \Lambda^1(\mathcal{Z}) \times \Lambda^1(\mathcal{Z}) \times \Lambda^1(\mathcal{Z}) \to \Lambda^0(\mathcal{Z}), \quad (26)$$

satisfying the same symmetries as the Riemann tensor. Such an object can be constructed, for example, by composing the fully covariant curvature tensor R with any tangent-cotangent isomorphism, i.e., raising and lowering operation, the metric being one obvious choice. We have a natural bracket on functions f, k, g, and n by

$$(f, k; g, n) := R(\mathbf{d}f, \mathbf{d}k, \mathbf{d}g, \mathbf{d}n), \quad (27)$$

which is our metriplectic 4-bracket. Restricted to a coordinate neighborhood, the metriplectic 4-bracket can be expressed in index form as

$$(f, k; g, n) = R^{ijkl}(z)\frac{\partial f}{\partial z^i}\frac{\partial k}{\partial z^j}\frac{\partial g}{\partial z^k}\frac{\partial n}{\partial z^l}. \quad (28)$$

From the above construction leading to (27) or (28), the following algebraic properties are immediately evident:
(i) Linearity in all arguments, e.g.,

$$(f + h, k; g, n) = (f, k; g, n) + (h, k; g, n) \quad (29)$$

(ii) The algebraic identities/symmetries

$$(f, k; g, n) = -(k, f; g, n), \quad (30)$$
$$(f, k; g, n) = -(f, k; n, g), \quad (31)$$
$$(f, k; g, n) = (g, n; f, k), \quad (32)$$
$$(f, k; g, n) + (f, g; n, k) + (f, n; k, g) = 0 \quad (33)$$

(iii) Derivation in all arguments, e.g.,

$$(fh, k; g, n) = f(h, k; g, n) + (f, k; g, n)h, \quad (34)$$

which is manifest when written in coordinates as in (28). Here, as usual, fh denotes pointwise multiplication.

Using the above definitions, we can define the contravariant analog of many constructions in standard Riemannian geometry. Of particular dynamical interest is the contravariant sectional curvature defined on 1-forms, say, $\sigma, \eta \in \Lambda^1(\mathcal{Z})$, by

$$K(\sigma, \eta) := R(\sigma, \eta, \sigma, \eta,) = (f, g, f, g), \quad (35)$$

where the second equality follows if $\sigma = \mathbf{d}f$ and $\eta = \mathbf{d}g$. Here we choose to forsake the conventional normalization of $|\sigma \wedge \eta|$ for simplicity. Throughout this paper, we will assume that

$$K(\sigma, \eta) \geqslant 0. \quad (36)$$

In Sec. II D 2 we will give a construction that ensures this positive semidefiniteness.

By the above construction, it is clear that a large class of metriplectic 4-brackets exist on Riemannian manifolds. In Sec. II D 4, we will show that the addition of a Poisson structure leads naturally to such a bracket. For our purposes, it is often convenient to part ways with the underlying geometry requiring only that metriplectic 4-brackets satisfy the algebraic properties of (30), (31), and (32) as well as the linearity and Leibnitz properties. This is equivalent to saying that the metriplectic 4-bracket derives from a 4-tensor satisfying (22), (23), and (24). In addition we will assume the positivity condition of (36) and refer to 4-tensors and associated 4-brackets that have these properties as being *minimal metriplectic*. In Sec. II C 4 we will see that the cyclic identity of (25) does not play a role in the dynamics. The differential Bianchi identity has ramifications akin to those of the Jacboi identity of Hamiltonian dynamics. These will be elucidated in a future publication.

2. Dynamics generated by the metriplectic 4-bracket

From the metriplectic 4-bracket of (27) we construct the symmetric, yet degenerate bracket:

$$(f, g)_H := (f, H; g, H) = R^{ijkl}\frac{\partial f}{\partial z^i}\frac{\partial H}{\partial z^j}\frac{\partial g}{\partial z^k}\frac{\partial H}{\partial z^l}. \quad (37)$$

Interchanging f and g amounts to interchanging i and k, and because we have symmetrical contraction in j and l, we get by (32)

$$(f, g)_H = (g, f)_H. \quad (38)$$

Thus, the G metric follows from (37), *viz.*,

$$G^{ik} = R^{ijkl}\frac{\partial H}{\partial z^j}\frac{\partial H}{\partial z^l}, \quad (39)$$

and with this bracket, the dissipative dynamics is generated by

$$\dot{z}^i = (z^i, H; S, H) = (z^i, S)_H = G^{ij}\frac{\partial S}{\partial z^j}, \quad (40)$$

where, for the full metriplectic dynamics, we would add the Poisson bracket contribution to the above.

Energy conservation comes automatically upon using (30) and (31), i.e.,

$$(f, H)_H = (H, f)_H = 0 \quad \forall f. \quad (41)$$

Then the entropy dynamics would be governed by

$$\dot{S} = (S, S)_H = (S, H; S, H) \geqslant 0, \quad (42)$$

where the inequality comes from (36) and ensures entropy production.

Thus, we see how metriplectic 2-brackets first given in [7,9] arise from metriplectic 4-brackets.

3. Lie-metriplectic manifolds: A metriplectic 4-bracket view

Since the Poisson manifolds of metriplectic dynamics usually arise from the standard picture of reduction [39], we give here some comments on this case. We will refer to manifolds of this type as *Lie-metriplectic* manifolds.

Hamiltonian systems with a configuration space being a Lie group G can lead to a reduced phase space $\mathcal{Z} = T^*G/G \cong \mathfrak{g}^*$ where $\mathfrak{g}^*$ is the dual of the Lie algebra $\mathfrak{g}$ of G. To endow G with a metriplectic structure, we need a metric g [not to be confused with $g \in \Lambda^0(\mathcal{Z})$]. While many such metrics may exist, left-invariant metrics (metrics constant with respect to the vector field module basis of left-invariant vector fields) are the only ones that respect the reduced phase space $\mathcal{Z}$. By left translation, these metrics are globally defined by their action at the identity of G. Given such a metric, we may consider the left-invariant curvature tensor R_G, which restricts to, and is in fact entirely encoded by the constant tensor $R_G|_e$ on $\mathfrak{g}$. Metrically raising the indices of R_G, the metriplectic 4-bracket on $\mathfrak{g}^*$ takes the form of (28) with $R^{ijkl} = R_G|_e(z^i, z^j, z^k, z^l)$ and z^i being the coordinates of $\mathfrak{g}^*$.

Because cases like the above, where the metriplectic 4-tensor is independent of the coordinate z, have special properties, we call these *Lie-metriplectic* 4-brackets. For such brackets, the z dependance of the associated metriplectic 2-bracket is determined by H. For example, when the Hamiltonian is quadratic, say, $H = H_{ij}z^i z^j$, the metriplectic 2-bracket is a quadratic form.

4. Torsion removal: Uniqueness of metriplectic 4-brackets

A minimal metriplectic 4-tensor A^{ijkl} obeying (30), (31), and (32) but not the cyclic symmetry of (33) is said to have torsion because this cyclic symmetry can be traced to the symmetry in two of the Christoffel symbol indices (see Sec. II D 1) in Riemannian geometry. Tensors that satisfy (30), (31), (32), and (33) are often called algebraic curvature tensors.

Minimal metriplectic tensors like A^{ijkl} can have their torsion removed (see, e.g., [40]) by defining the antisymmetric tensor

$$T^{ijkl} = \tfrac{1}{3}(A^{ijkl} + A^{iklj} + A^{iljk}) \tag{43}$$

and using it to construct

$$R := A - T, \tag{44}$$

which does indeed satisfy the algebraic Bianchi identity of (33), and thus is an algebraic curvature tensor.

Because T is totally antisymmetric, for any choice of functions f, g, H,

$$R^{ijkl}\frac{\partial f}{\partial z^i}\frac{\partial H}{\partial z^j}\frac{\partial g}{\partial z^k}\frac{\partial H}{\partial z^l} = A^{ijkl}\frac{\partial f}{\partial z^i}\frac{\partial H}{\partial z^j}\frac{\partial g}{\partial z^k}\frac{\partial H}{\partial z^l}, \tag{45}$$

where R is given by (44). Therefore, the metriplectic 2-bracket

$$(f, g)_H = (f, H; g, H) \tag{46}$$

does not see the torsion. Although the metriplectic 2-bracket and the concomitant dynamics it generates do not see torsion, the geometrical structure of the manifold without torsion is decidedly different from that with torsion. Consequently, the global understanding of the dynamics is facilitated by knowing the torsion can be removed.

In light of (45) we can use the procedure above to define a chain of isomorphisms between unique zero torsion metriplectic 4-brackets and minimal metriplectic tensors obeying (30), (31), and (32). Suppose we are given a minimal metriplectic 4-bracket, then, with T according to (43), $A - T$ defines the same metriplectic system as A. Hence, to define a better space of metriplectic 4-brackets we define the following equivalence relation (quotient): Given two minimal metriplectic 4-brackets A and B with respective T tensors T_A and T_B, we say A is equivalent to B, denoted symbolically $A \sim B$, if

$$A - T_A = B - T_B. \tag{47}$$

In the set of equivalent curvature tensors, there is a unique one that is torsion free for defining a metriplectic 4-bracket. To see this, suppose R and A are torsion-free algebraic curvature tensors that define the same metriplectic 2-bracket,

$$R^{ijkl}\frac{\partial H}{\partial z^j}\frac{\partial H}{\partial z^l} = A^{ijkl}\frac{\partial H}{\partial z^j}\frac{\partial H}{\partial z^l},$$

for every choice of some function H. Let $A = R + T$. Since algebraic curvature tensors form a vector space, it follows that T is an algebraic curvature tensor satisfying

$$T^{ijkl}\frac{\partial H}{\partial z^j}\frac{\partial H}{\partial z^l} = 0,$$

for all functions H. Upon choosing $H = z^r$, it follows that

$$T^{irkr} = 0, \tag{48}$$

where r is arbitrary and not summed over. Upon choosing $H = z^r + z^s$ it follows that

$$T^{irkr} + T^{isks} + T^{irks} + T^{iskr} = T^{irks} + T^{iskr} = 0, \tag{49}$$

where the first equality follows by using (48) in the first and second terms. If T did satisfy (33), then

$$T^{ijkl} = -T^{iljk} - T^{iklj} = T^{ilkj} - T^{iklj},$$

while antisymmetry in the second and fourth slot, because of (49), would further imply that

$$T^{ijkl} = -T^{ijkl} + T^{ijlk} = -T^{ijkl} - T^{ijkl} \Rightarrow T = 0.$$

Thus, $R = A$ and we see why the algebraic Bianchi identity of (33) is important and desirable. It removes redundancy in the theory. Also we note it allows R to be written as a sum of the Kulkarni-Nomizu products described in Sec. II D 2, which we will see is a quite useful tool.

D. Special metriplectic 4-bracket constructions

We consider now some natural 4-bracket constructions.

1. Affine and Levi-Civita forms

Given any affine manifold we can define the Riemann-Christoffel curvature tensor

$$R^i{}_{jkl} = \Gamma^i{}_{rk}\Gamma^r{}_{jl} - \Gamma^i{}_{rl}\Gamma^r{}_{jk} + \frac{\partial \Gamma^i{}_{jl}}{\partial z^k} - \frac{\partial \Gamma^i{}_{jk}}{\partial z^l}, \tag{50}$$

and further if our manifold is Riemannian, we have the usual Levi-Civita connection

$$\Gamma^l{}_{jk} = \frac{1}{2}g^{lr}\left(\frac{\partial g_{rk}}{\partial z^j} + \frac{\partial g_{rj}}{\partial z^k} - \frac{\partial g_{jk}}{\partial z^r}\right). \tag{51}$$

Thus, using the metric g we can construct

$$R^{ijkl} = g^{jr}g^{ks}g^{lt}R^i{}_{rst} \tag{52}$$

and hence obtain a metriplectic 4-bracket of the form of (28). This 4-bracket has the associate G-metric tensor

$$G^{ij} = R^{ikjl}\frac{\partial H}{\partial z^k}\frac{\partial H}{\partial z^l}. \tag{53}$$

Using (50), (51), and (52) we see that the G metric of (53) is trivially zero for Euclidean space, but in general it is a complicate expression in terms of the Riemannan metric g, designed to have $\partial H/\partial z$ in its kernel.

As explained in Sec. II C 1, this class of 4-brackets motivated our theory. However, our metriplectic construction is based on the algebraic properties of the bracket of (27). We point out that not all 4-brackets are based on such Riemann curvature tensors. Below we give some other constructions of metriplectic 4-brackets.

2. Kulkarni-Nomizu construction

Curvature 4-brackets with the requisite symmetries can be easily constructed by making use of the Kulkarni-Nomizu (K-N) product [41,42] (anticipated in [40]). Consistent with the bracket formulation of Sec. II C 1 we deviate from convention for K-N products and work on the dual space. Given two symmetric bivector fields, say, σ and μ, operating on 1-forms $\mathbf{d}f$, $\mathbf{d}k$, and $\mathbf{d}g$, $\mathbf{d}n$, the K-N product is defined by

$$\begin{aligned}\sigma \owedge \mu\,(\mathbf{d}f, \mathbf{d}k, \mathbf{d}g, \mathbf{d}n) = {}& \sigma(\mathbf{d}f, \mathbf{d}g)\,\mu(\mathbf{d}k, \mathbf{d}n)\\ &- \sigma(\mathbf{d}f, \mathbf{d}n)\,\mu(\mathbf{d}k, \mathbf{d}g)\\ &+ \mu(\mathbf{d}f, \mathbf{d}g)\,\sigma(\mathbf{d}k, \mathbf{d}n)\\ &- \mu(\mathbf{d}f, \mathbf{d}n)\,\sigma(\mathbf{d}k, \mathbf{d}g). \end{aligned} \tag{54}$$

Thus, we may define a 4-bracket according to

$$(f, k; g, n) = \sigma \owedge \mu(\mathbf{d}f, \mathbf{d}k, \mathbf{d}g, \mathbf{d}n). \tag{55}$$

In coordinates this gives a 4-bracket of the form (28) with

$$R^{ijkl} = \sigma^{ik}\mu^{jl} - \sigma^{il}\mu^{jk} + \mu^{ik}\sigma^{jl} - \mu^{il}\sigma^{jk}. \tag{56}$$

It is easy to show that such a bracket has all of the algebraic symmetries described in Sec. II C 1. In addition, it can be shown using the Cauchy-Schwarz inequality that positivity of the sectional curvature is satisfied if both σ and μ are positive semidefinite. Moreover, if one of σ or μ is positive definite, thus defining an inner product, then the sectional curvature of (36) satisfies $K(\sigma, \eta) \geqslant 0$ with equality if and only if $\sigma \propto \eta$. (See [43] for additional results along these lines, including theorems about completeness of K-N types of bases.) Thus, it is easy to build minimal metriplectic 4-brackets.

If one chooses both σ and μ to be proportional to the metric of a Riemannian manifold, then (56) reduces to

$$R^{ijkl} = K(g^{ik}g^{jl} - g^{il}g^{jk}), \tag{57}$$

which is the curvature associated with a form of metriplectic bracket first given in [22] [cf. Eq. (38) of that reference]. In the case were g is Euclidean this yields the metriplectic 4-bracket

$$(f, k; g, n) = K(\delta^{ik}\delta^{jl} - \delta^{il}\delta^{jk})\frac{\partial f}{\partial z^i}\frac{\partial k}{\partial z^j}\frac{\partial g}{\partial z^k}\frac{\partial n}{\partial z^l}, \tag{58}$$

whence $(f, H; g, H)$ produces the metriplectic bracket for the rigid body given in [9]. A Riemannian manifold is called a *space form* [44] if its sectional curvature is equal to a constant, say, K. The above are thus space form metriplectic 4-brackets.

In the cases of (57) and (58), the metric tensor G^{ij} of (14) is simply the projector that projects out $\partial H/\partial z^i$ using g^{ij} and δ^{ij}, respectively.

3. Lie algebra-based metriplectic 4-brackets

Metriplectic 2-brackets associated with Lie algebras were first investigated in [22] and later in [24] and [45]. Here we give two natural Lie algebra-related constructions for metriplectic 4-brackets.

First, given any Lie algebra $\mathfrak{g}$ with structure constants $c^{ij}_{\ \ k}$ and a symmetric semidefinite tensor g^{rs} one can construct a 4-bracket based on these quantities as follows:

$$(f, k; g, n) = c^{ij}_{\ \ r}c^{kl}_{\ \ s}\,g^{rs}\frac{\partial f}{\partial z^i}\frac{\partial k}{\partial z^j}\frac{\partial g}{\partial z^k}\frac{\partial n}{\partial z^l}. \tag{59}$$

It is easy to see that this bracket is minimal metriplectic, obeying (30), (31), and (32), but it does not have the cyclic symmetry of (33). However, this symmetry can be obtained, i.e., the torsion removed, by the procedure of Sec. II C 4, where $A^{ijkl} = c^{ij}_{\ \ r}c^{kl}_{\ \ s}\,g^{rs}$. Using the notation of Sec. II C 4, we have

$$T^{ijkl} = \tfrac{1}{3}g^{rs}\big(c^{ik}_{\ \ r}c^{lj}_{\ \ s} + c^{il}_{\ \ r}c^{jk}_{\ \ s} + c^{ij}_{\ \ r}c^{kl}_{\ \ s}\big).$$

Thus, it follows that A is equivalent to the following algebraic curvature tensor

$$B^{ijkl} = \frac{g^{rs}}{3}\big(2c^{ij}_{\ \ r}c^{kl}_{\ \ s} + c^{ik}_{\ \ r}c^{jl}_{\ \ s} - c^{il}_{\ \ r}c^{jk}_{\ \ s}\big). \tag{60}$$

Furthermore, A is also equivalent to the following minimal metriplectic tensors:

$$\begin{aligned}A^{ijkl} &\sim \frac{g^{rs}}{2}\big(c^{ij}_{\ \ r}c^{kl}_{\ \ s} - c^{ik}_{\ \ r}c^{lj}_{\ \ s} - c^{il}_{\ \ r}c^{jk}_{\ \ s}\big)\\ &\sim -g^{rs}\big(c^{ik}_{\ \ r}c^{lj}_{\ \ s} + c^{il}_{\ \ r}c^{jk}_{\ \ s}\big)\end{aligned} \tag{61}$$

and so on.

In this construction care must be taken in ensuring that the null space and signature of g^{rs}, so far only assumed symmetric, does not lead to undesirable effects, such as preventing the desired relaxation to equilibrium. The Euclidean metric $g^{rs} = \delta^{rs}$ is the simplest choice that alleviates these problems, but any metric is a possibility.

As a second case, a refinement of the first, suppose the tensor g^{rs} is proportional to the Cartan-Killing form, i.e., $g^{rs}_{CK} = \lambda c^{rm}_{\ \ \ n}c^{sn}_{\ \ \ m}$ for constant λ, as considered in [22]. Recall that for semisimple Lie algebras g_{CK} has no kernel and thus it possesses an inverse, and for compact semisimple Lie algebras like $\mathfrak{so}(3)$ it is in addition definite. With the choice of g_{CK} the bracket of (59) is naturally associated with any Lie algebra with no additional structure needed, akin to the bracket given in [22].

For the Lie algebra $\mathfrak{so}(3)$, $g^{rs}_{CK} \sim \delta^{rs}$, and the bracket using (60) reduces to the rigid body bracket of (58). In general, we find the following upon inserting g_{CK} into (59):

$$(f, k; g, n) = \lambda\, c^{ij}_{\ \ r}c^{kl}_{\ \ s}\,c^{rm}_{\ \ \ n}c^{sn}_{\ \ \ m}\frac{\partial f}{\partial z^i}\frac{\partial k}{\partial z^j}\frac{\partial g}{\partial z^k}\frac{\partial n}{\partial z^l}. \tag{62}$$

Using the Jacobi identity on the terms $c^{ij}_{\ \ r}\,c^{rm}_{\ \ \ n}$ and $c^{kl}_{\ \ s}\,c^{sn}_{\ \ \ m}$, gives an expression that can be manipulated into a tensor of the form

$$B^{ijkl}_{CK} = g^{rs}_{CK}\big(2c^{ij}_{\ \ r}c^{kl}_{\ \ s} - c^{ik}_{\ \ r}c^{lj}_{\ \ s} - c^{il}_{\ \ r}c^{jk}_{\ \ s}\big). \tag{63}$$

Thus, for this case, torsion is already removed. Moreover, it can further be shown to take the form of (59) with metric g_{CK}, as was the case for $\mathfrak{so}(3)$ resulting in (62). This follows from the fact that c^{ijk} is antisymmetric under any interchange of indices, i.e.,

$$g_{CK}^{rs} c^{ij}{}_s = c^{ijr} = -c^{irj} = -g_{CK}^{js} c^{ir}{}_s. \tag{64}$$

Using this and the Jacobi identity

$$c_r^{is} c_s^{jk} = c_r^{js} c_s^{ik} - c_r^{ks} c_s^{ij}$$

the middle term of (60) satisfies

$$g_{CK}^{rs} c_r^{ik} c_s^{jl} = g_{CK}^{rs} c_r^{il} c_s^{jk} + c_r^{ij} c_s^{kl} \tag{65}$$

and consequently

$$B^{ijkl} = g_{CK}^{rs} c_r^{ij} c_s^{kl}. \tag{66}$$

Recall that we referred to 4-brackets defined by such z-independent 4-tensors as Lie-metriplectic.

4. Metriplectic geometry

Given the form of metriplectic dynamics of (14) it is natural to explore manifolds with both Poissonian and Riemannian structure. Such manifolds are plentiful because a metric exists on any Poisson manifold (assuming Hausdorff, paracompactness). As a guiding principle of Poisson geometry, one often looks to generalize objects defined on the tangent bundle to the cotangent bundle. Following this principle, we introduce the cotangent analogs of connections and curvature. For the sake of brevity, we omit much of the motivation, mathematical detail, and theoretical importance of such objects. Instead, we introduce the ideas of contravariant connections and contravariant curvature axiomatically [46] and refer readers to [29], which served as a main motivation for us (see also [47]).

Let $\mathcal{Z}$ be a manifold with a Poisson bracket $\{\cdot,\cdot\}: C^\infty(\mathcal{Z}) \times C^\infty(\mathcal{Z}) \to C^\infty(\mathcal{Z})$. We will work locally on a coordinate patch, denoting the coordinate functions z^i as above. In order to define a curvature tensor on forms, it is desirable to extend the Poisson bracket to a Lie bracket on forms $[\cdot,\cdot]^J : \Lambda^1(\mathcal{Z}) \times \Lambda^1(\mathcal{Z}) \to \Lambda^1(\mathcal{Z})$. For exact forms, $[\cdot,\cdot]^J$ is given by

$$[\mathbf{d}f, \mathbf{d}g]^J := \mathbf{d}\{f, g\} = \mathbf{d}(J(\mathbf{d}f, \mathbf{d}g)), \tag{67}$$

where f, g are functions, i.e., 0-forms. Since we do not explicitly use the extension of this bracket to arbitrary 1-forms, we direct readers to [29] [Eq. (1.3)] or [48] [Eq. (2)] where a more complete formula can be found. The Poisson bracket provides the natural Poisson tensor of Sec. II A defined by $J^{ij} = \{z^i, z^j\}$, which is useful in that it allows us to define a map $J : \Lambda^1(\mathcal{Z}) \to \mathfrak{X}(\mathcal{Z})$ from 1-forms to vector fields by

$$(J\alpha)^j := \alpha_i J^{ij}, \tag{68}$$

where α is a 1-form.

In conventional Riemannian geometry, one defines a covariant connection as a map between vector fields, $\nabla : \mathfrak{X} \times \mathfrak{X} \to \mathfrak{X}$, satisfying some linearity properties. However, given the singular submanifold structures inherent to Poisson geometry, it is sometimes a matter of mathematical necessity to extend the notion of a connection to the cotangent bundle. There a contravariant connection is a map $D : \Lambda^1(\mathcal{Z}) \times \Lambda^1(\mathcal{Z}) \to \Lambda^1(\mathcal{Z})$ satisfying the same identities as the covariant connection. Namely, given $\alpha, \beta, \gamma \in \Lambda^1(\mathcal{Z})$ and $f \in \Lambda^0(\mathcal{Z})$,

$$D_{\alpha+\beta}\gamma = D_\alpha\gamma + D_\beta\gamma, \tag{69}$$

$$D_{f\alpha}\gamma = f D_\alpha \gamma, \tag{70}$$

$$D_\alpha(\beta + \gamma) = D_\alpha\beta + D_\alpha\gamma, \tag{71}$$

$$D_\alpha(f\gamma) = f D_\alpha\gamma + J(\alpha)[f]\gamma. \tag{72}$$

In (72), $J(\alpha)[f] = \alpha_i J^{ij} \partial f/\partial z^j$ is a 0-form that replaces the term $\mathbf{X}(f)$ in Koszul's algebraic Leibniz identity.

Given a choice of contravariant connection, it is natural to define the contravariant curvature $R : \Lambda^1(\mathcal{Z}) \times \Lambda^1(\mathcal{Z}) \times \Lambda^1(\mathcal{Z}) \to \Lambda^1(\mathcal{Z})$ by obvious analogy to standard Riemannian geometry

$$\begin{aligned} R(\alpha, \beta)\gamma &= D_\alpha D_\beta \gamma - D_\beta D_\alpha \gamma - D_{[\alpha,\beta]^J}\gamma \\ &= (R(\alpha, \beta)\gamma)_l \, \mathbf{d}z^l. \end{aligned} \tag{73}$$

To get the coordinate form of the contravariant curvature we simply define

$$R^{ijk}{}_l := (R(\mathbf{d}z^i, \mathbf{d}z^j)\mathbf{d}z^k)_l. \tag{74}$$

Of particular interest to us are metric connections.

Given a metric g, there is a Levi-Civita-like contravariant connection given by the formula

$$\begin{aligned} 2g(D_\alpha\beta, \gamma) = {} & J(\alpha)[g(\beta, \gamma)] - J(\gamma)[g(\alpha, \beta)] \\ & + J(\beta)[g(\gamma, \alpha)] + g([\alpha, \beta]^J, \gamma) \\ & - g([\beta, \gamma]^J, \alpha) + g([\gamma, \alpha]^J, \beta), \end{aligned} \tag{75}$$

which has the coordinate form

$$\begin{aligned} & 2g^{rs}(D_{\mathbf{d}f}\mathbf{d}h)_s \\ & = \frac{\partial f}{\partial z^i} J^{ij} \frac{\partial}{\partial z^j}\left[g^{kr}\frac{\partial h}{\partial z^k}\right] + J^{ir}\frac{\partial}{\partial z^i}\left[g^{kl}\frac{\partial f}{\partial z^k}\frac{\partial h}{\partial z^l}\right] \\ & \quad + \frac{\partial h}{\partial z^i} J^{ij}\frac{\partial}{\partial z^j}\left[g^{kr}\frac{\partial f}{\partial z^k}\right] + g^{kr}\frac{\partial}{\partial z^k}\left[J^{ij}\frac{\partial f}{\partial z^i}\frac{\partial g}{\partial z^j}\right] \\ & \quad - g^{kl}\frac{\partial}{\partial z^k}\left[J^{ir}\frac{\partial h}{\partial z^i}\right]\frac{\partial f}{\partial z^l} - g^{kl}\frac{\partial}{\partial z^k}\left[J^{jk}\frac{\partial f}{\partial z^j}\right]\frac{\partial h}{\partial z^l}. \end{aligned} \tag{76}$$

This formula is perhaps best understood as defining the contravariant Christoffel symbols:

$$\begin{aligned} \Gamma^{ij}_{\ \ l} & := (D_{\mathbf{d}z^i}\mathbf{d}z^j)_l \\ & = \frac{1}{2} g_{kl}\left[J^{is}\frac{\partial g^{jk}}{\partial z^s} - J^{ks}\frac{\partial g^{ij}}{\partial z^s} + J^{js}\frac{\partial g^{ik}}{\partial z^s}\right] \\ & \quad + \frac{1}{2} g_{kl}\left[g^{ks}\frac{\partial J^{ij}}{\partial z^s} - g^{si}\frac{\partial J^{jk}}{\partial z^s} - g^{sj}\frac{\partial J^{ik}}{\partial z^s}\right]. \end{aligned} \tag{77}$$

Just like its covariant analog, the contravariant Levi-Civita connection is the unique connection that is both torsion-free,

$$D_\alpha\beta - D_\beta\alpha = [\alpha, \beta]^J \ \longleftrightarrow \ \Gamma^{ij}_{\ \ k} - \Gamma^{ji}_{\ \ k} = \frac{\partial J^{ij}}{\partial z^k}, \tag{78}$$

and metric compatible (vanishing covariant derivative of the metric),

$$\begin{aligned} & J(\alpha)[g(\beta, \gamma)] = g(D_\alpha\beta, \gamma) + g(\beta, D_\alpha\gamma) \\ & \longleftrightarrow \ J^{is}\frac{\partial g^{jk}}{\partial z^s} = g^{ks}\Gamma^{ij}_{\ \ s} + g^{js}\Gamma^{ik}_{\ \ s}. \end{aligned} \tag{79}$$

Explicitly, for 1-forms $\alpha = \alpha_i dz^i$ and $\beta = \beta_j dz^j$ we compute

$$\begin{aligned} D_\alpha \beta &= \alpha_i D_{\mathbf{d}z^i}[\beta_j \mathbf{d}z^j] \\ &= \alpha_i \beta_j D_{\mathbf{d}z^i}[\mathbf{d}z^j] + \alpha_i J(\mathbf{d}z^i)[\beta_j]\mathbf{d}z^j \\ &= \mathbf{d}z^j \alpha_i J^{is}\frac{\partial \beta_j}{\partial z^s} + \alpha_i\beta_j \Gamma^{ij}_{\ \ k}\mathbf{d}z^k. \end{aligned} \quad (80)$$

Thus, (73) produces the tensor

$$\begin{aligned} R^{ijk}_{\ \ \ \ l} &= \Gamma^{jk}_{\ \ s}\Gamma^{is}_{\ \ l} - \Gamma^{ik}_{\ \ s}\Gamma^{js}_{\ \ l} - \frac{\partial J^{ij}}{\partial z^s}\Gamma^{sk}_{\ \ l} \\ &\quad + J^{is}\frac{\partial \Gamma^{jk}_{\ \ l}}{\partial z^s} - J^{js}\frac{\partial \Gamma^{ik}_{\ \ l}}{\partial z^s}. \end{aligned} \quad (81)$$

The addition of a metric structure allows us to raise the indices and obtain the fully contravariant 4-tensor from (81) according to

$$R^{ijkl} := R^{ijk}_{\ \ \ \ s}\, g^{sl}. \quad (82)$$

Provided we use the contravariant connection described above in (73), R^{ijkl} obeys all the symmetries of the normal Riemann tensor, including the first and second Bianchi identities. Further, by raising an index, we have a map $R : \Lambda^1(\mathcal{Z}) \times \Lambda^1(\mathcal{Z}) \times \Lambda^1(\mathcal{Z}) \times \Lambda^1(\mathcal{Z}) \to \Lambda^0(\mathcal{Z})$ from forms to scalars. This induces a 4-bracket on functions $(\cdot,\cdot;\cdot,\cdot) : C^\infty(\mathcal{Z}) \times C^\infty(\mathcal{Z}) \times C^\infty(\mathcal{Z}) \times C^\infty(\mathcal{Z}) \to C^\infty(\mathcal{Z})$ under the association of a function with its differential

$$\begin{aligned} (f,g;k,n) &:= R(\mathbf{d}f, \mathbf{d}g, \mathbf{d}k, \mathbf{d}n) \\ &= R^{ijkl}\frac{\partial f}{\partial z^i}\frac{\partial k}{\partial z^j}\frac{\partial g}{\partial z^k}\frac{\partial n}{\partial z^l}. \end{aligned}$$

We note that the contravariant connections can behave quite differently from what one might expect in Riemannian geometry. For example, consider the Poisson manifold $\mathcal{Z} = \mathfrak{so}(3)$ with the standard Poisson bracket $\{z^i, z^j\} = -\epsilon^{ijk}z^k$. Even with a seemingly flat metric $g^{ij} = \delta^{ij}$, we have the nontrivial Christoffel symbol and nontrivial curvature tensors

$$\Gamma^{ij}_{\ \ k} = -\tfrac{1}{2}\epsilon^{ijk} \quad \text{and} \quad R^{ijkl} = \tfrac{1}{4}(\delta^{ik}\delta^{jl} - \delta^{il}\delta^{jk}). \quad (83)$$

Thus, we see again the emergence of the metriplectic 4-bracket whose associated 2-bracket was given [9] and later used in [25] to model a controlling torque for the free rigid body.

As an aside, we note the structure outlined above can be a powerful tool in the study of Poisson manifolds. For example, if the metric and Poisson structure are compatible (vanishing covariant derivative of J) for all $\alpha, \beta, \gamma \in \Lambda^1(\mathcal{Z})$

$$J(D_\alpha\beta, \gamma) + J(\beta, D_\alpha\gamma) = 0, \quad (84)$$

then the symplectic leaves of $\mathcal{Z}$ become Kähler manifolds (see, e.g., [48]).

While the metriplectic dynamics generated by compatible Poisson and Riemannian structures promises to be very theoretically interesting, such a condition is too strong for our current purposes. For example, when J is Lie-Poisson it is easy to verify that the corresponding Cartan-Killing metric is never compatible with J.

Special cases of the metriplectic manifolds of this section come to mind: Lie-Poisson manifolds with an unidentified metric tensor, Poisson manifolds with a constant metric tensor, Lie-Poisson manifolds with a constant Euclidean metric tensor, or the Cartan-Killing metric, g_{CK}.

The first case follows upon inserting the Lie-Poisson form for J into (77) and the result into (81). This yields an interesting expression that we will not record here. The connection for case of constant metric follows immediately from (77), *viz.*,

$$\Gamma^{ij}_{\ \ l} = \frac{1}{2}g_{kl}\left[g^{ks}\frac{\partial J^{ij}}{\partial z^s} - g^{si}\frac{\partial J^{jk}}{\partial z^s} - g^{sj}\frac{\partial J^{ik}}{\partial z^s}\right]. \quad (85)$$

If J is Lie-Poisson, this becomes

$$\Gamma^{ij}_{\ \ l} = \tfrac{1}{2}g_{kl}\left[g^{ks}c^{ij}_{\ \ s} - g^{si}c^{jk}_{\ \ s} - g^{sj}c^{ik}_{\ \ s}\right], \quad (86)$$

and if the Euclidean metric $g^{rs} = \delta^{rs}$ is inserted into (86), the following simplified connection is obtained:

$$\Gamma^{ij}_{\ \ k} = \tfrac{1}{2}\left(c^{ij}_{\ \ k} - c^{jk}_{\ \ i} + c^{ki}_{\ \ j}\right) \quad (87)$$

(cf. [49] where a similar formula is found). Note here and henceforth that index placement purity is returned by inserting appropriate factors of the metric. The curvature tensor following from (87) is

$$\begin{aligned} R^{kij}_{\ \ \ \ b} &= R_b(dx^i, dx^j)dx^k \\ &= \tfrac{1}{4}\left(c^{jk}_a - c^{ka}_j + c^{aj}_k\right)\left(c^{ia}_b - c^{ab}_i + c^{bi}_a\right) \\ &\quad - \tfrac{1}{4}\left(c^{ik}_a - c^{ka}_i + c^{ai}_k\right)\left(c^{ja}_b - c^{ab}_j + c^{bj}_a\right) \\ &\quad - \tfrac{1}{2}c^{ij}_a\left(c^{ak}_b - c^{kb}_a + c^{ba}_k\right), \end{aligned} \quad (88)$$

where we would raise b with δ^{bl} to obtain the 4-tensor for the corresponding metriplectic 4-bracket. Finally, if the g_{CK} metric, as discussed in Sec. II D 3 is assumed, then a simple form for the 4-tensor is obtained,

$$R^{lkij} = \tfrac{1}{4}c^{jk}_a c^{ial} - \tfrac{1}{4}c^{ik}_a c^{jal} + \tfrac{1}{2}c^{ij}_a c^{kal}, \quad (89)$$

which as we have noted reduces to

$$R^{ijkl} = \tfrac{3}{4}c^{ij}_a\, c^{kl}_a.$$

For later use in Sec. II E 2 we record some useful lemmas about Casimirs. If S is a Casimir and $f, g \in \Lambda_0$ are arbitrary functions, then

$$\begin{aligned} D_{\mathbf{d}S}\mathbf{d}f &= D_{\mathbf{d}S}\left[\frac{\partial f}{\partial z^j}\mathbf{d}z^j\right] \\ &= \mathbf{d}z^j J(\mathbf{d}S)\left[\frac{\partial f}{\partial z^j}\right] + \frac{\partial f}{\partial z^j}D_{\mathbf{d}S}[\mathbf{d}z^j] \\ &= \frac{\partial S}{\partial z^i}\frac{\partial f}{\partial z^j}D_{\mathbf{d}z^i}\mathbf{d}z^j = \frac{\partial S}{\partial z^i}\frac{\partial f}{\partial z^j}\Gamma^{ij}_{\ \ k}\mathbf{d}z^k. \end{aligned} \quad (90)$$

Thus, we have the symmetry

$$D_{\mathbf{d}S}\mathbf{d}f = \mathbf{d}\{S, f\} + D_{\mathbf{d}f}\mathbf{d}S = D_{\mathbf{d}f}\mathbf{d}S, \quad (91)$$

where $\{S, f\} = 0$ because S is a Casimir. Furthermore, we have the antisymmetric bracket

$$\begin{aligned} g(D_{\mathbf{d}S}\mathbf{d}f, \mathbf{d}g) &= -g(\mathbf{d}f, D_{\mathbf{d}S}\mathbf{d}g) + J(\mathbf{d}S)[g(\mathbf{d}f, \mathbf{d}g)] \\ &= -g(\mathbf{d}f, D_{\mathbf{d}S}\mathbf{d}g) \end{aligned} \quad (92)$$

and

$$\begin{aligned} R(\mathbf{d}S, \mathbf{d}f)\mathbf{d}g &= D_{\mathbf{d}S}D_{\mathbf{d}f}\mathbf{d}g - D_{\mathbf{d}f}D_{\mathbf{d}S}\mathbf{d}g - D_{\mathbf{d}\{S,f\}}\mathbf{d}g \\ &= D_{\mathbf{d}S}D_{\mathbf{d}f}\mathbf{d}g - D_{\mathbf{d}f}D_{\mathbf{d}S}\mathbf{d}g\,. \end{aligned} \quad (93)$$

These properties make Casimirs very special function with respect to the contravariant Riemann tensor and hence the metriplecic 4-bracket.

E. Relation to other dissipation bracket formalisms

The metriplectic 4-bracket provides a unifying picture, tying together other brackets for dissipation that have previously appeared in the literature.

1. Reduction to Kaufman-Morrison dynamics

In [4] a bracket for describing the so-called quasilinear theory of plasma physics, a dissipative relaxation theory, was proposed. Brackets that have the properties of this example will be referred to as KM brackets: they are bilinear and antisymmteric, consequently degenerate, with dynamics generated by a Hamiltonian, H. We denote the bracket of this theory by $[\cdot,\cdot]_S : \Lambda^0(\mathcal{Z}) \times \Lambda^0(\mathcal{Z}) \to \Lambda^0(\mathcal{Z})$, where the subscript S will become clear momentarily. The KM bracket generates dynamics according to

$$\dot{z}^i = [z^i, H]_S, \tag{94}$$

with the properties that $\dot{H} = 0$ and $\dot{S} \geqslant 0$. Energy conservation follows from the antisymmetry of the KM bracket, while the entropy production was built into the theory.

It is apparent that the KM bracket emerges naturally from any metriplectic 4-bracket as follows:

$$\begin{aligned} [f, g]_S := (f, g; S, H) &= \frac{\partial f}{\partial z^i}\frac{\partial g}{\partial z^j}\frac{\partial S}{\partial z^k}\frac{\partial H}{\partial z^l}R^{ijkl} \\ &= J^{ij}_{KM}\frac{\partial f}{\partial z^i}\frac{\partial g}{\partial z^j}, \end{aligned} \tag{95}$$

where J^{ij}_{KM} is the antisymmetric bivector is given by

$$J^{ij}_{KM} = \frac{\partial S}{\partial z^k}\frac{\partial H}{\partial z^l}R^{ijkl}. \tag{96}$$

Thus, clearly, we have the following:

$$[f, g]_S = -[g, f]_S \tag{97}$$

by (30) or (22), consequently,

$$\dot{H} = [H, H]_S = (H, H; S, H) = 0 \tag{98}$$

and

$$\dot{S} = [S, H]_S = (S, H; S, H) \geqslant 0, \tag{99}$$

by (42), as was proposed in [4].

2. Reduction to double bracket dynamics

Double brackets were proposed in [10,11] as a computational means of relaxing to equilibria by extremizing a Hamiltonian at fixed Casimirs by using the square of the Poisson tensor J to generate dynamics. The formalism was improved in [14] and subsequently used in a variety of magnetohydrodynamics contexts in [50–52]. With the improvements given in [14] we can write this dynamics as follows:

$$\dot{z}^i = ((z^i, H)), \tag{100}$$

where the double bracket $((.,.)) : C^\infty(\mathcal{Z}) \times C^\infty(\mathcal{Z}) \to C^\infty(\mathcal{Z})$ has the coordinate represention

$$((f, g)) = J^{ik}g_{kl}J^{jl}\frac{\partial f}{\partial z^i}\frac{\partial g}{\partial z^j}. \tag{101}$$

For metric g, evidently

$$\dot{H} = ((H, H)) \geqslant 0 \quad \text{and} \quad \dot{C} = 0, \tag{102}$$

where C is any Casimir of J. A commonly used case of (101) is that for Lie-Poisson systems, where it takes the form

$$((f, g)) = c^{ik}{}_r c^{jl}{}_s\, g_{kl}\, z^s z^r \frac{\partial f}{\partial z^i}\frac{\partial g}{\partial z^j}. \tag{103}$$

One connection between double brackets and the metriplectic 4-bracket can be made by simply interchanging the role of H with a Casimir S, i.e., considering dynamics generated by the symmetric bracket

$$(f, g)_S = (f, S; g, S), \tag{104}$$

which for a Casimir S will satisfy the conditions of (102). However, if C is another Casimir, distinct from S, then there is no guarantee it is conserved. We will see in a moment that the development of Sec. II D 4 provides a way to improved upon this.

A direct relationship between Lie-Poisson double brackets of the form of (103) and metriplectic 4-brackets of the form of (59) can be made for Cartan-Killing metrics by choosing the entropy

$$S_{LP} = \tfrac{1}{2} z^a \bar{g}_{ab} z^b \tag{105}$$

and inserting this into (59), yielding

$$\begin{aligned} (f, g)_{S_{LP}} &= (f, S_{LP}; g, S_{LP}) \\ &= c^{ij}{}_r c^{kl}{}_s\, g^{rs} \frac{\partial f}{\partial z^i}\frac{\partial S_{LP}}{\partial z^j}\frac{\partial g}{\partial z^k}\frac{\partial S_{LP}}{\partial z^l} \\ &= c^{ij}{}_r c^{kl}{}_s\, g^{rs}\, \bar{g}_{ja} z^a\, \bar{g}_{lb} z^b \frac{\partial f}{\partial z^i}\frac{\partial g}{\partial z^k}. \end{aligned} \tag{106}$$

Now if we suppose g is the Cartan-Killing metric and $\bar{g}$ is its assumed inverse, (106) becomes upon using (64)

$$\begin{aligned} (f, g)_{S_{LP}} &= c^{ij}{}_r c^{rk}{}_s\, g^{ls}\, g_{ja} z^a\, g_{lb} z^b \frac{\partial f}{\partial z^i}\frac{\partial g}{\partial z^k} \\ &= c^{ij}{}_r c^{rk}{}_s\, g_{ja}\, z^a z^s \frac{\partial f}{\partial z^i}\frac{\partial g}{\partial z^k} \\ &= c^{ij}{}_r J^{rk}\, g_{ja}\, z^a \frac{\partial f}{\partial z^i}\frac{\partial g}{\partial z^k} \\ &= J^{ir} g_{rs} J^{ks} \frac{\partial f}{\partial z^i}\frac{\partial g}{\partial z^k}. \end{aligned} \tag{107}$$

Now let us see what transpires when we use the identities at the end of Sec. II D 4. If S and C are any Casimirs and our manifold has the metric and Poisson bracket as described, then we can compute as follows:

$$
\begin{aligned}
(S, C; S, C) &= g(dC, D_{dS}D_{dC}dS - D_{dC}D_{dS}dS) \\
&= -g(D_{dS}dC, D_{dC}dS) + g(D_{dC}dC, D_{dS}dS) \\
&= -g(D_{dC}dS, D_{dC}dS) + g(D_{dC}dC, D_{dS}dS) \\
&= \left(\frac{\partial C}{\partial z^i}\frac{\partial C}{\partial z^j}\frac{\partial S}{\partial z^k}\frac{\partial S}{\partial z^l} - \frac{\partial C}{\partial z^i}\frac{\partial C}{\partial z^k}\frac{\partial S}{\partial z^j}\frac{\partial S}{\partial z^l}\right) \\
&\quad \times \Gamma^{ij}_{\ \ a} g^{ab} \Gamma^{kl}_{\ \ b},
\end{aligned} \tag{108}
$$

a perspicuous form.

We now specialize to Lie-Poisson systems with such constant Cartan-Killing metrics, which gives a prevalent case that we termed Lie-metriplectic. For any Lie-metriplectic system, the Christoffel symbol takes the form of (86). Trivially, now, we see that when both S and C are Casimirs we are left with

$$
(S, C; S, C) = 0. \tag{109}
$$

Thus, for these Lie-metriplectic systems, double brackets emerge nicely from our metriplectic 4-bracket.

3. GENERIC is metriplectic

In this section we place the ideas given in [27] for a bracket for the Boltzmann collision operator into a finite-dimensional setting. Since this work was the origin of a trail leading to what was later referred to as GENERIC, we call this bracket the GENERIC bracket. We will show given assumptions how to transform the GENERIC bracket, which is not symmetric and not bilinear, into a metriplectic 2-bracket that has these properties.

Apparently, the latest rendition of GENERIC [53] is written in terms of a dissipation potential $\Xi(z, z_*)$, where the shorthand $z_{*i} = \partial S/\partial z^i$, for some entropy function S, is used. Using $\Xi(z, z_*)$ the components of the dissipative vector field are generated via

$$
Y_S^i = \left.\frac{\partial\, \Xi(z, z_*)}{\partial z_{*i}}\right|_{z_*=\partial S/\partial z}. \tag{110}
$$

Thus, in the special case where

$$
\Xi(z, z_*) = \frac{1}{2}\frac{\partial S}{\partial z^i} G^{ij}(z) \frac{\partial S}{\partial z^j}, \tag{111}
$$

the dissipative vector field is

$$
Y_S^i = G^{ij}(z)\frac{\partial S}{\partial z^j}, \tag{112}
$$

which is equivalent to that generated by a metriplectic 2-bracket as originally proposed in [7–9]. Thus, it has been said that metriplectic dynamics is a special case of GENERIC. We will show that this is not the case by showing how vector fields of the form of (110) can be generated by a metriplectic 2-bracket.

As before, we suppose our phase space is some finite-dimensional manifold $\mathcal{Z}$ on which lives a dynamical system, and suppose smooth functions $f, g, H, S \in \Lambda^0(\mathcal{Z})$ with, as usual, H being the Hamiltonian and S the entropy. Then, in coordinates, GENERIC dissipative dynamics is generated by

$$
(f, g)) = \frac{\partial f}{\partial z^i} Y_S^i(z, \partial g/\partial z), \tag{113}
$$

a bracket that is a linear derivation in the first slot but not in its second slot; at this point Y_S^i is considered an arbitrary function of its arguments. Dissipative dynamics is generated with an entropy function as follows:

$$
\dot{z}^i = (z^i, S)) = Y_S^i(z, \partial S/\partial z), \tag{114}
$$

and energy conservation is assumed to be satisfied because

$$
\dot{H} = \frac{\partial H}{\partial z^i} Y_S^i(z, \partial S/\partial z) = 0. \tag{115}
$$

Finally, entropy production requires

$$
\dot{S} = (S, S)) \geqslant 0, \tag{116}
$$

a property built into $\mathbf{Y}_s$ by $Y_S^i\, \partial S/\partial z^i \geqslant 0$.

In the above we could identify

$$
Y_S^i(z, z_*) = \frac{\partial\, \Xi(z, z_*)}{\partial z_{*i}}, \tag{117}
$$

but the linearization procedure does not require the dissipative vector field to have this form in terms of a dissipation potential.

Given that at the outset one has in mind a dynamical system with a particular entropy function S, one can turn the bracket of (113) into a bilinear form by solving

$$
\hat{G}^{ij}\frac{\partial S}{\partial z^j} = Y_S^i(z, \partial S/\partial z), \tag{118}
$$

which requires $Y_S^i(z, 0) = 0$. If one can solve for $\hat{G}$, then the bracket

$$
(f, g\rangle = \frac{\partial f}{\partial z^i}\hat{G}^{ij}\frac{\partial g}{\partial z^j}, \tag{119}
$$

will yield the dissipative vector field of (118) in the form $Y_S^i = (z^i, S\rangle$. The bracket of (119) is clearly bilinear in f and g, but symmetry is not guaranteed.

Since any $\hat{G}^{ij}$ that satisfies (118) will do, we assume the following direct product form:

$$
\hat{G}^{ij} = Y_S^i(z, \partial S/\partial z)\, M^j(z, \partial S/\partial z), \tag{120}
$$

which upon insertion into (118) yields

$$
Y_S^i\left(1 - M^j\frac{\partial S}{\partial z^j}\right) = 0. \tag{121}
$$

Upon choosing

$$
M^j = \delta^{jk}\frac{\partial S}{\partial z_k}\bigg/\frac{\partial S}{\partial z^l}\frac{\partial S}{\partial z_l}, \tag{122}
$$

where $\delta^{jk}\partial S/\partial z^k = \partial S/\partial z_j$, we obtain

$$
\begin{aligned}
\hat{G}^{ij}(z) &= Y_S^i(z, \partial S/\partial z) M^j(\partial S/\partial z) \\
&= Y_S^i\,\frac{\partial S}{\partial z_j}\bigg/\frac{\partial S}{\partial z^l}\frac{\partial S}{\partial z_l},
\end{aligned} \tag{123}
$$

where now we interpret $\hat{G}$ to be a given matrix function of the coordinated z. With this choice we have

$$
(H, g\rangle = \frac{\partial H}{\partial z^i} Y_S^i(z, \partial S/\partial z)\, M^j(z, \partial S/\partial z)\frac{\partial g}{\partial z^j} = 0,
$$

for all functions g, which builds in degeneracy in the first argument of the bracket.

Thus, any GENERIC vector field (110) can be generated by the bilinear bracket of the form of (119). However, in general $\hat{G}$ is not symmetric, so we do not yet have a metriplectic 2-bracket. So next we show, given some assumptions, how to symmetrize $(f, g\rangle$. Although above we considered only the dissipative dynamics, consider the full dynamics for some observable $o \in \Lambda^0(\mathcal{Z})$ to take the form

$$\dot{o} = J(\mathbf{d}o, \mathbf{d}H) + \hat{G}(\mathbf{d}o, \mathbf{d}S),$$

where J is a Poisson bracket, $\hat{G}$ is a rank 2-tensor, such as that of (123), which we assume satisfies

$$\hat{G}(\mathbf{d}H, \cdot) = 0,$$

and S and H are fixed and have never vanishing differentials.

Lemma 1. Suppose $\mathbf{d}S$ is nonvanishing. There exists a symmetric tensor G such that

$$\begin{aligned}\dot{o} &= J(\mathbf{d}o, \mathbf{d}H) + \hat{G}(\mathbf{d}o, \mathbf{d}S)\\ &= J(\mathbf{d}o, \mathbf{d}H) + G(\mathbf{d}o, \mathbf{d}S)\end{aligned} \tag{124}$$

and

$$G(\mathbf{d}f, \mathbf{d}g) = G(\mathbf{d}g, \mathbf{d}f), \tag{125}$$

for all $f, g \in \Lambda^0(\mathcal{Z})$, Thus, we can make an **equivalent** dynamical system generated by a symmetric bilinear form with symmetric tensor G.

Proof. Let $\{U_n\}_{n\in\mathbb{N}}$ be a cover of $\mathcal{Z}$ with coordinate neighborhoods. Let $\{\psi_n\}$ be a partition of unity subordinate to this cover. Since $\mathbf{d}S$ is assumed nonvanishing, on every open set U_n we can choose coordinates (z_n^i) such that $S = z^1$. On U_n we define the symmetric contravariant 2-tensor

$$G_n^{ij} := G^{ij}, \quad \text{for } i \geqslant j,$$

with the rest of the tensor determined by symmetry. This local construction patches together to give the globally defined symmetric tensor

$$\tilde{G} = \sum_n \psi_n G_n.$$

One can confirm that for all $\alpha \in \Lambda^1(\mathcal{Z})$ we have the desired property that

$$\tilde{G}(\alpha, \mathbf{d}S) = G(\alpha, \mathbf{d}S).$$

■

Care should be taken near the vanishing points of $\mathbf{d}S$ since rectification arguments fail at the vanishing points of covector fields.

The discussion above addresses both the dissipation potential form and nonbilinear nonsymmetric bracket form of GENERIC. However, there is an alternative proof that proceeds directly from the dissipation potential form. Let $\Xi \in C^\infty(T^*\mathcal{Z})$ be the dissipation potential with a corresponding dissipative vector field, which we write as

$$Y_S^i(z) = \frac{\partial \Xi}{\partial \xi_i}(z, \mathbf{d}S(z)),$$

where (ξ_i) are the fiber coordinates. If any bilinear bracket exists giving Y_S^i upon the insertion of $\mathbf{d}S$, then it must be the case that

$$\frac{\partial \Xi}{\partial \xi_i}(z, 0) = 0. \tag{126}$$

Given (126) we may write

$$Y^i(z) = \left[\int_0^1 d\lambda \frac{\partial^2 \Xi}{\partial \xi_i \partial \xi_j}(z, \lambda \mathbf{d}S(z))\right]\frac{\partial S}{\partial x^j}. \tag{127}$$

This immediately yields the symmetric contravariant 2-tensor

$$\tilde{G}^{ij} = \int_0^1 d\lambda \frac{\partial^2 \Xi}{\partial \xi_i \partial \xi_j}(z, \lambda \mathbf{d}S(z)) \tag{128}$$

satisfying the desired relation that $\tilde{G}(\cdot, \mathbf{d}S) = Y$. We note that $\tilde{G}$ implicitly depends on both Ξ and S. One can check that $\tilde{G}$ is independent of the choice of coordinates and hence globally defined.

F. Finite-dimensional examples

In this subsection we discuss finite-dimensional systems of dimension three. However, we note that it is easy to construct metriplectic 4-brackets for systems of arbitrary dimension. For example, this can be done by using the Lie algebra construction of Sec. II D 3. Moreover, one can begin with a Lie-Poisson system and construct extensions on n-tuples as in [54] by direct product, semidirect product, etc. (see [55] for the heavy top). Here for simplicity we restrict to three dimensions.

1. Free rigid body

The Euler's equations for the free rigid body have a Hamiltonian structure in terms of a Lie-Poisson bracket as discussed in Sec. II A (see Chap. 17 of [32]). For this three-dimensional system the coordinates are the three components of the angular momenta (L^1, L^2, L^3) and the Lie algebra is $\mathfrak{so}(3)$. Thus, the Lie-Poisson bracket has the form of (10) with the coordinates z^k being L^k and $c_k^{ij} = -\epsilon_{ijk}$. The Hamiltonian and Casimir of the system are given by

$$H = \frac{(L^1)^2}{2I_1} + \frac{(L^2)^2}{2I_2} + \frac{(L^3)^2}{2I_3} \tag{129}$$

and

$$C = (L^1)^2 + (L^3)^2 + (L^3)^2, \tag{130}$$

respectively. Here the parameters I_i are the principal moments of inertia.

In [9] the metriplectic 2-bracket was given for this system, so as to create a system that removes (or adds) angular momentum C of (131) while preserving the energy H of (130) as it approaches an equilibrium of rotation about one of its principal axes. With slight reformatting the bracket of Eq, (31) of [9] becomes the following:

$$\begin{aligned}(f, g)_H &= (f, H; g, H)\\ &= -\lambda\left[\frac{\partial H}{\partial L^k}\frac{\partial H}{\partial L^l}(\delta^{ik}\delta^{jl} - \delta^{ij}\delta^{lk})\frac{\partial f}{\partial L^i}\frac{\partial g}{\partial L^j}\right].\end{aligned} \tag{131}$$

Thus, we see easily that the rigid body metriplectic 4-bracket is of the form of (58) and is in fact the simple K-N construction of Sec. II D 2 with Euclidean metric.

Our sectional curvature for entropy production generated by (131) is

$$\dot{S} = (S, H; S, H) \\ = -\lambda((\nabla_L H \cdot \nabla_L S)^2 - |\nabla_L H|^2 |\nabla_L S|^2) \geqslant 0, \quad (132)$$

where $\nabla_L = \partial/\partial\mathbf{L}$ and the inequality follows for $\lambda > 0$. We point out that the conventional sectional curvature in Riemannian geometry is normalized by a denominator, $|\mathbf{Y}|^2|\mathbf{X}|^2 - (\mathbf{X}\cdot\mathbf{Y})^2$ for vectors $\mathbf{X}, \mathbf{Y} \in \mathfrak{X}(\mathcal{Z})$. With this normalization in the present context we would divide (132) by $|\nabla_L H|^2|\nabla_L S|^2 - (\nabla_L H \cdot \nabla_L S)^2$, arriving at the constant production rate $\dot{S} = \lambda$. At the outset, we could have defined the metriplectic 4-bracket with this normalization; however, we chose not to do this in order to preserve multilinearity of the 4-bracket.

2. Kirchhoff-Kida ellipse

Kida generalized Kirchhoff's reduction of the two-dimensional Euler equations of fluid mechanics (see Sec. III C 2), obtaining a reduction to a set of ordinary differential equations. The reduced dynamical system describes a constant patch of vorticity enclosed by a elliptical boundary, where as time proceeds the boundary remains an ellipse. Like the free rigid body, this system is a three-dimensional Lie-Poisson system, but instead of $\mathfrak{so}(3)$ it has the Lie algebra $\mathfrak{sl}(2, 1)$.

In [56] it was shown that quadratic moments of the vorticity, say, $\omega(x, y)$ constitute a subalgebra of the two-dimensional Euler fluid Poisson bracket [see Eq. (165) below and [2]]. The coordinates for the Kirchhoff-Kida system, say, (z^1, z^2, z^3), are linearly related to the vorticity moments $(\int d^2x\, \omega x^2, \int d^2x\, \omega y^2, \int d^2x\, \omega xy)$. The Poisson tensor for this case is

$$J = \begin{pmatrix} 0 & z^3 & -z^2 \\ -z^3 & 0 & -z^1 \\ z^2 & z^1 & 0 \end{pmatrix}, \quad (133)$$

which has the associated Casimir invariant,

$$C = (z^1)^2 - (z^2)^2 - (z^3)^2, \quad (134)$$

and this Casimir is a measure of the area of the Kirchhoff ellipse raised to the fourth power. We refer the reader to [14] for the Hamiltonian for this system, but note the level sets of H are curved sheets with a symmetry direction, because they are independent of the coordinate z^2. Thus, orbits of this Hamiltonian system can be understood in terms of intersection of the sheets with the Casimir hyperboloid defined by (134), similarly to how the free rigid body can be understood in terms of the intersection of the angular momentum sphere with the Hamiltonian ellipsoid. However, for the Kida case one has three classes of orbits, corresponding to an elliptical patch rotating, librating, or stretching to infinite aspect ratio, which are easily delineated by examining these intersections.

Again, using a metriplectic 4-bracket of the form of (58) gives our desired result. This 4-bracket produces dynamics that will either increase or decrease the Casimir, implying growth or shrinkage of the area of the ellipse, while the energy is preserved. This is essentially a finite-dimensional version of the selective decay hypothesis (e.g., [57]) and is, in a sense, dual to the double bracket dynamics of [14,58], where the Hamiltonian is extremized at fixed Casimir (area).

3. Other three-dimensional systems

The examples of Secs. II F 2 and II F 1 are based on the three-dimensional Lie algebras $\mathfrak{so}(3)$ and $\mathfrak{sl}(2, 1)$, respectively. According to the Bianchi classification, there are nine real three-dimensional Lie algebras, and one can construct finite-dimensional Lie-Poisson systems (see [59] for a listing) and then construct metriplectic 4-brackets from each of these. Many of the Lie-Poisson systems have physical realizations; e.g., Type IV was shown in [60] to underlie a simple model for the rattleback toy that defies the normal understanding of chirality. Also, outside of Lie-Poisson dynamics, there is a three-dimensional system that describes the invicid interaction of tilted fluid vortex rings [61], a system that diverges in finite time. A natural energy-conserving metriplectic 4-bracket can easily be constructed for this system as well.

III. INFINITE-DIMENSIONAL METRIPLECTIC 4-BRACKETS: FIELD THEORIES

A. General Hamiltonian and metriplectic field theories

Here we briefly review some general properties of brackets for field theories. For further development see, e.g., [3,14] and in a somewhat more mathematical setting [62].

For field theory, we replace a discrete index i of finite-dimensional theories with labeling by a continuous variable z and a field component index i, with the degrees of freedom denoted by χ, a multicomponent field. The functions on phase space are replaced by functionals of the dynamical degrees of freedom, which are maps of $\chi \mapsto \mathbb{R}$. More specifically, we consider the dynamics of classical field theories with multicomponent fields

$$\chi(z, t) = (\chi^1(z, t), \chi^2(z, t), \dots, \chi^M(z, t))$$

defined on $z \in \mathcal{D}$ for times $t \in \mathbb{R}$, i.e., $\chi : \mathcal{D} \times \mathbb{R} \to \mathbb{R}$. Here we use z to be a label space coordinate unlike in the previous section where it was a dynamical variable or phase space coordinate. In fluid mechanics $\mathcal{D}$ would be the three-dimensional domain occupied by the fluid and z the coordinates of this point. We will assume in general $\mathcal{D}$ has dimension N.

The time rate of change of functionals, maps from fields to real numbers, will be generated by making use of various brackets, and these involve a notion of functional or variational derivative. These are defined via the first variation, which for a functional F is

$$\delta F[\chi; \eta] = \frac{d}{d\epsilon} F[\chi + \epsilon\eta]\bigg|_{\epsilon=0} = \int_{\mathcal{D}} d^N z\, \frac{\delta F[\chi]}{\delta \chi^i} \eta^i, \quad (135)$$

where again repeated indices are to be summed. Here the variation $\delta F[\chi; \eta]$ acts on $\eta(z)$ with the integral over z providing the pairing between the quantity $\delta F/\delta\chi$ (the gradient) and $\eta(z)$ (the displacement). The function that is the evaluation of χ at a point $\hat{z}$ satisfies $\delta\chi(\hat{z})/\delta\chi(z) = \delta(\hat{z} - z)$, with δ being the Dirac delta function.

With this notation, a general noncanonical Poisson bracket is a binary operator on functionals, say, F and G, of the form

$$\{F,G\}=\int_{\mathcal{D}}d^Nz'\int_{\mathcal{D}}d^Nz''\,\frac{\delta F[\chi]}{\delta\chi^i(z')}\mathcal{J}^{ij}(z',z'')\frac{\delta G[\chi]}{\delta\chi^j(z'')}\,. \quad (136)$$

Here $\mathcal{J}$ is the Poisson operator (replacing the Poisson tensor of Sec. II A) that must ensure that the Poisson bracket satisfies antisymmetry, $\{F,G\}=-\{G,F\}$, and the Jacobi identity, $\{\{F,G\},H\}+\{\{G,H\},F\}+\{\{H,F\},G\}=0$, for all functionals F,G,H. As before, this form builds in bilinearity and the Leibnitz derivation properties.

For the dynamics on infinite-dimensional Poisson manifolds $\mathcal{J}$ is degenerate, as was the case in finite dimensions. When it is degenerate the nontrivial null space gives rise to the Casimir invariants, C, that satisfy $\{C,G\}=0$ for all functionals G. Thus, all is formally, if not rigorously, equivalent to the finite-dimensional development of Sec. II (see, e.g., [3,58] for review).

General symmetric brackets were given in [14], ones that can generate double bracket or metriplectic 2-bracket dynamics,

$$(F,G)=\int_{\mathcal{D}}d^Nz'\int_{\mathcal{D}}d^Nz''\,\frac{\delta F[\chi]}{\delta\chi^i(z')}\,\mathcal{G}^{ij}(z',z'')\frac{\delta G[\chi]}{\delta\chi^j(z'')}, \quad (137)$$

where the metric operator $\mathcal{G}$, analogous to the G metric of Sec. II, is chosen to ensure $(F,G)=(G,F)$ and to be semidefinite. We may also want to build degeneracies into $\mathcal{G}$ so that there exist distinguished functionals D that satisfy $(D,G)=0$ for all G.

A specific form of (138) was given in [14], which is a generalization of the symmetric brackets given in previous works [9,11–13,63],

$$(F,G)=\int_{\mathcal{D}}d^Nz'\int_{\mathcal{D}}d^Nz''\,\{F,\chi^i(z')\}\mathcal{K}_{ij}(z',z'')\{\chi^j(z''),G\}, \quad (138)$$

with $\{F,G\}$ being any Poisson bracket and $\mathcal{K}$ is a symmetric kernel that can be chosen at will, e.g., to effect smoothing. With this form, the Casimir invariants of $\{F,G\}$ will automatically be distinguished functionals D.

Proceeding we define a general form for the field-theoretic metriplectic 4-bracket by replacing the 4-tensor by a 4-tensor-functional with coordinate form given by the following integral kernel:

$$\begin{aligned}&\hat{R}^{ijkl}(z,z',z'',z''')[\chi(z)]\\&\quad=\hat{R}(\mathbf{d}\chi^i(z),\mathbf{d}\chi^j(z'),\mathbf{d}\chi^k(z''),\mathbf{d}\chi^l(z'''))[\chi(z)]. \quad (139)\end{aligned}$$

Formally identifying the functional derivative with the exterior derivative we are led naturally to the following metriplectic 4-bracket on functionals:

$$\begin{aligned}(F,G;K,N)=&\int d^Nz\int d^Nz'\int d^Nz''\int d^Nz'''\hat{R}^{ijkl}(z,z',z'',z''')\\&\times\frac{\delta F}{\delta\chi^i(z)}\frac{\delta G}{\delta\chi^j(z')}\frac{\delta K}{\delta\chi^k(z'')}\frac{\delta N}{\delta\chi^l(z''')}\,, \quad (140)\end{aligned}$$

with properties built in making it minimally metriplectic as discussed in Sec. II.

Because we are dealing with field theory, the quantity $\hat{R}^{ijkl}(z,z',z'',z''')$ of (140) should be defined distributionally and in general is an operator acting on the functional derivatives. In particular, we could write $\hat{R}$ in terms of its Fourier transform. For pseudodifferential operators

$$\begin{aligned}&\hat{R}^{ijkl}(z,z',z'',z''')\\&\quad=\int d^Np\,e^{ip\cdot z}\int d^Np'\,e^{ip'\cdot z'}\int d^Np''\,e^{ip''\cdot z''}\int d^Np'''\,e^{ip'''\cdot z'''}\\&\qquad\times\tilde{R}^{ijkl}(p,p',p'',p'''). \quad (141)\end{aligned}$$

Analogous to Sec. II C 3, we say $\hat{R}$ is Lie-metriplectic if $\hat{R}$ does not depend directly on the values of the field variable χ, although it can depend on the label z. When such a dependence is present, we can interpret it as a location dependent "curvature" in the manifold of functions.

B. Reduction to special cases in infinite dimensions

Reductions of the field theoretic metriplectic 4-bracket of (141) follow in the same manner as the finite-dimensional reductions of Sec. II E. The metric 2-bracket follows as expected, $(F,G)_H=(F,H;G,H)$, the KM bracket according to $[F,G]_S=(F,G;S,H)$, various double brackets follow from $(F,G)_S=(F,S;G,S)$, etc. In Secs. III C and III D we will give many examples that demonstrate these reductions. Rather than treating a general case of linearizing and symmetrizing a GENERIC bracket, in Sec. III D 2 we do so for the specific case of bracket for the Boltzmann equation given in [27].

C. Fluidlike examples

1. 1 + 1 fluidlike theories

Now consider the case were we have a single real-valued field variable depending on one space- and one time-independent variable, $u(x,t)$. We will give three examples of dissipation generated by a 4-bracket. We do so by using a version of the K-N decomposition of Sec. II D 2, where the tensors σ and μ are in this field-theoretic context replaced by symmetric operators Σ and M. Using these operators a field theoretic version of the K-N product gives a 4-bracket of the following form:

$$\begin{aligned}(F,K;G,N)&=\int_{\mathbb{R}}dx\,W(\Sigma\owedge M)(F_u,K_u,G_u,N_u)\\&=\int_{\mathbb{R}}dx\,W(\Sigma(F_u,G_u)M(K_u,N_u)\\&\quad-\Sigma(F_u,N_u)M(K_u,G_u)\\&\quad+M(F_u,G_u)\Sigma(K_u,N_u)\\&\quad-M(F_u,N_u)\Sigma(K_u,G_u)), \quad (142)\end{aligned}$$

where W is an arbitrary weight, depending on u and x, that multiplies $\Sigma\owedge M$ and for convenience we define $F_u=\delta F/\delta u$. We are free to choose W without destroying the 4-bracket algebraic symmetries. We note, as we will show, the form of 4-bracket of (142) can be generalized in various ways to higher dimensions of both the dependent and independent variables.

For our *first example* we assume the following symmetric operators:

$$\Sigma(F_u,G_u)=-\frac{d}{dx}\frac{\delta F}{\delta u}\frac{d}{dx}\frac{\delta G}{\delta u}=-\partial F_u\partial G_u, \quad (143)$$

$$M(F_u,G_u)=\frac{\delta F}{\delta u}\frac{\delta G}{\delta u}=F_uG_u, \quad (144)$$

where again for convenience we simplify the notation by defining $\partial = \partial/\partial x$. In addition we assume $W = \nu$, some constant, and the Hamiltonian and Casimir are given by

$$H = \int_{\mathbb{R}} dx\, u \quad \text{and} \quad S = \frac{1}{2}\int_{\mathbb{R}} dx\, u^2. \tag{145}$$

Inserting these into (142) gives

$$(F, G)_H = (F, H; G, H) = \nu \int_{\mathbb{R}} dx\, F_u \partial^2 G_u, \tag{146}$$

which produces in an equation of motion

$$(u, S)_H = \nu\, \partial^2 u, \tag{147}$$

the usual form for viscous dissipation of a one-dimensional fluid.

In light of the above result and the metriplectic formalism, it is natural to ask which Hamiltonian theory has the Hamiltonian and Casimir of (145)? Although not so well known, one can construct a 1 + 1 Poisson bracket that has any desired Casimir. To this end, consider

$$\{F, G\} = \int_{\mathbb{R}} dx\, h(u)(F_u \partial G_u - G_u \partial F_u), \tag{148}$$

where the function $h(u)$ is unspecified. Using a theorem of [2] it is easy to show that (148) satisfies the Jacobi identity. A bracket of the form of (148) that has $\int_{\mathbb{R}} dx\, u^2/2$ as a Casimir must satisfy

$$\{F, C\} = 0\ \forall F \Rightarrow 2h\partial C_u + C_u \partial u = 0, \tag{149}$$

which easily solved to yield $h = 1/u^2$. Ignoring the singularity, we proceed and obtain the Hamiltonian dynamics with Hamiltonian of (145), *viz.*,

$$\frac{\partial u}{\partial t} = \{u, H\} = \partial(u^{-2}). \tag{150}$$

Thus, our metriplectic system of this example is

$$\frac{\partial u}{\partial t} = \{u, H\} = \partial(u^{-2}) + \nu \partial^2 u. \tag{151}$$

As with the Harry Dym equation [64], the singularity of (150) can be removed by a coordinate change. For example, setting $w = 2/u^3$ takes the Hamiltonian system of (150) into

$$\frac{\partial w}{\partial t} = -w\partial w, \tag{152}$$

the inviscid Burger's equation. The bracket of (148) can be transformed via the chain rule into many forms: the form where $h = u$ is the Lie-Poisson form and the form where h is constant is Gardner's bracket [65],

$$\{F, G\} = \int_{\mathbb{R}} dx\, G_u \partial F_u. \tag{153}$$

Our *second example* uses Gardner's bracket with the Hamiltonian

$$H = \frac{1}{2}\int_{\mathbb{R}} dx \left(\frac{u^3}{6} - \frac{(\partial u)^2}{2} + c\frac{u^2}{2}\right), \tag{154}$$

which together generate the Korteweg-De Vries equation in a frame boosted by speed c. Gardner's bracket has the Casimir

$$S = \int_{\mathbb{R}} dx\, u. \tag{155}$$

Thus, we have all the ingredients needed to construct a metriplectic system with dissipation that conserves (154). Using (142), again with (143) and (144), this dissipation is generated by

$$\begin{aligned}(u, S)_H &= (u, H; S, H)\\ &= -\partial(W H_u \partial H_u) - W(\partial H_u)^2,\end{aligned} \tag{156}$$

where

$$H_u = cu + \frac{u^2}{2} + \partial^2 u, \tag{157}$$

we leave W arbitrary, and

$$\dot{S} = -\int_{\mathbb{R}} dx\, W(\partial H_u)^2. \tag{158}$$

By design, the right-hand side of (158) vanishes when evaluated on $a\,\mathrm{sech}^2(\alpha x)$, the boosted single soliton solution, with appropriate a and α.

In our *third example* of this subsection, our final example, we choose for Σ,

$$\Sigma(F_u, G_u)(x) = \partial F_u(x)\mathcal{H}[G_u](x) + \partial G_u(x)\mathcal{H}[F_u](x), \tag{159}$$

where $\mathcal{H}$ is the Hilbert transform

$$\mathcal{H}[u] = \frac{1}{\pi} ⨍_{\mathbb{R}} dx' \frac{u(x')}{x - x'}, \tag{160}$$

with ⨍ denoting the Cauchy principal value integral (see, e.g., [66]). For M we choose again that of (144) and again we choose the Hamiltonian and entropy of (145). Note that $\Sigma(F_u, H_u) = 0$ for all functionals F because $H_u = 1$, $\partial H_u = 0$, and $\mathcal{H}[1] = 0$. Thus, we obtain

$$\begin{aligned}(F, G)_H = (F, H; G, H) &= \int_{\mathbb{R}} dx\, W\, \Sigma(F_u, G_u)\\ &= \int_{\mathbb{R}} dx\, W(\partial F_u \mathcal{H}[G_u] + \partial G_u \mathcal{H}[F_u]).\end{aligned} \tag{161}$$

Using the formal anti-self-adjoint property of the Hilbert transform,

$$\int_{\mathbb{R}} dx\, f\, \mathcal{H}[g] = -\int_{\mathbb{R}} dx\, g\, \mathcal{H}[f], \tag{162}$$

assuming W is constant, and noting that $\partial\mathcal{H}[u] = \mathcal{H}[\partial u]$, we obtain

$$\begin{aligned}(u, S)_H &= -W(\partial\mathcal{H}[u] + \mathcal{H}[\partial u])\\ &= -2W\, \mathcal{H}[\partial u].\end{aligned} \tag{163}$$

Upon choosing $W = \alpha_1/(4\sqrt{2\pi})$ we see this is precisely Ott and Sudan dissipation [67] proposed for modeling electron Landau damping in a fluid model. This form has been used extensively in the magnetic fusion literature, based on a later paper [68].

In this section we have seen how a variety of dissipation mechanisms in 1 + 1 models can be generated by 4-brackets. Indeed, there is considerable room for generalization, e.g., in our last example by replacing the opertors ∂ and $\mathcal{H}$ by any formally anti-self-adjoint operators. In [24] (see Sec. 4.4) a large family of dissipative structures were given in terms of multilinear forms with symmetries that build in invariance of a chosen set of quantities. As 4-brackets build in the invariance of H, we can extend to other quantities generalizing the present framework. Proceeding along these lines is beyond the scope of the present paper.

2. 2 + 1 plasma and fluidlike theories

A large class of 2 + 1 Hamiltonian fluidlike theories exist in the fluid mechanics and plasma physics literature. These include the two-dimensional Euler equation for the dynamics of scalar vorticity and, for example, generalizations including quasigeostrophic dynamics of the potential vorticity which have a single scalar field defined on some two-dimensional domain, say, with coordinates (x, y). Another example is the one-dimensional Vlasov-Poisson system of plasma physics, for which the domain is the two-dimensional phase space with coordinates, say, with (x, v). These theories all have a noncanonical Poisson bracket with a Lie-Poisson bracket based on the Lie-algebra realization on functions (see [2,3,34]), often called the symplectomorphism algebra,

$$[f, g] = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}, \tag{164}$$

with the infinite-dimensional Lie-Poisson bracket being

$$\{F, G\} = \int d^2x\, \omega[F_\omega, G_\omega], \tag{165}$$

where we use the shorthand $F_\omega = \delta F/\delta\omega$.

Quite naturally the infinite-dimensional metriplectic 4-bracket akin to the finite-dimensional 4-bracket of (59) is the following:

$$(F, K; G, N) = \int d^2x \int d^2x'\, \mathcal{G}(\mathbf{x}, \mathbf{x}') \times [F_\omega, K_\omega](\mathbf{x})\, [G_\omega, N_\omega](\mathbf{x}'), \tag{166}$$

which for symmetric $\mathcal{G}(\mathbf{x}, \mathbf{x}')$ has the minimal metriplectic symmetries. In the special case where $\mathcal{G} = \lambda\delta(\mathbf{x} - \mathbf{x}')$ with $\lambda \in \mathbb{R}$, this reduces to

$$(F, K; G, N) = \lambda \int d^2x\, [F_\omega, K_\omega][G_\omega, N_\omega]. \tag{167}$$

If we insert the enstrophy

$$S = \frac{1}{2}\int d^2x\, \omega^2 \tag{168}$$

into (167) as follows, we obtain

$$\begin{aligned}(F, S; G, S) &= \lambda \int d^2x\, [F_\omega, S][G_\omega, S] \\ &= \lambda \int d^2x\, [F_\omega, \omega][G_\omega, \omega] \\ &= \lambda \int d^2x\, F_\omega[\omega, [G_\omega, \omega]].\end{aligned} \tag{169}$$

Next, with the two-dimensional Euler Hamiltonian

$$H = \frac{1}{2}\int d^2x\, \omega\psi, \tag{170}$$

where the stream function ψ satisfies $\nabla^2\psi = -\omega$ and $H_\omega = \psi$, we obtain

$$\frac{\partial\omega}{\partial t} = (\omega, S; H, S) = -\lambda[\omega, [\omega, \psi]], \tag{171}$$

which gives the double bracket dynamics first proposed in [11], which was generalized and used extensively in a variety of contexts in [14,50–52,58,69]. In light of the development of Sec. II E 2, this was to be expected.

Next, it is natural to ask what is the metriplectic 2-bracket that results from (167). We find

$$\begin{aligned}(F, G)_H &= (F, H; G, H) \\ &= \lambda \int d^2x\, [F_\omega, H][G_\omega, H],\end{aligned} \tag{172}$$

which is the metriplectic 2-bracket recorded in [24,45]. Extensive calculations using this bracket appeared in the context of two-dimensional Euler flows and a generalization to magnetohydrodynamics in Bressan's Ph.D. thesis [70]. Preliminary results were published in [71], and the main results will appear in a paper under preparation [38]. Our results in these works reveal a caveat: Because of degeneracy, the system may not relax to what one expects! This problem is remedied by using a bracket based on that given in Sec. III D 1 below.

3. 3 + 1 fluidlike theories

Next we consider two 3 + 1 fluidlike systems. We choose our set of dynamical variables to be composed of densities, $\chi = \{\rho, \sigma, \mathbf{M}\}$, where ρ is the mass density, σ is entropy per unit volume, and $\mathbf{M} = (M_1, M_2, M_3)$ is momentum density, all of which depend on $\mathbf{x} = (x_1, x_2, x_3)$, a Cartesian coordinate. In our first example we choose our entropy Casimir to be the actual entropy of a fluid system, while for the second we choose the helicity, which is a Casimir, to be our entropy.

In our *first example* we desire a theory that conserves total mass, momentum, and energy, with thermodynamics depending on only two thermodynamic variables, say, ρ and σ. Generalizations where we include the chemical potential are possible, but we won't consider such now. Thus, we expect our 4-bracket to not depend on functional derivatives with respect to ρ, which might produce density diffusion. We build a theory out of a K-N pair.

The simplest choice imaginable for M is given by

$$M(F_\chi, G_\chi) = F_\sigma G_\sigma, \tag{173}$$

where as before $F_\sigma = \delta F/\delta\sigma$. Since Σ will involve pairs of functional derivatives $F_{\mathbf{M}} = \delta F/\delta\mathbf{M}$ and, analogous to (144), derivatives so as to assure a diffusive nature, we are thus led to the general isotropic (invariant under rotations) Cartesian tensor of order 4,

$$\hat{\Lambda}_{ikst} = \alpha\delta_{ik}\delta_{st} + \beta(\delta_{is}\delta_{kt} + \delta_{it}\delta_{ks}) + \gamma(\delta_{is}\delta_{kt} - \delta_{it}\delta_{ks}) \tag{174}$$

as an ingredient for creating Σ, which might build in Galilean symmetry. Given the above we assume

$$\Sigma(F_\chi, G_\chi) = \hat{\Lambda}_{ijkl}\, \partial_j F_{M_i} \partial_k G_{M_l} + a\, \nabla F_\sigma \cdot \nabla G_\sigma\,, \tag{175}$$

where $\partial_i := \partial/\partial x_i$, $F_{M_i} = \delta F/\delta M_i$, and we assume α, β, γ, a are arbitrary functions of the thermodynamics variables ρ and σ. Putting this all together in the 3 + 1 context we obtain

$$
\begin{aligned}
(F, K; G, N) &= \int d^3x\,(\Sigma \owedge M)(F_\chi, K_\chi, G_\chi, N_\chi) \\
&= \int d^3x\,[\Sigma(F_\chi, G_\chi)M(K_\chi, N_\chi) \\
&\quad - \Sigma(F_\chi, N_\chi)M(K_\chi, G_\chi) \\
&\quad + M(F_\chi, G_\chi)\Sigma(K_\chi, N_\chi) \\
&\quad - M(F_\chi, N_\chi)\Sigma(K_\chi, G_\chi)]. \qquad (176)
\end{aligned}
$$

Now, choosing the parameters with a specific target in mind we pick

$$
\begin{aligned}
\Sigma(F_\chi, G_\chi) &= \frac{(\xi - 2\eta/3)}{\lambda T}(\nabla \cdot F_{\mathbf{M}})(\nabla \cdot G_{\mathbf{M}}) \\
&\quad + \frac{\eta}{\lambda T}\left(\partial_i F_{M_k}\partial_i G_{M_k} + \partial_i F_{M_k}\partial_k G_{M_i}\right) \\
&\quad + \frac{\kappa}{\lambda T^2}(\nabla F_\sigma \cdot \nabla G_\sigma), \qquad (177)
\end{aligned}
$$

where choices for the parameters α, β, γ of (174) and a of (175) have been made, giving the parameter temperature T, viscosities ξ and η, and thermal conductivity κ. This leads to the following complicated 4-bracket:

$$
\begin{aligned}
(F, K; G, N) &= \int \frac{d^3x}{T}\,\frac{(\xi - 2\eta/3)}{\lambda} \\
&\quad \times [K_\sigma \nabla \cdot F_{\mathbf{M}} - F_\sigma \nabla \cdot K_{\mathbf{M}}] \\
&\quad \times [N_\sigma \nabla \cdot G_{\mathbf{M}} - G_\sigma \nabla \cdot N_{\mathbf{M}}] \\
&\quad + \int \frac{d^3x}{T}\,\frac{\eta}{\lambda} \\
&\quad \times \left[F_\sigma G_\sigma\left(\partial_i K_{M_k}\partial_i N_{M_k} + \partial_i K_{M_k}\partial_k N_{M_i}\right)\right. \\
&\quad + K_\sigma N_\sigma\left(\partial_i G_{M_k}\partial_i F_{M_k} + \partial_i G_{M_k}\partial_k F_{M_i}\right) \\
&\quad - K_\sigma G_\sigma\left(\partial_i F_{M_k}\partial_i N_{M_k} + \partial_i F_{M_k}\partial_k N_{M_i}\right) \\
&\quad \left. - F_\sigma N_\sigma\left(\partial_i G_{M_k}\partial_i K_{M_k} + \partial_i G_{M_k}\partial_k K_{M_i}\right)\right] \\
&\quad + \int \frac{d^3x}{T^2}\,\frac{\kappa}{\lambda}[F_\sigma G_\sigma(\nabla K_\sigma \cdot \nabla H_\sigma) \\
&\quad + K_\sigma N_\sigma(\nabla F_\sigma \cdot \nabla G_\sigma) - N_\sigma G_\sigma(\nabla F_\sigma \cdot \nabla H_\sigma) \\
&\quad - F_\sigma N_\sigma(\nabla K_\sigma \cdot \nabla G_\sigma)]. \qquad (178)
\end{aligned}
$$

With the ideal fluid Hamiltonian

$$
H = \int d^3x\left[\frac{|\mathbf{M}|^2}{2\rho} + \rho U(\rho, s)\right], \qquad (179)
$$

where $\mathbf{M} = \rho\mathbf{v}$ and $\sigma = \rho s$ with s being the specific entropy and ρ the mass density, the 4-bracket of (178) yields the following metriplectic 2-bracket:

$$
\begin{aligned}
(F, G)_H &= (F, H; G, H) \\
&= \frac{1}{\lambda}\int d^3x\, T\,\Lambda_{ikmn}\left[\frac{\partial}{\partial x_i}\left(\frac{\delta F}{\delta M_k}\right) - \frac{1}{T}\frac{\partial v_i}{\partial x_k}\frac{\delta F}{\delta \sigma}\right] \\
&\quad \times \left[\frac{\partial}{\partial x_m}\left(\frac{\delta G}{\delta M_n}\right) - \frac{1}{T}\frac{\partial v_m}{\partial x_n}\frac{\delta G}{\delta \sigma}\right] \\
&\quad + \int d^3x\,\kappa T^2\frac{\partial}{\partial x_k}\left[\frac{1}{T}\frac{\delta F}{\delta \sigma}\right]\frac{\partial}{\partial x_k}\left[\frac{1}{T}\frac{\delta G}{\delta \sigma}\right], \qquad (180)
\end{aligned}
$$

where

$$
\Lambda_{ikmn} = \eta\left(\delta_{ni}\delta_{mk} + \delta_{nk}\delta_{mi} - \tfrac{2}{3}\delta_{ik}\delta_{mn}\right) + \xi\,\delta_{ik}\delta_{mn}. \qquad (181)
$$

We have written (180) without abbreviations so it is easy to see it is precisely the metriplectic bracket first given in [8]. The dynamics generated by this bracket follows upon inserting the entropy functional

$$
S[\sigma] = \int d^3x\,\sigma\,; \qquad (182)
$$

accordingly $(\mathbf{M}, S)_H$ produces a kind of viscous dissipation, while $(\sigma, S)_H$ gives an entropy equation with thermal conduction and viscous heating. Together with the ideal fluid Hamiltonian bracket given in [33], the metriplectic system so generated is a version of the Navier-Stokes equation that conserves the energy of (179) while producing entropy; i.e., it produces a fluid dynamical realization of the first and second laws of thermodynamics. See [8] and [26] for details.

In our *second example* we choose the helicity

$$
S[\mathbf{v}] = \int d^3x\,\mathbf{v} \cdot \nabla \times \mathbf{v}, \qquad (183)
$$

which is known to be a Casimir for the ideal barotropic fluid [3], to be our entropy. We insert this into the metriplectic 4-bracket of (178) along with the Hamiltonian of (179), to obtain

$$
\begin{aligned}
(F, S)_H &= (F, H, S, H) \\
&= \frac{1}{\lambda}\int d^3x\left(T(\xi - 2\eta/3)\left[\nabla\frac{1}{\rho} \cdot (\nabla \times \mathbf{v})\right]\nabla \cdot F_{\mathbf{M}}\right. \\
&\quad + T\eta\{\partial_i[(\nabla \times \mathbf{v})_k/\rho](\partial_i F_{M_k} + \partial_k F_{M_i})\} \\
&\quad - F_\sigma\{(\xi - 2\eta/3)\left[\nabla\frac{1}{\rho} \cdot (\nabla \times \mathbf{v})\right]\nabla \cdot \mathbf{v} \\
&\quad \left. + \eta\,\partial_i[(\nabla \times \mathbf{v})_k/\rho](\partial_i v_k + \partial_k v_i)\}\right). \qquad (184)
\end{aligned}
$$

This bracket will make entropy helicity, while conserving the energy H of (179). This is an interesting system in its own right, which will be further investigated elsewhere.

D. Kinetic theory examples

1. Landau-like collision operator

In [7,9] the metriplectic 2-bracket for the Landau-Lenard-Balescu (LLB) collision operator was given, one that generated a gradient flow using the standard entropy. Here we show how this 2-bracket dynamics comes from a metriplectic 4-bracket. The basic variable of this theory is the phase space density $f(z,t)$ (the distribution function), where a six-dimensional phase space coordinate is $z = (\mathbf{x}, \mathbf{v}) = (x_1, x_2, x_3, v_1, v_2, v_3)$, which is typically a point in T^*Q, where Q is a configuration manifold. Here we won't emphasize the geometry and think of this as $\mathbb{R}^6$. By $\int d^6z$ we will mean an integration over this phase space.

For functionals defined on f, as before, we abbreviate $\delta F/\delta f = F_f$, and for convenience we define for functions $w : \mathbb{R}^6 \to \mathbb{R}$, the operator P,

$$P[w]_i = \frac{\partial w(z)}{\partial v_i} - \frac{\partial w(z')}{\partial v_i'}, \tag{185}$$

which is a linear operator mapping functions on some subset of $\mathbb{R}^6$ to functions from $\mathbb{R}^{12} \to \mathbb{R}^3$ via the expression. Also, we define $\mathbf{g} = \mathbf{v} - \mathbf{v}'$ and

$$\omega_{ij} = \frac{1}{|\mathbf{g}|^3}(|\mathbf{g}|^2\delta_{ij} - g_i g_j)\delta(\mathbf{x} - \mathbf{x}') = \delta(\mathbf{x} - \mathbf{x}')\frac{\partial^2}{\partial v_i \partial v_j}|\mathbf{v} - \mathbf{v}'| = \delta(\mathbf{x} - \mathbf{x}')\frac{\partial^2 |\mathbf{g}|}{\partial v_i \partial v_j}. \tag{186}$$

From (186) it follows that

$$\omega_{ij}(z, z') = \omega_{ji}(z, z'), \tag{187}$$

$$\omega_{ij}(z, z') = \omega_{ij}(z', z), \tag{188}$$

$$g_i\, \omega^{ij} = 0. \tag{189}$$

Given the above, we can write the metriplectic 2-bracket that produces the LLB collision operator, the bracket that was given in [7,9],

$$(F, G)_H = \int d^6z \int d^6z'\, P[F_f]_i\, T^{ij}\, P[G_f]_j, \tag{190}$$

where

$$T^{ij} = \tfrac{1}{2} f(z) f(z')\, \omega^{ij}(z, z'), \tag{191}$$

with ω_{ij} given by (186). The symmetric metriplectic 2-bracket of (190) together with the Poisson bracket for the Vlasov-Poisson system [2,34],

$$\{F, G\} = \int d^6z\, f\, [F_f, G_f], \tag{192}$$

with

$$[f, g] := \frac{1}{m}\left(\frac{\partial f}{\partial \mathbf{x}} \cdot \frac{\partial g}{\partial \mathbf{v}} - \frac{\partial g}{\partial \mathbf{x}} \cdot \frac{\partial f}{\partial \mathbf{v}}\right), \tag{193}$$

generates the collisional Vlasov-Poisson system. This follows if the Hamiltonian for the Vlasov Poisson system,

$$H[f] = \frac{1}{2}\int d^6z\, v^2 f(z) + \frac{1}{2}\int d^6z \int d^6z'\, V(z, z') f(z) f(z'), \tag{194}$$

where a special choice for V gives the Coulomb interaction potential, is inserted along with an appropriate Casimir chosen from the set of Casimirs of (192), *viz.*, $\int d^6z\, \mathcal{C}(f)$ with $\mathcal{C}$ an arbitrary function of f. Choosing the following, which is proportional to the physical entropy:

$$S[f] = \int d^6z\, f \log f, \tag{195}$$

we have the results of [7,9], where the system is generated by

$$\frac{\partial f}{\partial t} = \{f, H\} + (f, S)_H = \{f, \mathcal{F}\} + (f, \mathcal{F})_H, \tag{196}$$

where $\mathcal{F} = H + S$.

Now we construct the metriplectic 4-bracket, which comes quite naturally upon using a generalization of the 4-bracket of (142). Let $\mathcal{G}(z, z')$ be any kernel and suppose Σ and M are symmetric (under the integral) maps from functions to functions of z and z'. Given such Σ and M, a K-N product on functional derivatives can be defined as follows:

$$\begin{aligned}(\Sigma \owedge M)\ (F_f, K_f, G_f, N_f)(z, z') &= \Sigma(F_f, G_f)(z, z')\, M(K_f, N_f)(z, z') \\ &\quad - \Sigma(F_f, N_f)(z, z') M(K_f, G_f)(z, z') \\ &\quad + M(F_f, G_f)(z, z')\Sigma(K_f, N_f)(z, z') \\ &\quad - M(F_f, N_f)(z, z')\Sigma(K_f, G_f)(z, z'),\end{aligned} \tag{197}$$

from which we define a bracket on functionals by

$$(F, K; G, N) = \int d^6z \int d^6z'\, \mathcal{G}(z, z')(\Sigma \owedge M) \times (F_f, K_f, G_f, N_f)(z, z'). \tag{198}$$

This form of 4-bracket can be generalized to higher dimensions of both the dependent and independent variables.

We find the metriplectic 4-bracket for the LLB collision operator has the following simple symmetric form:

$$(F, K; G, N) = \int d^6z \int d^6z'\, \mathcal{G}(z, z')(\delta \owedge \delta)_{ijkl} \times\ P[F_f]_i P[K_f]_j P[G_f]_k P[N_f]_l, \tag{199}$$

where

$$(\delta \owedge \delta)_{ijkl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} + \delta_{jl}\delta_{ik} - \delta_{jk}\delta_{il} = 2(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) \tag{200}$$

and

$$\mathcal{G}(z, z') = \frac{\delta(\mathbf{x} - \mathbf{x}') f(z) f(z')}{4|\mathbf{g}|^3}. \tag{201}$$

Inserting the H of (194) into the bracket of (199), we find $(F, G)_H = (F, H; G, H)$ is that given by (190).

In general any metriplectic 2-bracket of the form of (190), with any T, using a metric g and a Hamiltonian H such that $\mathbf{d}H$ never vanishes, we can always define a parent metriplectic 4-bracket using the formula

$$(F, K; G, N) = \int \frac{1}{g(\mathbf{d}H, \mathbf{d}H)}(T \owedge g)(\mathbf{d}F, \mathbf{d}K, \mathbf{d}G, \mathbf{d}N)$$

that satisfies the relation

$$(F, G)_H = (F, H; G, H).$$

For the case of Landau, this bracket is given by

$$(F, K; G, N) = \int d^6z \int d^6z' \frac{1}{|\mathbf{g}|^2}[T \owedge \delta]_{ijkl} P\left[\frac{\delta F}{\delta f}\right]_i \times P\left[\frac{\delta K}{\delta f}\right]_j P\left[\frac{\delta G}{\delta f}\right]_k P\left[\frac{\delta N}{\delta f}\right]_l. \tag{202}$$

One of the advantages of the 4-bracket formalism is it allows various forms of dissipation to be effortlessly created and interchanged. For example, suppose we replace (201) by

$$\mathcal{G}_M(z,z') = \frac{\delta(\mathbf{x}-\mathbf{x}')M(f(z))M(f(z'))}{4|\mathbf{g}|^3}, \tag{203}$$

where M is an arbitrary function of f. The metriplectic 4-bracket thus defined, with this kernel, the Hamiltonian (194), and entropy given by

$$S[f] = \int d^6z\, s(f), \tag{204}$$

can be designed to relax to a desired stable equilibrium. If we choose $Ms'' = 1$, then $(F,G)_H = (F,H;G,H)$ is the metriplectic 2-bracket of [9], which yields a gradient flow that relaxes to the state determined by

$$H_f = -s'(f). \tag{205}$$

A rigorous Lyapunov stability argument would require s' monotonic and suitable convexity of s.

A special case of the above construction occurs for the choice

$$M(f) = f(1-f), \tag{206}$$

$$s(f) = [f\ln f + (1-f)\ln(1-f)]. \tag{207}$$

The metriplectic 2-bracket $(F,G)_H = (F,H;G,H)$ was shown in [9] to produce a collision operator given in [72], which was designed to relax to a Fermi-Dirac-like equilibrium state proposed in [73].

As a final example of this subsection, we show how to covert the metriplectic theory for the LLP collision operator into a KM bracket, i.e., one with the properties discussed in Sec. II E 1. This is a theory generated by the Hamiltonian with an antisymmetric bracket. Here we suppose $S = \int f\log(f)$ and obtain

$$\begin{aligned}
[F,K]_S &= (F,K;S,H)\\
&= \int d^6z\int d^6z'\,\frac{1}{2|\mathbf{g}|^3}\,f(z)f(z')\,\delta(x-x')\\
&\quad\times\left(P\left[\frac{\delta F}{\delta f}\right]\times P\left[\frac{\delta K}{\delta f}\right]\right)\cdot\left(P\left[\frac{\delta S}{\delta f}\right]\times P\left[\frac{\delta H}{\delta f}\right]\right)\\
&= \int d^6z\int d^6z'\,\frac{1}{2|\mathbf{g}|^3}\,\delta(x-x')\\
&\quad\times\left(P\left[\frac{\delta F}{\delta f}\right]\times P\left[\frac{\delta K}{\delta f}\right]\right)\\
&\quad\cdot\{[f(z')\nabla_v f(z) - f(z)\nabla_{v'}f(z')]\times\mathbf{g}\},
\end{aligned} \tag{208}$$

were $\times$ in the third, fifth, and sixth lines is the usual vector cross product.

2. *Symmetrizing and linearizing GENERIC for Boltzmann*

As a final example we show how to symmetrize and linearize a bracket given by Grmela in [27] for the Boltzmann collision operator. Thus, showing how this system is a metriplectic system. Then we show how it can be obtained from a metriplectic 4-bracket.

Grmela's bracket is

$$\begin{aligned}
(A,S)^{Gr} &= \frac{1}{4}\int d^6z_2'\int d^6z_1'\int d^6z_2\int d^6z_1\, W(z_1',z_2',z_1,z_2)\\
&\quad\times[A_f(z_1)+A_f(z_2)-A_f(z_1')-A_f(z_2')]\\
&\quad\times\{\exp[S_f(z_1')+S_f(z_2')]-\exp[S_f(z_1)+S_f(z_2)]\},
\end{aligned} \tag{209}$$

where $f(z)$ is again the phase space density and W is an integral kernel with the following symmetries:

$$\begin{aligned}
W(z_1',z_2',z_1,z_2) &= W(z_1,z_2,z_1',z_2')\\
&= W(z_2,z_1,z_1',z_2')\\
&= W(z_1,z_2,z_2',z_1').
\end{aligned} \tag{210}$$

Furthermore, $W(z_1,z_2,z_1',z_2')$ is assumed to vanish unless the following conditions are met:

(i) $\mathbf{v}_1^2+\mathbf{v}_2^2 = \mathbf{v}_1'^2+\mathbf{v}_2^{2'}$

(ii) $\mathbf{v}_1+\mathbf{v}_2 = \mathbf{v}_1+\mathbf{v}_2$

(iii) $\mathbf{x_1} = \mathbf{x_2} = \mathbf{x_1}' = \mathbf{x_2}'$.

Dynamics generated by the entropy functional

$$S[f] = \int d^6z\, f\log f \tag{211}$$

in the bracket of (209) gives according to $(f,S)^{Gr}$ the Boltzmann equation.

While this bracket works for obtaining the correct equations of motion, it is neither bilinear nor symmetric. To rectify both of these problems we define a new bracket,

$$\begin{aligned}
(F,S) &= \frac{1}{2}\int d^Nz_1\int d^Nz_2\int d^Nz_1'\int d^Nz_2'\,\mathcal{W}(z_1,z_2,z_1',z_2')\\
&\quad\times[F_f(z_2')+F_f(z_1')-F_f(z_2)-F_f(z_1)]\\
&\quad\times[S_f(z_2')+S_f(z_1')-S_f(z_2)-S_f(z_1)],
\end{aligned} \tag{212}$$

with a new kernel,

$$\mathcal{W} := \frac{W(z_1,z_2,z_1',z_2')}{\log\left(\frac{f(z_2')f(z_1')}{f(z_2)f(z_1)}\right)}\, f(z_1)f(z_2). \tag{213}$$

Notice that the bracket of (212) has the metriplectic properties of being bilinear and symmetric, and that it recovers the Boltzmann equation when S is the entropy of (211). Furthermore, it is appropriately degenerate, i.e., because of the properties of W, it follows that

$$(F,H) = 0,$$

where the Hamiltonian H satisfies

$$H_f(z) = \tfrac{1}{2}\mathbf{v}^2 + V(\mathbf{x}).$$

This allows the Boltzmann-Vlasov kinetic equation to be put into metriplectic form, with the dynamics generated by the free energy.

If we define the symmetric maps

$$P[F_f, G_f] = [F_f(z_2') + F_f(z_1') - F_f(z_2) - F_f(z_1)] \times [G_f(z_2') + G_f(z_1') - G_f(z_2) - G_f(z_1)] \tag{214}$$

and

$$G[F_f, G_f] = F_f(z_1)G_f(z_1) + F_f(z_2)G_f(z_2)$$

and the integral kernel by

$$U(z_1, z_2, z_1, z_1', z_2') = \frac{W(z_1, z_2, z_1', z_2')}{\log\left(\frac{f(z_2')f(z_1')}{f(z_2)f(z_1)}\right)}\left(\frac{1}{\frac{1}{2}\mathbf{v}_1^2 + V(\mathbf{x}_1) + \frac{1}{2}\mathbf{v}_2^2 + V(\mathbf{x}_2)}\right)^2 \times f(z_1)f(z_2), \tag{215}$$

we can define the corresponding 4-bracket as

$$(F, K; G, N) = \frac{1}{2}\int d^N z_1 \int d^N z_2 \int d^N z_1' \int d^N z_2' \times U(z_1, z_2, z_1, z_1', z_2')(P \owedge I)(F_f, K_f; G_f, N_f),$$

which gives the desired result.

IV. CONCLUSION

The main contribution of this work is the idea of endowing a manifold, finite or infinite, with the metriplectic 4-bracket structure, a bracket like the Poisson bracket but with slots for four functions and properties motivated by those of curvature tensors. Dynamics, flows on the manifold, are generated by two phase space functions, a Hamiltonian/energy H and a Casimir/entropy S, in such a way as to conserve energy and produce entropy. The formalism naturally mates noncanonical Hamiltonian dynamics, whose set of Casimirs includes candidate entropies, with dissipative dynamics generated by the metriplectic 4-bracket. The formalism encompasses previous dissipative bracket formalisms as special cases and has rich geometrical structure; in fact, there exists much structure that was not covered in the present paper that will be treated in a future work.

Many avenues for further develpment are apparent. For example, given a Lie-Poisson bracket, there are a variety of theories based on Lie-algebra extensions, the original paper [34] and the nondissipative fluid model of Sec. III C 3 being examples. Many other magnetofluid models for plasma dynamics follow this framework (see, e.g., [54]). A thorough geometric analysis and classification in the metriplectic 4-bracket framework remains to be done. As is well known, noncanonical Hamiltonian dynamics arises via reduction; e.g., for fluids this is embodied in the mapping from Lagrangian to Eulerian variables, with Lagrangian variables having standard canonical form and Eulerian being Lie-Poisson. Thus, the question arises of what happens on the unreduced level as metriplectic dynamics transpires. This was investigated in [25,26], but a thorough understanding of how metriplectic 4-bracket dynamics relates to unreduced dynamics deserves attention. Last, we mention that a more complete understanding of symmetry and conservation in the metriplectic 4-bracket context would be helpful; e.g., in previous work [7,24] this was done by considering multilinear brackets of various types in order to maintain Casimir or other dynamical invariants.

In closing we suggest two practical uses for the metriplectic 4-bracket formalism: as an aid or framework for model building and as a kind of structure to be preserved for computation.

Fundamental Hamiltonian theories, e.g., with microscopic interactions involving many degrees of freedom, tend to be difficult to analyze and to extract predictions. Consequently, one resorts to model building. Sometimes models are obtained by identifying small parameters and performing rigorous asymptotics starting from a fundamental theory, resulting in reduced systems that contain both Hamiltonian and dissipative parts. Good asymptotics will lead to systems that respect the laws of energy conservation and entropy production. Alternatively, often models are based on phenomenology, using some known or believed properties, constraints, and the like in order to produce a model with desired behavior. In the course of such an endeavor, one should obtain a model with clearly identifiable dissipative and nondissipative parts. Upon setting the nondissipative parts to zero, the remaining part should be Hamiltonian with a conserved Hamiltonian having a clear physical interpretation as energy. Similarly, the complete system should respect the law of entropy production in addition to energy conservation, although sometimes the amount of heat produced may be so small so as to neglect energy conservation on large scales, as is the case for turbulence studies with the Navier-Stokes equations. However, such a model should come from a more complete model including entropy dynamics like that given in Sec. III C 3. So our claim is that the metriplectic 4-bracket formalism serves as a kind of paradigm, akin to roles the Hamiltonian or Lagrangian formalisms have played for obtaining fundamental theories. It provides a convenient framework for building models with good thermodynamic properties. The K-N product of Sec. II D 2, although not the only tool available, can be useful in this regard.

Finally, we suggest that the metriplectic 4-bracket formulation introduces an avenue for structure-preserving numerics (see, e.g., [74] for an overview). Just as symplectic integrators (see, e.g., [75]) preserve Hamiltonian form by time stepping with canonical transformations, Poisson integrators do the same while preserving Casimir leaves (e.g., [76,77]), and various dissipative brackets have been used and proposed for a variety of numerical schemes. For example, the original goal of the double bracket of [11–13] and the improvements in [14] were to calculate vortex states, while additional calculations of fluid and magnetofluid stationary states were given in [50–52,69]. The metriplectic 2-bracket formulation already has been used or proposed for computation [38,70,71,78], while some exploratory metriplectic 4-bracket computations have been done in [55].

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