Transient temperature dynamics of reservoirs connected through an open quantum system

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The dynamics of open quantum systems connected with several reservoirs attract great attention due to their importance in quantum optics, biology, quantum thermodynamics, transport phenomena, etc. In many problems, the Born approximation is applicable, which implies that the influence of the open quantum system on the reservoirs can be neglected. However, in the case of long-time dynamics or mesoscopic reservoirs, the reverse influence can be crucial. In this paper, we investigate the transient dynamics of several bosonic reservoirs connected through an open quantum system. We use an adiabatic approach to study the temporal dynamics of temperatures of the reservoirs during relaxation to thermodynamic equilibrium. We show that there are various types of temperature dynamics that strongly depend on the values of dissipative rates and initial temperatures. We demonstrate that temperatures of the reservoirs, including the hottest and coldest ones, can exhibit nonmonotonic behavior. Moreover, there are moments of time during which the reservoir with an initially intermediate temperature becomes the hottest or coldest reservoir. The obtained results pave the way for managing energy flows in mesoscale and nanoscale systems.

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I. INTRODUCTION

Open quantum systems have gained a lot of attention over several decades. A system can be treated as an open quantum system if it consists of a subsystem (qubit, molecule, etc.) that we are interested in that interacts with its environment, e.g., photons of free space, phonons of host medium. If the interaction between the subsystem and its environment is weak, one can exclude the environmental degrees of freedom from the consideration using Born approximation [1–3] that assumes that the state of the environment is not changed due to interaction with the subsystem. If one additionally uses Markov approximation that implies that the open quantum system dynamics is local in time [1–5], one obtains master equation for the subsystem density matrix in Gorini-Kossakowsky-Sudarshan-Lindblad (GKSL) form [6–8]. Different ratios between relaxation rates and open quantum system eigenfrequencies require different approaches for GKSL master equation application, namely, local approach [9–12], global approach [13–16], partial-secular approach [17–20], modified local approach [21,22]. Some of these works have been dedicated to the thermodynamics of the open quantum system and to the dynamics of the energy flows [10,12,13,16,20]. These approaches have shown to be appropriate for the description of the dissipation in weak coupling limit. Usually these are applied to the description of qubits [23–26], quantum dots [27], molecules [28], nanostructures [29–31], nanolasers [32–34].

The Born-Markov approximation assumes that the state of the reservoir—it is the Gibbs state with the given temperature—does not change [33,35,36]. This approximation is natural when one considers a single reservoir with a large number of degrees of freedom. However, there are many

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situations when the subsystem is connected to two reservoirs with different temperatures. Such an approximation can not be valid, at least at long-time dynamics. Indeed, in such a case the reservoirs interact through the subsystem, and their temperatures should get asymptotically equal.

The evolution of energy flows through the open quantum system as well as the evolution of the environment, especially when logical elements, diodes, or transistors are considered, are very important in phononics and photonics [37–40] to prevent undesired energy flows. This evolution plays a significant role also for circuit elements of thermal computing needed for the realization of artificial intelligence [41–44]. In addition, it is important to optimize the temperature regime of coupled qubits due to the existence of an optimal coupling constant between them, which maximizes their stationary entanglement at a given temperature [45]. Thus, understanding of the reservoirs' temperatures dynamics at nanoscale is very important.

The direct solving (in particular, numerical) of the von Neumann equation for the reservoir is complicated and can be done only in the case of not too many modes (usually less than 100) [46–49]. Another possibility for solving this problem is to use the Zwanzig projection operator method to construct the master equation for the reservoir density matrix [50–52]. This approach implies solving complicated integral-differential equations. One more option is to get equations on a number of quanta in the reservoir at some frequency under non-Markovian approaches, which are usually used in the limit of strong coupling with the reservoir. This can be done via the Zwanzig projection operator method [53,54] or via time evolving density operator with orthogonal polynomials (TEDOPA mapping) [55–58]. The first approach implies solving the integral-differential master equation for the open quantum system along with finding the number of quanta in a mode of the reservoir. The second approach implies solving a system of a large number of equations that describe a set of connected oscillators. Both mentioned approaches do not take into account the processes of thermalization of the reservoirs. Because the terms of the reservoirs' Hamiltonians that are responsible for the thermalization usually are not directly taken into account. The absence of these terms leads to a non-Gibbs distribution of excitations in the reservoir [59,60]. However, if dynamics of an open quantum system and thermalization of reservoirs occur at different time scales [18,33], it is possible to effectively exclude thermalization processes from consideration.

In this paper, we consider an open quantum system coupled to n bosonic reservoirs in the weak coupling limit. We study the transient behavior of the reservoirs' temperatures in the framework of developed adiabatic approximation. In this approximation, we suppose the existence of short, intermediate, and long time scales. At the short time scale, internal thermalizations of the reservoirs happen and temperatures of the reservoirs are established. At the intermediate time scale, the dynamics of the open quantum system take place, while the temperature of the reservoirs is considered fixed. On this time scale, the open quantum system reaches its nonequilibrium stationary state with nonzero energy flow through it. At the long time scale, the energy flow through the open quantum system results in a change in the reservoirs' temperatures. Thus, each reservoir is in thermal equilibrium at the intermediate time scale, and their temperatures are changed due to the energy flow through the open quantum system at the long time scale. We develop a general theory for the description of the reservoirs' temperatures dynamics and apply it to the case of an open quantum system consisting of a set of quantum harmonic oscillators. We show that at certain values of dissipative rates and initial temperatures of reservoirs, nonmonotonic dynamics of reservoirs' temperatures take place. This involves the ability of a reservoir with an intermediate initial temperature to become the hottest or the coldest reservoir. We calculate the thermal conductivity of the open quantum system in the case of two reservoirs, find characteristic time of temperature equilibration, and show that this characteristic time can be minimized by adjusting the ratio of open quantum system eigenfrequencies to reservoirs' initial temperatures. We show that nonmonotonic dynamics of temperatures can take place.

II. MODEL

We consider n bosonic reservoirs describing by the Hamiltonians $\hat{H}_{Rj}$, $j = 1, \ldots, n$ interacting with a open quantum system (hereinafter, the open system) with Hamiltonian $\hat{H}_S$. The total Hamiltonian reads

$$\hat{H} = \hat{H}_S + \sum_{j=1}^{n} \hat{H}_{Rj} + \sum_{j=1}^{n} \hat{H}_{SRj}, \tag{1}$$

$$\hat{H}_{SRj} = \epsilon_j \hat{S}_j \hat{R}_j. \tag{2}$$

Here $\hat{S}_j$ and $\hat{R}_j$ are operators of the open system and the jth reservoir, respectively. The operator $\hat{H}_{SRj}$ describes interaction between them with coupling constant ϵ_j. In the Born-Markov approximation, the open system dynamics are governed by the GKSL master equation [6–8,16]

$$\frac{\partial \hat{\rho}_S}{\partial t} = -i[\hat{H}_S, \hat{\rho}_S] + \sum_{j=1}^{n} \Lambda_j[\hat{\rho}_S], \tag{3}$$

where $\hbar = 1$, $\Lambda_j[\hat{\rho}_S]$ is Lindblad superoperator describing interaction with jth reservoir. In the Born approximation, the state of each reservoir is constant during the evolution of the open system, and it is thermal equilibrium with temperature T_j (Gibbs distribution). For this assumption to be valid, the temperature and chemical potential of the reservoir should be constant at the time scale τ_S: characteristic dissipation time of the open system. From this, it follows that τ_S should be much greater than $\tau_{Rj} \sim \hbar/k_B T_j$: characteristic time of jth reservoir thermalization (Markov approximation [1]), and the initial energy of the open system should be much smaller than the initial energies of reservoirs. Otherwise, the dissipation of the open system will affect the temperatures of the reservoirs. Furthermore, τ_S should be much smaller than t_{eq}: characteristic time of the temperature equilibration between the different reservoirs due to energy flows J_j through the open system.

Let us first discuss the conditions needed for the Born approximation to be used. The first condition implies that the reservoirs have a large number of modes. The energy exchange between the open system and reservoirs happens when the open system transits from one eigenstate to another. The

energy quantum with the frequency that equals the difference of the mentioned eigenstates' eigenfrequencies is consumed by the reservoir. This means that the energy exchange between the open system and reservoir happens only at some set of frequencies, defined by the difference between eigenstates of the open system with allowed transitions. Thus, for the reservoir to be considered to consist of an infinite number of modes, it is sufficient that the density of states of the reservoir to be dense near the mentioned frequencies. The characteristic difference between eigenfrequencies of the reservoir is of the order $\Delta\nu = c/L$, where L is the characteristic size of the reservoir (i.e., cavity size). If the open system has the minimal transition frequency ω, then the mentioned condition can be written as $\Delta\nu/\omega = (c/L)/\omega \sim \lambda/L \ll 1$, where λ is maximal radiation wavelength.

The second condition implies that the reservoirs have sufficient energies. The energy of the jth reservoir is proportional to the number of particles N_{R_j} in the reservoir and its temperature T_j: $E_j \sim N_{R_j} k_B T_j$. The energy of an open system is proportional to the number of particles in the open system N_S and its characteristic frequency ω: $E_S \sim N_S \hbar\omega$. Thus, condition $N_{R_j} k_B T_j \gg N_S \hbar\omega$ guarantees that the dissipation of the open system will not affect the reservoirs. Hence, the influence of open system initial states on the states of the reservoirs can be neglected, if the mentioned condition is satisfied.

Now, let us discuss the conditions needed for the Markov approximation to be used. The characteristic time of the open system evolution $\tau_S \sim 1/\gamma_{\max}$ where $\gamma_{\max}$ is the largest dissipative rate of the open system. For $\tau_S \gg \tau_{R_j}$ ($\tau_R \sim 1/T_{\min}$) it is sufficient $k_B T_{\min}/\hbar \gg \gamma_{\max}$, where $T_{\min}$ is minimal temperature among reservoirs. Thus, if $k_B T_{\min}/\hbar \gg \gamma_{\max}$, the establishment of the Gibbs distributions in reservoirs happens faster than the open system dissipates. Hence, it is natural to consider reservoirs to be in Gibbs states at every moment of time. Mathematically, this means that the derivative in the left side of Eq. (3) is a coarse-grained derivative with step Δt such that $\tau_S \gg \Delta t \gg \tau_{R_j}$.

For the condition $t_{\text{eq}} \gg \tau_S$ to be satisfied, the following condition should be fulfilled. The energy flow from a reservoir (through the open system) is proportional to the coupling strength between the reservoir and open system γ and the energy of a quantum ω that can be absorbed by the open system. If reservoirs have a difference in energies proportional to $\Delta(Nk_BT)$, the time of thermal equilibrium establishment can be evaluated as $t_{\text{eq}} \sim \Delta(Nk_BT)/(\hbar\omega\gamma)$. Using that $\tau_S \sim 1/\gamma$ we arrive at the conclusion that the condition $t_{\text{eq}}/\tau_S \gg 1$ implies $\Delta(Nk_BT)/\hbar\omega \gg 1$. This expression has clear physical meaning: to the condition for $t_{\text{eq}} \gg \tau_S$ be satisfied, the difference in energies between reservoirs should be much greater than the energy of a quantum they are exchanging.

Summing up, for the Eq. (3) with fixed reservoirs' temperatures to be applicable four conditions should be satisfied: $k_B T_{\min}/\hbar \gg \gamma_{\max}$, $\lambda/L \ll 1$ and $E_j \gg E_S$, $|\Delta E_{kj}|/\hbar\omega \gg 1$, where ΔE_{kj} is difference in energies between kth and jth reservoir. All of them are usually fulfilled in optic region.

The fulfillment of established conditions implies that there are three time scales mentioned in the description to Eq. (3) that are in the following relation $\tau_R \ll \tau_S \ll t_{\text{eq}}$. Using Eq. (3), the total energy flow into the open system can be defined as follows [13,16,20]

$$\langle \dot{\hat{H}}_S \rangle = \frac{d}{dt}\langle \hat{\rho}_S \hat{H}_S \rangle = \langle \dot{\hat{\rho}}_S \hat{H}_S \rangle = \sum_{j=1}^{n} \mathrm{tr}(\Lambda_j[\hat{\rho}_S]\hat{H}_S) \equiv \sum_{j=1}^{n} J_j, \tag{4}$$

where $J_j = \mathrm{tr}(\Lambda_j[\hat{\rho}_S]\hat{H}_S)$ is energy flow from jth reservoir to the open system. If the open system losses energy via J_j then $J_j < 0$, if the open system gains energy via J_j then $J_j > 0$.

As $\tau_R \ll \tau_S \ll t_{\text{eq}}$, the temperatures of reservoirs are constant on the time scale τ_S. Consequently, the energy flows J_j are the functions of reservoirs' temperatures T_j, $J_j = J_j(T_1, \ldots, T_n, t)$ (in this work we are focused on the case of reservoirs with fixed volumes and zero chemical potentials). On this time scale the density matrix tends to the stationary state for given temperatures of the reservoirs, $\hat{\rho}_S = \overline{\hat{\rho}}_S(T_1, \ldots, T_n)$ (the overline denotes the stationary value of the overlined expression). Thus $\overline{\langle \dot{\hat{H}}_S \rangle} = \sum_j \overline{J}_j = 0$. The energy of each reservoir is conserved on this time scale, but stationary energy flows from the reservoirs are not zero after the open system reaches $\hat{\rho}_S = \overline{\hat{\rho}}_S(T_1, \ldots, T_n)$.

In turn, on the time scale t_{eq}, the energy flow from jth reservoir to the open system changes the energy of the reservoir E_j as follows:

$$\frac{dE_j}{dt} = -\overline{J}_j(T_1, \ldots, T_n). \tag{5}$$

Thus, $dE_j = (dE_j/dT_j)|_{T_j} dT_j = C_j(T_j) dT_j$ where $C_j(T_j)$ is the heat capacity of the jth reservoir. As reservoirs are considered in thermal equilibrium with temperatures T_j, their heat capacities can be derived, when reservoirs' Hamiltonians are determined. The heat capacity of the jth reservoir is a function of T_j. Thus, we get the equations on the reservoirs' temperature dynamics

$$\frac{dT_j}{dt} = -\overline{J}_j(T_1, \ldots, T_n)/C(T_j). \tag{6}$$

The stationary solution to these equations implies that all J_j should be zero. Generally, the equilibrium temperatures, as well as their time dependencies, strongly depend on the type of interaction of the open system with each reservoir. In the subsequent section, we investigate time dynamics of T_j for the case of a set of harmonic oscillator interacting with reservoirs consisting of bosonic quasiparticles.

III. CASE OF BOSONIC RESERVOIRS

In this section, we apply the general Eq. (6) for the case of n bosonic reservoirs interacting with a set of harmonic oscillators (see Fig. 1). Such an open system may be considered as model of a multimode cavity [61,62], surface or bulk plasmons [63,64], polaritons [65], vibrational modes in solid bodies and molecules [66,67], magnons [68,69], modes of separated resonators [70]. For these systems, the Hamiltonians of the open system and the bosonic reservoirs have the form

$$\hat{H}_S = \sum_{\kappa} \omega_\kappa \hat{a}_\kappa^\dagger \hat{a}_\kappa, \tag{7}$$

$$\hat{H}_{Rj} = \sum_{m} \tilde{\omega}_{mj} \hat{e}_{mj}^\dagger \hat{e}_{mj}, \tag{8}$$

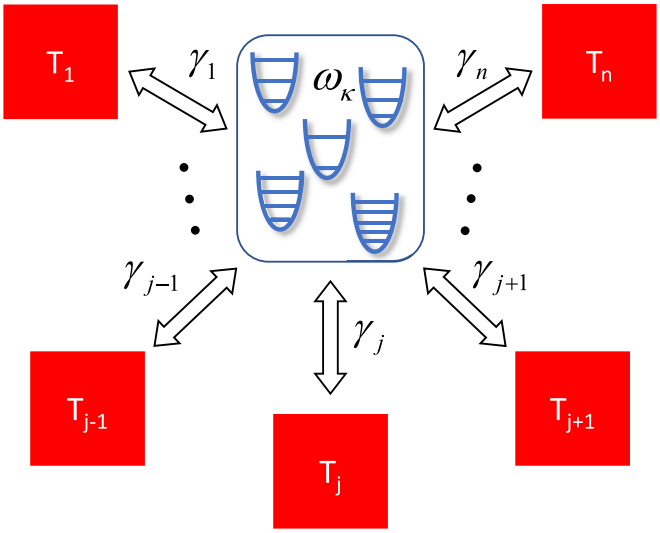

FIG. 1. The system considered in Sec. III: n bosonic reservoirs of quasiparticles at temperature $T_1, \ldots, T_n$ interacting through an open system represented by a set of oscillators. Oscillators' frequencies are indexed with ω_κ [see Eq. (7)]. The interaction between jth reservoir and oscillator with frequency ω_κ is described via function $\gamma_j(\omega_\kappa)$ [see Eq. (13)].

$$\hat{H}_{SRj} = \sum_\kappa \sum_m \chi_{\omega_\kappa,j}(\tilde{\omega}_{mj})(\hat{a}_\kappa^\dagger + \hat{a}_\kappa)(\hat{e}_{mj}^\dagger + \hat{e}_{mj}). \quad (9)$$

Here, ω_κ are oscillator frequencies, $\hat{a}_\kappa$ is the annihilation operator for the oscillator in the open system with frequency ω_κ. Reservoirs are represented as sets of oscillators, $\hat{e}_{mj}$ is annihilation operator of the mth oscillator in the jth reservoir, $\tilde{\omega}_{mj}$ is the frequency of the mth oscillator in the jth reservoir. Coefficients $\chi_{\omega_\kappa,j}(\tilde{\omega}_{mj})$ describe the strength of interaction between oscillator with frequency ω_κ in the open system and mth oscillator in the jth reservoir.

Hereinafter, we assume that there are three time scales, $\tau_R \ll \tau_S \ll t_{\text{eq}}$ (see Sec. II). Therefore, the dynamics of the system with Hamiltonian (7)–(9) on the time scale τ_S obey Eq. (3) with the following Lindblad superoperators [18]

$$\Lambda_j[\hat{\rho}_S] = \sum_\kappa \Lambda_{\kappa,j}[\hat{\rho}_S],$$
$$\Lambda_{\kappa,j}[\hat{\rho}_S] = \frac{G_j(-\omega_\kappa, T_j)}{2}\hat{L}[\hat{a}_\kappa, \hat{a}_\kappa^\dagger] + \frac{G_j(\omega_\kappa, T_j)}{2}\hat{L}[\hat{a}_\kappa^\dagger, \hat{a}_\kappa]. \quad (10)$$

Here $\hat{L}[\hat{X}, \hat{Y}] = 2\hat{X}\hat{\rho}_S\hat{Y} - \hat{Y}\hat{X}\hat{\rho}_S - \hat{\rho}_S\hat{Y}\hat{X}$, and ω denote the frequency of one of the oscillators from the open system. The coefficients $G_j(\pm\omega_\kappa, T_j)$ determine the rates of transitions between eigenstates of the harmonic oscillator with frequency ω_α due to its interaction with jth reservoir. For the considered Hamiltonian Eqs. (7)–(9), they equal [52]

$$G_j(\pm\omega, T_j) = \gamma_j(\omega)[n_j(\omega, T_j) + 1/2 \mp 1/2]. \quad (11)$$

Here

$$n_j(\omega, T_j) = 1/[\exp(\omega/T_j) - 1], \quad (12)$$

it is the mean occupancy of the jth reservoir's states with eigenfrequency ω, and

$$\gamma_j(\omega) = \pi g_j(\omega)|\chi_{\omega,j}(\omega)|^2, \quad (13)$$

where $g_j(\omega)$ is the jth reservoir's density of states at the frequency ω, $j = 1, \ldots, n$. From Eq. (11) the fulfillment of Kubo-Martin-Schwhinger condition, namely, $G_j(\omega)/G_j(-\omega) = e^{-\omega/T_j}$ follows [52]. Thus, the first term in $\Lambda_{\kappa,j}[\hat{\rho}_S]$ is responsible for the downward transitions and the second term is responsible for the upward transitions in the oscillator with frequency ω_κ.

Note, that such dependence of $G_j(\pm\omega, T_j)$ coefficients on ω and T_j strongly relies on that $\tau_R \ll \tau_S$. These coefficients are calculated considering, that each of the reservoirs is in a Gibbs state, which is valid only if $\tau_R \ll \tau_S$ [52,71].

The equations for the dynamics of the mean number of quanta can be found through the identity $d\langle\hat{A}\rangle/dt = \text{tr}(\dot{\hat{\rho}}_S\hat{A})$. Using commutation relation $[\hat{a}_\kappa, \hat{a}_\kappa^\dagger] = \hat{1}$ we obtain the following equation for the mean number of quanta $\langle\hat{a}_\kappa^\dagger\hat{a}_\kappa\rangle$ in oscillator:

$$\frac{\partial\langle\hat{a}_\kappa^\dagger\hat{a}_\kappa\rangle}{\partial t} = \sum_{j=1}^n -G_j(-\omega_\kappa, T_j)\langle a_\kappa^\dagger a_\kappa\rangle + G_j(\omega_\kappa, T_j)(1 + \langle a_\kappa^\dagger a_\kappa\rangle). \quad (14)$$

The stationary solution to this equation is

$$\overline{\langle a_\kappa^\dagger a_\kappa\rangle} = \frac{\sum_{j=1}^n G_j(\omega_\kappa, T_j)}{\sum_{j=1}^n [G_j(-\omega_\kappa, T_j) - G_j(\omega_\kappa, T_j)]}. \quad (15)$$

Using the notations $\gamma_j(\omega)$, $n_j(\omega, T_j)$, Eq. (15) can be rewritten in the form

$$\overline{\langle a_\kappa^\dagger a_\kappa\rangle} = \frac{\sum_{j=1}^n \gamma_j(\omega_\kappa) n_j(\omega_\kappa, T_j)}{\sum_{j=1}^n \gamma_j(\omega_\kappa)}. \quad (16)$$

From general expression (4), the stationary energy flow, $\bar{J}_j$, that is formed on the timescale τ_S is found to be

$$\begin{aligned}\bar{J}_j &= \sum_\kappa \omega_\kappa \{[G_j(\omega_\kappa, T_j) - G_j(-\omega_\kappa, T_j)]\overline{\langle a_\kappa^\dagger a_\kappa\rangle} + G_j(\omega_\kappa, T_j)\} \\ &= \sum_\kappa \omega_\kappa \frac{\sum_{q=1}^n \gamma_j(\omega_\kappa)\gamma_q(\omega_\kappa)[n_j(\omega_\kappa, T_j) - n_q(\omega_\kappa, T_q)]}{\sum_{q=1}^n \gamma_q(\omega_\kappa)} \\ &= \sum_\kappa \omega_\kappa \gamma_j(\omega_\kappa)\left(n_j(\omega_\kappa, T_j) - \sum_{q=1}^n p_q(\omega_\kappa) n_q(\omega_\kappa, T_q)\right). \end{aligned} \quad (17)$$

Here $p_q(\omega_\kappa) = \gamma_q(\omega_\kappa)/\sum_{m=1}^n \gamma_m(\omega_\kappa)$. In other words, the absolute value of the energy flow from jth reservoir is defined by the difference between the $n_j(\omega_\kappa, T_j)$, occupancy of the state with frequency ω_κ in the jth reservoir, and mean occupancy of the state with frequency ω_κ among all reservoirs: according to Eq. (17), normalized dissipation rates per unit time, $p_j(\omega)$ can be interpreted as probabilities of absorption of the energy quantum ω by the jth reservoir.

Now, using the dependence of the stationary energy flows on the reservoirs' temperatures, we can apply Eq. (6) to determine the dynamics of the reservoirs' temperatures on the time scale $t_{\text{eq}} \gg \tau_S$. For the n D-dimensional bosonic

reservoirs with zero chemical potential, spectrum $\varepsilon(p) = cp^d$, and density of states $g(\varepsilon) = S_D \varepsilon^{D/d-1}/(2\pi\hbar)^D$, where $S_D = D\pi^{D/2}/\Gamma(D/2+1)$ is the surface area of unit sphere, the energy of each reservoir reads [72]

$$E = \int_\Gamma \frac{d^D x d^D p}{(2\pi\hbar)^D} \frac{\varepsilon}{\exp(\varepsilon/T)-1} = AVT^{\alpha+1}. \tag{18}$$

Here $A = S_D\Gamma(\alpha+1)\zeta(\alpha+1)/(2\pi\hbar)^D$ is a constant (ζ is the Riemann zeta function), $\alpha = D/d$ (dimensionless), and V is the volume of the reservoir: length, area, or volume depending on the D.

On the time scale $t_{\rm eq} \gg \tau_S$, energy flows Eq. (17) sufficiently change the energies of reservoirs. On this time scale, Eq. (5) represent temperatures' dynamics

$$\frac{dE_j}{dt} = (\alpha_j+1)A_jV_jT_j^{\alpha_j}\frac{dT_j}{dt} = C_j(T_j)\frac{dT_j}{dt} = -\bar{J}_j, \tag{19}$$

or, alternatively,

$$\frac{dT_j}{dt} = -\sum_\kappa \frac{\omega_\kappa \gamma_j(\omega_\kappa)\left(n_j(\omega_\kappa,T_j) - \sum_{q=1}^n p_q(\omega_\kappa)n_q(\omega_\kappa,T_q)\right)}{(\alpha_j+1)A_jV_jT_j^{\alpha_j}}. \tag{20}$$

From Eq. (20) it follows that the hottest reservoir is always cooling down. For that, consider a moment of time with reservoirs having certain temperatures, and T_m to be a maximal temperature among all of them at this moment of time. As $\partial n_j(\omega,T_j)/\partial T_j > 0$, from $T_m = \max T_j$ it follows that $n_m(\omega,T_m) \geqslant n_j(\omega,T_j)$. Thus, $n_m(\omega,T_m) \geqslant \sum_k p_k(\omega)n_k(\omega,T_k)$, because the maximal number $n_m(\omega,T_m)$ from the set $\{n_1(\omega,T_1),\ldots,n_n(\omega,T_n)\}$ is greater or equal than the average value of this set. Thus, from Eq. (20) it follows that $dT_m/dt \leqslant 0$. If there are some not equal reservoirs' occupancies (i.e., temperatures of some reservoirs are not equal), then $n_m(\omega,T_m) > \sum_k p_k(\omega)n_k(\omega,T_k)$, and $dT_m/dt < 0$. Thus, in general, the hottest reservoir is cooling down at all moments of time. Analogously, the coldest reservoir is heating up at all moments of time.

Let us consider a function that represents the maximal instant temperature difference between the reservoirs: $f(t) = \max_j T_j(t) - \min_j T_j(t)$. By definition, $f(t) \geqslant 0$. Also, because the coldest reservoir is heating up and the hottest reservoir is cooling down, $f(t)$ is a decreasing function of time, if there are reservoirs with different temperatures. We are interested in the limit of $f(t)$ when $t \to \infty$.

Next, we prove by contradiction that $f(t) \to 0$, when $t \to \infty$. Let us suppose that $f(t)$ tends to a finite value $\delta > 0$. Then $\max_j n_j(\omega,T_j) > \sum_k p_k(\omega)n_k(\omega,T_k) > \min_j n_j(\omega,T_j)$ when $t \to \infty$. Thus, the energy flows out of the hottest reservoir and in the coldest reservoir should tend to a finite value according to Eq. (20) when $t \to \infty$. This means that the heat capacities of these reservoirs should tend to infinity at some temperatures. Because, by consideration, the reservoirs have finite heat capacities at all temperatures, this is impossible. Thus, $f(t) \to 0$, when $t \to \infty$. Consequently, in the stationary state, all reservoirs' temperatures are the same. From the arguments presented above, it follows that the stationary state of Eq. (20) is stable with respect to small perturbations.

An easy way to find the equilibrium temperature is to establish some integrals of motion of the Eq. (20). From Eq. (20) and the definition of $p_q(\omega_\kappa)$ it follows that total energy of reservoirs is conserved

$$\sum_j \frac{dE_j}{dt} = \sum_j (\alpha_j+1)A_jV_jT_j^{\alpha_j}\frac{dT_j}{dt} = 0. \tag{21}$$

Thus, the equilibrium temperatures can be found from total energy conservation

$$\sum_{j=1}^n A_jV_jT_j^{\alpha_j+1} = \sum_{j=1}^n A_jV_jT_{\rm eq}^{\alpha_j+1}. \tag{22}$$

Note, that from $\partial E_j(T_j)/\partial T_j > 0$ it follows that Eq. (22) has only one positive real solution $T_{\rm eq}$.

If α_j is the same for all reservoirs, then the equilibrium temperature can be found as

$$T_{\rm eq} = \sqrt[\alpha_j+1]{\sum_{j=1}^n A_jV_jT_j^{\alpha_j+1}(0)/\sum_{j=1}^n A_jV_j}. \tag{23}$$

IV. TRANSIENT DYNAMICS OF RESERVOIRS' TEMPERATURES

A. Evaluation of time of temperature equalization

Let us consider the case of two reservoirs. In this case, Eqs. (16)–(17) are reduced to

$$\overline{\langle a_\kappa^\dagger a_\kappa\rangle} = \frac{\gamma_1(\omega_\kappa)n_1(\omega_\kappa,T_1) + \gamma_2(\omega_\kappa)n_2(\omega_\kappa,T_2)}{\gamma_1(\omega_\kappa)+\gamma_2(\omega_\kappa)} \tag{24}$$

$$\bar{J}_1 = \sum_\kappa \omega_\kappa \Gamma(\omega_\kappa)[n_1(\omega_\kappa,T_1) - n_2(\omega_\kappa,T_2)], \tag{25}$$

where $\Gamma(\omega) = \gamma_1(\omega)\gamma_2(\omega)/[\gamma_1(\omega)+\gamma_2(\omega)]$.

It is possible to define the energy flow through the equality $\bar{J} = -\varkappa\Delta T$, where $\varkappa(T_1,T_2)$ is the thermal conductivity and $\Delta T = T_2 - T_1$. A characteristic time of the establishment of thermal equilibrium is $t_{\rm eq} = C(T)/\varkappa$. Using Eqs. (24) and (25), the thermal conductivity $\varkappa(T_1,T_2)$ is found to be

$$\begin{aligned}\varkappa(T_1,T_2) &= -\sum_\kappa \frac{\omega_\kappa\Gamma(\omega_\kappa)}{T_1-T_2}\frac{e^{\omega_\kappa/T_1}-e^{\omega_\kappa/T_2}}{(e^{\omega_\kappa/T_1}-1)(e^{\omega_\kappa/T_2}-1)} \\ &= -\sum_\kappa \frac{2\omega_\kappa\Gamma(\omega_\kappa)}{T_1-T_2}\frac{\sinh\left(\frac{\omega_\kappa}{2T_1}-\frac{\omega_\kappa}{2T_2}\right)}{\sinh\left(\frac{\omega_\kappa}{2T_1}\right)\sinh\left(\frac{\omega_\kappa}{2T_2}\right)}.\end{aligned} \tag{26}$$

If $\Delta T = T_2 - T_1 \ll T_1 \equiv T$, then

$$\begin{aligned}&\omega\Gamma(\omega)[n(\omega,T) - n(\omega,T+\Delta T)] \\ &= -\omega\Gamma(\omega)\frac{\partial n(\omega,T)}{\partial T}\Delta T \\ &= -\Gamma(\omega)\frac{e^{\omega/T}\omega^2/T^2}{(e^{\omega/T}-1)^2}\Delta T = -\Gamma(\omega)\frac{(\omega/2T)^2}{\sinh^2(\omega/2T)}\Delta T.\end{aligned} \tag{27}$$

Thus, the thermal conductivity $\varkappa(T_1, T_2) = \kappa(T)$ equals

$$\varkappa(T) = \sum_\kappa \Gamma(\omega_\kappa) \frac{e^{\omega_\kappa/T} \omega_\kappa^2/T^2}{(e^{\omega_\kappa/T} - 1)^2} = \sum_\kappa \Gamma(\omega_\kappa) \frac{(\omega_\kappa/2T)^2}{\sinh^2(\omega_\kappa/2T)} = \sum_\kappa \varkappa(\omega_\kappa, T). \quad (28)$$

Here $\varkappa(\omega_\kappa, T)$ is thermal conductivity at the temperature T associated with the contribution of oscillator with the frequency ω_κ. For $\omega_\kappa/T \gg 1$ we have $\varkappa(\omega_\kappa, T) \approx \Gamma(\omega_\kappa)(\omega_\kappa^2/T^2)e^{-\omega_\kappa/T}$, while for $\omega_\kappa/T \ll 1$ we obtain $\varkappa(\omega_\kappa, T) \approx \Gamma(\omega_\kappa)$.

Consequently, the characteristic time of temperature equalization between two reservoirs with $|\Delta T(0)|/T_{1,2}(0) \ll 1$ can be estimated as

$$t_{\rm eq}(T) = \frac{C_{\rm min}(T)}{\varkappa(T)} = C_{\rm min}(T) \left(\sum_\kappa \Gamma(\omega_\kappa) \frac{(\omega_\kappa/2T)^2}{\sinh^2(\omega_\kappa/2T)} \right)^{-1}, \quad (29)$$

where $C_{\rm min}(T) = \min[C_1(T), C_2(T)]$.

For a certain frequency ω from the set ω_κ

$$t_{\rm eq}(\omega, T) = \frac{C_{\rm min}(T)}{\Gamma(\omega)} \frac{\sinh^2(\omega/2T)}{(\omega/2T)^2}. \quad (30)$$

For $T \ll \omega$, the characteristic time of temperature equalization can be evaluated as $t_{\rm eq}(\omega, T) \simeq [C_{\rm min}(T)/\Gamma(\omega)][e^{\omega/T}/(\omega/T)^2]$, and thus, $t_{\rm eq}(\omega, T)$ increases with decrease of T. For $T \gg \omega$, we have $t_{\rm eq}(\omega, T) \simeq C_{\rm min}(T)/\Gamma(\omega)$, and $t_{\rm eq}(\omega, T)$ increases with an increase in T. Thus, there is a value ω/T that minimizes the time of temperature equalization determined by the equation

$$\frac{\omega/T}{\alpha_{\rm min} + 2} = \tanh\left(\frac{\omega}{2T}\right), \quad (31)$$

where $\alpha_{\rm min}$ equals α of the reservoir with minimal heat capacity.

The dependence of $t_{\rm eq}(\omega, T)$ from Eq. (30) on the ratio ω/T is shown on Fig. 2. It is a function with the minimum at the ratio $\omega/T \simeq 5$ and a wide flat range near this minimum.

Thus, $t_{\rm eq}(T) \simeq \min_\kappa t_{\rm eq}(\omega_\kappa, T)$ provides a good upper bound approximation for $t_{\rm eq}(T)$. This approximation works well unless oscillators' frequencies in the open system are concentrated in the frequency region corresponding to the flat region in Fig. 2, because outside the flat region $t_{\rm eq}(\omega_\kappa, T)$ grows fast. In the case of oscillators' frequencies concentration in the flat region, temperature equilibration speeds up proportionally to the number of such oscillators due to the mentioned reason.

Note that $\tau_S \sim \Gamma(\omega)^{-1}$. For the model to be applicable, the condition $t_{\rm eq}/\tau_S \gg 1$ should be satisfied. This means that $C_{\rm min}[T(t)] \gg 1$. For the considered reservoirs of quasiparticles with zero chemical potential, this condition means that the number of quasiparticles in the reservoir should be much greater than 1. This matches the energy condition discussed in Sec. II.

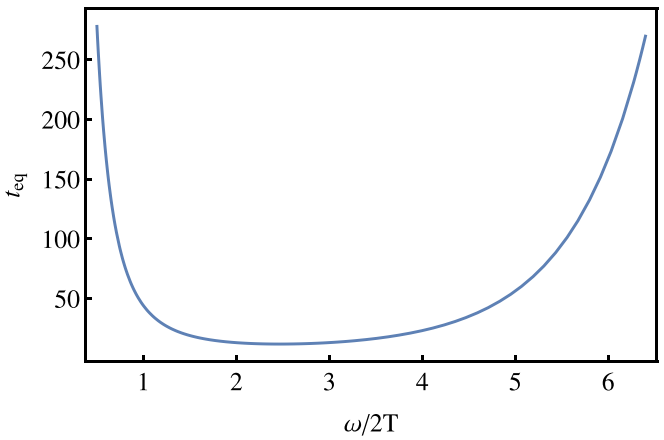

FIG. 2. The dependence of $t_{\rm eq}(\omega, T)$ on the ratio $\omega/2T$. Parameters of the reservoirs are set to $A_1 = A_2 = 1$, $V_1 = 1$, $V_2 = 1$, $\alpha_1 = \alpha_2 = 3$ [product AV is not dimensionless, however, here and after we consider A and V to be measured in such units, that temperature in Eq. (18) is measured in units of energy].

B. Variation of temperatures' sequence

In this section, we model Eq. (20) in the case of three reservoirs. We use a numerical method with stiffness detection (explicit midpoint method with double-harmonic extrapolation as a nonstiff solver, implicit Euler method with harmonic extrapolation as a stiff solver [73]).

The developed model predicts that the hottest and coldest reservoirs may cease to be so during evolution. As has been mentioned, from the Eq. (17) it follows that if $n_j(T_j)$ is greater than the mean occupancy, the reservoir is cooling down, and if $n_j(T_j)$ is smaller than the mean occupancy, the reservoir is heating up. Thus, the coldest reservoir can never become the hottest reservoir, and vice versa. However, other pairs of reservoirs do not obey this restriction. For example, if the dissipative rate of the open system in the first reservoir is much less than the dissipative rates of the open system in the second and third reservoirs, then the second and third reservoirs become to have equal temperatures, and only after that, they become to have equal temperatures with the first reservoir. As a result, the temperature of the first reservoir, being initially intermediate, becomes the highest before temperature equalization with the second and third reservoirs, see solid blue line in Fig. 3.

Finally, the temperatures of all reservoirs get equal to the temperature that can be found as the real positive solution of Eq. (22) that lies between the maximal and minimal initial temperatures of the reservoirs. For the considered case, the equilibrium temperature is determined by Eq. (23) with $n = 3$ and is depicted with a horizontal dot-dashed gray line in Fig. 3.

C. Transient nonmonotonic temperature behavior

In this section, we model Eq. (20) in the case of five reservoirs using the same numerical method. The developed theory also predicts the possibility of nonmonotonic evolution of reservoirs' temperatures. For example, the temperature of

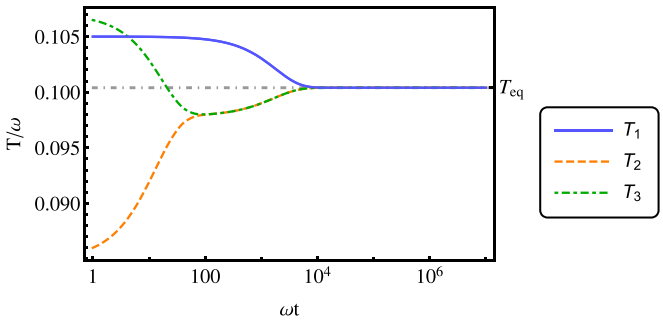

FIG. 3. The time dependence of temperature of three reservoirs connected through a set of seven oscillators with frequencies $\omega_1 = 1$, $\omega_2 = 1.1\omega_1$, $\omega_3 = 1.5\omega_1$, $\omega_4 = 0.9\omega_1$, $\omega_5 = 2\omega_1$, $\omega_6 = 2.2\omega_1$, $\omega_7 = 3\omega_1$. The parameters: $\alpha_j = \alpha = 3$, $\gamma_1/\omega_1 = 10^{-4}(\omega/\omega_1)^\alpha$, $\gamma_2/\omega_1 = 2 \times 10^{-2}(\omega/\omega_1)^\alpha$, $\gamma_3/\omega_1 = 2 \times 10^{-2}(\omega/\omega_1)^\alpha$, $A_j = 1$, $V_j = 1$. Initial temperatures of the reservoirs are $T_1/\omega_1 = 0.105$, $T_2/\omega_1 = 0.085$, $T_3/\omega_1 = 0.107$. The horizontal dot-dashed gray line depicts T_{eq} from Eq. (23).

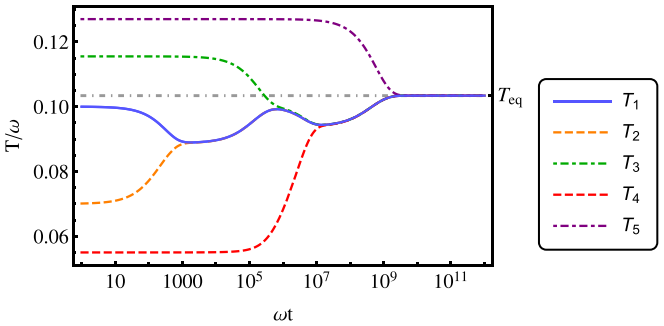

FIG. 4. The time dependence of temperatures of the five reservoirs connected through a set of seven oscillators with frequencies $\omega_1 = 1$, $\omega_2 = 1.1\omega_1$, $\omega_3 = 1.5\omega_1$, $\omega_4 = 0.9\omega_1$, $\omega_5 = 2\omega_1$, $\omega_6 = 2.2\omega_1$, $\omega_7 = 3\omega_1$. The parameters: $\alpha_j = \alpha = 3$, $\gamma_1/\omega_1 = 10^{-2}(\omega/\omega_1)^\alpha$, $\gamma_2/\omega_1 = 10^{-3}(\omega/\omega_1)^\alpha$, $\gamma_3/\omega_1 = 10^{-6}(\omega/\omega_1)^\alpha$, $\gamma_4/\omega_1 = 10^{-7}(\omega/\omega_1)^\alpha$, $\gamma_5/\omega_1 = 3 \times 10^{-9}(\omega/\omega_1)^\alpha$, $A_j = 1$, $V_j = 1$. Initial temperatures of the reservoirs are $T_1/\omega_1 = 0.1$, $T_2/\omega_1 = 0.07$, $T_3/\omega_1 = 0.1155$, $T_4/\omega_1 = 0.055$, $T_5/\omega_1 = 0.127$. The horizontal dot-dashed gray line depicts T_{eq} from Eq. (23).

the second reservoir in Fig. 3 exhibits nonmonotonic behavior (dashed orange line).

In general, such behavior is possible in the following case. Different pairs of reservoirs can have different t_{eq} time. As we consider n reservoirs, then we have $n(n-1)/2$ values of t_{eq} times. The set of reservoirs that have a minimal value of time of temperature equalization, t_{eq}, among these $n(n-1)/2$ t_{eq} values, become with equal temperatures first. Suppose that there are m reservoirs in this set. All of these m reservoirs have the same temperature after their temperatures become equal to each other and can be considered as a single reservoir with effective heat capacity. This reduces the number of reservoirs in the system. Now we have $(n-m+1)(n-m)/2$ values of t_{eq} times, and the process repeats until only one reservoir is left. Thus, temperature equalization between all rese rvoirs can be divided into the consequent temperature equalization of reservoirs in each subset.

In the case when initial subsets of reservoirs have sufficient differences in initial temperatures and t_{eq}, nonmonotonic dynamics of temperatures are possible. An example of such behavior is represented in Fig. 4 (solid blue line). Here, the first and second reservoirs become to have equal temperatures first. The initial temperature of the first reservoir is higher than the initial temperature of the second reservoir. As a result, the temperature of the first reservoir decreases.

After the first and second reservoirs become to have equal temperatures, they both become to have equal temperatures with the third reservoir having a greater initial temperature than they both. Thus, the equal temperature of these three reservoirs is greater than the equal temperature of the first two reservoirs. As a consequence, the temperature of the first reservoir, which initially decreased, has increased now. In other words, nonmonotonic temperature dynamics take place. Note that not only the first reservoir exhibits nonmonotonic behavior. Temperatures' dynamics of the second and third reservoirs are also nonmonotonic (see Fig. 4, dashed orange line starting from $T/\omega = 0.07$ and dot-dashed green line starting from $T/\omega = 0.1155$).

After all stages of nonmonotonic dynamics, the temperatures of all reservoirs get equal. The equilibrium temperature can be found similarly to the case of three reservoirs. For the considered case of five reservoirs, the equilibrium temperature is determined by Eq. (23) with $n = 5$ and is depicted with the horizontal dot-dashed gray line in Fig. 4.

V. DISCUSSION AND CONCLUSION

In this work, we considered an open quantum system consisting of an arbitrary set of oscillators connected to several reservoirs. We studied the transient temperature regimes of reservoirs during temperature equalization via the open quantum system. We showed that the interplay between dissipative rates and occupancies of reservoirs results in various transitional temperature regimes. We showed that it is possible to achieve nonmonotonic dynamics of the reservoir temperature by right choosing the initial temperatures and dissipative rates of the reservoirs. Moreover, by lowering the dissipative rate of one reservoir, one can increase the time that is needed for this reservoir to equalize temperature with other ones. This can make a reservoir with an intermediate temperature the hottest or the coldest one after some moment of time.

The explicit expression for the stationary energy flow from a reservoir to the open quantum system, revealing the interplay between dissipative rates and occupancies of reservoirs, was derived using the GKSL equation. The energy flow from the jth reservoir was shown to be proportional to the difference between the occupancy of the jth reservoir at the eigenfrequency of the open quantum system and the mean occupancy at this eigenfrequency among all reservoirs. The mean occupancy is equal to the sum of the reservoirs' occupancies, with weights equal to normalized dissipative rates.

In the case of two reservoirs, we calculated the thermal conductivity of the considered open quantum system, with the help of which we estimated the time needed for the temperature equalization of these reservoirs. Moreover, we showed that there exists an optimal ratio of the transitional

frequency to reservoir temperature that minimizes the time of temperature equalization between reservoirs.

The proposed model enables the use of different methods for calculating energy flows in an open quantum system. Namely, the heat flow from a reservoir can be calculated with the aid of Green's functions [74–77] instead of the GKSL equation. For example, in the case of an open quantum system consisting of one bosonic or fermionic mode, one can obtain an equation on the energy flow from jth reservoir similar to Eq. (17) by using the convolution of energy flow from Eq. (17) with the Lorentzian transmission function [74,78]. The careful comparison of energy flows obtained in the GKSL approach and Green's function formalism we leave for future works.

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[1] H. Carmichael, *An Open Systems Approach to Quantum Optics: Lectures Presented at the Université Libre de Bruxelles, October 28–November 4 1991*, Vol. 18 (Springer Science & Business Media, Berlin, 2009).

[2] F. Haake, Statistical treatment of open systems by generalized master equations, in *Quantum Statistics in Optics and Solid-State Physics*, Springer Tracts in Modern Physics, Vol. 66 (Springer, Berlin, Heidelberg, 1973), pp. 98–168.

[3] F. Haake, Non-Markoffian effects in the laser, Z. Phys. A **227**, 179 (1969).

[4] W. H. Louisell, *Quantum Statistical Properties of Radiation* (John Wiley and Sons, New York, 1973).

[5] C. Cohen-Tannoudji, J. Dupont-Roc, and G. Grynberg, *Atom-Photon Interactions: Basic Processes and Applications* (John Wiley & Sons, New York, 1998).

[6] G. Lindblad, On the generators of quantum dynamical semigroups, Commun. Math. Phys. **48**, 119 (1976).

[7] V. Gorini, A. Kossakowski, and E. C. G. Sudarshan, Completely positive dynamical semigroups of N-level systems, J. Math. Phys. **17**, 821 (1976).

[8] E. B. Davies, Markovian master equations, Commun. Math. Phys. **39**, 91 (1974).

[9] V. Y. Shishkov, E. S. Andrianov, A. A. Pukhov, A. P. Vinogradov, and A. A. Lisyansky, Perturbation theory for Lindblad superoperators for interacting open quantum systems, Phys. Rev. A **102**, 032207 (2020).

[10] A. Levy and R. Kosloff, The local approach to quantum transport may violate the second law of thermodynamics, Europhys. Lett. **107**, 20004 (2014).

[11] M. Konopik and E. Lutz, Local master equations may fail to describe dissipative critical behavior, Phys. Rev. Res. **4**, 013171 (2022).

[12] A. Trushechkin and I. Volovich, Perturbative treatment of inter-site couplings in the local description of open quantum networks, Europhys. Lett. **113**, 30005 (2016).

[13] P. P. Hofer, M. Perarnau-Llobet, L. D. M. Miranda, G. Haack, R. Silva, J. B. Brask, and N. Brunner, Markovian master equations for quantum thermal machines: local versus global approach, New J. Phys. **19**, 123037 (2017).

[14] J. O. González, L. A. Correa, G. Nocerino, J. P. Palao, D. Alonso, and G. Adesso, Testing the validity of the 'Local' and 'Global' GKLS master equations on an exactly solvable model, Open Sys. Info. Dyn. **24**, 1740010 (2017).

[15] M. T. Naseem, A. Xuereb, and Ö. E. Müstecaplıoğlu, Thermodynamic consistency of the optomechanical master equation, Phys. Rev. A **98**, 052123 (2018).

[16] V. Y. Shishkov, E. S. Andrianov, A. A. Pukhov, A. P. Vinogradov, and A. A. Lisyansky, Relaxation of interacting open quantum systems, Phys. Usp. **62**, 510 (2019).

[17] M. Cattaneo, G. L. Giorgi, S. Maniscalco, and R. Zambrini, Local versus global master equation with common and separate baths: superiority of the global approach in partial secular approximation, New J. Phys. **21**, 113045 (2019).

[18] I. V. Vovchenko, V. Y. Shishkov, A. A. Zyablovsky, and E. S. Andrianov, Model for the description of the relaxation of quantum-mechanical systems with closely spaced energy levels, JETP Lett. **114**, 51 (2021).

[19] T. T. Sergeev, I. V. Vovcenko, A. A. Zyablovsky, and E. S. Andrianov, Environment-assisted strong coupling regime, Quantum **6**, 684 (2021).

[20] I. V. Vovcenko, A. A. Zyablovsky, A. A. Pukhov, and E. S. Andrianov, Energy transport induced by transition from the weak to the strong coupling regime between non-Hermitian optical systems, J. Opt. Soc. Am. B **40**, 2990 (2023).

[21] A. Trushechkin, Unified Gorini-Kossakowski-Lindblad-Sudarshan quantum master equation beyond the secular approximation, Phys. Rev. A **103**, 062226 (2021).

[22] P. P. Potts, A. A. S. Kalaee, and A. Wacker, A thermodynamically consistent Markovian master equation beyond the secular approximation, New J. Phys. **23**, 123013 (2021).

[23] F. Reiter, L. Tornberg, G. Johansson, and A. S. Sørensen, Steady-state entanglement of two superconducting qubits engineered by dissipation, Phys. Rev. A **88**, 032317 (2013).

[24] Z. Leghtas, U. Vool, S. Shankar, M. Hatridge, S. M. Girvin, M. H. Devoret, and M. Mirrahimi, Stabilizing a Bell state of two superconducting qubits by dissipation engineering, Phys. Rev. A **88**, 023849 (2013).

[25] D. Martín-Cano, A. González-Tudela, L. Martín-Moreno, F. J. García-Vidal, C. Tejedor, and E. Moreno, Dissipation-driven generation of two-qubit entanglement mediated by plasmonic waveguides, Phys. Rev. B **84**, 235306 (2011).

[26] I. Vovcenko, V. Y. Shishkov, and E. Andrianov, Dephasing-assisted entanglement in a system of strongly coupled qubits, Opt. Express **29**, 9685 (2021).

[27] L. Contreras-Pulido, M. Bruderer, S. F. Huelga, and M. B. Plenio, Dephasing-assisted transport in linear triple quantum dots, New J. Phys. **16**, 113061 (2014).

[28] M. B. Plenio and S. F. Huelga, Dephasing-assisted transport: quantum networks and biomolecules, New J. Phys. **10**, 113019 (2008).

[29] B. Buča and T. Prosen, A note on symmetry reductions of the Lindblad equation: transport in constrained open spin chains, New J. Phys. **14**, 073007 (2012).
[30] C. Flindt, T. Novotný, and A.-P. Jauho, Full counting statistics of nano-electromechanical systems, Europhys. Lett. **69**, 475 (2005).
[31] T. H. Jonsson, A. Manolescu, H.-S. Goan, N. R. Abdullah, A. Sitek, C.-S. Tang, and V. Gudmundsson, Efficient determination of the Markovian time-evolution towards a steady-state of a complex open quantum system, Comput. Phys. Commun. **220**, 81 (2017).
[32] A. A. Zyablovsky, I. A. Nechepurenko, E. S. Andrianov, A. V. Dorofeenko, A. A. Pukhov, A. P. Vinogradov, and A. A. Lisyansky, Optimum gain for plasmonic distributed feedback lasers, Phys. Rev. B **95**, 205417 (2017).
[33] G. M. Moy, J. J. Hope, and C. M. Savage, Born and Markov approximations for atom lasers, Phys. Rev. A **59**, 667 (1999).
[34] O. Jonasson, F. Karimi, and I. Knezevic, Partially coherent electron transport in terahertz quantum cascade lasers based on a Markovian master equation for the density matrix, J. Comput. Electron. **15**, 1192 (2016).
[35] G. Schaller, *Open Quantum Systems far from Equilibrium*, Vol. 881 (Springer, Berlin, 2014).
[36] S. van Smaalen and T. F. George, On the Born and Markov approximations: Phonon relaxation and coherent excitation of adsorbed molecules, J. Chem. Phys. **87**, 5504 (1987).
[37] N. Li, J. Ren, L. Wang, G. Zhang, P. Hänggi, and B. Li, Colloquium: Phononics: Manipulating heat flow with electronic analogs and beyond, Rev. Mod. Phys. **84**, 1045 (2012).
[38] C. W. Chang, D. Okawa, A. Majumdar, and A. Zettl, Solid-state thermal rectifier, Science **314**, 1121 (2006).
[39] B. Li, L. Wang, and G. Casati, Negative differential thermal resistance and thermal transistor, Appl. Phys. Lett. **88**, 143501 (2006).
[40] Y. Dubi and M. Di Ventra, Colloquium: Heat flow and thermoelectricity in atomic and molecular junctions, Rev. Mod. Phys. **83**, 131 (2011).
[41] P. J. Coles, C. Szczepanski, D. Melanson, K. Donatella, A. J. Martinez, and F. Sbahi, Thermodynamic AI and the fluctuation frontier, in *2023 IEEE International Conference on Rebooting Computing (ICRC)* (IEEE, New York, 2023), pp. 1–10.
[42] T. Hylton, Thermodynamic computing: An intellectual and technological frontier, in *Proceedings of IS4SI SUMMIT Berkeley 2019*, edited by M. Burgin, G. Dodig-Crnkovic, and M. J. Schroeder, Vol. 47 (MDPI, Basel, 2020), p. 23.
[43] T. Hylton, Thermodynamic neural network, Entropy **22**, 256 (2020).
[44] N. Ganesh, A thermodynamic treatment of intelligent systems, in *2017 IEEE International Conference on Rebooting Computing (ICRC)* (IEEE, New York, 2017), pp. 1–4.
[45] M. Scala, R. Migliore, A. Messina, and L. Sánchez-Soto, Robust stationary entanglement of two coupled qubits in independent environments, Europhys J. D **61**, 199 (2011).
[46] G. Katz, D. Gelman, M. A. Ratner, and R. Kosloff, Stochastic surrogate Hamiltonian, J. Chem. Phys. **129**, 034108 (2008).
[47] R. Baer and R. Kosloff, Quantum dissipative dynamics of adsorbates near metal surfaces: A surrogate Hamiltonian theory applied to hydrogen on nickel, J. Chem. Phys. **106**, 8862 (1997).
[48] D. Gelman and R. Kosloff, Simulating dissipative phenomena with a random phase thermal wavefunctions, high temperature application of the surrogate Hamiltonian approach, Chem. Phys. Lett. **381**, 129 (2003).
[49] T. T. Sergeev, A. A. Zyablovsky, E. S. Andrianov, and Y. E. Lozovik, Signature of exceptional point phase transition in Hermitian systems, Quantum **7**, 982 (2023).
[50] S. Nakajima, On quantum theory of transport phenomena: Steady diffusion, Prog. Theor. Phys. **20**, 948 (1958).
[51] R. Zwanzig, Ensemble method in the theory of irreversibility, J. Chem. Phys. **33**, 1338 (1960).
[52] H.-P. Breuer and F. Petruccione, *The Theory of Open Quantum Systems* (Oxford University Press, New York, 2002).
[53] A. S. Trushechkin, Dynamics of reservoir observables within the Zwanzig projection operator method in the theory of open quantum systems, Proc. Steklov Inst. Math. **306**, 257 (2019).
[54] M. te Vrugt, Understanding probability and irreversibility in the Mori-Zwanzig projection operator formalism, Euro. J. Philos. Sci. **12**, 41 (2022).
[55] A. Nüßeler, D. Tamascelli, A. Smirne, J. Lim, S. F. Huelga, and M. B. Plenio, Fingerprint and universal Markovian closure of structured bosonic environments, Phys. Rev. Lett. **129**, 140604 (2022).
[56] A. W. Chin, Á. Rivas, S. F. Huelga, and M. B. Plenio, Exact mapping between system-reservoir quantum models and semi-infinite discrete chains using orthogonal polynomials, J. Math. Phys. **51**, 092109 (2010).
[57] R. Rosenbach, J. Cerrillo, S. F. Huelga, J. Cao, and M. B. Plenio, Efficient simulation of non-Markovian system-environment interaction, New J. Phys. **18**, 023035 (2016).
[58] D. Tamascelli, A. Smirne, J. Lim, S. F. Huelga, and M. B. Plenio, Efficient simulation of finite-temperature open quantum systems, Phys. Rev. Lett. **123**, 090402 (2019).
[59] F. Borgonovi, F. M. Izrailev, L. F. Santos, and V. G. Zelevinsky, Quantum chaos and thermalization in isolated systems of interacting particles, Phys. Rep. **626**, 1 (2016).
[60] T. Mori, T. N. Ikeda, E. Kaminishi, and M. Ueda, Thermalization and prethermalization in isolated quantum systems: a theoretical overview, J. Phys. B: At., Mol. Opt. Phys. **51**, 112001 (2018).
[61] N. M. Sundaresan, Y. Liu, D. Sadri, L. J. Szőcs, D. L. Underwood, M. Malekakhlagh, H. E. Türeci, and A. A. Houck, Beyond strong coupling in a multimode cavity, Phys. Rev. X **5**, 021035 (2015).
[62] V. D. Vaidya, Y. Guo, R. M. Kroeze, K. E. Ballantine, A. J. Kollár, J. Keeling, and B. L. Lev, Tunable-range, photon-mediated atomic interactions in multimode cavity QED, Phys. Rev. X **8**, 011002 (2018).
[63] J. Pitarke, V. Silkin, E. Chulkov, and P. Echenique, Theory of surface plasmons and surface-plasmon polaritons, Rep. Prog. Phys. **70**, 1 (2007).
[64] M. Bagherian, A. Passian, S. Kouchekian, and G. Siopsis, Quantum Hamiltonian for the surface charge density on a ring torus and radiative decay of plasmons, Phys. Rev. B **102**, 085422 (2020).
[65] A. Rahimi-Iman, *Polariton Physics*, Springer Series in Optical Sciences (Springer, Cham, 2020), Vol. 229.
[66] P. O. J. Scherer and S. F. Fischer, On the theory of vibronic structure of linear aggregates. Application to pseudoisocyanin (PIC), Chem. Phys. **86**, 269 (1984).
[67] R. P. Feynman, *Statistical Mechanics: A Set of Lectures* (CRC Press, Boca Raton, 2018).

[68] S.-F. Qi and J. Jing, Magnon-assisted photon-phonon conversion in the presence of structured environments, Phys. Rev. A **103**, 043704 (2021).

[69] C. Kittel and C.-Y. Fong, *Quantum Theory of Solids* (Wiley, New York, 1987).

[70] M. Abdi, P. Tombesi, and D. Vitali, Entangling two distant non-interacting microwave modes, Ann. Phys. **527**, 139 (2015).

[71] H. J. Carmichael, *Statistical Methods in Quantum Optics 1: Master Equations and Fokker-Planck Equations* (Springer, Berlin, 2013).

[72] L. D. Landau and E. M. Lifshitz, *Statistical Physics*, Vol. 5 (Elsevier, Amsterdam, 2013).

[73] G. Wanner and E. Hairer, *Solving Ordinary Differential Equations II*, Vol. 375 (Springer, Berlin, 1996).

[74] H. Haug and A. P. Jauho, *Quantum Kinetics in Transport and Optics of Semiconductors*, Vol. 2 (Springer, Berlin, 2008).

[75] Y. Meir and N. S. Wingreen, Landauer formula for the current through an interacting electron region, Phys. Rev. Lett. **68**, 2512 (1992).

[76] J.-S. Wang, B. K. Agarwalla, H. Li, and J. Thingna, Nonequilibrium Green's function method for quantum thermal transport, Front. Phys. **9**, 673 (2014).

[77] W. Zhang, T. S. Fisher, and N. Mingo, The atomistic Green's function method: An efficient simulation approach for nanoscale phonon transport, Numer. Heat Transf. B **51**, 333 (2007).

[78] B. K. Agarwalla and D. Segal, Assessing the validity of the thermodynamic uncertainty relation in quantum systems, Phys. Rev. B **98**, 155438 (2018).