Two-dimensional open quantum system in dissipative bosonic heat bath, external magnetic field, and two time-dependent electric fields

E. Kh. Alpomishev [1,2] G. G. Adamian [1] and N. V. Antonenko [1,3]
[1] *Joint Institute for Nuclear Research, 141980 Dubna, Russia*
[2] *Institute of Nuclear Physics, 100214 Tashkent, Uzbekistan*
[3] *Tomsk Polytechnic University, 634050 Tomsk, Russia*

(Received 15 October 2023; accepted 21 March 2024; published 9 April 2024)

Non-Markovian dynamics of a charged particle in a two-dimensional harmonic oscillator linearly coupled to a neutral bosonic heat bath is investigated in an external uniform magnetic field and two perpendicular time-dependent electric fields. The analytical expressions for the time-dependent and asymptotic angular momentum are derived for the Markovian and non-Markovian dynamics. The dependence of the angular momentum on the frequency of the electric field, cyclotron frequency, collective frequency, and anisotropy of the heat bath is studied. The angular momentum (or magnetization) of a charged particle can be ruled by varying the frequency of the electric field.

DOI: 10.1103/PhysRevE.109.044125

I. INTRODUCTION

Nowadays, the field of two-dimensional (2D) materials and two-dimensional electron gas (2DEG) is being rapidly developed. A big impetus for this was the discovery of the highly topical 2DEG of graphene, which accelerated the theoretical and experimental research of all 2D materials [1–12]. Several important properties of 2D materials and 2DEGs allow their application in nanotechnology, telecommunications, biotechnology, electronics, and optoelectronics, as an active medium for light amplification, electrode materials, electrocatalysts in energy storage, and many other applications. Therefore, it is important to study the dependence of their electromagnetic, mechanical, and optical properties on the electric field, magnetic field, temperature, pressure, etc. In Ref. [13], a phase transition from a paramagnetic state to a ferromagnetic state on the surface of the Pd monolayer by an applied electric field is predicted. The electric-field effect is also utilized to control magnetism in ferromagnets [14]. There are theoretical and experimental studies of the effects of polarized electromagnetic waves on 2D materials and 2DEG [15–18]. An analysis of the nonlinear electric conductivity of graphene beyond the Kubo regime under the influence of a dissipative bosonic heat bath was carried out in Refs. [17,18]. Electric conductivity and nonlinear optical and optoelectronic properties of 2D materials have been extensively studied, but the effect of a polarized electric field on their magnetic properties has been relatively little studied.

The well-known Landau theory of diamagnetism [19,20] stems from the solution of the quantum-mechanical problem of a charged particle in the presence of a constant magnetic field. The physics of Landau levels is of great interest in many physical systems, e.g., the quantum Hall effect and high-temperature superconductivity [21]. A generalization of Landau diamagnetism in a dissipative 2D system was obtained in the Markovian limit in Ref. [22]. As predicted in Refs. [23,24], a dynamic magnetic moment also appears in a 2D dissipative asymmetric harmonic oscillator affected by a linear-polarized monochromatic electromagnetic wave, even without an external magnetic field. This effect was revealed in the Markovian and dipole approximations in the case of the symmetric friction tensor (isotropic environment). As shown in Refs. [23,24], a linear-polarized microwave field creates stationary magnetization in mesoscopic ballistic quantum dots with 2D electron gas being at thermal equilibrium. Magnetization is proportional to a number of electrons in a quantum dot and to the microwave power and does not depend on temperature and, generally, on the form of the initial distribution function. Microwave fields of moderate strength create magnetization in the quantum dot of a few microns in size that is several orders of magnitude larger than the magnetization produced by persistent currents [24]. However, in symmetric (isotropic) quantum dots (symmetric 2D harmonic oscillator) a dynamic magnetic moment does not appear.

In this work, we will significantly expand the study of induced magnetization taking into account the anisotropy of the environment, the external uniform magnetic field, and non-Markovian effects. The formalism presented allows us to explore the magnetization of 2D materials under influence of external magnetic and electric fields and coupling with the environment. The challenge is to indicate how we can control this magnetization with the electric field and cyclotron frequency depending on the anisotropy of the environment in which the system is embedded. We use the theory of open quantum systems, which is widely utilized to identify the effects of fluctuations and dissipation in macroscopic systems [25–41]. To calculate dynamic magnetization, we consider the charge carrier as a quantum particle coupled to a neutral bosonic environment (heat bath) through particle-phonon interactions and develop a general approach based on the quantum non-Markovian Langevin model for a 2D harmonic oscillator with an asymmetric (anisotropic) friction tensor in the field of a linearly polarized monochromatic electromagnetic wave and an uniform magnetic field. So asymmetry is also created by the heat bath and the formalism suggested generalizes the previous results on induced magnetization. We show that a nonzero magnetic moment appears even in the case of the symmetric 2D harmonic oscillator and zero magnetic field. We also extend our analysis to the case of zero potential (a free particle) and anisotropic environment. Note that the results presented here are relevant for any nondegenerate 2D quantum objects, in which the influence of Fermi-Dirac statistics can be neglected.

The paper is organized as follows. In Sec. II we introduce the Hamiltonian of the total system consisting of a 2D quantum oscillator embedded in a neutral bosonic medium in the presence of a constant magnetic field and time-dependent electric fields. Solving the second-order Heisenberg equations for the heat bath degrees of freedom, the generalized non-Markovian Langevin equations are explicitly obtained for a quantum particle. Using the solutions of these equations (Appendix A), we derive the z component of time-dependent and asymptotic angular moments L_z for the two-dimensional charged oscillator and explore how the values of L_z can be controlled with the external field and properties of the environment in the cases of non-Markovian and Markovian dynamics. In Sec. III the obtained results are analyzed and the resonance conditions of $L_z(\infty)$ are shown. A summary is given in Sec. IV. In addition, angular momentum and conditions for its resonance were obtained for a free charged particle in a dissipative bosonic heat bath, an external constant magnetic field, and two nonstationary electric fields (Appendix C).

II. NON-MARKOVIAN LANGEVIN EQUATIONS FOR A DISSIPATIVE HARMONIC OSCILLATOR IN EXTERNAL MAGNETIC AND ELECTRIC FIELDS

In Ref. [37] a 2D quantum harmonic oscillator in a heat bath was considered in a constant magnetic field to study the induction of angular momentum. The quantum non-Markovian Langevin model was also used to consider the dynamics of this oscillator. Here we are interested in the open quantum system affected by an electromagnetic wave, not only by a constant magnetic field. Thus, the equations of motion should be derived from the very beginning starting from the total Hamiltonian.

In order to investigate the influence of external fields on the dynamics of an open quantum system, we consider the motion of a charged particle with effective mass m and positive charge e in a 2D parabolic potential (in the xy plane) surrounded by a neutral bosonic heat bath in the presence of a perpendicular axisymmetric magnetic field (along the z axis). In the case of a linear coupling in coordinates between this particle and the heat bath, the total Hamiltonian of the collective subsystem + heat bath is as follows:

$$
\begin{aligned}
H = {} & \frac{1}{2m}\left(\pi_x^2+\pi_y^2\right)+\frac{m}{2}\left(\omega_x^2x^2+\omega_y^2y^2\right)+\sum_\nu \hbar\omega_\nu b_\nu^+ b_\nu \\
& + exE_x(t)+eyE_y(t)+\sum_\nu (x\alpha_\nu+yg_\nu)(b_\nu^+ + b_\nu) \\
& + \sum_\nu \frac{1}{\hbar\omega_\nu}(x\alpha_\nu+yg_\nu)^2,
\end{aligned}
\tag{1}
$$

where

$$
\pi_x = p_x+\tfrac{1}{2}m\omega_c y,\quad \pi_y = p_y-\tfrac{1}{2}m\omega_c x, \tag{2}
$$

$[\pi_x,\pi_y] = -[\pi_y,\pi_x] = i\hbar m\omega_c$, $\omega_c = eB/m$ is the cyclotron frequency [here $\mathbf{A} = (-\frac{1}{2}yB, \frac{1}{2}xB, 0)$ is the vector potential of a magnetic field with the strength $B = |\mathbf{B}|$], $\mathbf{p} = (p_x, p_y)$ is the canonically conjugated momentum, the time-dependent electric fields $E_x(t)$ and $E_y(t)$ act, respectively, in the x and y directions, ω_x and ω_y are the collective frequencies, b_ν^+ and b_ν are the phonon creation and annihilation operators of the heat bath, and α_ν and g_ν are the coupling parameters. The values of physical quantities in Eqs. (1) and (2) are given in SI units. The bosonic heat bath is modeled by an ensemble of noninteracting harmonic oscillators with frequencies ω_ν. The coupling between the heat bath and the collective subsystem is linear in coordinates. The coupling terms, external magnetic field, and time-dependent electric field do not affect each other. The last term in Hamiltonian (1) compensates for the normalization of the potential due to the coupling between the heat bath and the collective subsystem. This term is necessary to restore the translational invariance of an open system. We should stress that Eq. (1) is set up in the classical Hall effect geometry.

Using Hamiltonian (1), we obtain the system of Langevin equations of motion for the collective variables:

$$
\begin{aligned}
\dot{x}(t) &= \frac{\pi_x(t)}{m},\\
\dot{y}(t) &= \frac{\pi_y(t)}{m},\\
\dot{\pi}_x(t) &= \pi_y(t)\omega_c - m\omega_x^2x(t) - eE_x(t)\\
&\quad -\frac{1}{m}\int_0^t d\tau K_\alpha(t-\tau)\pi_x(\tau)\\
&\quad -\frac{1}{m}\int_0^t d\tau K_{\alpha g}(t-\tau)\pi_y(\tau) - F_\alpha(t),\\
\dot{\pi}_y(t) &= -\pi_x(t)\omega_c - m\omega_y^2y(t) - eE_y(t)\\
&\quad -\frac{1}{m}\int_0^t d\tau K_{\alpha g}(t-\tau)\pi_x(\tau)\\
&\quad -\frac{1}{m}\int_0^t d\tau K_g(t-\tau)\pi_y(\tau) - F_g(t),
\end{aligned}
\tag{3}
$$

where $K_{\alpha,g,\alpha g}$ and $F_{\alpha,g}$ are the dissipative kernels and random forces, respectively (see Appendix A). The presence of the integral parts in Eqs. (3) indicates non-Markovian dynamics of the system. The dissipative kernels have the form of memory functions since they make the equations of motion at time t dependent on the values of $\dot{x}$ and $\dot{y}$ for previous times. The Langevin equation for a charged oscillator in the magnetic field and dissipative bosonic heat bath was tackled earlier in Ref. [31] using the Heisenberg picture. The inclusion of nonstationary electric fields $E_{x,y}(t)$ is new in Eqs. (3).

The system of Eqs. (3) is solved by applying the Laplace transform. After the tedious algebra we obtain the solution

$$\begin{aligned}
x(t) &= A_1(t)x(0) + A_2(t)y(0) + A_3(t)\pi_x(0) + A_4(t)\pi_y(0) - I_x(t) - I'_x(t) - I_{ex}(t) - I'_{ex}(t),\\
y(t) &= B_1(t)x(0) + B_2(t)y(0) + B_3(t)\pi_x(0) + B_4(t)\pi_y(0) - I_y(t) - I'_y(t) - I_{ey}(t) - I'_{ey}(t),\\
\pi_x(t) &= C_1(t)x(0) + C_2(t)y(0) + C_3(t)\pi_x(0) + C_4(t)\pi_y(0) - I_{\pi_x}(t) - I'_{\pi_x}(t) - I_{e\pi_x}(t) - I'_{e\pi_x}(t),\\
\pi_y(t) &= D_1(t)x(0) + D_2(t)y(0) + D_3(t)\pi_x(0) + D_4(t)\pi_y(0) - I_{\pi_y}(t) - I'_{\pi_y}(t) - I_{e\pi_y}(t) - I'_{e\pi_y}(t)
\end{aligned} \tag{4}$$

of this system of equations of motion. The time-dependent coefficients A_i, B_i, C_i, and D_i ($i = 1, 2, 3, 4$) are given in Appendix A.

A. Non-Markovian dynamics

For simplicity, we choose the time-dependent electric fields $E_x(t) = E_{x0}\cos(\omega_{ex}t)$ and $E_y(t) = E_{y0}\cos(\omega_{ey}t)$ (the linear-polarized monochromatic electromagnetic wave at $\omega_{ex} = \omega_{ey}$) in Hamiltonian (1). Employing Eqs. (4), and the correlations of random forces at different times (see Appendix A), we derive the z component of angular momentum [or magnetic moment per unit volume $M(t) = neL_z(t)/(2m)$, where n is the concentration of charged particles]

$$L_z(t) = L_z(t) = \langle x(t)\pi_y(t) - y(t)\pi_x(t)\rangle = L_z^0(t) + L_{z1}(t) + L_{z2}(t), \tag{5}$$

where the symbol $\langle\cdot\rangle$ denotes averaging over the whole system of the heat bath and oscillator, the value of $L_z^0(t)$ depends on the initial second moments (see Appendix B) and does not contribute to the asymptotic value of L_z ($L_z^0(\infty) = 0$),

$$\begin{aligned}
L_{z1}(t) = \frac{m\hbar\gamma^2}{\pi}\int_0^\infty d\omega \int_0^t d\tau \int_0^t d\tilde{\tau}\, \frac{\omega\coth\left[\frac{\hbar\omega}{2T_0}\right]}{\omega^2+\gamma^2}\cos(\omega[\tau-\tilde{\tau}])\\
\times \{\lambda_x[A_3(\tau)D_3(\tilde{\tau}) - B_3(\tau)C_3(\tilde{\tau})] + \lambda_y[A_4(\tau)D_4(\tilde{\tau}) - B_4(\tau)C_4(\tilde{\tau})]\}
\end{aligned} \tag{6}$$

and

$$L_{z2}(t) = [I_{ex}(t) + I'_{ex}(t)][I_{e\pi_y}(t) + I'_{e\pi_y}(t)] - [I_{ey}(t) + I'_{ey}(t)][I_{e\pi_x}(t) + I'_{e\pi_x}(t)]. \tag{7}$$

Here the temperature T_0 of the bosonic heat bath is given in the energy units,

$$\begin{aligned}
I_{ex}(t) &= \frac{eE_{x0}}{m}\sum_{i=1}^{6}\frac{\beta_i(s_i+\gamma)}{s_i^2+\omega_{ex}^2}\left[(s_i+\gamma)\left(s_i^2+\omega_y^2\right)+\lambda_y\gamma s_i\right]a_i^x(t) = I'_{ey}(t)|_{x\leftrightarrow y},\\
I'_{ex}(t) &= \frac{eE_{y0}\omega_c}{m}\sum_{i=1}^{6}\frac{\beta_i s_i(s_i+\gamma)^2}{s_i^2+\omega_{ey}^2}a_i^y(t) = -I_{ey}(t)|_{x\leftrightarrow y},\\
I_{e\pi_x}(t) &= eE_{x0}\sum_{i=1}^{6}\frac{\beta_i s_i(s_i+\gamma)}{s_i^2+\omega_{ex}^2}\left[(s_i+\gamma)\left(s_i^2+\omega_y^2\right)+\lambda_y\gamma s_i\right]a_i^x(t) = I'_{e\pi_y}(t)|_{x\leftrightarrow y},\\
I'_{e\pi_x}(t) &= eE_{y0}\omega_c\sum_{i=1}^{6}\frac{\beta_i s_i^2(s_i+\gamma)^2}{s_i^2+\omega_{ey}^2}a_i^y(t) = -I_{e\pi_y}(t)|_{x\leftrightarrow y},\\
a_i^{x,y}(t) &= s_i[e^{s_it} - \cos(\omega_{ex,ey}t)] + \omega_{ex,ey}\sin(\omega_{ex,ey}t),
\end{aligned} \tag{8}$$

s_i are the roots of the equation

$$\begin{aligned}
D(s) = (s_i+\gamma)\left\{\left[s_i^4+\omega_x^2\omega_y^2+s_i^2\left(\omega_c^2+\omega_x^2+\omega_y^2\right)\right](s_i+\gamma)+s_i\gamma\lambda_x\left(s_i^2+\omega_y^2\right)\right\}\\
+ s_i\gamma\lambda_y\left[(s_i^2+\omega_x^2)(s_i+\gamma)+s_i\gamma\lambda_x\right] = 0,
\end{aligned} \tag{9}$$

and $\beta_i = [\prod_{j\neq i}(s_i - s_j)]^{-1}$ with $i, j = 1, 2, \ldots, 6$ (see Appendix A).

The numerical calculations show that if $\omega_{ex} \neq \omega_{ey}$, then the angular momentum has no asymptotic value and oscillates at large t. Therefore, we chose the frequencies of the electric fields to be the same, i.e., $\omega_{ex} = \omega_{ey} = \omega_e$, and we find the asymptotic z component of angular momentum

$$L_z(\infty) = L_{z1}(\infty) + L_{z2}(\infty), \tag{10}$$

where

$$L_{z1}(\infty) = -\frac{2\hbar\omega_c\gamma^2}{\pi}\int_0^\infty d\omega\omega^3\coth\left[\frac{\hbar\omega}{2T_0}\right]\frac{(\omega^2+\gamma^2)\left[(\omega^2-\omega_y^2)\lambda_x+\left(\omega^2-\omega_x^2\right)\lambda_y\right]-2\omega^2\gamma\lambda_x\lambda_y}{\left(s_1^2+\omega^2\right)(s_2^2+\omega^2)\left(s_3^2+\omega^2\right)\left(s_4^2+\omega^2\right)\left(s_5^2+\omega^2\right)\left(s_6^2+\omega^2\right)} \tag{11}$$

and

$$L_{z2}(\infty) = \big[e^2\omega_e^2(\gamma^2+\omega_e^2)\big(\gamma^2\big[\omega_c\big(\omega_x^2E_{y0}^2+\omega_y^2E_{x0}^2\big)+E_{x0}E_{y0}\big(\omega_x^2\lambda_y-\omega_y^2\lambda_x\big)\big] + \big\{E_{x0}^2\omega_c\big[\omega_y^2-\gamma(\gamma-\lambda_y)\big]+E_{y0}^2\omega_c\big[\omega_x^2-\gamma(\gamma-\lambda_x)\big]+E_{x0}E_{y0}\gamma^2(\lambda_x-\lambda_y)\big\}\omega_e^2-\big(E_{x0}^2+E_{y0}^2\big)\omega_c\omega_e^4\big)\big]/N \quad (12)$$

with

$$N = m\big(s_1^2+\omega_e^2\big)\big(s_2^2+\omega_e^2\big)\big(s_3^2+\omega_e^2\big)\big(s_4^2+\omega_e^2\big)\big(s_5^2+\omega_e^2\big)\big(s_6^2+\omega_e^2\big).$$

Employing the residue theorem, closing the contour in the upper half-plane, and using the cotangent function poles at $\hbar\omega/(2T_0) = i\pi n$ with an integer n, we also calculate analytically the integral over ω in Eq. (11) (see Appendix B). As seen from Eqs. (11) and (12), in the presence of two perpendicular electric fields, angular momentum or orbital magnetization arises in the system $[L_{z2}(\infty)\neq 0]$, even if no external magnetic field is applied ($\omega_c = 0$ and $L_{z1}(\infty) = 0$), in the case of $\omega_x \neq \omega_y$ and/or $\lambda_x \neq \lambda_y$. For a free charged particle, we also obtain that $L_{z2}(\infty)\neq 0$ at $\lambda_x\neq\lambda_y$ (Appendix C).

B. Markovian dynamics

For Markovian dynamics ($\gamma\to\infty$), the asymptotic z component of angular momentum $L_z(\infty) = L_{z1}(\infty)+L_{z2}(\infty)$ contains the following terms:

$$L_{z1}(\infty) = -\frac{2\hbar\omega_c}{\pi}\int_0^\infty d\omega\omega^3\coth\left[\frac{\hbar\omega}{2T_0}\right]\frac{\lambda_x\big(\omega^2-\omega_y^2\big)+\lambda_y\big(\omega^2-\omega_x^2\big)}{\big(s_1^2+\omega^2\big)\big(s_2^2+\omega^2\big)\big(s_3^2+\omega^2\big)\big(s_4^2+\omega^2\big)} \quad (13)$$

and

$$L_{z2}(\infty) = \frac{e^2\omega_e^2\Psi(\omega_{x,y},\lambda_{x,y})}{m\big(s_1^2+\omega_e^2\big)\big(s_2^2+\omega_e^2\big)\big(s_3^2+\omega_e^2\big)\big(s_4^2+\omega_e^2\big)}. \quad (14)$$

Here

$$\Psi(\omega_{x,y},\lambda_{x,y}) = \omega_c\big(E_{x0}^2\omega_y^2+E_{y0}^2\omega_x^2\big)-\big[\big(E_{x0}^2+E_{y0}^2\big)\omega_c-E_{x0}E_{y0}(\lambda_x-\lambda_y)\big]\omega_e^2+E_{x0}E_{y0}\big(\omega_x^2\lambda_y-\omega_y^2\lambda_x\big)$$

and s_i ($i = 1, 2, 3, 4$) are the roots of the following equation:

$$D(s) = \big(\omega_x^2+s^2+s\lambda_x\big)\big(\omega_y^2+s^2+s\lambda_y\big)+s^2\omega_c^2 = 0. \quad (15)$$

Calculating analytically the integral over ω in Eq. (13) and employing the Vieta theorem in Eq. (14), we derive

$$L_{z1}(\infty) = -\hbar\omega_c[2\mathrm{Re}(J)-J_s] \quad (16)$$

and

$$L_{z2}(\infty) = \frac{e^2\omega_e^2\Psi(\omega_{x,y},\lambda_{x,y})}{m\Phi_{\omega_e}(\omega_{x,y},\lambda_{x,y})}, \quad (17)$$

where

$$J = \frac{s_1^2\big[\lambda_x\big(s_1^2+\omega_y^2\big)+\lambda_y\big(s_1^2+\omega_x^2\big)\big]}{\big(s_1^2-s_1^{*2}\big)\big(s_1^2-s_2^2\big)\big(s_1^2-s_2^{*2}\big)}\cot\left[\frac{\hbar s_1}{2T_0}\right]+\frac{s_2^2\big[\lambda_x\big(s_2^2+\omega_y^2\big)+\lambda_y\big(s_2^2+\omega_x^2\big)\big]}{\big(s_2^2-s_1^2\big)\big(s_2^2-s_1^{*2}\big)\big(s_2^2-s_2^{*2}\big)}\cot\left[\frac{\hbar s_2}{2T_0}\right],$$

$$J_s = 32\pi^3\frac{T_0^4}{\hbar^4}\sum_{n=1}^{\infty}\frac{\big[\lambda_x\big(x_n^2+\omega_y^2\big)+\lambda_y\big(x_n^2+\omega_x^2\big)\big]n^3}{\big|x_n^2-s_1^2\big|^2\big|x_n^2-s_2^2\big|^2},$$

and

$$\Phi_{\omega_e}(\omega_{x,y},\lambda_{x,y}) = \omega_x^4\omega_y^4-\big[2\omega_c^2\omega_x^2\omega_y^2+\omega_x^4\big(2\omega_y^2-\lambda_y^2\big)+\omega_y^4\big(2\omega_x^2-\lambda_x^2\big)\big]\omega_e^2+\big[\omega_c^4+\omega_x^4+\omega_y^4-\lambda_x^2\big(2\omega_y^2-\lambda_y^2\big) + 2\omega_c^2\big(\omega_x^2+\omega_y^2+\lambda_x\lambda_y\big)+2\omega_x^2\big(2\omega_y^2-\lambda_y^2\big)\big]\omega_e^4-\big[2\big(\omega_c^2+\omega_x^2+\omega_y^2\big)-\lambda_x^2-\lambda_y^2\big]\omega_e^6+\omega_e^8.$$

Here $\mathrm{Re}(s_1) < 0$ and $\mathrm{Re}(s_2) < 0$. As seen, in the case of $\omega_x \neq \omega_y$ and/or $\lambda_x\neq\lambda_y$ and the absence of an external magnetic field but the presence of two perpendicular electric fields, orbital magnetization arises in the system ($L_{z2}(\infty)\neq 0$). The same behavior is observed for a free charged particle (Appendix C). The expression for $L_{z1}(\infty)$ was also derived in Ref. [22]. In the case of an isotropic oscillator and an isotropic heat bath ($\omega_x = \omega_y = \omega_0$ and $\lambda_x = \lambda_y = \lambda$), the expression for angular momentum

$$L_{z2}(\infty) = \frac{e^2\omega_e^2\omega_c\big(E_{x0}^2+E_{y0}^2\big)\big(\omega_0^2-\omega_e^2\big)}{m\Phi_{\omega_e}(\omega_0,\lambda)} \quad (18)$$

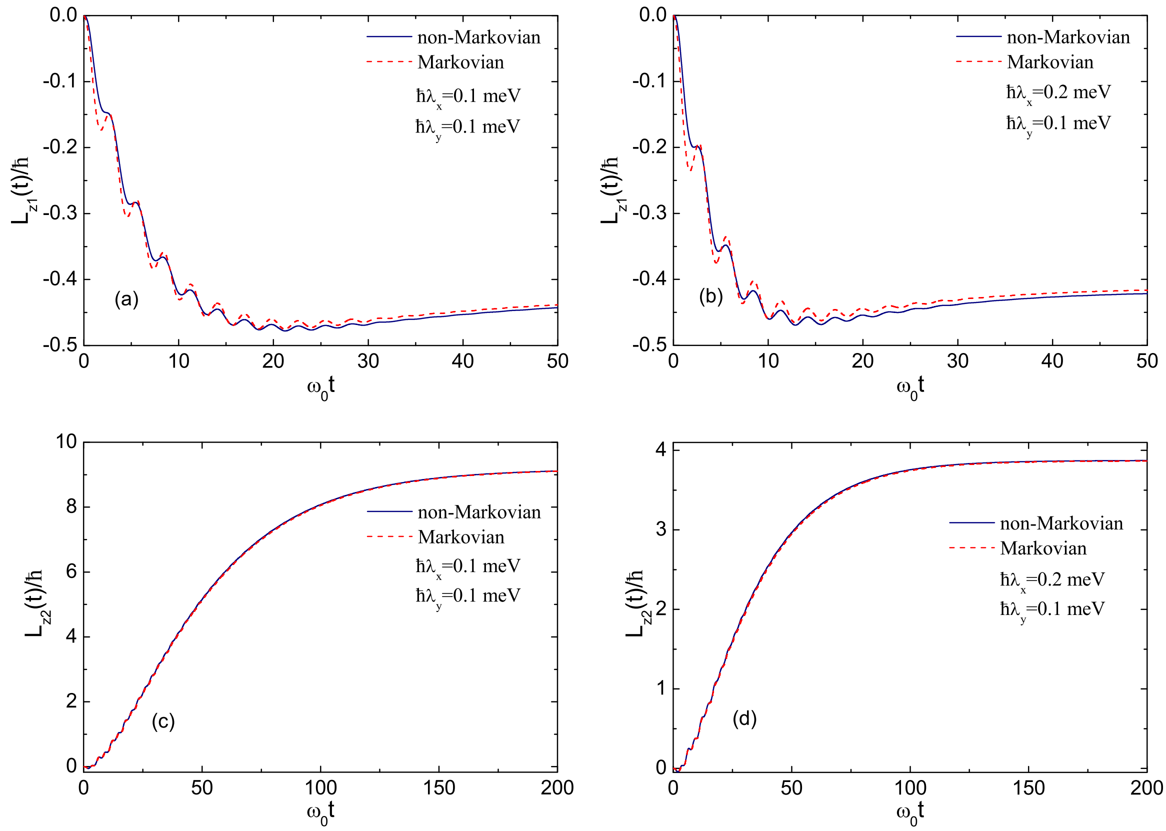

FIG. 1. In the Markovian (red dashed lines) and non-Markovian (black solid lines) cases, the calculated z components $L_{z1}(t)$ and $L_{z2}(t)$ of angular momentum as functions of time t at the indicated friction coefficients. Here $T_0 = 0.1\,\text{meV}$, $\hbar\omega_c = 1\,\text{meV}$, $\hbar\omega_0 = 1\,\text{meV}$, $\hbar\omega_e = \hbar\omega_e^{\max} = 0.62\,\text{meV}$, and $\hbar\gamma = 12\,\text{meV}$.

has a fairly simple shape. As seen, $L_{z2}(\infty) = 0$ if the electric field frequency is $\omega_e = 0$ or $\omega_e = \omega_0$.

III. CALCULATED RESULTS

The GaAs/AlGaAs heterojunction was chosen as a 2DEG [8]. For this material, the effective mass of electrons is equal to $m = 0.067m_0$ (m_0 is the mass of an electron at rest) [15,24]. Note that if the amplitude of the electric fields is about $E_{x0} = E_{y0} \approx 1$ V/cm, then $L_{z1}(\infty) \gg L_{z2}(\infty)$ in the mean and high magnetic field ($B \geqslant 1$ T). If sufficiently large electric fields act on the 2DEG (for example, $E_{x0} = E_{y0} \approx 10^3$ V/cm), the absolute value of $L_{z2}(\infty)$ becomes comparable or even larger than $|L_{z1}(\infty)|$. Therefore, for numerical calculation, we set $E_{x0} = E_{y0} = 10^3$ V/cm. The frequency of electric fields ($\hbar\omega_e$) is selected according to the resonance conditions for observing resonant states. Note that the applied electric fields do not destroy the electric properties of the 2DEG. The frequencies $\omega_{x,y}$ of the 2D harmonic oscillator are chosen of the order of (1–5) meV, which is typical for 2D materials and 2DEG [8,42]. Since $\gamma \gg \omega_{x,y}$, the value of $\hbar\gamma$ was set to 12 meV. The friction coefficients $\lambda_{x,y}$ are significantly lower than the frequencies $\omega_{x,y}$, which holds at low temperatures $T_0 \leqslant 0.1$ meV (0.1 meV corresponds to 1.16 K).

For Markovian and non-Markovian dynamics of the 2D harmonic oscillator, the time dependences of the z components $L_{z1}(t)$ and $L_{z2}(t)$ of angular momentum are shown in Fig. 1. As seen, the absolute values of $L_{z1}(t)$ and $L_{z2}(t)$ increase with small oscillations with increasing time t and reach their asymptotic values. The amplitude of these oscillations decreases with increasing damping. For $L_{z1}(t)$, non-Markovian effects are more noticeable only in the initial short time interval ($\omega_0 t \leqslant \hbar\omega_0/T_0 = 10$), which can be probed by ultrafast spectroscopy. For $L_{z2}(t)$, non-Markovian and Markovian dynamics are almost indistinguishable throughout the entire process.

Figure 2 shows the dependence of the asymptotic z component of angular momentum $L_{z1}(\infty)$ on the magnetic field for the Markovian and non-Markovian cases. As seen, the value of $L_{z1}(\infty)$ is almost the same for Markovian and non-Markovian dynamics at temperature $T_0 = 0.1$ meV. So the asymptotic z component of angular momentum $L_{z1}(\infty)$ [or the magnetization $M = neL_{z1}/(2m)$] is weakly affected by non-Markovian effects. In Ref. [40] we came to the same conclusion at $\omega_x = \omega_y = \omega_0$ and $\lambda_x = \lambda_y = \lambda$. The results of calculations for different collective frequencies and friction coefficients are also presented in Fig. 2. The largest $|L_{z1}(\infty)|$ corresponds to the axial symmetric case ($\omega_x/\omega_y = 1$ and $\lambda_x/\lambda_y = 1$). If the ratio of the friction coefficients $\lambda_x/\lambda_y = 5$, the value of $L_{z1}(\infty)$ increases with the ratio ω_x/ω_y of collective frequencies [Fig. 2(a) and 2(b)]. In a high magnetic field ($\hbar\omega_c \gg T_0$), we obtain

$$L_{z1}(\infty) = -\hbar, \quad M(\infty) = -\frac{ne\hbar}{2m}, \tag{19}$$

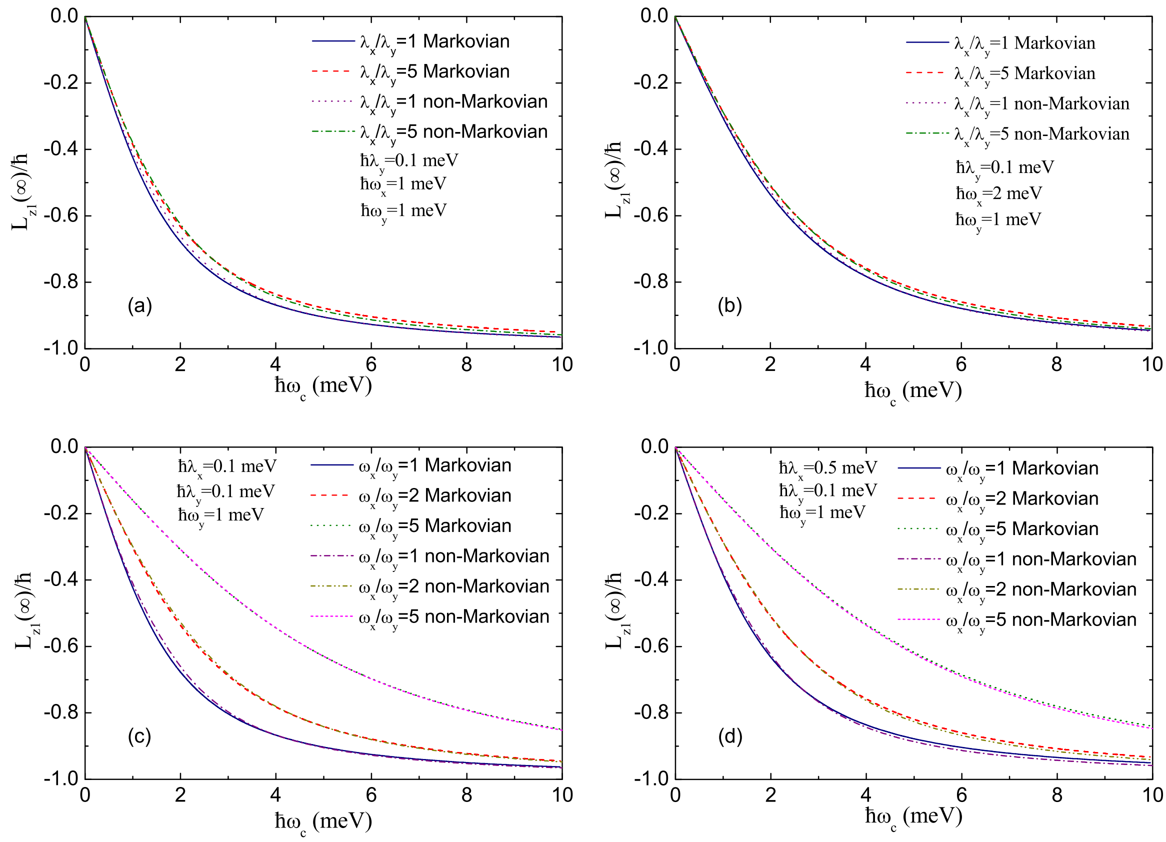

FIG. 2. The calculated asymptotic z component $L_{z1}(\infty)/\hbar$ of angular momentum as a function of $\hbar\omega_c$. Here $T_0 = 0.1$ meV and $\hbar\gamma = 12$ meV for non-Markovian dynamics. In the Markovian and non-Markovian cases, Eq. (16) and Eq. (B2) are used, respectively.

which means quantization of angular momentum or magnetization.

Figure 3 shows the dependence of the asymptotic z component $L_{z2}(\infty)$ of angular momentum as a function of the electric field frequency $\hbar\omega_e$ at $\hbar\omega_x = \hbar\omega_y = \hbar\omega_0 = 1$ meV and $\hbar\lambda_y = 0.1$ meV. In Fig. 3(a) there are a maximum and minimum of $L_{z2}(\infty)$. At $\omega_{c,0} \gg \lambda$, we find the positions of these extremes:

$$\omega_e^{\max} \approx \sqrt{\omega_0^2 + \frac{\omega_c^2}{4}} - \frac{\omega_c}{2}, \quad \omega_e^{\min} \approx \sqrt{\omega_0^2 + \frac{\omega_c^2}{4}} + \frac{\omega_c}{2}. \tag{20}$$

So the dependence of $L_{z2}(\infty)$ and thus $L_z(\infty)$ on $\hbar\omega_e$ has a resonance behavior. If the friction coefficients λ_x and λ_y are close to each other (for example, $\lambda_x = 1.1\lambda_y$), then the maximum value of $L_{z2}(\infty)$ is relatively large and the half-width of the peak is rather small. With increasing ratio of the friction coefficients (for example, $\lambda_x = 2\lambda_y$), the absolute value of $L_{z2}(\infty)$ decreases at extremes and the half-width of the peaks increases. Numerical calculations show that if λ_x/λ_y increases n_0 times, then $|L_{z2}(\infty)|$ decreases almost n_0 times. One can see that $L_{z2}(\infty) = 0$ at $\omega_e = \omega_0$, $L_{z2}(\infty) > 0$ at $\omega_e < \omega_0$, and $L_{z2}(\infty) < 0$ at $\omega_e > \omega_0$. This means that we can control angular momentum (magnetization) of charged particles through the frequency of the electric field.

In Fig. 3(b) with the same parameters as in Fig. 3(a) but without the magnetic field ($\hbar\omega_c = 0$), $L_{z2}(\infty) < 0$ at $\omega_e < \omega_0$, $L_{z2}(\infty) = 0$ at $\omega_e = \omega_0$, and $L_{z2}(\infty) > 0$ at $\omega_e > \omega_0$. The maximum value of $|L_{z2}(\infty)|$ is much smaller than that at $\hbar\omega_c = 1$ meV in Fig. 3(a). If $\lambda_x = \lambda_y$, $L_{z2}(\infty) = 0$ at $\omega_c = 0$ as in Refs. [23,24]. So the maximum value of $|L_{z2}(\infty)|$ increases with $\lambda_x/\lambda_y > 1$.

In Fig. 3(c) the component of angular momentum

$$L_{z2}^a(\infty) = L_{z2}(\omega_c = 0) = \frac{e^2\omega_e^2 E_{x0}E_{y0}\left[\omega_x^2\lambda_y - \omega_y^2\lambda_x + (\lambda_x - \lambda_y)\omega_e^2\right]}{m\left[\omega_x^4 - \left(2\omega_x^2 - \lambda_x^2\right)\omega_e^2 + \omega_e^4\right]\left[\omega_y^4 - \left(2\omega_y^2 - \lambda_y^2\right)\omega_e^2 + \omega_e^4\right]} \tag{21}$$

is presented. Note that Eq. (21) at $\lambda_x = \lambda_y$ transforms into the expression obtained in Refs. [23,24]. So we obtain a more general formula for angular momentum or magnetic moment in the case of the anisotropic 2D harmonic oscillator and anisotropic heat bath.

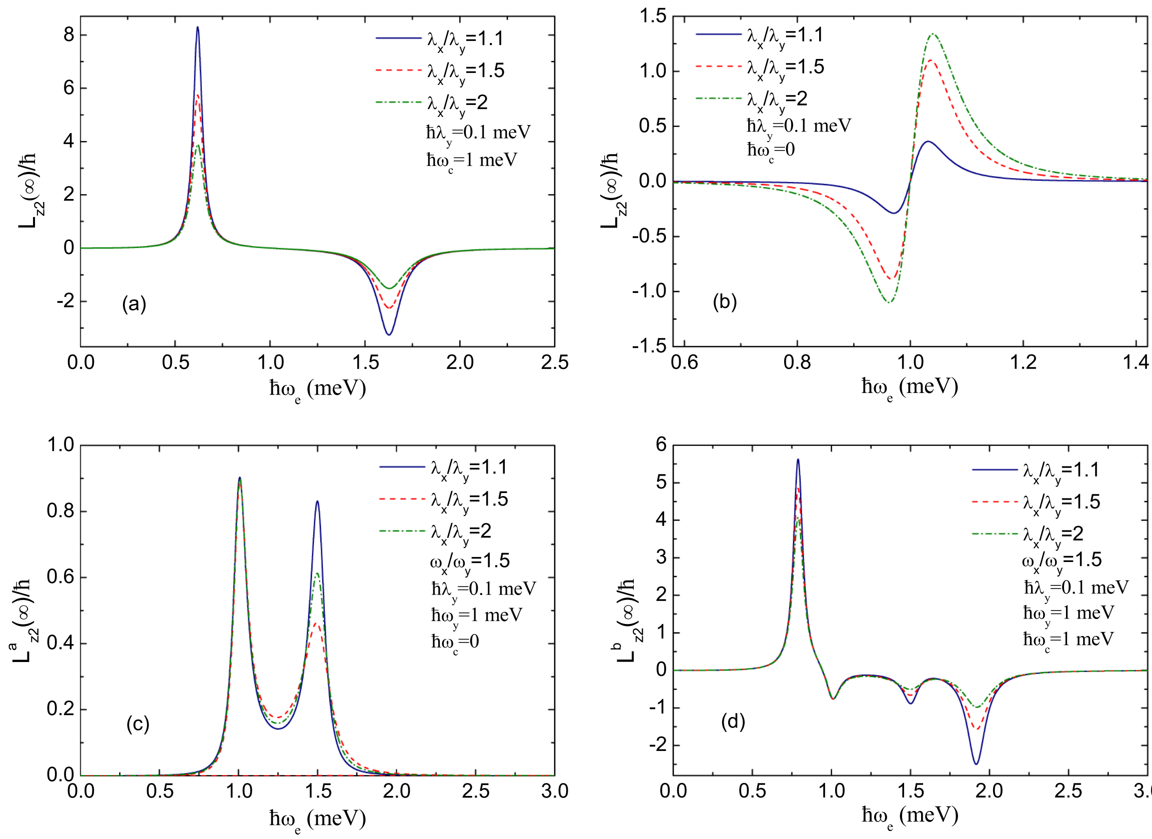

FIG. 3. The calculated asymptotic z component $L_{z2}(\infty)/\hbar$ of angular momentum as a function of $\hbar\omega_e$ for non-Markovian dynamics at (a) $\hbar\omega_c = 1$ meV and (b) $\hbar\omega_c = 0$ for $\hbar\omega_0 = 1$ meV. The calculated $L^a_{z2}(\infty)/\hbar$ at (c) $\hbar\omega_c = 0$ and $L^b_{z2}(\infty)/\hbar$ at (d) $\hbar\omega_c = 1$ meV for $\hbar\omega_x = 1.5\hbar\omega_y = 1.5$ meV. Here $\hbar\lambda_y = 0.1$ meV, $\hbar\gamma = 12$ meV, $E_{x0} = E_{y0} = 10^5$ V/m, and $\mu = 0.067m_0$. The solid, dashed, and dash-dotted lines correspond to the cases with $\lambda_x/\lambda_y = 1.1$, 1.5, and 2, respectively.

Figure 3(d) shows $L^b_{z2}(\infty) = L_{z2}(\infty) - L^a_{z2}(\infty)$. If $\omega_x = \omega_y = \omega_0$,

$$L^a_{z2}(\infty) = \frac{e^2\omega_e^2 E_{x0}E_{y0}(\lambda_x - \lambda_y)(\omega_e^2 - \omega_0^2)}{m[\omega_0^4 - (2\omega_0^2 - \lambda_x^2)\omega_e^2 + \omega_e^4][\omega_0^4 - (2\omega_0^2 - \lambda_y^2)\omega_e^2 + \omega_e^4]}. \tag{22}$$

There are two maxima in $L^a_{z2}(\infty)$ as a function of $\hbar\omega_e$ and the ratio of these maxima crucially depends on λ_x/λ_y. So one can estimate λ_x/λ_y from the ratio of these two maxima. There are two major and two minor extremes in $L^b_{z2}(\infty)$. Two minor minima are opposite to the maxima of $L^a_{z2}(\infty)$ and, thus, are not expressed in Fig. 3(a). Therefore, there are two extremes of $L_{z2}(\infty)$ in the general case [see Fig. 3(a)]. Two major extremes of $L^b_{z2}(\infty)$ cause these extremes of $L_{z2}(\infty)$ [Fig. 3(a)]. As seen from Eq. (22), at $\lambda_x = \lambda_y$ (or $\omega_e = \omega_0$) $L^a_{z2}(\infty) = 0$. For a free particle ($\omega_0 \to 0$), we obtain

$$L^a_{z2}(\infty) = \frac{e^2 E_{x0}E_{y0}(\lambda_x - \lambda_y)}{m(\lambda_x^2 + \omega_e^2)(\lambda_y^2 + \omega_e^2)}. \tag{23}$$

As seen from Eq. (23), the angular momentum is positive along the z axis at $\lambda_x > \lambda_y$. If $\lambda_x < \lambda_y$, the projection of angular momentum is negative. If $\omega_e = 0$ (constant electric field), we have nonzero angular momentum

$$L^a_{z2}(\infty) = \frac{e^2 E_{x0}E_{y0}(\lambda_x - \lambda_y)}{m\lambda_x^2\lambda_y^2}. \tag{24}$$

If the collective frequencies are close to each other, the value of $L_{z2}(\infty)$ deviates more strongly from zero at extremes. For example, the absolute value of $L_{z2}(\infty)$ at extremes in Fig. 4 is larger at $\omega_x/\omega_y = 1.1$ than at $\omega_x/\omega_y = 2$.

In Fig. 5 the dependence of the asymptotic z component of angular momentum is shown as a function of $\hbar\omega_e$ for a free particle. At $\hbar\omega_c = 0$, the maximum value of $L_{z2}(\infty)$ is near $\omega_e^{\max} = 0$. With increasing anisotropy in the system (the ratio λ_x/λ_y increases), the value of $L_{z2}(\infty)$ becomes larger. If $\hbar\omega_c = 1$ meV, the maximum value of $|L_{z2}(\infty)|$ is at $\omega_e^{\max} = \omega_c$, and as the ratio λ_x/λ_y increases, the value of $|L_{z2}(\infty)|$ decreases. It should be noted that the value of $L_{z2}(\infty)$ changes sign in the presence of a magnetic field, the particle moves in the opposite direction. At the same time, $|L_{z2}(\infty)|$ is larger at extremum at $\hbar\omega_c = 0$ for a given λ_x/λ_y. In order to obtain a large angular momentum (or a large magnetic moment) in the system at $\omega_c = 0$, the frequency of the electric field should be small or it can be affected by a constant electric field. In this case, the angular momentum changes as $L_{z2}(\infty) \sim 1/\omega_e^4$ [see Eq. (23)].

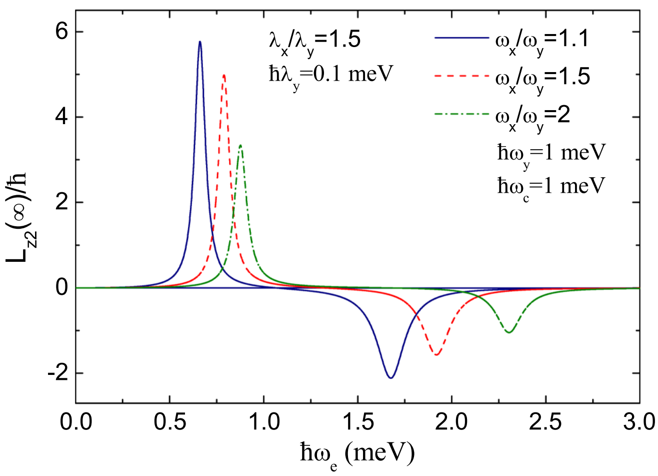

FIG. 4. The calculated asymptotic z component $L_{z2}(\infty)/\hbar$ of angular momentum as a function of $\hbar\omega_e$ at different ratios of collective frequencies. Here $\hbar\lambda_x = 1.5\hbar\lambda_y = 0.15$ meV, $\hbar\omega_c = 1$ meV, $\hbar\omega_y = 1$ meV, and $\hbar\gamma = 12$ meV.

The dependence of $L_{z2}(\infty)$ on the magnetic field is shown in Fig. 6 for various friction coefficients and electric field frequencies. The $L_{z2}(\infty)$ is almost the same in the Markovian and non-Markovian cases. Therefore, the value of $L_{z2}(\infty)$ is weakly affected by non-Markovian effects. Here we set $\omega_x = \omega_y = \omega_0$ and $\lambda_x = \lambda_y = \lambda$ and numerical calculations show that $L_{z2}(\infty)$ reaches an extremum at certain values of the magnetic field:

$$\omega_c^{\max} = \sqrt{\frac{\omega_0^4 - (2\omega_0^2 + \lambda^2)\omega_e^2 + \omega_e^4 + 2\sqrt{M}}{3\omega_e^2}}, \qquad (25)$$

where

$$M = \omega_0^8 - \omega_0^4\omega_e^2(4\omega_0^2 - \lambda^2) + (6\omega_0^4 - 2\omega_0^2\lambda^2 + \lambda^4)\omega_e^4 - (4\omega_0^2 - \lambda^2)\omega_e^6 + \omega_e^8. \qquad (26)$$

For a free particle, we obtain

$$\omega_c^{\max} = \sqrt{\frac{\omega_e^2 - \lambda^2 + 2\sqrt{\omega_e^4 + \omega_e^2\lambda^2 + \lambda^4}}{3}}. \qquad (27)$$

If $\lambda \to 0$, Eq. (25) results in

$$\omega_c^{\max} \approx \frac{|\omega_0^2 - \omega_e^2|}{\omega_e}. \qquad (28)$$

For a free particle and $\lambda = 0$, we obtain $\omega_c^{\max} \approx \omega_e$. As the friction coefficient decreases, the half-width of $L_{z2}(\infty)$ decreases and the absolute value increases. For example, at $\hbar\omega_e = 0.1$ meV, $L_{z2}(\infty) \approx 57\hbar$ and $57 \times 10^4\hbar$ for $\hbar\lambda = 0.1$ and 0.001 meV, respectively (Fig. 6). Thus, at extremes $L_{z2}(\infty) \sim 1/\lambda^2$. As the frequency of the electric field increases, the absolute value of $L_{z2}(\infty)$ decreases.

The dependence of $L_{z2}(\infty)$ on the magnetic field is shown in Fig. 7 for a free particle. As seen, the maximum value of $|L_{z2}(\infty)|$ corresponds to $\omega_c^{\max} = \omega_e$ for small friction coefficients.

Figure 8 presents the dependence of $L_{z2}(\infty)$ on the oscillator frequency at different friction coefficients, cyclotron, and electric field frequencies. Calculations show that there is one extremum of $L_{z2}(\infty)$ at large $\hbar\omega_c$ and small $\hbar\omega_e$. If the electric field frequency is larger, for example, $\hbar\omega_e = 2$ meV, then $L_{z2}(\infty)$ has two extremes. However, their amplitudes decrease as $\hbar\omega_e$ increases. At given electric field frequency, the maximum value of $|L_{z2}(\infty)|$ decreases with magnetic field. If $\omega_x = \omega_y = \omega_0$ and $E_{x0} = E_{y0}$, $L_{z2}(\infty) = 0$ at $\omega_c = \frac{|\lambda_x - \lambda_y|}{2}$ [see Eq. (17)]. If $\omega_c > \frac{|\lambda_x - \lambda_y|}{2}$, then $L_{z2}(\infty) < 0$ at $\omega_0 < \omega_e$ and $L_{z2}(\infty) > 0$ at $\omega_0 > \omega_e$. If $\omega_c < \frac{|\lambda_x - \lambda_y|}{2}$, then $L_{z2}(\infty) > 0$ at $\omega_0 < \omega_e$ and $L_{z2}(\infty) < 0$ at $\omega_0 > \omega_e$. So we can control magnetization through the frequency of the electric field and the magnetic field acting on the charged particle embedded in the heat bath.

IV. SUMMARY

The effects of an external uniform magnetic and two perpendicular time-dependent electric fields on magnetization of the open quantum system have been studied in the non-Markovian limit. The explicit expression for the z component of angular momentum $L_z(t) = L_{z1}(t) + L_{z2}(t)$ was obtained for the 2D charged quantum harmonic oscillator, where $L_{z1}(t)$ does not depend on electric fields, and $L_{z2}(t)$ is the component

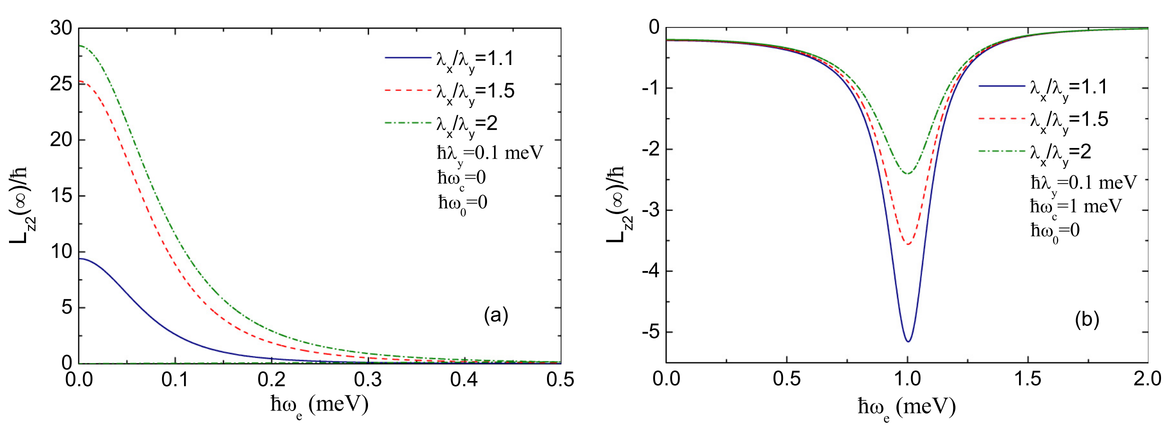

FIG. 5. The calculated asymptotic z component $L_{z2}(\infty)/\hbar$ of angular momentum as a function of $\hbar\omega_e$ for a free particle ($\hbar\omega_0 = 0$) at (a) $\hbar\omega_c = 0$ and (b) $\hbar\omega_c = 1$ meV in the non-Markovian case.

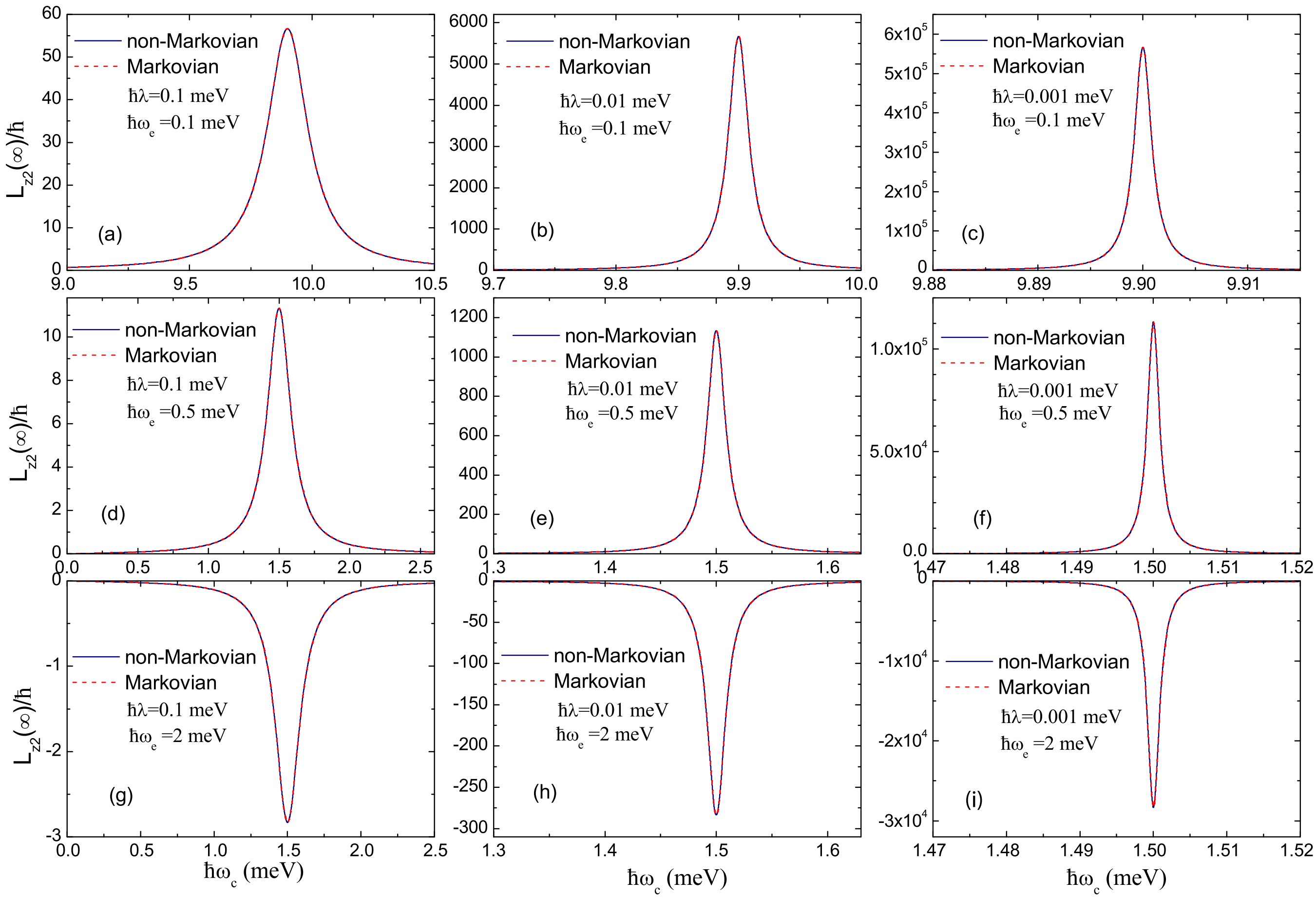

FIG. 6. The calculated asymptotic z component $L_{z2}(\infty)/\hbar$ of angular momentum as a function of $\hbar\omega_c$ for the Markovian and non-Markovian dynamics. Here $\hbar\omega_0 = 1$ meV and $\hbar\gamma = 12$ meV.

depending on electric fields. Based on the numerical calculations, we showed that at different frequencies of electric fields ($\omega_{ex} \neq \omega_{ey}$), the $L_{z2}(t)$ does not have an asymptotic value $L_{z2}(\infty)$ and oscillates at large t. For $L_{z2}(t)$ at $\omega_e = \omega_{ex} = \omega_{ey}$, non-Markovian and Markovian dynamics are almost indistinguishable throughout the entire process. For $L_{z1}(t)$, non-Markovian effects are more noticeable only in the initial short time interval. These facts indicate that non-Markovian effects have a weak influence on the diamagnetic properties of the system.

In the case of $\omega_e = \omega_{ex} = \omega_{ey}$, the analytical formula for the asymptotic z component of angular momentum was derived. The values of $L_{z1}(\infty)$ and $L_{z2}(\infty)$ are almost the same as in the Markovian and non-Markovian cases. If the anisotropy of the environment increases, for example, λ_x/λ_y deviates from 1, then the value of $|L_{z1}(\infty)|$ slightly decreases for a certain magnetic field. The value of $|L_{z1}(\infty)|$ significantly decreases with increasing the ratio ω_x/ω_y of collective harmonic oscillator frequencies. The $L_{z1}(\infty) = -\hbar$ is quantized in strong magnetic fields, $\hbar\omega_c \gg T_0$. As shown, $L_{z2}(\infty)$ has two extremes, and the frequencies of the electric field corresponding to these extremes depend on the collective frequency and magnetic field (Fig. 2). If $E_{x0} = E_{y0}$ and $\omega_x = \omega_y = \omega_0$, then $L_{z2}(\infty) = 0$ at $\omega_c = |\lambda_x - \lambda_y|/2$. By varying the electric field frequency $\omega_e = \omega_{ex} = \omega_{ey}$ or magnetic field we can change the sign and value of $L_{z2}(\infty)$ and, thus, control the diamagnetic properties of 2DEG.

In the anisotropic environment with $\lambda_x \neq \lambda_y$, the component of angular momentum appears, $L_{z2} \neq 0$, even with $\omega_c = 0$ and $\omega_x = \omega_y$. This means that magnetization arises because of the asymmetry in the system. The asymmetry can be imposed in the system not only through different frequencies of oscillators in x and y but also through the asymmetry of the heat bath in which the 2D harmonic oscillator is embedded. In this case, magnetization appears even for a free particle and a symmetric harmonic oscillator and increases with the anisotropy of the environment. It is possible to propose a method for determining the ratio λ_x/λ_y for the environment by measuring the dependence of L_{z2} on the frequency of the electric field at $\omega_c = 0$ and $\omega_x = \omega_y$.

In the absence of a magnetic field for a free particle, the asymptotic z component of angular momentum decreases as $L_{z2}(\infty) \sim 1/\omega_e^4$. If $\omega_c \neq 0$, then $|L_{z2}(\infty)|$ has a maximum value and this corresponds to $\omega_c^{\max} = \omega_e$. One can see that $|L_{z2}(\infty)|$ decreases as the anisotropy of the environment increases. As in the case of a harmonic oscillator, the influence of the non-Markovian effect on dynamics of a free particle is very weak.

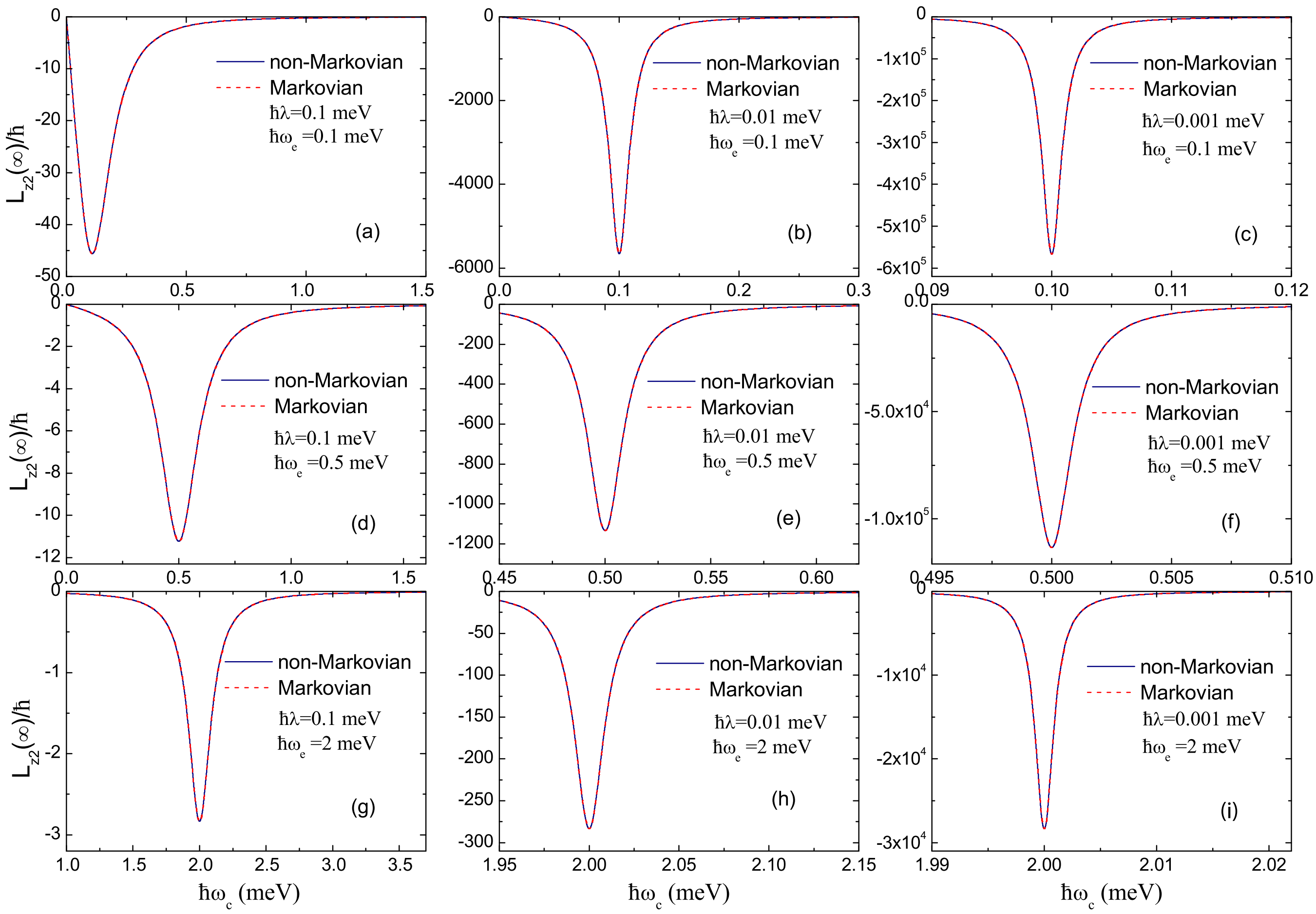

FIG. 7. The calculated asymptotic z component $L_{z2}(\infty)/\hbar$ of angular momentum as a function of $\hbar\omega_c$ for a free particle ($\hbar\omega_0 = 0$).

APPENDIX A: DERIVATION AND SOLUTION OF THE SYSTEM OF EQS. (3)

The system of the Heisenberg equations for the collective coordinates

$$\begin{aligned}
\dot{x}(t) &= \frac{i}{\hbar}[H, x] = \frac{\pi_x(t)}{m},\\
\dot{y}(t) &= \frac{i}{\hbar}[H, y] = \frac{\pi_y(t)}{m},\\
\dot{\pi}_x(t) &= \frac{i}{\hbar}[H, \pi_x]\\
&= \pi_y(t)\omega_c - m\omega_x^2 x(t) - eE_x(t) - \sum_\nu \alpha_\nu[b_\nu^+(t) + b_\nu(t)] - 2\sum_\nu \frac{\alpha_\nu}{\hbar\omega_\nu}[\alpha_\nu x(t) + g_\nu y(t)],\\
\dot{\pi}_y(t) &= \frac{i}{\hbar}[H, \pi_y]\\
&= -\pi_x(t)\omega_c - m\omega_y^2 y(t) - eE_y(t) - \sum_\nu g_\nu[b_\nu^+(t) + b_\nu(t)] - 2\sum_\nu \frac{g_\nu}{\hbar\omega_\nu}[\alpha_\nu x(t) + g_\nu y(t)]
\end{aligned} \tag{A1}$$

and the bath phonon operators

$$\begin{aligned}
\dot{b}_\nu^+(t) &= \frac{i}{\hbar}[H, b_\nu^+] = i\omega_\nu b_\nu^+(t) + \frac{i}{\hbar}[\alpha_\nu x(t) + g_\nu y(t)],\\
\dot{b}_\nu(t) &= \frac{i}{\hbar}[H, b_\nu] = -i\omega_\nu b_\nu(t) - \frac{i}{\hbar}[\alpha_\nu x(t) + g_\nu y(t)]
\end{aligned} \tag{A2}$$

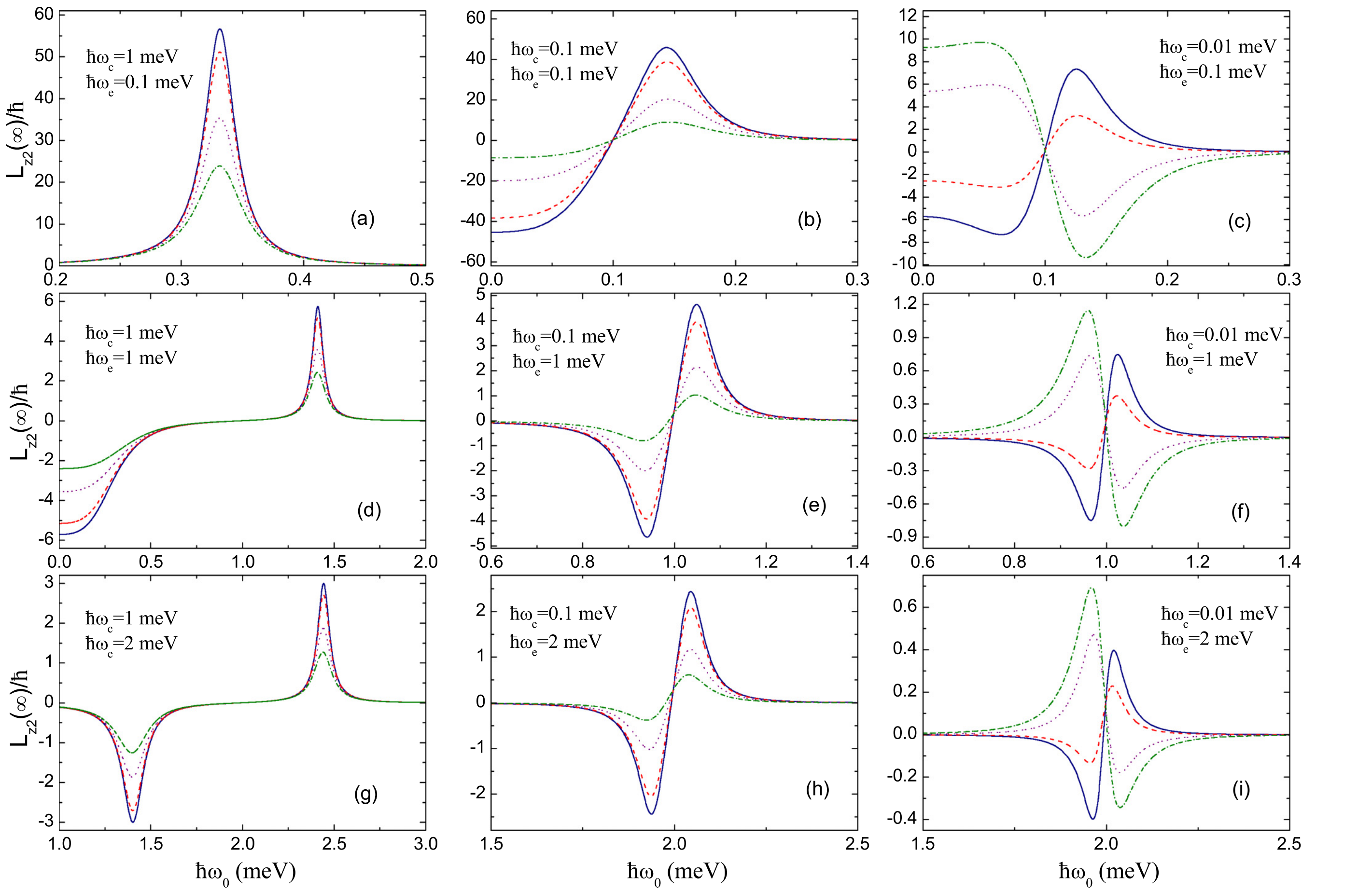

FIG. 8. The calculated asymptotic z component $L_{z2}(\infty)/\hbar$ of angular momentum as a function of $\hbar\omega_0$. Here $\hbar\lambda_y = 0.1$ meV, and the solid, dashed, dotted, and dash-dotted lines correspond to the cases with $\lambda_x/\lambda_y = 1$, 1.1, 1.5, and 2, respectively.

is obtained by commuting them with Hamiltonian (1). The solution of Eq. (A2) is

$$b_\nu^+(t) = f_\nu^+(t) - \frac{\alpha_\nu x(t) + g_\nu y(t)}{\hbar\omega_\nu} + \frac{\alpha_\nu}{\hbar\omega_\nu}\int_0^t d\tau \dot{x}(\tau) e^{i\omega_\nu(t-\tau)} + \frac{g_\nu}{\hbar\omega_\nu}\int_0^t d\tau \dot{y}(\tau) e^{i\omega_\nu(t-\tau)},$$
$$b_\nu(t) = f_\nu(t) - \frac{\alpha_\nu x(t) + g_\nu y(t)}{\hbar\omega_\nu} + \frac{\alpha_\nu}{\hbar\omega_\nu}\int_0^t d\tau \dot{x}(\tau) e^{-i\omega_\nu(t-\tau)} + \frac{g_\nu}{\hbar\omega_\nu}\int_0^t d\tau \dot{y}(\tau) e^{-i\omega_\nu(t-\tau)}, \tag{A3}$$

where

$$f_\nu(t) = \left[b_\nu(0) + \frac{\alpha_\nu x(0) + g_\nu y(0)}{\hbar\omega_\nu}\right] e^{-i\omega_\nu t}.$$

Substituting (A3) into (A1), we eliminate the bath variables from the equations of motion of the collective subsystem and obtain the nonlinear integro-differential stochastic dissipative Eqs. (3) for the collective coordinates. In Eqs. (3),

$$K_\alpha(t-\tau) = \sum_\nu \frac{2\alpha_\nu^2}{\hbar\omega_\nu}\cos[\omega_\nu(t-\tau)], \quad K_g(t-\tau) = \sum_\nu \frac{2g_\nu^2}{\hbar\omega_\nu}\cos[\omega_\nu(t-\tau)],$$
$$K_{\alpha g}(t-\tau) = K_{g\alpha}(t-\tau) = \sum_\nu \frac{2\alpha_\nu g_\nu}{\hbar\omega_\nu}\cos[\omega_\nu(t-\tau)], \tag{A4}$$

and

$$F_\alpha(t) = \sum_\nu F_\alpha^\nu(t) = \sum_\nu \alpha_\nu[f_\nu^+(t) + f_\nu(t)],$$
$$F_g(t) = \sum_\nu F_g^\nu(t) = \sum_\nu g_\nu[f_\nu^+(t) + f_\nu(t)] \tag{A5}$$

are the dissipative kernels and the random forces in the coordinates, respectively. The random force operators $F_\alpha^\nu(t)$ and $F_g^\nu(t)$ are identified as fluctuations due to the uncertainty of the initial conditions for the thermostat operators. We consider an ensemble of initial states in which the operators of the collective subsystem are fixed at the values $x(0)$ and $y(0)$, and the initial bath operators are drawn from an ensemble that is canonical relative to the collective subsystem [27,37]. The initial distribution is then the conditional density matrix $\rho_0(\{b_\nu^+(0), b_\nu(0)\}|\mathbf{q}(0)) = \exp(-\sum_\nu \hbar\omega_\nu[b_\nu^+ + \frac{\alpha_\nu x + g_\nu y}{\hbar\omega_\nu}][b_\nu + \frac{\alpha_\nu x + g_\nu y}{\hbar\omega_\nu}]/T_0)/Z(T_0)$, where $Z(T_0)$ is the conditional partition function and T_0 (in the energy units) is the temperature of the heat bath. In an ensemble of initial states for the bath operators, the fluctuations $F_\alpha^\nu(t)$ and $F_g^\nu(t)$ have the Gaussian distributions with zero average value

$$\langle\langle F_\alpha^\nu(t)\rangle\rangle = \langle\langle F_g^\nu(t)\rangle\rangle = 0, \tag{A6}$$

where the symbol $\ll \cdot \gg$ denotes the average over the bath. The temperature T_0 of the heat bath is included in the analysis through the distribution of initial conditions. We use the Bose-Einstein statistics for the heat bath:

$$\begin{aligned} \langle\langle f_\nu^+(t) f_{\nu'}^+(t')\rangle\rangle &= \langle\langle f_\nu(t) f_{\nu'}(t')\rangle\rangle = 0, \\ \langle\langle f_\nu^+(t) f_{\nu'}(t')\rangle\rangle &= \delta_{\nu,\nu'} n_\nu e^{i\omega_\nu(t-t')}, \\ \langle\langle f_\nu(t) f_{\nu'}^+(t')\rangle\rangle &= \delta_{\nu,\nu'}(n_\nu+1) e^{-i\omega_\nu(t-t')} \end{aligned} \tag{A7}$$

with occupation numbers for phonons $n_\nu = [\exp(\hbar\omega_\nu/T_0) - 1]^{-1}$ depending on T_0. Using the properties of random forces, we obtain the quantum fluctuation-dissipation relations

$$\sum_\nu \varphi_{\alpha\alpha}^\nu(t,t') \frac{\tanh\left[\frac{\hbar\omega_\nu}{2T_0}\right]}{\hbar\omega_\nu} = K_\alpha(t-t'), \quad \sum_\nu \varphi_{gg}^\nu(t,t') \frac{\tanh\left[\frac{\hbar\omega_\nu}{2T_0}\right]}{\hbar\omega_\nu} = K_g(t-t'),$$
$$\sum_\nu \varphi_{\alpha g}^\nu(t,t') \frac{\tanh\left[\frac{\hbar\omega_\nu}{2T_0}\right]}{\hbar\omega_\nu} = K_{\alpha g}(t-t'),$$

where

$$\begin{aligned} &\varphi_{\alpha\alpha}^\nu(t,t') = 2\alpha_\nu^2[2n_\nu+1]\cos(\omega_\nu[t-t']), \quad \varphi_{gg}^\nu(t,t') = 2g_\nu^2[2n_\nu+1]\cos(\omega_\nu[t-t']), \\ &\varphi_{\alpha g}^\nu(t,t') = 2\alpha_\nu g_\nu[2n_\nu+1]\cos(\omega_\nu[t-t']) \end{aligned}$$

are the symmetrized correlation functions $\varphi_{kk'}^\nu(t,t') = \langle\langle F_k^\nu(t) F_{k'}^\nu(t') + F_{k'}^\nu(t') F_k^\nu(t)\rangle\rangle$, $k, k' = \alpha, g$. The quantum fluctuation-dissipation relations differ from the classical ones and are reduced to them in the limit of high temperature T_0 (or $\hbar \to 0$): $\sum_\nu \varphi_{\alpha\alpha}^\nu(t,t') = 2T_0 K_\alpha(t-t')$, $\sum_\nu \varphi_{gg}^\nu(t,t') = 2T_0 K_g(t-t')$, and $\sum_\nu \varphi_{\alpha g}^\nu(t,t') = 2T_0 K_{\alpha g}(t-t')$.

The Laplace transform $\hat{L}$ of Eqs. (3) leads to the system of linear equations:

$$\begin{aligned} x(s)s &= x(0) + \frac{\pi_x(s)}{m}, \\ y(s)s &= y(0) + \frac{\pi_y(s)}{m}, \\ \pi_x(s)s &= \pi_x(0) + \omega_c\pi_y(s) - m\omega_x^2 x(s) - eE_x(s) - K_\alpha(s)\frac{\pi_x(s)}{m} - K_{\alpha g}(s)\frac{\pi_y(s)}{m} - F_\alpha(s), \\ \pi_y(s)s &= \pi_y(0) - \omega_c\pi_x(s) - m\omega_y^2 y(s) - eE_y(s) - K_g(s)\frac{\pi_y(s)}{m} - K_{\alpha g}(s)\frac{\pi_x(s)}{m} - F_g(s). \end{aligned} \tag{A8}$$

Here $K_\alpha(s)$, $K_g(s)$, $K_{\alpha g}(s)$, and $F_\alpha(s)$, $F_g(s)$ are the Laplace transforms of the dissipative kernels and random forces, respectively. The system of Eqs. (A8) is easy to solve and performs the inverse Laplace transform $\hat{L}^{-1}$ using the residue theorem and the roots of the determinant

$$D = s^4 + (\omega_c^2 + \omega_x^2 + \omega_y^2)s^2 + \omega_x^2\omega_y^2 + (s^3 + s\omega_y^2)\frac{K_\alpha(s)}{m} + (s^3 + s\omega_x^2)\frac{K_g(s)}{m} + \frac{s^2 K_\alpha(s) K_g(s)}{m^2} - \frac{s^2 K_{\alpha g}^2(s)}{m^2} = 0. \tag{A9}$$

Finally, we obtain Eqs. (4) with the following time-dependent coefficients:

$$\begin{aligned} &I_x(t) = \int_0^t A_3(\tau) F_\alpha(t-\tau)\,d\tau, I_x'(t) = \int_0^t A_4(\tau) F_g(t-\tau)\,d\tau, I_{ex}(t) = e\int_0^t A_3(\tau) E_x(t-\tau)\,d\tau, \\ &I_{ex}'(t) = e\int_0^t A_4(\tau) E_y(t-\tau)\,d\tau, I_y(t) = \int_0^t B_3(\tau) F_\alpha(t-\tau)\,d\tau, I_y'(t) = \int_0^t B_4(\tau) F_g(t-\tau)\,d\tau, \\ &I_{ey}(t) = e\int_0^t B_3(\tau) E_x(t-\tau)\,d\tau, I_{ey}'(t) = e\int_0^t B_4(\tau) E_y(t-\tau)\,d\tau, I_{\pi_x}(t) = \int_0^t C_3(\tau) F_\alpha(t-\tau)\,d\tau, \end{aligned}$$

$$
\begin{aligned}
&I'_{\pi_x}(t)=\int_0^t C_4(\tau)F_g(t-\tau)\,d\tau,\, I_{e\pi_x}(t)=e\int_0^t C_3(\tau)E_x(t-\tau)\,d\tau,\, I'_{e\pi_x}(t)=e\int_0^t C_4(\tau)E_y(t-\tau)\,d\tau,\\
&I_{\pi_y}(t)=\int_0^t D_3(\tau)F_\alpha(t-\tau)\,d\tau,\, I'_{\pi_y}(t)=\int_0^t D_4(\tau)F_g(t-\tau)\,d\tau,\, I_{e\pi_y}(t)=e\int_0^t D_3(\tau)E_x(t-\tau)\,d\tau,\\
&I'_{e\pi_y}(t)=e\int_0^t D_4(\tau)E_y(t-\tau)\,d\tau,
\end{aligned}
$$

and

$$
\begin{aligned}
A_1(t)&=\hat{L}^{-1}\left[\frac{m^2s^3+m^2\left(\omega_c^2+\omega_y^2\right)s+m\left(s^2+\omega_y^2\right)K_\alpha(s)+ms^2K_g(s)+sK_\alpha(s)K_g(s)-sK_{\alpha g}^2(s)}{m^2D(s)}\right]\\
&=B_2(t)|_{x,\alpha\leftrightarrow y,g},\quad A_2(t)=\hat{L}^{-1}\left[-\frac{\omega_y^2[m\omega_c-K_{\alpha g}(s)]}{mD(s)}\right],\\
A_3(t)&=\hat{L}^{-1}\left[\frac{ms^2+m\omega_y^2+sK_g(s)}{m^2D(s)}\right]=B_4(t)|_{x,\alpha\leftrightarrow y,g},\quad A_4(t)=\hat{L}^{-1}\left[\frac{s[m\omega_c-K_{\alpha g}(s)]}{m^2D(s)}\right],\\
B_1(t)&=\hat{L}^{-1}\left[\frac{\omega_x^2[m\omega_c+K_{\alpha g}(s)]}{mD(s)}\right],\quad B_3(t)=\hat{L}^{-1}\left[-\frac{s[m\omega_c+K_{\alpha g}(s)]}{m^2D(s)}\right],\\
C_1(t)&=-m^2\omega_x^2A_3(t)=D_2(t)|_{x,\alpha\leftrightarrow y,g},\quad C_2(t)=\hat{L}^{-1}\left[-\frac{s\omega_y^2[m\omega_c-K_{\alpha g}(s)]}{D(s)}\right],\\
C_3(t)&=\hat{L}^{-1}\left[\frac{ms^3+m\omega_y^2s+s^2K_g(s)}{mD(s)}\right]=D_4(t)|_{x,\alpha\leftrightarrow y,g},\quad C_4(t)=\hat{L}^{-1}\left[\frac{s^2[m\omega_c-K_{\alpha g}(s)]}{mD(s)}\right],\\
D_1(t)&=\hat{L}^{-1}\left[\frac{s\omega_x^2[m\omega_c+K_{\alpha g}(s)]}{D(s)}\right],\quad D_3(t)=\hat{L}^{-1}\left[-\frac{s^2[m\omega_c+K_{\alpha g}(s)]}{mD(s)}\right].
\end{aligned}\tag{A10}
$$

In general, the diagonal dissipative kernels are much larger than off-diagonal ones. For simplicity, we assume that there is no correlation between the operators F_α^ν and F_g^ν, so that $K_{\alpha g}=K_{g\alpha}=0$. It is convenient to introduce the spectral density D_ω of the heat bath excitations, which allows us to replace the sum over different oscillators, ν, by the integral over frequency: $\sum_\nu\cdots\to\int_0^\infty d\omega D_\omega\cdots$. This is accompanied by the following replacements: $\alpha_\nu\to\alpha_\omega$, $g_\nu\to g_\omega$, $\omega_\nu\to\omega$, and $n_\nu\to n_\omega$. Let us consider the following spectral functions [26,27]:

$$
D_\omega\frac{\alpha_\omega^2}{\omega}=\frac{\lambda_x^2}{\pi}\frac{\gamma^2}{\gamma^2+\omega^2},\quad D_\omega\frac{g_\omega^2}{\omega}=\frac{\lambda_y^2}{\pi}\frac{\gamma^2}{\gamma^2+\omega^2},\tag{A11}
$$

where the memory time γ^{-1} of dissipation is inverse to the phonon bandwidth of the heat bath excitations which are coupled with the collective oscillator and the coefficients

$$
\lambda_x=\frac{1}{m}\int_0^\infty d\tau K_\alpha(t-\tau),\quad \lambda_y=\frac{1}{m}\int_0^\infty d\tau K_g(t-\tau)
$$

are the friction coefficients in the Markovian limit. This Ohmic dissipation with the Lorentzian cutoff (Drude dissipation) results in the dissipative kernels

$$
K_\alpha(t)=m\lambda_x\gamma e^{-\gamma|t|},\quad K_g(t)=m\lambda_y\gamma e^{-\gamma|t|}.
$$

The relaxation time of the heat bath should be much less than the period of the collective oscillator, i.e., $\gamma\gg\omega_{x,y}$. Using all assumptions mentioned above, we obtain the explicit expressions for the time-dependent coefficients

$$
\begin{aligned}
A_1(t)&=\sum_{i=1}^{6}\beta_i\{(s_i+\gamma)\left[s_i(s_i+\gamma)\left(s_i^2+\omega_y^2+\omega_c^2\right)+\lambda_x\gamma\left(s_i^2+\omega_y^2\right)\right]+s_i\lambda_y\gamma[s_i(s_i+\gamma)+\lambda_x\gamma]\}e^{s_it},\\
A_2(t)&=-\omega_c\omega_y^2\sum_{i=1}^{6}\beta_i(s_i+\gamma)^2e^{s_it},\\
A_3(t)&=\frac{1}{m}\sum_{i=1}^{6}\beta_i(s_i+\gamma)[(s_i+\gamma)\left(s_i^2+\omega_y^2\right)+s_i\lambda_y\gamma]e^{s_it},
\end{aligned}
$$

$$
\begin{aligned}
&A_4(t)=\frac{\omega_c}{m}\sum_{i=1}^{6}\beta_i s_i(s_i+\gamma)^2e^{s_it},\\
&B_1(t)=-A_2(t)|_{x\leftrightarrow y},\;\; B_2(t)=A_1(t)|_{x\leftrightarrow y},\;\; B_3(t)=-A_4(t)|_{x\leftrightarrow y},\;\; B_4(t)=A_3(t)|_{x\leftrightarrow y},\\
&C_1(t)=-m^2\omega_x^2A_3(t),\;\; C_2(t)=m\dot{A}_2(t),\;\; C_3(t)=m\dot{A}_3(t),\;\; C_4(t)=m\dot{A}_4(t),\\
&D_1(t)=m\dot{B}_1(t),\;\; D_2(t)=-m^2\omega_y^2B_4(t),\;\; D_3(t)=m\dot{B}_3(t),\;\; D_4(t)=m\dot{B}_4(t)
\end{aligned}
\tag{A12}
$$

of Eqs. (4). The roots s_i of the determinant (9) arise in the time-dependent coefficients when we apply the residue theorem to perform the integration in the inverse Laplace transform.

APPENDIX B: THE ORBITAL ANGULAR MOMENTA $L_z^0(t)$ AND $L_{z1}(\infty)$

In Sec. II A, Eq. (5) contains the term

$$
\begin{aligned}
L_z^0(t)&=\alpha_{11}(t)\langle x^2(0)\rangle+\alpha_{12}(t)\langle x(0)y(0)\rangle+\alpha_{13}(t)\langle x(0)\pi_x(0)\rangle+\alpha_{14}(t)\langle x(0)\pi_y(0)\rangle\\
&+\alpha_{21}(t)\langle y(0)x(0)\rangle+\alpha_{22}(t)\langle y^2(0)\rangle+\alpha_{23}(t)\langle y(0)\pi_x(0)\rangle+\alpha_{24}(t)\langle y(0)\pi_y(0)\rangle\\
&+\alpha_{31}(t)\langle \pi_x(0)x(0)\rangle+\alpha_{32}(t)\langle \pi_x(0)y(0)\rangle+\alpha_{33}(t)\langle \pi_x^2(0)\rangle+\alpha_{34}(t)\langle \pi_x(0)\pi_y(0)\rangle\\
&+\alpha_{41}(t)\langle \pi_y(0)x(0)\rangle+\alpha_{42}(t)\langle \pi_y(0)y(0)\rangle+\alpha_{43}(t)\langle \pi_y(0)\pi_x(0)\rangle+\alpha_{44}(t)\langle \pi_y^2(0)\rangle
\end{aligned}
\tag{B1}
$$

depending on the initial second moments of the collective coordinates. Here

$$
\alpha_{kl}(t)=A_k(t)D_l(t)-B_k(t)C_l(t),\qquad k,l=1,2,3,4.
$$

Analytical integration over ω in Eq. (11) leads to [40]

$$
L_{z1}(\infty)=-\hbar\omega_c\gamma^2[2\mathrm{Re}(I)-I_s],
\tag{B2}
$$

where

$$
\begin{aligned}
I&=\frac{s_1^2\{(\gamma^2-s_1^2)[\lambda_x(s_1^2+\omega_y^2)+\lambda_y(s_1^2+\omega_x^2)]-2s_1^2\gamma\lambda_x\lambda_y\}}{(s_1^2-s_1^{*2})(s_1^2-s_2^2)(s_1^2-s_2^{*2})(s_1^2-s_3^2)(s_1^2-s_3^{*2})}\cot\left[\frac{\hbar s_1}{2T_0}\right]\\
&+\frac{s_2^2\{(\gamma^2-s_2^2)[\lambda_x(s_2^2+\omega_y^2)+\lambda_y(s_2^2+\omega_x^2)]-2s_2^2\gamma\lambda_x\lambda_y\}}{(s_2^2-s_1^2)(s_2^2-s_1^{*2})(s_2^2-s_2^{*2})(s_2^2-s_3^2)(s_2^2-s_3^{*2})}\cot\left[\frac{\hbar s_2}{2T_0}\right]\\
&+\frac{s_3^2\{(\gamma^2-s_3^2)[\lambda_x(s_3^2+\omega_y^2)+\lambda_y(s_3^2+\omega_x^2)]-2s_3^2\gamma\lambda_x\lambda_y\}}{(s_3^2-s_1^2)(s_3^2-s_1^{*2})(s_3^2-s_2^2)(s_3^2-s_2^{*2})(s_3^2-s_3^{*2})}\cot\left[\frac{\hbar s_3}{2T_0}\right]
\end{aligned}
$$

and

$$
I_s=32\pi^3\frac{T_0^4}{\hbar^4}\sum_{n=1}^{\infty}\frac{\{(\gamma^2-x_n^2)[\lambda_x(x_n^2+\omega_y^2)+\lambda_y(x_n^2+\omega_x^2)]-2x_n^2\gamma\lambda_x\lambda_y\}n^3}{|x_n^2-s_1^2|^2|x_n^2-s_2^2|^2|x_n^2-s_3^2|^2}.
$$

Here $x_n=2\pi nT_0/\hbar$, $\mathrm{Re}(s_1)<0$, $\mathrm{Re}(s_2)<0$, and $\mathrm{Re}(s_3)<0$.

APPENDIX C: FREE CHARGED PARTICLE IN A DISSIPATIVE BOSONIC HEAT BATH IN AN EXTERNAL CONSTANT MAGNETIC FIELD AND TWO NONSTATIONARY ELECTRIC FIELDS

Let us consider the action of two perpendicular time-dependent electric fields in the xy plane and the transverse (along the z axis) constant magnetic field on a free charged particle. Employing the Markovian limit and a trick originally due to Darwin [43], we find the value of

$$
\begin{aligned}
L_{z1}(\infty)&=\frac{8T_0\omega_c}{4\omega_c^2+(\lambda_x+\lambda_y)^2}-\frac{\hbar\sinh[\hbar\omega_c/T_0]}{2\{\sinh^2[\hbar\omega_c/(2T_0)]+\sin^2[\hbar(\lambda_x+\lambda_y)/(4T_0)]\}}\\
&+\sum_{n=1}^{\infty}\frac{128n\pi T_0^2\hbar^3\omega_c(\lambda_x+\lambda_y)}{|4n\pi T_0+\hbar(2i\omega_c-\lambda_x-\lambda_y)|^2|4n\pi T_0-\hbar(2i\omega_c-\lambda_x-\lambda_y)|^2}
\end{aligned}
\tag{C1}
$$

from Eq. (13) in the limit $\omega_x = \omega_y \to 0$ (after the integration over ω). If the friction is isotropic, $\lambda_x = \lambda_y = \lambda$, then we derive the expression

$$L_{z1}(\infty) = \frac{2T_0\omega_c}{\omega_c^2+\lambda^2} + \frac{\hbar\sinh[\hbar\omega_c/T_0]}{\cos[\hbar\lambda/T_0]-\cosh[\hbar\omega_c/T_0]} + \sum_{n=1}^{\infty}\frac{16n\pi T_0^2\hbar^3\lambda\omega_c}{16n^4\pi^4T_0^4+8\hbar^2n^2\pi^2T_0^2(\omega_c^2-\lambda^2)+\hbar^4(\omega_c^2+\lambda^2)^2}. \quad \text{(C2)}$$

As found, the contribution of the sums in Eqs. (C1) and (C2) to $L_{z1}(\infty)$ is negligible at high temperatures. As seen from Eqs. (C1) and (C2), the Bohr-Van Leeuwen theorem that states that diamagnetism does not exist in classical statistical mechanics is restored for large damping (as $\lambda_{x,y} \to \infty$).

In the limit of zero friction ($\lambda \to 0$), Eq. (C2) results in the Landau formula or the Langevin function

$$L_{z1}(\infty) = \hbar\left(\frac{2T_0}{\hbar\omega_c} - \coth\left[\frac{\hbar\omega_c}{2T_0}\right]\right). \quad \text{(C3)}$$

So Eq. (C1) or (C2) naturally generalizes the Landau formula (C3) in the case of dissipative system. These formulas are applicable in a situation in which the system has scattering processes that can lead to decoherence of Landau orbits [21]. Formula (C2) for dissipative orbital magnetization was first obtained in Ref. [22], where the starting point is the quantum Langevin equation of a charged particle in a magnetic field and an isotropic bosonic heat bath. The case of $\lambda_x \neq \lambda_y$ was studied more extensively in Ref. [40].

In the non-Markovian limit, we find the asymptotic z component of angular momentum

$$L_{z2}(\infty) = -e^2(\gamma^2+\omega_e^2)\{\gamma[E_{x0}^2\omega_c(\gamma-\lambda_y)+E_{y0}^2\omega_c(\gamma-\lambda_x)-E_{x0}E_{y0}\gamma(\lambda_x-\lambda_y)] + (E_{x0}^2+E_{y0}^2)\omega_c\omega_e^2\}/N, \quad \text{(C4)}$$

at $\omega_{ex} = \omega_{ey} = \omega_e$, where

$$N = m\Big(\gamma^4(\omega_c^2+\lambda_x\lambda_y)^2 + \gamma^2\Big\{[2\omega_c^2+\gamma(\lambda_x+\lambda_y)]^2 - 2(\omega_c^2+\lambda_x\lambda_y)[\omega_c^2+\gamma(\gamma+\lambda_x+\lambda_y)]\Big\}\omega_e^2$$
$$+\{\omega_c^4 - 2\omega_c^2\gamma(2\gamma-\lambda_x-\lambda_y) + \gamma^2[\gamma^2+\lambda_x^2+4\lambda_x\lambda_y+\lambda_y^2-2\gamma(\lambda_x+\lambda_y)]\}\omega_e^4 - 2[\omega_c^2-\gamma(\gamma-\lambda_x-\lambda_y)]\omega_e^6 + \omega_e^8\Big).$$

To find the maximum of the asymptotic z component of angular momentum, we take the derivatives of Eq. (C4) with respect to E_{y0} and obtain

$$E_{y0}^{\max} = \frac{E_{x0}\gamma^2(\lambda_x-\lambda_y)}{2\omega_c(\gamma^2-\gamma\lambda_x+\omega_e^2)}. \quad \text{(C5)}$$

With this ratio of $E_{y0}^{\max}/E_{x0}$ we obtain the maximum value of $L_{z2}(\infty)$

$$L_{z2}^{\max}(\infty) = -\frac{e^2E_{x0}^2(\gamma^2+\omega_e^2)Z}{4\omega_c[\gamma(\gamma-\lambda_x)]N}, \quad \text{(C6)}$$

where

$$Z = \{\gamma^2[4\omega_c^2(\gamma-\lambda_x)(\gamma-\lambda_y)-\gamma^2(\lambda_x-\lambda_y)^2] + 4\omega_c^2\omega_e^2[\omega_e^2+\gamma(2\gamma-\lambda_x-\lambda_y)]\}.$$

In the isotropic environment ($\lambda_x = \lambda_y = \lambda$), $E_{y0}^{\max} = 0$, and only one electric field is sufficient. In the constant electric fields ($\omega_e = 0$), we obtain

$$L_{z2}(\infty) = -\frac{e^2[E_{x0}^2\omega_c(\gamma-\lambda_y)+E_{y0}^2\omega_c(\gamma-\lambda_x)-E_{x0}E_{y0}\gamma(\lambda_x-\lambda_y)]}{m\gamma(\omega_c^2+\lambda_x\lambda_y)^2}. \quad \text{(C7)}$$

If $\omega_c = 0$,

$$L_{z2}(\infty) = \frac{e^2E_{x0}E_{y0}(\lambda_x-\lambda_y)}{m\lambda_x^2\lambda_y^2}. \quad \text{(C8)}$$

As seen, for $\lambda_x = \lambda_y = \lambda$ the asymptotic z component of angular momentum equals zero. The absolute value of $L_{z2}(\infty)$ does not depend on γ and reaches the maximum

$$L_{z2}^{\max}(\infty) = \frac{e^2E_{x0}E_{y0}}{4m\lambda_y^3} \quad \text{(C9)}$$

at $\lambda_x = 2\lambda_y$. The absolute values of $L_{z2}^{\max}(\infty)$ decreases rapidly with increasing λ_y. Frequency $\omega_c^{\max}$ corresponding to the maximum value of the asymptotic z component of angular momentum at $\lambda_x \neq \lambda_y$ and nonzero $E_{x0} \neq E_{y0}$ depends on the frictional coefficients λ_x and λ_y, amplitude electrical fields E_{x0} and E_{y0}, and γ as

$$\omega_c^{\max} = \sqrt{\frac{2E_{x0}E_{y0}\gamma(\lambda_x-\lambda_y)+\sqrt{3\lambda_x\lambda_y\big[E_{x0}^2(\gamma-\lambda_y)+E_{y0}^2(\gamma-\lambda_x)\big]^2+4E_{x0}^2E_{y0}^2\gamma^2(\lambda_x-\lambda_y)^2}}{3\big[E_{x0}^2(\gamma-\lambda_y)+E_{y0}^2(\gamma-\lambda_x)\big]}}. \tag{C10}$$

At $E_{x0} = E_{y0} \neq 0$, we derive the following expression:

$$\omega_c^{\max} = \sqrt{\frac{2\gamma(\lambda_x-\lambda_y)+\sqrt{4\gamma^2(\lambda_x-\lambda_y)^2+3\lambda_x\lambda_y(2\gamma-\lambda_x-\lambda_y)^2}}{3(2\gamma-\lambda_x-\lambda_y)}}. \tag{C11}$$

If the external constant electric field affects the charged particle in only one direction $E_{x0} = 0$, (or $E_{y0} = 0$), we obtain

$$\omega_c^{\max} = \sqrt{\frac{\lambda_x\lambda_y}{3}}. \tag{C12}$$

If the friction coefficients are equal, $\lambda_x = \lambda_y = \lambda$, regardless of the external electric fields, $\omega_c^{\max}$ is equal to

$$\omega_c^{\max} = \frac{\lambda}{\sqrt{3}}. \tag{C13}$$

At $\lambda_x = \lambda_y = \lambda$, we find from Eq. (C7) the asymptotic z component of angular momentum

$$L_{z2}(\infty) = -\frac{e^2\big(E_{x0}^2+E_{y0}^2\big)\omega_c(\gamma-\lambda)}{m\gamma\left(\omega_c^2+\lambda^2\right)^2}. \tag{C14}$$

In the case of $E_{y0} = 0$, $\gamma \gg \lambda$ and $\omega_c \gg \lambda$, it is transformed into

$$L_{z2}(\infty) = -\frac{e^2E_{x0}^2}{m\omega_c^3}. \tag{C15}$$

In the Markovian limit ($\gamma \to \infty$), we obtain from Eq. (C14)

$$L_{z2}(\infty) = -\frac{e^2\big(E_{x0}^2+E_{y0}^2\big)\omega_c}{m\big(\omega_c^2+\lambda^2\big)^2}. \tag{C16}$$

[1] K. S. Novoselov, A. K. Giem, S. V. Morozov, D. Jiang, Y. Zhang, S. V. Dubonos, I. V. Grigorieva, and A. A. Firsov, Science **306**, 666 (2004).

[2] K. S. Novoselov, E. McCann, S. V. Morozov, V. I. Fal'ko, M. I. Katsnelson, U. Zeitler, D. Jiang, F. Schedin, and A. K. Geim, Nat. Phys. **2**, 177 (2006).

[3] A. H. Castro Neto, F. Guinea, N. M. R. Peres, K. S. Novoselov, and A. K. Geim, Rev. Mod. Phys. **81**, 109 (2009).

[4] A. K. Geim and I. V. Grigorieva, Nature (London) **499**, 419 (2013).

[5] D. R. Cooper, B. D'Anjou, N. Ghattamaneni, B. Harack, M. Hilke, A. Horth, N. Majlis, M. Massicote, L. Vandsburger, E. Whiteway, and V. Yu, Int. Scholarly Res. Notic., Condens. Matter Phys. **2012**, 501686 (2012).

[6] N. Deng, H. Tian, J. Zhang, J. Jian, F. Wu, Y. Shen, and T.-L. Ren, J. Semicond. **42**, 081001 (2021).

[7] X.-X. Xue, H. Meng, Z. Huang, Y. Feng, and X. Qi, J. Phys. Energy **3**, 042002 (2021).

[8] L. Jacak, P. Hawrylaka, and A. Wójs, *Quantum Dots* (Springer-Verlag, Berlin, 1998).

[9] W. D. Heiss and R. G. Nazmitdinov, Phys. Rev. B **55**, 16310 (1997); Pis'ma Zh. Eksp. Teor. Fiz. **68**, 870 (1998) [JETP Lett. **68**, 915 (1998)]; M. Dineykhan and R. G. Nazmitdinov, Phys. Rev. B **55**, 13707 (1997); R. G. Nazmitdinov, N. S. Simonovic, and J. M. Rost, *ibid.* **65**, 155307 (2002).

[10] Yu. Demidenko, A. Kuzyk, V. Lozovski, and O. Tretyak, J. Phys. C **16**, 543 (2004); A. Matulis and E. Anisimovas, *ibid.* **17**, 3851 (2005).

[11] L. Langer, V. Bayot, E. Grivei, J. P. Issi, J. P. Heremans, C. H. Olk, L. Stockman, C. Van Haesendonck, and Y. Bruynseraede, Phys. Rev. Lett. **76**, 479 (1996); O. Bourgeois, P. Gandit, J. Lesueur, R. Mélin, A. Sulpice, X. Grison, and J. Chaussy, arXiv:cond-mat/9901045; E. L. Nagaev, Usp. Fiz. Nauk **166**, 833 (1996).

[12] K.-C. Chuang, A. M. R. Baker, and R. J. Nicholas, Phys. Rev. B **80**, 161410(R) (2009); M. Orlita, C. Faugeras, J. M. Schneider, G. Martinez, D. K. Maude, and M. Potemski, Phys. Rev. Lett. **102**, 166401 (2009).

[13] Y. Sun, J. D. Burton, and E. Y. Tsymbal, Phys. Rev. B **81**, 064413 (2010).

[14] F. Matsukura, Y. Tokura, and H. Ohno, Nat. Nanotechnol. **10**, 209 (2015).

[15] T. Arakawa, T. Oka, S. Kon, and Y. Niimi, Phys. Rev. Lett. **129**, 046801 (2022).

[16] M. M. Glazov and S. D. Ganichev, Phys. Rep. **535**, 101 (2014).

[17] E. G. Mishchenko, Phys. Rev. Lett. **103**, 246802 (2009).

[18] S. Dattagupta, Phys. Rev. B **104**, 085411 (2021).

[19] L. Landau, Z. Phys. **64**, 629 (1930).

[20] D. Shoenberg, *Magnetic Oscillations in Metals* (Cambridge University Press, Cambridge, 1984).

[21] S. Dattagupta and S. Puri, *Dissipative Phenomena in Condensed Matter* (Springer-Verlag, Berlin, 2004).

[22] S. Dattagupta and J. Singh, Phys. Rev. Lett. **79**, 961 (1997).

[23] L. I. Magarill and A. V. Chaplik, JETP Lett. **70**, 615 (1999).

[24] A. D. Chepelianskii and D. L. Shepelyansky, Eur. Phys. J. B **55**, 261 (2007).

[25] A. O. Caldeira and A. J. Leggett, Phys. Rev. Lett. **46**, 211 (1981); Ann. Phys. (N.Y.) **149**, 374 (1983).

[26] N. G. van Kampen, *Stochastic Processes in Physics and Chemistry* (North-Holland, Amsterdam, 1981).

[27] K. Lindenberg and B. J. West, *The Nonequilibrium Statistical Mechanics of Open and Closed Systems* (VCH Publishers, New York, 1990); Phys. Rev. A **30**, 568 (1984).

[28] A. Isar, A. Sandulescu, H. Scutaru, E. Stefanescu, and W. Scheid, Int. J. Mod. Phys. E **03**, 635 (1994).

[29] D. Zubarev, V. Morozov, and G. Röpke, *Statistical Mechanics of Nonequilibrium Processes*, Vols. 1 and 2 (Academie Verlag, Berlin, 1997).

[30] V. V. Dodonov and V. I. Man'ko, *Density Matrices and Wigner Functions of Quasiclassical Quantum Systems* (Proc. Lebedev Phys. Inst. Sci., Vol. 167, A. A. Komar, ed.) (Nova Science, Commack, NY, 1987).

[31] G. W. Ford, J. T. Lewis, and R. F. O'Connell, Phys. Rev. A **36**, 1466 (1987); **37**, 4419 (1988).

[32] H. P. Breuer and F. Petruccione, *The Theory of Open Quantum Systems* (Oxford University Press, New York, 2002).

[33] Th. M. Nieuwenhuizen and A. E. Allahverdyan, Phys. Rev. E **66**, 036102 (2002).

[34] B. Bellomo, A. De Pasquale, G. Gualdi, and U. Marzolino, Phys. Rev. A **82**, 062104 (2010); J. Phys. A: Math. Theor. **43**, 395303 (2010).

[35] M. Carlesso and A. Bassi, Phys. Rev. A **95**, 052119 (2017).

[36] Sh. A. Kalandarov, Z. Kanokov, G. G. Adamian, and N. V. Antonenko, Phys. Rev. E **75**, 031115 (2007).

[37] Sh. A. Kalandarov, I. B. Abdurakhmanov, Z. Kanokov, G. G. Adamian, and N. V. Antonenko, Phys. Rev. A **99**, 062109 (2019).

[38] I. B. Abdurakhmanov, G. G. Adamian, N. V. Antonenko, and Z. Kanokov, Physica A **508**, 613 (2018).

[39] I. B. Abdurakhmanov, Z. Kanokov, G. G. Adamian, and N. V. Antonenko, Physica A **514**, 957 (2019).

[40] E. Kh. Alpomishev, G. G. Adamian, and N. V. Antonenko, Phys. Rev. E **104**, 054120 (2021).

[41] E. Kh. Alpomishev, G. G. Adamian, and N. V. Antonenko, Symmetry **16**, 7 (2024).

[42] R. C. Ashoori, H. L. Stormer, J. S. Weiner, L. N. Pfeiffer, K. W. Baldwin, and K. W. West, Phys. Rev. Lett. **71**, 613 (1993).

[43] C. D. Darwin, Proc. Cambridge Philos. Soc. **27**, 86 (1930).