Quest for the golden ratio universality class

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(Received 1 November 2023; accepted 1 March 2024; published 5 April 2024)

Using mode-coupling theory, the conditions for all allowed dynamical universality classes for the conserved modes in one-dimensional driven systems are presented in closed form as a function of the stationary currents and their derivatives. With an eye on the search for the golden ratio universality class, the existence of some families of microscopic models is ruled out *a priori* by using an Onsager-type macroscopic current symmetry. In particular, if the currents are symmetric or antisymmetric under the interchange of the conserved densities, then at equal mean densities the golden modes can only appear in the antisymmetric case and if the conserved quantities are correlated, but not in the symmetric case where at equal densities one mode is always diffusive and the second may be either Kardar-Parisi-Zhang (KPZ), modified KPZ, 3/2-Lévy, or also diffusive. We also show that the predictions of mode-coupling theory for a noisy chain of harmonic oscillators are exact.

DOI: 10.1103/PhysRevE.109.044111

I. INTRODUCTION

The golden ratio $\varphi = (1+\sqrt{5})/2 \approx 1.618$ is undoubtedly one of the most fascinating irrational numbers. It shows up in completely different circumstances, from the arrangement of seeds in sunflowers or petals in flowers [1–3], the shape of snowflakes [4], and the statistics of signals in the human brain [5] to art and architectural masterpieces [6]. From the mathematical viewpoint, the golden ratio is the most irrational number, in the sense that the number for which rational approximations like the continued fraction [7] perform the worst. In the realm of physics, the golden ratio is somewhat less prominent. In particular, in the theory of phase transitions—where *rational* numbers are ubiquitous in the characteristics of universality classes—specific irrational numbers have not been encountered until recently. Indeed, all critical indices coming from Landau theory of phase transitions [8] are rational numbers. Also, the critical exponents produced in conformal field theories are all rational (unless continuously varying). Generically, since various scaling relations relating different critical exponents have the form of a ratio, the rationality of critical exponents seems to be a robust feature, leaving little room for irrationality. It has therefore come as a surprise that the golden ratio can characterize universal fluctuations of long-lived relaxation modes of interacting many-body systems with conservation laws, governed by the dynamical exponent z, i.e., with stationary space-time correlations of the form

$$\langle \sigma(0,0)\sigma(x,t)\rangle \sim f\left(\frac{|x-vt|^z}{t}\right), \tag{1}$$

at large space and timescales. Here $\sigma(x,t)$ is a conserved fluctuation field and v is its characteristic mode velocity. The theory of the Fibonacci universality classes [9], based on mode-coupling theory for nonlinear fluctuating hydrodynamics, predicts the possibility of the coexistence of modes with different dynamical exponents from the discrete infinite set of Kepler ratios $z_k = C_{k+1}/C_k$, where C_k are the Fibonacci numbers 1, 1, 2, 3, 5, ... The ubiquitous diffusive Edwards-Wilkinson (EW) mode $k = 2$ [10] and the celebrated superdiffusive Kardar-Parisi-Zhang (KPZ) mode $k = 3$ [11] already appear in systems with one conservation law, while to reach the more exotic modes with $k \geqslant 4$ at least $k-2$ conservation laws (and hence relaxation modes) are necessary. The golden ratio $\varphi = \lim_{k\to\infty} z_k = (1+\sqrt{5})/2$ turns out to be a remarkable exception: This truly irrational dynamical exponent can be generated (1) already in a system with just two long-lived relaxation modes.

With this knowledge at hand, and using fine-tuning of interaction parameters, we were able to demonstrate the existence of the irrational critical exponent φ in driven lattice gases with two and three conservation laws [9,12]. Also, in anharmonic chains with two conservation laws, the golden universality class appears for fine-tuned parameter values [13]. However, ideally, one would like to have the means to produce a system from the remarkable golden universality class in a robust way, based on symmetries, instead of fine-tuning. Our present paper presents a systematic study of properties of the interactions needed to generate one or another universality class, including the golden ratio, for a system with two conservation laws. To emphasize that under the given assumptions some of the results are mathematically rigorous, we use a mathematical presentation in terms of propositions, theorems, and corollaries when appropriate.

II. FLUCTUATIONS IN SYSTEMS WITH LOCAL CONSERVATION LAWS

To set the stage, we describe the setting that we have in mind. In this section, we keep the number of conserved quantities arbitrary (but finite) and only point to the case of two conservation laws in some instances to facilitate the detailed discussion of two conservation laws in the subsequent sections.

A. Nonequilibrium steady state

Consider a one-dimensional many-body dynamics with translation invariant short-range interactions with n locally conserved densities ρ_α, $\alpha \in \{1, \dots, n\}$ where a driving force produces macroscopic stationary currents denoted by j^α. The resulting nonequilibrium steady state (NESS) is assumed to be unique for given values of the conserved densities, to have nonvanishing global fluctuations of the conserved densities, and exhibit a decay of local density correlations that is sufficiently fast to guarantee that the total stationary density fluctuations are finite. These properties are generic since in one space dimension, stationary correlations between local observables usually decay rapidly with distance [14]. Indeed, for a single conservation law and translation invariant short-range interactions no counter examples to these properties are known except for some very special cases, viz., in systems with long-range interactions [15–17] or with facilitated dynamics [18–22]. For two conservation laws, long-range correlations accompanied by phase separation have been reported and analyzed numerically and by mean-field theories [23–25]. All these atypical types of behavior as well as frozen systems without density fluctuations are ruled out in the present investigation.

The total stationary density fluctuations are encoded in the compressibility matrix K in terms of the variances $K_{\alpha\alpha}$ of the conserved quantities and their covariances $K_{\alpha\beta}$. Hence K is symmetric and all matrix elements are, in general, functions of all the conserved quantities ρ_α. The cross correlations given by the covariances may be positive or negative or even vanish for all pairs of densities while for the variances $K_{\alpha\alpha}$ strictly positive real numbers (excluding the atypical situations mentioned above). Specifically, for two conservation laws we denote the matrix elements of the compressibility matrix K by

$$K = \begin{pmatrix} \kappa_1 & \bar{\kappa} \\ \bar{\kappa} & \kappa_2 \end{pmatrix} \tag{2}$$

rather than by $K_{\alpha\beta}$ and we usually omit the dependence of these quantities on ρ_1 and ρ_2.

Also, the stationary currents j^α are, in general, functions of all densities ρ_α. Their derivatives with respect to the densities are denoted by subscripts j^α_β, $j^\alpha_{\beta\gamma}$, and so on. The current Jacobian J has matrix elements $J_{\alpha\beta} = j^\alpha_\beta$ and the Hessians H^α associated with the current-density relation have matrix elements $H^\alpha_{\beta\gamma} = j^\alpha_{\beta\gamma}$. Thus, for two conservation laws, the current Jacobian and the Hessians are the matrices

$$J = \begin{pmatrix} j^1_1 & j^1_2 \\ j^2_1 & j^2_2 \end{pmatrix}, \quad H^\alpha = \begin{pmatrix} j^\alpha_{11} & j^\alpha_{12} \\ j^\alpha_{21} & j^\alpha_{22} \end{pmatrix}, \tag{3}$$

with the dependence on ρ_1 and ρ_2 also usually omitted.

B. Large-scale dynamics

The long-time relaxation of the system is assumed to be determined by the long-wave-length Fourier modes of the conserved quantities in the sense that at sufficiently large times the system is locally (on a microscopic scale) in a stationary state with densities ρ_α, which vary only slowly with space and time.

The generic decay of correlations ensures bounded density fluctuations in the NESS and, thus, in driven nonequilibrium systems with nonvanishing stationary currents, the validity of a coarse-grained hydrodynamic description of the conserved densities $\rho_\alpha(x,t)$ in terms of a continuity equation $\partial_t \rho_\alpha(x,t) + \partial_x J_\alpha(x,t) = 0$, where the currents $J_\alpha(x,t)$ depend on macroscopic space and time only via the densities $\rho_1(x,t), \dots, \rho_n(x,t)$ [14,26]. Thus the hydrodynamic limit is given by a system of hyperbolic conservation laws

$$\partial_t \rho_\alpha(x,t) + \sum_{\beta=1}^{n} J_{\alpha\beta} \partial_x \rho_\beta(x,t) = 0, \tag{4}$$

where the current Jacobian $J_{\alpha\beta} = \partial_{\rho_\beta} j^\alpha(\rho_1, \dots, \rho_n)$ is given by the stationary current-density relation $j^\alpha(\rho_1, \dots, \rho_n)$ evaluated at the space-time point (x,t).

Fluctuations are not captured in this deterministic large-scale description. They are expected to be described by the theory of nonlinear fluctuating hydrodynamics [27,28] for fluctuation fields which are centered around a fixed stationary mean ρ_α of the conserved quantities. To analyze the large-scale behavior of these fluctuations, it is most convenient to express them as the eigenmodes $\sigma_\alpha(x,t)$ that appear in generic form in (1) without mode index α, i.e., as those linear combinations of the conserved fields for which the current Jacobian J is diagonal for the chosen stationary densities ρ_α. According to nonlinear fluctuating hydrodynamics, these eigenmodes then satisfy the system of coupled stochastic Burgers equations

$$\partial_t \sigma_\alpha + \partial_x \left(v_\alpha \sigma_\alpha + \sum_{\beta,\gamma=1}^{2} G^\alpha_{\beta\gamma} \sigma_\beta \sigma_\gamma - \sum_{\beta=1}^{2} D_{\alpha\beta} \partial_x \sigma_\beta + \xi_\alpha \right) = 0, \tag{5}$$

with the mode velocities v_α, which are the eigenvalues of the current Jacobian, with phenomenological diffusion matrix D and with Gaussian white noises ξ_α. The strength of the nonlinearity is given by the mode-coupling coefficients $G^\alpha_{\beta\gamma}$ which depend primarily on the current Jacobian and also—in an insubstantial way—on the static compressibilites $K_{\alpha\beta}$ (see below).

These nonlinear stochastic partial differential equations (SPDEs) can be treated by mode-coupling theory as described in Ref. [27]. With this approach, we found in Refs. [9,29] together with our collaborators the exact analytical scaling solution for the dynamical structure function (1) for all modes α. This yields the Fibonacci dynamical universality classes as follows:

(1) In the absence of the diagonal mode-coupling term, i.e., $G^\alpha_{\beta\beta} = 0$ for all modes β (including the mode $\beta = \alpha$), the mode α is diffusive and belongs to the EW universality class with dynamical exponent $z_\alpha = 2$.

(2) For the nonvanishing self-coupling term, i.e., $G^\alpha_{\alpha\alpha} \neq 0$, the mode α is superdiffusive and belongs to the KPZ universality class or a modification thereof, both with dynamical exponent $z_\alpha = 3/2$.

(3) If $G^{\alpha}_{\alpha\alpha} = 0$ but $G^{\alpha}_{\beta\beta} \neq 0$ for some mode $\beta \neq \alpha$, then mode α is in a subdiffusive Lévy universality class with the dynamical exponent z_α being a Kepler ratio of two consecutive Fibonacci numbers or the golden ratio.

(4) The nondiagonal mode-coupling terms $G^{\alpha}_{\beta\gamma}$ with $\beta \neq \gamma$ are irrelevant for these scaling properties.

The results of Refs. [9,29] for the scaling functions are valid for strict hyperbolicity of the underlying deterministic PDE (4), i.e., when the eigenvalues of the current Jacobian are nondegenerate.

It is worth stressing that the mode-coupling coefficients $G^{\alpha}_{\beta\beta}$ appearing in the macroscopic phenomenological SPDE (4) are fully given by the stationary distribution of the underlying dynamics, viz., by the current Jacobian J and by the static compressibility matrix K. In fact, whether $G^{\alpha}_{\beta\beta}$ vanishes or not is determined by the current Jacobian J alone, the compressibilities encoded in K only renormalize the amplitudes. This implies the very remarkable fact that all the *dynamical* universality classes of a system can be read from the *stationary* currents alone. An explicit example is provided in Appendix B.

Specifically for two conservation laws, treated independently in Refs. [12,13] in the same issue of the *Journal of Statistical Physics*, the elusive golden ratio occurs if and only if $G^1_{11} = G^2_{22} = 0$, $G^1_{22} \neq 0$, and $G^2_{11} \neq 0$. In this case, both modes belong to the golden universality class with $z_1 = z_2 = \varphi$. However, the question that has been left open and which is addressed here are the circumstances under which these conditions may be satisfied or not, i.e., which current-density relations that arise from some microscopic dynamics admit or forbid the occurrence of two golden modes.

C. Onsager-type current symmetry and microscopic dynamics

Before attacking this problem directly, we point out a further general consequence of the generic assumption of sufficiently rapidly decaying stationary correlations. This is the Onsager-type current symmetry,

$$JK = KJ^T, \tag{6}$$

proved rigorously first for a family of Markovian particle systems with fully factorized stationary product measure in Ref. [30] and later generally under very mild assumptions concerning the decay of stationary correlations and range of microscopic interactions for classical dynamics in Ref. [31] and subsequently for quantum systems in Refs. [32,33]. This current symmetry appears in many contexts in hydrodynamic theory, see, e.g., Refs. [27,28] for a review and Refs. [34–36] for recent applications in generalized hydrodynamics for integrable quantum systems. As pointed out in Ref. [30], the symmetry (6) can be seen as a nonequilibrium version of the Onsager reciprocity relations insofar as its validity relies only on general time-reversal properties of the time evolution and on an extremely mild assumption on the decay of correlations as discussed in the rigorous proofs of Refs. [31,33]. Indeed, the proof of (6) follows arguments analogous to those for the Onsager relations. We shall call one-dimensional physical systems with these generic properties *regular one-dimensional hydrodynamic systems*.

Remarkably and perhaps also surprisingly, this *macroscopic* current symmetry directly imposes constraints on the existence of specific *microscopic* dynamics: For a given family of invariant measures (parametrized by the conserved densities ρ_α which fix K), the only physically permissible microscopic dynamics are such that the transition rates yield currents (which fix J) compatible with (6).

To appreciate the significance of this constraint in the search for physical systems that exhibit the golden universality class (or any other specific Fibonacci universality class), one must recall that for predicting universality classes for some microscopic dynamics one needs to know which diagonal mode-coupling matrix elements $G^{\alpha}_{\beta\beta}$ vanish. This requires knowledge of the *exact* stationary current-density relation, which in turn requires knowledge of the exact invariant measure of the dynamics. However, away from thermal equilibrium, detailed balance does not hold and there is no simple recipe for obtaining the exact invariant measure for a given dynamics. Therefore, computing the invariant measure for some arbitrary (even if nicely-looking) physical dynamics is usually a hopeless endeavor.

Instead, in the search for specific universality classes one usually works in opposite direction:

(i) One starts by defining a measure with a simple structure (such as product or matrix product measures [37,38]) that allows for computing expectation values like the stationary currents in explicit form as functions of the densities, (ii) proposes a physically motivated dynamics (such as short-range interactions) that respects the conservation laws, (iii) proves invariance of the measure with respect to this dynamics and/or adjusts parameters accordingly, and (iv) computes stationary currents and compressibilities from (i) and (ii). However, there is no obvious starting point for guessing such a hypothetical dynamics (which might not even exist) and then checking invariance with respect to the measure may be cumbersome due to the absence of detailed balance. Therefore, being able to rule out *a priori* certain microscopic dynamics as unphysical for a given invariant measure can be useful in the search for dynamical universality classes.

Here the current symmetry helps. It does not provide a sufficient condition for a microscopic dynamics to have the specified measure as an invariant measure, but a necessary condition that can be used to rule out a proposed dynamics. This property can be utilized by taking step (iv) *before* step (iii): If step (iv) yields a current-density relation that is incompatible with (6), then there is no need to go through the potentially daunting task of step (iii). Unlike step (iii) (which may actually be an unsoluble problem), step (iv) is easy since the measure has been proposed in step (i) so as to facilitate the computation of the stationary currents.

To illustrate the point, we focus on two conservation laws and note that the current symmetry (6) then becomes

$$J_{21}\kappa_1 - J_{12}\kappa_2 = (J_{11} - J_{22})\bar{\kappa}. \tag{7}$$

Evidently, this relation constitutes a necessary condition for the existence of microscopic dynamics that would be compatible with a predefined invariant measure—without the need to perform an explicit check of invariance of the measure under the proposed dynamics. Only the static compressibilities and the stationary currents of the model need to be computed using the invariant measure.

This is highlighted in the following theorem for partially factorized invariant measures of the form $\pi_{\rho_1,\rho_2} = \pi^1_{\rho_1}\pi^2_{\rho_2}$, where $\pi^\alpha_{\rho_\alpha}$ are measures that do not necessarily further factorize, such as matrix product measures. Frequently studied models of this type are two-lane systems [39–41], where particles of species α jump along a one-dimensional lattice called lane α like in a system with only one conservation law, but with a rate that depends on the particle configuration on the other lane.

Theorem II.1. Any regular hydrodynamic two-component system with a partially factorized invariant measure

$$\pi_{\rho_1,\rho_2} = \pi^1_{\rho_1}\pi^2_{\rho_2} \tag{8}$$

has stationary currents of the form

$$j^1(\rho_1,\rho_2) = f_1(\rho_1) + \kappa_1(\rho_1)\int^{\rho_2} dy\, g(\rho_1, y), \tag{9}$$

$$j^2(\rho_1,\rho_2) = f_2(\rho_2) + \kappa_2(\rho_2)\int^{\rho_1} dx\, g(x, \rho_2), \tag{10}$$

with the compressibilities $\kappa_\alpha(x) \in \mathbb{R}^+$ for $x \in \mathbb{R}$, functions $f_\alpha : \mathbb{R} \to \mathbb{R}$ and interaction kernel $g(x, y) \in \mathbb{R}$ for $(x, y) \in \mathbb{R}^2$ that is the same for both currents. Moreover, for any pair of densities (ρ_1, ρ_2), the off-diagonal elements of the current Jacobian either both vanish or have equal signs.

Proof: The lower integration limits in (9) and (10) do not need to be specified as they can be absorbed into the functions $f_\alpha(\cdot)$. By writing the currents in a generic fashion as

$$j^1(\rho_1,\rho_2) = f_1(\rho_1) + \kappa_1(\rho_1)\int^{\rho_2} dy\, g_1(\rho_1, y),$$

$$j^2(\rho_1,\rho_2) = f_2(\rho_2) + \kappa_2(\rho_2)\int^{\rho_1} dx\, g_2(x, \rho_2),$$

in terms of functions $f_\alpha(\cdot)$, $g_\alpha(\cdot,\cdot)$ yields

$$J_{12}(\rho_1,\rho_2) = \kappa_1(\rho_1)g_1(\rho_1,\rho_2),$$

$$J_{21}(\rho_1,\rho_2) = \kappa_2(\rho_2)g_2(\rho_1,\rho_2).$$

For a factorized invariant measure, one has $\bar{\kappa} = \partial_1\kappa_2 = \partial_2\kappa_1 = 0$, so the current symmetry reduces to

$$J_{21}(\rho_1,\rho_2)\kappa_1(\rho_1) = J_{12}(\rho_1,\rho_2)\kappa_2(\rho_2). \tag{11}$$

Since $\kappa_\alpha(\cdot) > 0$ by assumption, this implies $g_2(\rho_1,\rho_2) = g_1(\rho_1,\rho_2)$ for all $(\rho_1,\rho_2) \in \mathbb{R}^2$ and proves also the assertion regarding the off-diagonal elements of the current Jacobian. ■

For a monotone increase or decrease of the current j^α as a function of the other density ρ_β with $\beta \neq \alpha$ as in the models of Refs. [39–41] and also assuming the existence of symmetries between the densities, the current symmetry Theorem II.1 provides the following rather general no-go corollary.

Corollary II.2. A regular one-dimensional hydrodynamic two-component system with the three properties

(i) partially factorized invariant measure $\pi_{\rho_1,\rho_2} = \pi^1_{\rho_1}\pi^2_{\rho_2}$,

(ii) antisymmetric currents $j^2(\rho_1,\rho_2) = -j^1(\rho_2,\rho_1)$,

(iii) monotonicity $j^1_2 > 0$ and $j^2_1 > 0$ (or $j^1_2 < 0$ and $j^2_1 < 0$) for all densities ρ_1, ρ_2

does not exist.

This holds since, on the one hand, monotonicity for all (ρ_1,ρ_2) requires that $g(\cdot,\cdot)$ does not change signs, while on the other hand, the antisymmetry requires $g(\rho_1,\rho_2)\kappa_2(\rho_2) = -g(\rho_2,\rho_1)\kappa_1(\rho_2)$, which is contradictory since κ_1 and κ_2 are both strictly positive. How the current symmetry rules out specific microscopic dynamics for a product measure is demonstrated for a concrete example in Appendix A.

III. DYNAMICAL UNIVERSALITY CLASSES FOR TWO CONSERVATION LAWS

In the remainder of this paper, we restrict ourselves to two conservation laws and discuss criteria for the occurrence of the golden and other universality classes. To this end, the mode-coupling coefficients $G^\alpha_{\beta\beta}$ are computed in terms of the matrix elements of K and J. Related results of Refs. [12,13] do not allow for immediate predictions from the stationary currents for a given particle system, since in these works the mode-coupling matrices are not directly expressed in term of the stationary currents. This is achieved below.

A. Eigenmodes

The system of stochastic Burgers equations (5) is expressed in terms of eigenmodes for which the current Jacobian J is diagonal. For the diagonalization of J, we use as parameters the reduced trace θ, the asymmetry ω, and the signed square root δ of the discriminant which are defined by

$$\theta := J_{11} - J_{22}, \quad \omega := \frac{J_{12}}{J_{21}}, \quad \delta := \begin{cases} \theta\sqrt{1 + \frac{4J_{12}J_{21}}{\theta^2}} & \theta \neq 0 \\ \sqrt{4J_{12}J_{21}} & \theta = 0. \end{cases} \tag{12}$$

Only current Jacobians with nondegenerate eigenvalues,

$$\lambda_\pm = \tfrac{1}{2}(J_{11} + J_{22} \pm \delta), \tag{13}$$

i.e., with no-zero discriminant, are considered below which ensures a strictly hyperbolic conservation law (4).

The following proposition concerning the diagonalization of a 2×2 matrix is high school math. It serves to introduce a notation for the eigenmodes and some of their basic properties.

Proposition III.1. Let j^α be the stationary currents of a two-component system satisfying the current symmetry (6) with compressibility matrix K (2) with strictly positive diagonal elements and current Jacobian J with nonvanishing discriminant. With the functions

$$u_+ := \frac{\delta - \theta}{2J_{21}}, \quad u_- := -\frac{u_+}{\omega}, \quad v := 1 - u_+u_-, \tag{14}$$

the normalization factors

$$z_+^2 := \kappa_1 + 2u_+\bar{\kappa} + u_+^2\kappa_2, \quad z_-^2 := u_-^2\kappa_1 + 2u_-\bar{\kappa} + \kappa_2, \tag{15}$$

and the matrix elements

$$u_1^+ := \frac{1}{z_+}, \quad u_2^+ := \frac{1}{z_+}u_+, \quad u_1^- := \frac{1}{z_-}u_-, \quad u_2^- := \frac{1}{z_-} \tag{16}$$

$$v_1^+ := \frac{z_+}{v}, \quad v_2^+ := -\frac{z_+}{v}u_-, \quad v_1^- := -\frac{z_-}{v}u_+, \quad v_2^- := \frac{z_-}{v}, \tag{17}$$

the matrices

$$R := \begin{pmatrix} u_1^+ & u_2^+ \\ u_1^- & u_2^- \end{pmatrix}, \quad R^{-1} := \begin{pmatrix} v_1^+ & v_1^- \\ v_2^+ & v_2^- \end{pmatrix} \tag{18}$$

satisfy

$$\mathbb{1} = RR^{-1} = RKR^T, \quad RJR^{-1} = \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix}, \tag{19}$$

where $\mathbb{1}$ is the two-dimensional unit matrix.

Proof: The matrices R, R^{-1} read in terms of the quantities $z_\pm$, $u_\pm$, and v:

$$R = \begin{pmatrix} z_+^{-1} & z_+^{-1}u_+ \\ z_-^{-1}u_- & z_-^{-1} \end{pmatrix}, \quad R^{-1} = v^{-1}\begin{pmatrix} z_+ & -z_-u_+ \\ -z_+u_- & z_- \end{pmatrix}.$$

The definition (14) of v immediately yields the first equality in (19). By the definitions (15) of the normalization factors, the diagonal elements of RKR^T are equal to 1. The function u_+ satisfies the quadratic equation

$$J_{21}u_+ - J_{12}u_+^{-1} + \theta = 0,$$

which implies the two equalities:

$$1 + u_+u_- = \frac{\theta u_+}{J_{12}}, \quad 1 - u_+u_- = \frac{J_{12}}{\delta u_+}.$$

The first equality together with the current symmetry (37) yields $(RKR^T)_{12} = (RKR^T)_{21} = 0$. The second equality proves the diagonalization of J. ■

Corollary III.2. The column and row vectors

$$\vec{v}_\pm := \begin{pmatrix} v_1^\pm \\ v_2^\pm \end{pmatrix}, \quad \vec{u}_\pm := (u_1^\pm,\, u_2^\pm)$$

are right and left eigenvectors of J with eigenvalues $\lambda_\pm$ and they satisfy the biorthogonality relation

$$\vec{u}_s \cdot \vec{v}_{s'} := u_1^s v_1^{s'} + u_2^s v_2^{s'} = \delta_{s,s'}$$

for $s, s' \in \{+, -\}$.

Since $\delta > 0$ by assumption, the current Jacobian is diagonalizable for all permissible parameter values. In particular, the tridiagonal case $J_{12}J_{21} = 0$ is covered by the proposition by taking appropriate limits: For $J_{12} = 0$, one gets $\delta = \theta$, $u_+ = 0$, $u_- = -J_{21}/\theta$, while for $J_{21} = 0$ one has $\delta = -\theta$, $u_+ = -J_{12}/\theta$, $u_- = 0$.

B. Mode-coupling matrices

To detail the conditions that lead to the various dynamical universality classes predicted by mode-coupling theory, we express in explicit form the diagonal elements of the mode-coupling matrices in terms of the stationary currents and compressibilities.

Definition III.3. The mode-coupling matrices G^γ are the linear combinations

$$G^\gamma := \frac{1}{2}\sum_\lambda R_{\gamma\lambda}(R^{-1})^T H^\lambda R^{-1} \tag{20}$$

of transformed Hessians H^λ.

They satisfy $G^\gamma_{\alpha\beta} = G^\gamma_{\beta\alpha}$ since the Hessians H^λ are symmetric by construction. It is also possible to express the Hessians in terms of the mode-coupling matrices; see Appendix C.

According to mode-coupling theory, only the diagonal elements of the mode-coupling matrices determine the large-scale behavior of the fluctuations. Hence, we ignore the off-diagonal elements and also recall that the normalization factors $z_\pm$ of the eigenvectors are nonzero for any permissible choice of parameters. Therefore, the ratios

$$g := \frac{\sqrt{z_+z_-}}{v^2}, \quad y := \sqrt{\frac{z_+}{z_-}} \tag{21}$$

are nonzero and well-defined. The theorem below expresses the diagonal elements $G^\alpha_{\beta\beta}$ of the mode-coupling matrices in terms of the stationary currents and compressibilities. We stress that the compressibilities enter only the nonzero overall amplitude given in terms of g and y. Whether a matrix element vanishes or not is solely determined by the current-density relation via the Hessians and the unnormalized eigenvector components of the current Jacobian.

Theorem III.4. The diagonal elements of the mode-coupling matrices are given by

$$G^1_{11} = \frac{g}{2}y\big[H^1_{11} + u_+H^2_{11} - 2u_-\big(H^1_{12} + u_+H^2_{12}\big) + u_-^2\big(H^1_{22} + u_+H^2_{22}\big)\big], \tag{22}$$

$$G^1_{22} = \frac{g}{2}y^{-3}\big[u_+^2\big(H^1_{11} + u_+H^2_{11}\big) - 2u_+\big(H^1_{12} + u_+H^2_{12}\big) + H^1_{22} + u_+H^2_{22}\big], \tag{23}$$

$$G^2_{11} = \frac{g}{2}y^3\big[u_-H^1_{11} + H^2_{11} - 2u_-\big(u_-H^1_{12} + H^2_{12}\big) + u_-^2\big(u_-H^1_{22} + H^2_{22}\big)\big], \tag{24}$$

$$G^2_{22} = \frac{g}{2}y^{-1}\big[u_+^2\big(u_-H^1_{11} + H^2_{11}\big) - 2u_+\big(u_-H^1_{12} + H^2_{12}\big) + u_-H^1_{22} + H^2_{22}\big]. \tag{25}$$

Proof: To compute G^γ, we note that with the diagonal normalization matrix

$$Z = \begin{pmatrix} z_+ & 0 \\ 0 & z_- \end{pmatrix}$$

and the rescaled diagonalization matrices

$$\tilde{R} = \begin{pmatrix} 1 & u_+ \\ u_- & 1 \end{pmatrix}, \quad \tilde{R}^{-1} = v^{-1}\begin{pmatrix} 1 & -u_+ \\ -u_- & 1 \end{pmatrix},$$

one has $R = Z^{-1}\tilde{R}$. Furthermore, defining

$$S := v\tilde{R}^{-1} = \begin{pmatrix} 1 & -u_+ \\ -u_- & 1 \end{pmatrix}$$

yields the modified mode-coupling matrix

$$\tilde{G}^\gamma := 2v^2Z^{-1}G^\gamma Z^{-1} = \sum_\lambda R_{\gamma\lambda}S^T H^\lambda S,$$

with matrix elements

$$\tilde{G}^\gamma_{11} = \frac{2v^2}{z_+^2}G^\gamma_{11}, \quad \tilde{G}^\gamma_{22} = \frac{2v^2}{z_-^2}G^\gamma_{22},$$

$$\tilde{G}^\gamma_{12} = \frac{2v^2}{z_+z_-}G^\gamma_{12}, \quad \tilde{G}^\gamma_{21} = \frac{2v^2}{z_+z_-}G^\gamma_{21}.$$

The matrix multiplication can now be performed in a straightforward fashion and yields the matrix elements

$$(S^T H^\lambda S)_{11} = H^\lambda_{11} - 2u_- H^\lambda_{12} + u_-^2 H^\lambda_{22},$$
$$(S^T H^\lambda S)_{12} = (1 + u_+ u_-) H^\lambda_{12} - u_+ H^\lambda_{11} - u_- H^\lambda_{22},$$
$$(S^T H^\lambda S)_{22} = H^\lambda_{22} - 2u_+ H^\lambda_{12} + u_+^2 H^\lambda_{11},$$

and $(S^T H^\lambda S)_{21} = (S^T H^\lambda S)_{12}$ by symmetry. Using the definition (20) therefore gives

$$G^1_{11} = \frac{z_+}{2v^2}\big[H^1_{11} - 2u_- H^1_{12} + u_-^2 H^1_{22} + u_+\big(H^2_{11} - 2u_- H^2_{12} + u_-^2 H^2_{22}\big)\big],$$
$$G^1_{22} = \frac{z_-^2}{2v^2 z_+}\big[H^1_{22} - 2u_+ H^1_{12} + u_+^2 H^1_{11} + u_+\big(H^2_{22} - 2u_+ H^2_{12} + u_+^2 H^2_{11}\big)\big],$$
$$G^2_{11} = \frac{z_+^2}{2v^2 z_-}\big[u_-\big(H^1_{11} - 2u_- H^1_{12} + u_-^2 H^1_{22}\big) + H^2_{11} - 2u_- H^2_{12} + u_-^2 H^2_{22}\big],$$
$$G^2_{22} = \frac{z_-}{2v^2}\big[u_-\big(H^1_{22} - 2u_+ H^1_{12} + u_+^2 H^1_{11}\big) + H^2_{22} - 2u_+ H^2_{12} + u_+^2 H^2_{11}\big].$$

Regrouping terms yield (22)–(25). ■

C. Scenarios for dynamical universality classes

Generically, i.e., for two vanishing self-coupling coefficients $G^1_{11} \neq 0$ and $G^2_{22} \neq 0$, mode-coupling theory predicts two KPZ modes. To search for the golden universality class, we consider the mode-coupling scenarios with less than two KPZ modes in more detail. We note that $u_+ u_- \neq 1$ since $v = 1 - u_+ u_- \neq 0$ by construction. It follows that $u_-^{-1} \neq u_+$.

1. One vanishing self-coupling coefficient

The following cases are permitted by mode-coupling theory:

(i) Mode 1 is KPZ and mode 2 is 5/3-Lévy: This requires $G^1_{11} \neq 0$, $G^2_{22} = 0$, $G^2_{11} \neq 0$, with arbitrary G^1_{22}. Thus,

$$2\big(H^1_{12} + u_+ H^2_{12}\big) \neq u_-^{-1}\big(H^1_{11} + u_+ H^2_{11}\big) + u_-\big(H^1_{22} + u_+ H^2_{22}\big),$$
$$0 \neq \omega\big(u_- H^1_{11} + H^2_{11}\big) + \big(u_- H^1_{22} + H^2_{22}\big),$$
$$2\big(u_- H^1_{12} + H^2_{12}\big) = u_+\big(u_- H^1_{11} + H^2_{11}\big) + u_+^{-1}\big(u_- H^1_{22} + H^2_{22}\big).$$

(ii) Mode 1 is KPZ and mode 2 is diffusive: This requires $G^1_{11} \neq 0$, $G^1_{22} = G^2_{22} = G^2_{11} = 0$. Thus,

$$H^1_{22} + u_+ H^2_{22} \neq -\omega\big(H^1_{11} + u_+ H^2_{11}\big),$$
$$2\big(H^1_{12} + u_+ H^2_{12}\big) = u_+\big(H^1_{11} + u_+ H^2_{11}\big) + u_+^{-1}\big(H^1_{22} + u_+ H^2_{22}\big),$$
$$u_- H^1_{22} + H^2_{22} = -\omega\big(u_- H^1_{11} + H^2_{11}\big),$$
$$2\big(u_- H^1_{12} + H^2_{12}\big) = \big(u_+ + u_-^{-1}\big)\big(u_- H^1_{11} + H^2_{11}\big).$$

(iii) Mode 1 is modified KPZ and mode 2 is diffusive: This requires $G^1_{11} \neq 0$, $G^1_{22} \neq 0$, and $G^2_{22} = G^2_{11} = 0$. Thus,

$$2\big(H^1_{12} + u_+ H^2_{12}\big) \neq u_-^{-1}\big(H^1_{11} + u_+ H^2_{11}\big) + u_-\big(H^1_{22} + u_+ H^2_{22}\big),$$
$$2\big(H^1_{12} + u_+ H^2_{12}\big) \neq u_+\big(H^1_{11} + u_+ H^2_{11}\big) + u_+^{-1}\big(H^1_{22} + u_+ H^2_{22}\big),$$
$$u_- H^1_{22} + H^2_{22} = -\omega\big(u_- H^1_{11} + H^2_{11}\big),$$
$$2\big(u_- H^1_{12} + H^2_{12}\big) = \big(u_+ + u_-^{-1}\big)\big(u_- H^1_{11} + H^2_{11}\big).$$

2. Two vanishing self-coupling coefficients

This requires $G^2_{22} = G^1_{11} = 0$ and therefore

$$2\big(H^1_{12} + u_+ H^2_{12}\big) = u_-^{-1}\big(H^1_{11} + u_+ H^2_{11}\big) + u_-\big(H^1_{22} + u_+ H^2_{22}\big), \tag{26}$$

$$2\big(u_- H^1_{12} + H^2_{12}\big) = u_+\big(u_- H^1_{11} + H^2_{11}\big) + u_+^{-1}\big(u_- H^1_{22} + H^2_{22}\big). \tag{27}$$

The remaining diagonal mode coupling coefficients take the form

$$G^1_{22} = \frac{z_-^2}{2v z_+}\big[\omega\big(H^1_{11} + u_+ H^2_{11}\big) + \big(H^1_{22} + u_+ H^2_{22}\big)\big], \tag{28}$$
$$G^2_{11} = \frac{z_+^2}{2v z_- \omega}\big[\omega\big(u_- H^1_{11} + H^2_{11}\big) + \big(u_- H^1_{22} + H^2_{22}\big)\big]. \tag{29}$$

(i) Two modes are golden Lévy:

$$0 \neq \omega\big(H^1_{11} + u_+ H^2_{11}\big) + \big(H^1_{22} + u_+ H^2_{22}\big), \tag{30}$$
$$0 \neq \omega\big(u_- H^1_{11} + H^2_{11}\big) + \big(u_- H^1_{22} + H^2_{22}\big). \tag{31}$$

(ii) Mode 1 is 3/2-Lévy and mode 2 is EW:

$$0 \neq \omega\big(H^1_{11} + u_+ H^2_{11}\big) + \big(H^1_{22} + u_+ H^2_{22}\big), \tag{32}$$
$$0 = \omega\big(u_- H^1_{11} + H^2_{11}\big) + \big(u_- H^1_{22} + H^2_{22}\big). \tag{33}$$

(iii) Two modes are EW:

$$0 = \omega\big(H^1_{11} + u_+ H^2_{11}\big) + \big(H^1_{22} + u_+ H^2_{22}\big), \tag{34}$$
$$0 = \omega\big(u_- H^1_{11} + H^2_{11}\big) + \big(u_- H^1_{22} + H^2_{22}\big). \tag{35}$$

We conclude that the conditions (26), (27), (30), and (31) must be met for the occurrence of the golden ratio. In this case, both modes are golden Lévy.

Remark III..5. The current symmetry imposes the constraints

$$H^1_{12}\det(K) = H^2_{11}\kappa_1^2 - H^1_{11}\kappa_1\bar\kappa + H^1_{22}\bar\kappa\kappa_2 - H^2_{22}\bar\kappa^2 + J_{21}(\kappa_1\partial_1 - \bar\kappa\partial_2)\kappa_1 - J_{12}(\kappa_1\partial_1 - \bar\kappa\partial_2)\kappa_2 - (J_{11} - J_{22})(\kappa_1\partial_1 - \bar\kappa\partial_2)\bar\kappa \tag{36}$$

and

$$H^2_{12}\det(K) = H^2_{11}\bar\kappa\kappa_1 - H^1_{11}\bar\kappa^2 + H^1_{22}\kappa_2^2 - H^2_{22}\bar\kappa\kappa_2 + J_{12}(\kappa_2\partial_2 - \bar\kappa\partial_1)\kappa_2 - J_{21}(\kappa_2\partial_2 - \bar\kappa\partial_1)\kappa_1 + (J_{11} - J_{22})(\kappa_2\partial_2 - \bar\kappa\partial_1)\bar\kappa \tag{37}$$

on the matrix elements of the Hessians. Hence, the diagonal elements (22)–(25) of the mode-coupling matrix and therefore the mode scenarios discussed above can be expressed entirely in terms of J, K, and the diagonal elements of the Hessians rather than in terms of J, K, and the full Hessians.

IV. SYMMETRIC OR ANTISYMMETRIC CURRENTS AT EQUAL DENSITIES

The golden universality class has been observed in Refs. [12,13] on a special parameter manifold that requires fine-tuning of densities and/or interaction parameters. We investigate here whether it can occur in more generic circumstances, viz., for the symmetry property of the current-density relation $j^2(\rho_1, \rho_2) = \pm j^1(\rho_2, \rho_1)$ at the point of equal densities $\rho_1 = \rho_2$, where

$$J_{11} = \pm J_{22}, \tag{38}$$

$$J_{12} = \pm J_{21}. \tag{39}$$

These properties of J are sufficient to show which mode-coupling coefficients vanish.

A. Antisymmetric case

In the antisymmetric case, one has for equal densities $\theta \neq 0$ and $\omega = -1$, which gives $u_- = u_+$ and $\lambda_- = -\lambda_+$. The current symmetry implies $\bar{\kappa} \neq 0$, which means that a microscopic physical dynamics with a partially factorized invariant measure of the form (8) and antisymmetric current-density-relation at equal densities does not exist.

However, nonvanishing cross correlations between the conserved densities are not ruled out. The Hessians then have the symmetry relations

$$H^2_{11} = -H^1_{22}, \quad H^2_{12} = -H^1_{12}, \quad H^2_{22} = -H^1_{11}.$$

This yields the mode-coupling coefficients:

$$\begin{aligned}
G^1_{11} &= \frac{u_+ z_+}{2v^2}\big[u_+^{-1}(1-u_+^3)H^1_{11} \\
&\quad - 2(1-u_+)H^1_{12} - (1-u_+)H^1_{22}\big],\\
G^1_{22} &= \frac{u_+ z_-^2}{2v^2 z_+}\big[-(1-u_+)H^1_{11} - 2(1-u_+)H^1_{12} \\
&\quad + u_+^{-1}(1-u_+^3)H^1_{22}\big],\\
G^2_{11} &= -\left(\frac{z_+}{z_-}\right)^3 G^1_{22},\\
G^2_{22} &= -\frac{z_-}{z_+} G^1_{11}.
\end{aligned}$$

Hence, the following scenarios are possible:

(i) two KPZ modes: $u_+ \neq 1$ and

$$u_+\big(H^1_{22} + 2H^1_{12}\big) \neq (1 + u_+ + u_+^2)H^1_{11}, \tag{40}$$

(ii) two golden modes: $u_+ \neq 1$ and

$$u_+\big(H^1_{22} + 2H^1_{12}\big) = (1 + u_+ + u_+^2)H^1_{11}, \quad \textit{and} \tag{41}$$

$$H^1_{11} \neq H^1_{22}. \tag{42}$$

(iii) two EW modes: All diagonal mode-coupling coefficients vanish if either $u_+ = 1$ or

$$2u_+ H^1_{12} = (1 + u_+^2)H^1_{11}, \tag{43}$$

$$H^1_{11} = H^1_{22}. \tag{44}$$

In all allowed cases, both modes are the same. These four cases correspond to the diagonal of the mode-coupling table of Ref. [12].

B. Symmetric case

In the symmetric case, the properties of the Jacobian correspond to $\theta = 0$ and $\omega = 1$ and therefore $u_\pm = \pm 1$. The nondegeneracy of the eigenvalues $\lambda_\pm$ implies $J_{12}J_{21} \neq 0$. There is no conflict with the current symmetry. The Hessians have the symmetry relations

$$H^2_{11} = H^1_{22}, \quad H^2_{12} = H^1_{12}, \quad H^2_{22} = H^1_{11}.$$

From (22)–(25), one gets

$$\begin{aligned}
G^1_{11} &= \frac{1}{4(z_+^{-1})}\big[H^1_{11} + H^1_{22} + 2H^1_{12}\big],\\
G^1_{22} &= \frac{(z_+^{-1})}{4(z_-^{-1})^2}\big[H^1_{11} + H^1_{22} - 2H^1_{12}\big],\\
G^2_{11} &= G^2_{22} = 0.
\end{aligned}$$

Hence, in all three cases, mode 2 is diffusive, thus ruling out the golden mode. Mode 1 can be KPZ, mKPZ, 3/2L, or EW. These four cases correspond to the bottom row (rightmost column) of the mode-coupling table of Ref. [12].

V. CONCLUSIONS

So far, the golden universality class for fluctuations in driven systems with two conservation laws has remained somewhat elusive. It was only found by fine-tuning the conserved densities or interaction parameters which determine the form of the stationary current-density relations [12,13]. In the quest for a natural occurrence that does not require fine-tuning, we have shown that systems with currents that are symmetric at equal densities cannot exhibit golden modes. In fact, in this case at least one of the two modes is diffusive with dynamical exponent $z = 2$ and belongs to the EW universality whereas the other mode is KPZ, modified KPZ or 3/2-Lévy with dynamical exponent $z = 3/2$ in each case.

However, the golden modes can appear with systems with two conservation laws if the currents are antisymmetric at equal densities. In this case, both modes belong to the same universality class, which may be KPZ, EW, or golden. However, such systems cannot have invariant measures that factorize over the two conserved quantities. This property may explain the rarity of analytical and simulation results for the golden modes, as one often considers such systems to make them amenable to explicit computations. On the other hand, from a physical perspective such a factorization is highly exceptional which gives hope that the golden universality class might be less unusual, as it appears at the present state of the art.

The nonexistence of physical systems with antisymmetric currents at equal densities and an invariant measure that factorizes over the conserved quantities derives from the current symmetry that arises like Onsager's reciprocity relations from time reversal. Thus, the current symmetry is not only of fundamental significance but also helps rule out microscopic dynamics in the search for other modes without having to check whether an invariant measure that is amenable to exact computation of the current is invariant for a candidate microscopic dynamics. Potential applications include two-lane versions of zero-range processes [39] or KLS models [42–45], which are exclusion processes with next-nearest-neighbor interactions with the invariant measure of a one-dimensional Ising model for each lane.

Note added. Recently, P. Gonçalves mentioned to us that the correctness of the scale factor C_0 was communicated to her by H. Spohn some years ago. Since we are not aware of a published version of this observation, we decided to keep it in this paper, without claiming originality.

ACKNOWLEDGMENTS

It is a pleasure to acknowledge stimulating discussions with P. Gonçalves. This work is financially supported by FCT/Portugal through CAMGSD, IST-ID, Projects No. UIDB/04459/2020 and No. UIDB/04459/2020 and by the FCT Grant No. 2022.09232.PTDC. V.P. acknowledges financial support from the Deutsche Forschungsgemeinschaft through DFG project KL No. 645/20-2, and thanks IST Lisbon for hospitality and support.

APPENDIX A: CURRENT SYMMETRY AND CONSTRAINTS ON THE MICROSCOPIC DYNAMICS

We illustrate how the current symmetry rules out microscopic dynamics for some two-lane exclusion processes. These driven lattice gases are defined on two one-dimensional lattices (that we call lanes), that are indexed by $\alpha \in \{1, 2\}$ and have L sites each. We view the two lanes as being arranged in parallel, with a pair of sites on the two lanes labeled by an integer k.

Particles jump along each lane (but not to the other) obeying the exclusion rule, i.e., a jump attempt fails if the target site is already occupied. Thus, the state of each lattice site k is described by a pair of occupation numbers $n_k^\alpha \in \{0, 1\}$, indicating whether lane α is occupied by a particle or not.

To be definite, we look for jump dynamics such that the NESS has a particularly simple form given by an invariant measure that is a product measure factorizing over all sites of the two lattices, i.e.,

$$\pi_{\rho_1,\rho_2}(n_1^1, \dots, n_L^1, n_1^2, \dots, n_L^2) = \prod_{\alpha=1}^{2}\prod_{k=1}^{L}\left[(1-\rho_\alpha)\left(1-n_k^\alpha\right)+\rho_\alpha n_k^\alpha\right], \tag{A1}$$

with average density ρ_α on each lane, compressibilities $\kappa_\alpha = \rho_\alpha(1-\rho_\alpha)$, and no correlations between different sites, which implies $\bar{\kappa} = 0$. Well-studied examples for single-lane dynamics with these property include the paradigmatic asymmetric simple exclusion process [46–48] and the k-step exclusion processes with long-range jumps [49,50], to mention just a few.

We denote the stationary currents arising from these dynamics in the absence of any coupling between the lanes as $j_0^\alpha(\rho_\alpha)$. It is well-known that fluctuations in such systems generically belong to the KPZ universality class [11]. To study nontrivial coupled systems that may exhibit other fluctuation patterns, we allow the transition rates in each lane to depend on the occupation numbers of the other lane and ask whether such models with the invariant product measure (A1) can exist.

To be specific, we consider three cases:

(i) Totally asymmetric dynamics with equal dynamics on both lanes such that the rate for a forward jump from site k to a site $k+r$ on lane 1 is enhanced by the number of particles on sites k and $k+r$ on lane 2 and the rate of a backward jump from site k to a site $k-r$ on lane 2 is enhanced by the number of particles on sites k and $k-r$.

(ii) Totally asymmetric dynamics with equal dynamics on both lanes but opposite directionality such that the rate for a forward jump from site k to a site $k+r$ on lane 1 is enhanced by the number of particles on sites k and $k+r$ on lane 2 and the rate of a backward jump from site k to a site $k-r$ on lane 2 is enhanced by the number of particles on sites k and $k-r$.

(iii) Facilitated totally asymmetric dynamics, with parallel directionality on both lanes and rates such that a forward jump from site k to a site $k+r$ on lane α is enhanced by a factor b_α when a site k' is occupied on both lanes (and k' different from k and the target site $k+r$).

In case (i), the factorization of the invariant measure yields the exact currents

$$\begin{aligned} j^1(\rho_1, \rho_2) &= j_0(\rho_1)(1+2b_1\rho_2), \\ j^2(\rho_1, \rho_2) &= j_0(\rho_2)(1+2b_2\rho_1), \end{aligned} \tag{A2}$$

and therefore $J_{12}(\rho_1, \rho_2) = 2b_1 j_0^1(\rho_1)$ and $J_{21}(\rho_1, \rho_2) = 2b_2 j_0^2(\rho_2)$. For such dynamics, the current symmetry (11) requires

$$b_2 j_0(\rho_2)\rho_1(1-\rho_1) = b_1 j_0(\rho_1)\rho_2(1-\rho_2). \tag{A3}$$

Therefore, such dynamics exist if and only if $b_1 = b_2$ and $j_0(x) = cx(1-x)$ as in the conventional TASEP. For $r = 1$ this is the two-lane model of [40]. One still needs to prove that for $b_1 = b_2$ the product measure (A1) is invariant (as in Ref. [40]) but there is no reason to check invariance in the more general case $b_1 \neq b_2$: The macroscopic current symmetry asserts that such a microscopic model does not exist.

In case (ii), the factorization of the invariant measure yields the exact currents

$$\begin{aligned} j^1(\rho_1, \rho_2) &= j_0(\rho_1)(1+2b_1\rho_2), \\ j^2(\rho_1, \rho_2) &= -j_0(\rho_2)(1+2b_2\rho_1), \end{aligned} \tag{A4}$$

and therefore $J_{12}(\rho_1, \rho_2) = 2b_1 j_0^1(\rho_1)$ and $J_{21}(\rho_1, \rho_2) = -2b_2 j_0^2(\rho_2)$. For such dynamics to exist, the current symmetry (11) requires

$$b_2 j_0(\rho_2)\rho_1(1-\rho_1) = -b_1 j_0(\rho_1)\rho_2(1-\rho_2). \tag{A5}$$

Therefore, such dynamics exist if and only if $b_1 = -b_2$ and $j_0(x) = cx(1-x)$, as in the conventional TASEP. Hence, a model with current enhancement on both lanes due to the presence of particles on the other lane need not be considered. The macroscopic current symmetry asserts that such a microscopic model does not exist.

In case (iii), the factorization of the invariant measure yields the exact currents:

$$j^1(\rho_1, \rho_2) = j_0^1(\rho_1)(1 + b_1\rho_1\rho_2),$$
$$j^2(\rho_1, \rho_2) = j_0^2(\rho_2)(1 + b_2\rho_1\rho_2). \quad \text{(A6)}$$

Thus, $J_{12}(\rho_1, \rho_2) = b_1\rho_1 j_0^1(\rho_1)$ and $J_{21}(\rho_1, \rho_2) = b_2\rho_2 j_0^2(\rho_2)$. For such dynamics to exist, the current symmetry (11) requires

$$b_2 j_0^2(\rho_2)(1 - \rho_1) = b_1 j_0^1(\rho_1)(1 - \rho_2). \quad \text{(A7)}$$

This is valid for all (ρ_1, ρ_2) if and only if $b_1 = b_2 = 0$, i.e., no such coupled facilitated dynamics for which the fully factorized measure is invariant exists.

APPENDIX B: ON THE RELIABILITY OF MODE-COUPLING THEORY FOR NONLINEAR FLUCTUATING HYDRODYNAMICS

Consider an infinite chain of particles of unit mass at positions q_j interacting through a pair potential $V(r_j)$, where $r_j = q_j - q_{j-1}$ is the interparticle distance. With the momentum p_j of the particles, the total energy is given by

$$E = \sum_{j\in\mathbb{Z}} \left[\frac{1}{2}p_j^2 + V(r_j)\right], \quad \text{(B1)}$$

and the deterministic Hamiltonian dynamics for such a system are given by

$$\frac{\mathrm{d}}{\mathrm{d}t} r_j(t) = p_j(t) - p_{j-1}(t), \quad \text{(B2)}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} p_j(t) = V'(r_{j+1}(t)) - V'(r_j(t)). \quad \text{(B3)}$$

Evidently, besides the energy E also the total momentum P and volume V, i.e., the quantities

$$P = \sum_{j\in\mathbb{Z}} p_j, \quad V = \sum_{j\in\mathbb{Z}} r_j \quad \text{(B4)}$$

are conserved under the dynamics.

For the existence of such dynamics for general potentials, see Ref. [51]. In the special case of harmonic oscillators with $V(x) = x^2/2$, the system is completely integrable. This is most easily seen by introducing the variables

$$\eta_{2j-1}(t) := r_j(t), \quad \eta_{2j}(t) := p_j(t). \quad \text{(B5)}$$

Then the two coupled Eqs. (B2) and (B3) can be expressed as the single linear equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\eta_j(t) = \eta_{j+1}(t) - \eta_{j-1}(t), \quad \text{(B6)}$$

which is explicitly solvable by Fourier transformation.

To introduce noise, an exchange of the variables $\eta_j(t), \eta_{j+1}(t)$ at exponential random times was introduced in Ref. [52] for more general potentials and studied in detail for the harmonic case in Ref. [53]. This random exchange violates momentum conservation but leaves the total energy E and the pseudovolume W, expressed in terms of the variables η_j as

$$E = \sum_{j\in\mathbb{Z}} \eta_j^2, \quad W = \sum_{j\in\mathbb{Z}} \eta_j, \quad \text{(B7)}$$

conserved.

To write the generator of the stochastic dynamics for this randomized chain of harmonic oscillators, we denote by $\boldsymbol{\eta} := (\dots, \eta_{j-1}, \eta_j, \eta_{j+1}, \dots)$ the infinite set of variables $\eta_j \in \mathbb{R}$ and introduce the swapped state variable $\boldsymbol{\eta}^{kk+1}$ by the local state variables

$$\eta_l^{kk+1} = \eta_l + (\eta_{l+1} - \eta_l)\delta_{l,k} + (\eta_{l-1} - \eta_l)\delta_{l,k+1}, \quad \text{(B8)}$$

which the state reached from $\boldsymbol{\eta}$ after an exchange of η_k and η_{k+1}. The deterministic part of the generator that yields the Hamiltonian dynamics (B6) is denoted by $\mathcal{A}$ and the generator for the random exchange is denoted by $\mathcal{S}$. The full generator $\mathcal{L}$ acting on measurable functions $f(\cdot)$ is then given by $\mathcal{L} = \mathcal{S} + \mathcal{A}$ with

$$\mathcal{S}f(\eta) = \sum_{k\in\mathbb{Z}} [f(\eta^{kk+1}) - f(\eta)],$$

$$\mathcal{A}f(\eta) = \sum_{k\in\mathbb{Z}} (\eta_{k+1} - \eta_{k-1})\frac{\partial}{\partial\eta_k} f(\eta). \quad \text{(B9)}$$

The invariant measure of the process is a product measure

$$\mu(\boldsymbol{\eta}) = \prod_{k\in\mathbb{Z}} I(\eta_k), \quad \text{(B10)}$$

with marginals

$$I(x) = \sqrt{\frac{\beta}{2\pi}}\mathrm{e}^{-\frac{\beta}{2}(x-\rho)^2}. \quad \text{(B11)}$$

The quantities β and ρ are free parameters that can be expressed in terms of the densities of the conserved quantities as

$$\rho_1 := \langle\, \eta_k \,\rangle = \rho, \quad \rho_2 := \langle\, \eta_k^2 \,\rangle = \rho^2 + \frac{1}{\beta}. \quad \text{(B12)}$$

The static compressibilities are given by

$$\kappa_{11} := \langle\, (\eta_k - \rho_1)W \,\rangle = \langle\, \eta_k^2 \,\rangle - \langle\, \eta_k \,\rangle^2$$
$$= \frac{1}{\beta} = \rho_2 - \rho_1^2, \quad \text{(B13)}$$

$$\kappa_{12} := \langle\, (\eta_k - \rho_1)E \,\rangle = \langle\, \eta_k^3 \,\rangle - \rho_1\rho_2$$
$$= 2\rho_1(\rho_2 - \rho_1^2), \quad \text{(B14)}$$

$$\kappa_{22} := \langle\, (\eta_k^2 - \rho_2)E \,\rangle = \langle\, \eta_k^4 \,\rangle - \rho_2^2$$
$$= 2(\rho_2 - \rho_1^2)(\rho_1^2 + \rho_2). \quad \text{(B15)}$$

Hence, the compressibilty matrix reads

$$K = (\rho_2 - \rho_1^2)\begin{pmatrix} 1 & 2\rho_1 \\ 2\rho_1 & 2(\rho_1^2 + \rho_2) \end{pmatrix}. \quad \text{(B16)}$$

From the action of the generator on the locally conserved quantities, one obtains the discrete continuity equations

$$\mathcal{L}\eta_k = 2\eta_{k+1} - 2\eta_k = j_k^1 - j_{k+1}^1, \quad \text{(B17)}$$

$$\mathcal{L}\eta_k^2 = \eta_{k+1}^2 + \eta_{k-1}^2 - 2\eta_k^2 + 2(\eta_{k+1} - \eta_{k-1})\eta_k = j_k^2 - j_{k+1}^2, \quad \text{(B18)}$$

with the instantaneous currents

$$j_k^1 = -2\eta_k, \quad j_k^2 = \eta_{k-1}^2 - \eta_k^2 - 2\eta_{k-1}\eta_k. \quad \text{(B19)}$$

The stationary currents are given by

$$j^1 := \langle j_k^v \rangle = -2\rho_1, \quad j^2 := \langle j_k^e \rangle = -2\rho_1^2, \tag{B20}$$

which yields the current Jacobian:

$$J = -2\begin{pmatrix} 1 & 0 \\ 2\rho_1 & 0 \end{pmatrix}. \tag{B21}$$

Notice that the action of $\mathcal{L}$ on η_k is linear and therefore the time evolution of $\rho_k(t) := \langle \eta_k(t) \rangle$ can be integrated by Fourier transformation.

Using the results derived above, these stationary currents yield the mode velocities

$$v_1 = -2, \quad v_2 = 0, \tag{B22}$$

and the mode-coupling matrices

$$G^{(1)} = 0, \tag{B23}$$

$$G^{(2)} = -\sqrt{2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{B24}$$

Mode-coupling theory then predicts that mode 1 is diffusive and mode 2 belongs to the 3/2-Lévy universality class.

To compute the scaling form of the dynamical structure function [9,12,13], one needs the diffusion coefficient of mode 1. This can be computed directly and rigorously from the linear evolution of the conserved density $\eta_k(t)$ by Fourier transformation and yields the large-scale behavior in the moving reference frame with velocity v_1 the scaling form

$$S_1(p, t) = \frac{1}{\sqrt{2\pi}} e^{-D_1 p^2 t}, \tag{B25}$$

with $D_1 = 1$. Mode-coupling theory then predicts the Fourier transform of the scaling function for mode 2 to be [9]

$$S_2(p, t) = \frac{1}{\sqrt{2\pi}} e^{-C_0 p^{3/2}[1 - i\,\mathrm{sign}(p(v_1 - v_2))]t}, \tag{B26}$$

which is a maximally asymmetric 3/2-Lévy distribution with the scale factor

$$C_0 = \frac{(G_{11}^2)^2}{2\sqrt{D_1 |v_1 - v_2|}} = \frac{1}{\sqrt{2}}. \tag{B27}$$

On the other hand, in Ref. [53] the dynamical structure function was proved rigorously to be a fundamental solution of the fractional PDE:

$$\partial_t u(x, t) = -\frac{1}{\sqrt{2}}[(-\Delta)^{3/4} - \nabla(-\Delta)^{1/4}]u(x, t). \tag{B28}$$

A Fourier transformation reproduces the mode-coupling prediction (B26), thus proving that the mode-coupling approximation is exact for this model and even yields the correct scale factor C_0 [1].

APPENDIX C: INVERSE RELATIONSHIP BETWEEN THE MODE-COUPLING MATRICES AND THE HESSIANS

The inverse relationship between the mode-coupling matrices and the Hessians is

$$H^\gamma = 2\sum_\lambda (R^{-1})_{\gamma\lambda} R^T G^\lambda R.$$

Decomposing the mode-coupling matrices into their two diagonal components $G_{\alpha\alpha}^\lambda$ and the symmetric off-diagonal part G_{12}^λ yields, after straightforward computation,

$$\begin{aligned} H^\gamma &= f_1^\gamma \begin{pmatrix} 1 & u_+ \\ u_+ & u_+^2 \end{pmatrix} + \bar{f}^\gamma \begin{pmatrix} 2u_- & 1 + u_+u_- \\ 1 + u_+u_- & 2u_+ \end{pmatrix} \\ &\quad + f_2^\gamma \begin{pmatrix} u_-^2 & u_- \\ u_- & 1 \end{pmatrix} \\ &= u_+ f_1^\gamma \begin{pmatrix} \frac{\delta+\theta}{2J_{12}} & 1 \\ 1 & \frac{\delta-\theta}{2J_{21}} \end{pmatrix} + \frac{2u_+}{J_{12}} \bar{f}^\gamma \begin{pmatrix} -J_{21} & \theta \\ \theta & J_{12} \end{pmatrix} \\ &\quad + \frac{u_+ J_{21}}{J_{12}} f_2^\gamma \begin{pmatrix} \frac{\delta-\theta}{2J_{12}} & -1 \\ -1 & \frac{\delta+\theta}{2J_{21}} \end{pmatrix}, \end{aligned}$$

where the amplitudes

$$\begin{aligned} f_1^\gamma &= \frac{2}{z_+^2} \sum_\lambda (R^{-1})_{\gamma\lambda} G_{11}^\lambda, \\ \bar{f}^\gamma &= \frac{2}{z_+ z_-} \sum_\lambda (R^{-1})_{\gamma\lambda} G_{12}^\lambda, \\ f_2^\gamma &= \frac{2}{z_-^2} \sum_\lambda (R^{-1})_{\gamma\lambda} G_{22}^\lambda \end{aligned}$$

depend on the mode-coupling coefficients.

[1] After completing this computation, P. Gonçalves mentioned to us that the correctness of the scale factor C_0 was communicated to her by H. Spohn some years ago. Since we are not aware of a published version of this observation we decided to keep it in this manuscript, without claiming originality.

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