Weak noise theory of the O'Connell-Yor polymer as an integrable discretization of the nonlinear Schrödinger equation

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We investigate and solve the weak noise theory for the semidiscrete O'Connell-Yor directed polymer. In the large deviation regime, the most probable evolution of the partition function obeys a classical nonlinear system which is a nonstandard discretization of the nonlinear Schrödinger equation with mixed initial-final conditions. We show that this system is integrable and find its general solution through an inverse scattering method and a non-standard Fredholm determinant framework that we develop. This allows us to obtain the large deviation rate function of the free energy of the polymer model from its conserved quantities and to study its convergence to the large deviations of the Kardar-Parisi-Zhang equation. Our model also degenerates to the classical Toda chain, which further substantiates the applicability of our Fredholm framework.

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I. INTRODUCTION

A. Overview

There has been much recent progress in obtaining exact solutions for the weak noise theory (WNT) of stochastic continuous systems in $1+1$ dimensions. This was achieved in the context of the Kardar-Parisi-Zhang (KPZ) equation [1–3], introduced to describes interface growth [4], and of the macroscopic fluctuation theory (MFT) [5–10], which describes diffusive particle systems [11–14]. The WNT describes the optimal configuration of height or density fields which realizes atypically large fluctuations in the presence of a weak noise [15–17]: this is an example of a large deviation problem [18]. The progress was achieved by noticing that the nonlinear systems of equations which arise in WNT are integrable. For the KPZ equation, where the WNT describes the short-time regime, these equations are a close cousin of the nonlinear Schrödinger equation (NLS), while for the MFT the connection is to the derivative nonlinear Schrödinger equation. Both equations are integrable using the inverse scattering method based on the existence of a Lax pair [19–21], but solving the WNT requires handling mixed initial-final time conditions. These connections to stochastic problems have renewed the interest in these classical integrable systems due to the new challenge of nonstandard mixed boundary conditions, together with the possibility to solve exactly the equations using Fredholm determinants. In fact, some of the large deviation rate functions obtained by these classical integrability methods have also been derived [6,22–24] from the Fredholm determinant formula stemming from quantum integrability. While complementary, the two approaches are quite distinct, and the connections between them are not elucidated.

B. Polymer model and weak noise limit

These studies have so far remained in the scope of continuous systems and the question whether this can be extended to discrete systems is still open. In this paper, we provide a positive answer to this question in the context of the semidiscrete O'Connell-Yor (OY) polymer [25–27], a discretization of the KPZ equation [28]. In its point-to-point version, it is defined by the partition sum $Z_N(t)$ defined as

$$Z_N(t)=\int_{s_0=0<s_1<\cdots<s_N=t} \mathrm{d}s_1\cdots \mathrm{d}s_{N-1}\, e^{\sqrt{\varepsilon}\sum_{j=1}^N[B_j(s_j)-B_j(s_{j-1})]}. \tag{1}$$

The directed polymer path lives on the horizontal lines $j=1,\dots,N$ and jump upward from line j to $j+1$ at time s_j; see Fig. 1. The $B_j(s)$ are independent standard Brownian motions and represent the noise. The endpoints are fixed at $(j,s)=(1,0)$ and (N,t). For $N=1$ this is just the geometric Brownian motion $Z_1(t)=e^{\sqrt{\varepsilon}B(t)}$ frequently studied in the Black-Scholes model in finance [29].

The weak noise limit amounts to take the (inverse temperature) parameter $\varepsilon\ll 1$. Consider first the simplest case of the geometric Brownian motion in the weak noise, i.e., small volatility, limit. Although its typical fluctuations are simple (they are Brownian by expanding the exponential), the large deviations are more interesting. Indeed, consider the

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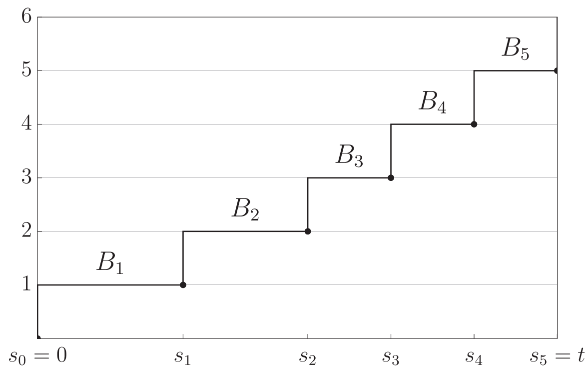

FIG. 1. Example of a configuration of the O'Connell-Yor polymer, here for $N = 5$, where independent Brownians live on each horizontal line.

observable

$$\overline{e^{-\frac{\hat{\Lambda}}{\varepsilon} Z_1(t=1)}} = \int_{\mathbb{R}} \frac{\mathrm{d}u}{\sqrt{2\pi\varepsilon}} e^{-\frac{u^2}{2\varepsilon} - \frac{\hat{\Lambda}}{\varepsilon} e^u} \underset{\varepsilon \to 0}{\sim} e^{-\frac{1}{\varepsilon}\Psi_1(\hat{\Lambda})}, \tag{2}$$

where the overbar denotes expectation values with respect to the noise, and the the rate function is

$$\Psi(\hat{\Lambda}) = \min_{u \in \mathbb{R}} \left(\frac{u^2}{2} + \hat{\Lambda} e^u \right) = \frac{W(\hat{\Lambda})^2}{2} + W(\hat{\Lambda}), \tag{3}$$

where W is the Lambert function [30] (see also [23], Supp. Material, Sec. 1). Hence it is already nontrivial for $N = 1$ and describes how the rare large fluctuations $B(t = 1) = O(1/\sqrt{\varepsilon})$ affect the geometric Brownian motion.

C. Aim of the paper

In this paper, we consider such large deviation observables for the full OY problem defined in Eq. (1) with arbitrary number of lines N. This now involves a nontrivial path-dependent optimization. We show that these large deviations are controlled by a system of deterministic nonlinear equations with mixed initial-final boundary conditions which arise from the saddle point of the dynamical action associated to the stochastic evolution of $Z_N(t)$. These equations are a discretization of the $\{P, Q\}$ system studied by us using inverse scattering in the context of the KPZ equation in Refs. [1,2,6], a close cousin of the NLS equation [19], and a member of the AKNS hierarchy [20] which also recently appeared in the context of the periodic TASEP and the KPZ fixed point [31,32]. Interestingly it is a nonstandard discretization, whose nonlinearity is local on the lattice, and which is closely related to other physical models such as the discrete self-trapping dimer model [33,34], the Toda lattice [35], and the periodic TASEP at large time [36]. These equations are integrable in the sense of the existence of a Lax pair. Our contribution is the derivation of an inverse scattering theory adapted to boundary conditions which are nonlocal in time, together with the explicit solution of these equations in terms of direct and inverse scattering. This allows us to obtain the analytical expression of the large deviation rate function for the OY model for arbitrary N as well as the optimal history of the partition function conditioned on the large deviations. We complement our study by performing an asymptotic analysis of a determinantal formula obtained for the OY model [28,37], which agrees with our results. We also show in detail how, in the limit of large N, these equations and their solutions recover the previous results for the Weak Noise Theory/short-time limit of the KPZ equation [1]. As an amusing byproduct, we obtain a contour integral representation of the Lambert function.

D. Outline

We start in Sec. II by studying the weak noise limit of the OY polymer from a field theoretical point of view. In Sec. II A we define the observables of interest in that limit, the details are given in Appendix A.

In Sec. II B we perform a saddle point calculation, which brings us to a semidiscrete nonlinear system of equations. The details are given in Appendix B. In some special cases $N = 1$ and $g = 0$ it is immediately solvable; see Appendix C. In the general case, this system is integrable as we discuss, and we obtain its Lax pair, which is further detailed in Appendix D.

In Sec. III we develop the scattering method to solve this system with mixed time boundary conditions. In Sec. III A we introduce the scattering basis and amplitudes. The details are given in Appendix E.

In Sec. III B we perform the inverse scattering analysis to obtain the general explicit solution of the system for general data with a Fredholm framework we develop. This is achieved through the triangular representation of the scattering solutions and the GLM equations that we derive explicitly in Appendixes K and L.

In Sec. IV we apply this method to the boundary conditions corresponding to the point-to-point OY polymer. We first obtain the scattering amplitudes for this case. The derivation is first detailed in Appendix G, and followed by a detailed Riemann-Hilbert analysis in Appendix H. The rate functions can then be computed using conserved quantities of the nonlinear systems, which are detailed in Appendix I. We find that there are two branches of solutions. The main branch is analyzed in Sec. IV A. The second branch is analyzed in Sec. IV B. This provides the full solution for the large deviation rate functions for the point-to-point OY polymer.

Next we show in Sec. IV C that the OY rate function converges smoothly at large N to the one of the weak noise theory of the KPZ equation. The details are given in Appendix J where we also show how that the nonlinear system, as well as its Lax pair, converges to the nonlinear Schrödinger equation in this limit.

In addition we check in Appendix N that the result for the rate functions can also be obtained by a weak noise asymptotic analysis of a Fredholm determinant formula for the OY derived in [28,37].

Finally we show in Sec. IV D that in yet another limit, the OY system converges to the classical Toda lattice, the details being given in Appendix M.

II. WEAK NOISE LIMIT

A. Observables

We start by recalling the coupled stochastic equations which are obeyed by the set of all the partition functions

$\{Z_n(t)\}_{n\in\mathbb{N}}$. It reads

$$\partial_t \mathsf{z}_n(t) = \mathsf{z}_{n-1}(t) - \mathsf{z}_n(t) + \sqrt{\varepsilon}\mathsf{z}_n(t)\eta_n(t) \tag{4}$$

in Itô discretization, where the $\eta_n(t)$ are independent white noises, and where $\mathsf{z}_n(t) = e^{-(1+\frac{\varepsilon}{2})t} Z_n(t)$ with the convention $\mathsf{z}_0(t) = 0$. Here we are interested in the large deviation form of the cumulant-generating function (CGF)

$$\overline{e^{-\frac{\Lambda}{\varepsilon}\mathsf{z}_N(t=1)}} \sim e^{-\frac{1}{\varepsilon}\Psi_N(\Lambda)}. \tag{5}$$

One of our aim is to compute the rate function $\Psi_N(\Lambda)$ explicitly for any N. From it one can extract the large deviation form of the PDF of $\mathcal{P}_N(H)$ of $\mathsf{H} = \log \mathsf{z}_N(t=1)$

$$\mathcal{P}_N(H) \sim e^{-\frac{1}{\varepsilon}\Phi_N(H)} \tag{6}$$

through Legendre inversion of the saddle point

$$\Psi_N(\Lambda) = \min_{H\in\mathbb{R}}[\Lambda e^H + \Phi_N(H)]. \tag{7}$$

The calculation of the CGF is done using the path-integral representation

$$\overline{e^{\frac{1}{\varepsilon}\int \mathrm{d}t \sum_n j_n(t) z_n(t)}} = \iint \mathcal{D}\tilde{z}\mathcal{D}z e^{-\frac{1}{\varepsilon}S[z,\tilde{z},j]} \tag{8}$$

in terms of the dynamical action

$$S[z,\tilde{z},j] = \int_0^\infty \mathrm{d}t \sum_{n=1}^N \left[\tilde{z}_n(\partial_t z_n - z_{n-1} + z_n) - \frac{1}{2}z_n^2\tilde{z}_n^2 - j_n z_n\right], \tag{9}$$

where the source field is here $j_n(t) = -\Lambda\delta_{n,N}\delta(t-1)$. More details about these observables and the dynamical action are provided in Appendixes A and B.

B. Saddle point equations: An integrable nonlinear system

In the limit $\varepsilon \ll 1$ we evaluate the right-hand side of (8) using a saddle point method which leads to the following nonlinear differential equations:

$$\begin{aligned} \partial_t z_n &= z_{n-1} - z_n + z_n^2\tilde{z}_n, \\ -\partial_t \tilde{z}_n &= \tilde{z}_{n+1} - \tilde{z}_n + z_n\tilde{z}_n^2. \end{aligned} \tag{10}$$

The derivation is standard and detailed in Appendix B. To study the point-to-point polymer one must impose the following initial and final conditions:

$$z_n(0) = \delta_{n,1}, \quad \tilde{z}_n(1) = -\Lambda\delta_{N,n}. \tag{11}$$

Note that one also has $z_n(t) = 0$ for all $n \leqslant 0$, and $\tilde{z}_n(t) = 0$ for all $n \geqslant N+1$. Also, for these boundary conditions, we expect the symmetry $\tilde{z}_n(t) = -\Lambda z_{N-n+1}(1-t)$ to hold.

It turns out that the equations (10) are integrable, as was noted in other contexts [33,36,38–40]. They enjoy a Lax pair representation, which reads

$$\partial_t v_n = U_n v_n, \quad v_{n+1} = L_n v_n, \tag{12}$$

where the Lax matrices are given explicitly by

$$U_n = \begin{pmatrix} \frac{\lambda^2-1}{2} & -z_{n-1} \\ \tilde{z}_n & \frac{1-\lambda^2}{2} \end{pmatrix}, \quad L_n = \begin{pmatrix} \frac{1}{\lambda} & \frac{z_n}{\lambda} \\ -\frac{1}{\lambda}\tilde{z}_n & \lambda - \frac{1}{\lambda}z_n\tilde{z}_n \end{pmatrix}. \tag{13}$$

Here n is the lattice index and λ is the spectral parameter and v_n is a two-component vector. One can check that the original system (10) is recovered from the compatibility equation $\partial_t L_n = U_{n+1}L_n - L_n U_n$. More details about these Lax pairs are given in Appendix D.

III. SCATTERING THEORY SOLUTION

A. Direct scattering

The general strategy that we use is to first solve the direct scattering problem (12). In a second stage we give the general solution of the inverse scattering problem and reconstruct the field $\{z_n(t), \tilde{z}_n(t)\}$ solution of the system (10). We first describe mathematically the general method to solve the system (10), before applying it to the specific boundary conditions (11) of interest for the polymer problem. To this aim, let us denote $v_n = e^{\frac{\lambda^2-1}{2}t}\phi_n$ with $\phi_n = (\phi_1, \phi_2)^\intercal$ and $v_n = e^{-\frac{\lambda^2-1}{2}t}\bar{\phi}_n$ two independent solutions of the linear Lax pair problem. Assuming that the fields $\{z_n, \tilde{z}_n\}$ vanish as $n \to \pm\infty$, which is the case for the boundary conditions (11), we can choose the solutions to behave asymptotically as $\phi_n \simeq \lambda^{-n}(1,0)^\intercal$ and $\bar{\phi}_n \simeq \lambda^n(0,-1)^\intercal$ at $n \to -\infty$. At $n \to +\infty$, each solution should be a particular linear combination of these elementary solutions. This allows us to define the scattering amplitudes $\{a, \tilde{a}, b, \tilde{b}\}$ as

$$\phi_n \underset{n\to+\infty}{\simeq} \begin{pmatrix} a(\lambda,t)\lambda^{-n} \\ b(\lambda,t)\lambda^n \end{pmatrix}, \quad \bar{\phi}_n \underset{n\to+\infty}{\simeq} \begin{pmatrix} \tilde{b}(\lambda,t)\lambda^{-n} \\ -\tilde{a}(\lambda,t)\lambda^n \end{pmatrix}. \tag{14}$$

Inserting Eqs. (14) in the Lax equation with U_n for $n \to +\infty$ yields the time dependence of the scattering amplitudes $a(\lambda,t) = a(\lambda)$, $\tilde{a}(\lambda,t) = \tilde{a}(\lambda)$, $\tilde{b}(\lambda,t) = \tilde{b}(\lambda)e^{(\lambda^2-1)t}$ and $b(\lambda,t) = b(\lambda)e^{(1-\lambda^2)t}$. Note that the representation of the Lax matrices (13) chosen here is particularly convenient as $\mathrm{Tr}(U_n) = 0$ and $\mathrm{Det}(L_n) = 1$. As a consequence, the Wronskian of $\{\phi_n, \bar{\phi}_n\}$ is constant in space and time; see Appendix E where more details are given about the scattering problem. This allows us to obtain the following normalization relation of the scattering amplitudes [41] $a(\lambda)\tilde{a}(\lambda) + b(\lambda)\tilde{b}(\lambda) = 1$. We expect $a(\lambda)$ to be analytic inside a contour enclosing the origin which we choose as a circle of radius $\mathtt{R}$, i.e., for $|\lambda| < \mathtt{R}$, and and $\tilde{a}(\lambda)$ to be analytic outside, i.e., for $|\lambda| > \mathtt{R}$. Although we will use that circle notation for simplicity, it is understood in this work that for $\Lambda > \Lambda_N^*$ (a large positive value defined later) the circle must actually be deformed into an ellipse $\mathcal{C}$; see Appendix H.

B. Inverse scattering and general solution of the nonlinear system

From the knowledge of the scattering amplitudes, it is possible to reconstruct explicitly the field $z_n(t)$ using a Fredholm operator formula that we now present. Here we give only the result; the derivation is lengthy and presented in Appendixes L and K. We first define the Fourier transforms of the ratio of scattering amplitudes called reflection coefficients as

$$F_t(n) = \oint_{|\lambda|=\mathtt{R}} \frac{d\lambda}{2\mathbf{i}\pi} \frac{b(\lambda,t)}{a(\lambda)} \lambda^{n-1}, \tag{15}$$

$$\tilde{F}_t(n) = \oint_{|\lambda|=\mathtt{R}} \frac{d\lambda}{2\mathbf{i}\pi} \frac{\tilde{b}(\lambda,t)}{\tilde{a}(\lambda)} \lambda^{-1-n}, \tag{16}$$

where the contour integrals are taken over the R circle. We then define two (space-time-dependent) Hankel operators $F_{n,t}$ and $\tilde{F}_{n,t}$ with the following kernels:

$$F_{n,t}(i,j) = F_t(2n+i+j), \tilde{F}_{n,t}(i,j) = \tilde{F}_t(2n+i+j), \tag{17}$$

where the indices are positive $i, j \geqslant 0$. The product of two such operators A, B is defined as

$$(AB)(i,j) = \sum_{k>0} A(i,k)B(k,j), \tag{18}$$

and we define the vector $\langle\delta|$ with component $\delta_{i,0}$ so that $\langle\delta|A|\delta\rangle = A(0,0)$ for any operator A. Then the general solution of Eq. (10) reads

$$z_n(t) = -\langle\delta|\tilde{F}_{n,t}(I+F_{n,t}\tilde{F}_{n,t})^{-1}|\delta\rangle. \tag{19}$$

The additive structure present in Eq. (17) arises from the integrability of the problem and is akin to the structure for the continuous space problem [1,42,43]. An analogous formula also exists for $\tilde{z}_n(t)$; see Appendix L. Note that this formula (19) can be easily evaluated numerically using a discretized version of the Bornemann algorithm [1].

IV. RIEMANN-HILBERT ANALYSIS FOR THE POINT-TO-POINT POLYMER AND LARGE DEVIATION FUNCTIONS

Quite remarkably for the boundary conditions of interest here (11), the calculation of the scattering amplitudes can be performed explicitly. Indeed, by solving the spatial Lax equation $\phi_{n+1} = L_n\phi_n$ and $\bar{\phi}_{n+1} = L_n\bar{\phi}_n$ at $t=0$ and $t=1$ (see Appendix G) one obtains

$$\tilde{b}(\lambda) = -\lambda^2, \quad b(\lambda) = \Lambda\lambda^{-2N-2}e^{\lambda^2-1}. \tag{20}$$

The normalization relation then implies that

$$a(\lambda)\tilde{a}(\lambda) = 1 + \Lambda\lambda^{-2N}e^{\lambda^2-1}. \tag{21}$$

To obtain the expression of $a, \tilde{a}$, we solve Eq. (21) as a scalar Riemann-Hilbert (RH) problem. We will summarize the results below, and the details are given in Appendix H. There are two families, or branches, of solutions of Eq. (21) which are relevant for our large deviation problem, i.e., needed to invert (7) to obtain $\Phi_N(H)$. One of the branch involves a solitonic component, which is typical for classical integrable models. Solitons have been shown to be important for weak noise large deviation problems, for instance they were discussed for the KPZ equation in [1–3,16,24] (they have also been called traveling front solutions). These solitons influence both the rate function and the saddle point solution, i.e., the optimal configuration of the polymer. Finally, the simpler cases $N=1$, mentioned in the Introduction, is treated in Appendix C.

A. Main branch, no solitons

The first family of solution does not involve solitons and determines *the main branch* of $\Psi_N(\Lambda)$. It assumes that $a(\lambda)$ has no zero for $|\lambda| < \mathtt{R}$ and that $\tilde{a}(\lambda)$ has no zero for $|\lambda| > \mathtt{R}$. Solving the RH problem Eq. (21) (see Appendix H) one obtains

$$\begin{aligned} \log\tilde{a}(\lambda) &= -\varphi(\lambda), \quad |\lambda| > \mathtt{R} \\ \log a(\lambda) &= \varphi(\lambda), \quad |\lambda| < \mathtt{R} \end{aligned} \tag{22}$$

with

$$\varphi(\lambda) = \oint_{\mathcal{C}} \frac{dw}{2\mathbf{i}\pi} \frac{w}{w^2-\lambda^2} \log(1+\Lambda w^{-2N}e^{w^2-1}), \tag{23}$$

where $\mathcal{C}$ is a closed contour around the origin which must avoid the branch cuts of the logarithm. For most values of interest, $\Lambda < \Lambda_N^*$, $\mathcal{C}$ can be chosen as the circle $\mathtt{R} = \sqrt{N}$. The threshold Λ_N^* is defined in Appendix H, it grows very fast with N (with $\Lambda_1^* = e^2$) and plays little role below, so we stick to the circle notation. From the knowledge of the scattering amplitudes, one additionally obtains the values taken by the conserved quantities of the problem; for details see Appendix F. This is achieved by Laurent or Taylor expanding the scattering amplitudes as

$$\log\tilde{a}(\lambda) = \sum_{n=1}^{\infty} \frac{\tilde{C}_n}{\lambda^{2n}}, \quad \log a(\lambda) = \sum_{n=0}^{\infty} \lambda^{2n} C_n. \tag{24}$$

In particular, expanding equivalently Eq. (23), we have that the value taken by the nth conserved quantity is

$$\tilde{C}_n = \oint_{|w|=\mathtt{R}} \frac{dw}{2\mathbf{i}\pi} w^{2n-1} \log(1+\Lambda w^{-2N}e^{w^2-1}). \tag{25}$$

Since the first conserved quantity is related to the fields $\{z_n, \tilde{z}_n\}$ as $\tilde{C}_1 = -\sum_{n=1}^{N} z_n(t)\tilde{z}_n(t)$ and is by definition time independent, it can be evaluated at $t=1$ where one has

$$\tilde{C}_1 = -\sum_{n=1}^{N} z_n(t=1)\tilde{z}_n(t=1) = \Lambda z_N(t=1) \tag{26}$$

due to the boundary conditions (11). From the derivative of (7) w.r.t. Λ one sees that $\Psi_N'(\Lambda) = e^H = z_N(t=1)$. Hence the large deviation rate function is determined by the first conserved quantity and one obtains

$$\Lambda\Psi_N'(\Lambda) = \oint_{|w|=\mathtt{R}} \frac{dw}{2\mathbf{i}\pi} w \log(1+\Lambda w^{-2N}e^{w^2-1}). \tag{27}$$

Note that for $N=1$, Eq. (27) provides a non-standard integral representation of the Lambert function as $\Lambda\Psi_1'(\Lambda) = W(\frac{\Lambda}{e})$ from Eq. (3). A direct integration of (27) finally yields

$$\Psi_N(\Lambda) = -\oint_{|v|=\mathtt{R}^2} \frac{dv}{2\mathbf{i}\pi} \mathrm{Li}_2(-\Lambda v^{-N}e^{v-1}), \tag{28}$$

where Li_2 refers to the dilogarithm [44] which domain of definition restricts the validity of this formula for

$$\Lambda \geqslant \Lambda_c, \quad \Lambda_c = -e^{1-N}N^N \leqslant 0. \tag{29}$$

It allows us to reconstruct the rate function $\Phi_N(H)$ with $H = \log z_N(t=1)$ in the range

$$0 \leqslant z_N(t=1) \leqslant \Psi_N'(\Lambda_c) = z_N^{(c)}, \tag{30}$$

and one obtains the parametric representation

$$\Phi_N(H) = \Psi_N(\Lambda) - \Lambda\Psi'_N(\Lambda)$$
$$H = \log \Psi'_N(\Lambda) \tag{31}$$

for $H \in (-\infty, H_c]$ as $\Lambda \in [\Lambda_c, +\infty)$. This range contains $\Lambda = 0$ which gives the typical value $\Psi'_N(0) = e^{H_{\rm typ}} = \overline{\mathsf{z}_N(t=1)} = \frac{e^{-1}}{(N-1)!}$, as well as the second cumulant of the partition sum $\overline{\mathsf{z}_N(t=1)^2}^c = \frac{2^{2N-2}}{(2N-1)!}e^{-2}\,\varepsilon$, and of its logarithm

$$\overline{\mathsf{H}^2}^c = \frac{\varepsilon}{\Phi''(H_{\rm typ})} = \frac{-\varepsilon\Psi''(0)}{\Psi'(0)^2} = \frac{2^{2N-2}(N-1)!^2}{(2N-1)!}\varepsilon. \tag{32}$$

B. Second branch, with solitons

The second family of solution involves solitons and allows us to obtain the rate function $\Phi_N(H)$ for the range $z_N(t=1) \geqslant z_N^{(c)}$, i.e., $H > H_c$. To see how it arises consider the right-hand side of Eq. (21). One can check that for $\Lambda \in (\Lambda_c, 0)$ it has four zeros on the real axis $\{\pm\lambda_0, \pm\lambda_{-1}\}$ with

$$\lambda^2_{0/-1} = -NW_{0/-1}\left(-\frac{1}{e}\left(\frac{\Lambda}{\Lambda_c}\right)^{\frac{1}{N}}\right) \tag{33}$$

and $|\lambda_{-1}| \geqslant \mathtt{R} = \sqrt{N}$ and $|\lambda_0| < \sqrt{N}$. Here W_0 and W_{-1} are the two main branches of the Lambert function [30]. The existence of these zeros allows for a modified solution to the RH problem (21), where $a(\lambda)$ has two zeros inside the $\mathtt{R}$ circle, $\pm\lambda_0$, and $\tilde{a}(\lambda)$ two zeros outside, $\pm\lambda_{-1}$. The solution then reads

$$\log\tilde{a}(\lambda) = -\varphi_{\rm soliton}(\lambda), \qquad |\lambda| > \mathtt{R}$$
$$\log a(\lambda) = \varphi_{\rm soliton}(\lambda), \qquad |\lambda| < \mathtt{R} \tag{34}$$

with

$$\varphi_{\rm soliton}(\lambda) = \varphi(\lambda) - \log\left(\frac{\lambda^2 - \lambda_{-1}^2}{\lambda^2 - \lambda_0^2}\right), \tag{35}$$

where $\varphi(\lambda)$ is given by the same formula as in (23). By either Taylor or Laurent expanding the new solitonic contribution in (35), one also obtains an additional (solitonic) contribution $\Delta\tilde{C}_n$ to the values of the conserved quantities. In particular one obtains $\Delta\tilde{C}_1 = \lambda_0^2 - \lambda_{-1}^2$ which leads to the following correction (see Appendix H):

$$\Lambda\Psi'_{N,\rm soliton}(\Lambda) = \Lambda\Psi'_N(\Lambda) + \lambda_0^2 - \lambda_{-1}^2. \tag{36}$$

It can be explicitly integrated over Λ and we obtain the *second branch* of the rate function $\Psi_N \to \Psi_{N,\rm soliton}$ with

$$\Psi_{N,\rm soliton}(\Lambda) = \Psi_N(\Lambda) + \Delta_N(\Lambda), \tag{37}$$

where the solitonic contribution $\Delta_N(\Lambda)$ is

$$\Delta_N(\Lambda) = \frac{N^2}{2}\left(W_{-1}\left[-\frac{1}{e}\left(\frac{\Lambda}{\Lambda_c}\right)^{\frac{1}{N}}\right]\left\{W_{-1}\left[-\frac{1}{e}\left(\frac{\Lambda}{\Lambda_c}\right)^{\frac{1}{N}}\right]+2\right\} - W_0\left[-\frac{1}{e}\left(\frac{\Lambda}{\Lambda_c}\right)^{\frac{1}{N}}\right]\left\{W_0\left[-\frac{1}{e}\left(\frac{\Lambda}{\Lambda_c}\right)^{\frac{1}{N}}\right]+2\right\}\right). \tag{38}$$

This second branch allows us to reconstruct $\Phi(H)$ for $H > H_c$, using the same parametric representation (31) where now one replaces everywhere $\Psi_N(\Lambda) \to \Psi_{N,\rm soliton}(\Lambda)$. As Λ increases from Λ_c to 0, the values of H increase from H_c to $+\infty$. The two branches of $\Psi'_N(\Lambda)$ are shown in Fig. 2, where one sees that the branches merge smoothly.

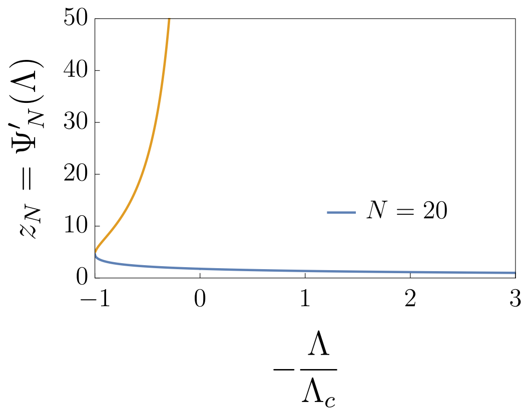

FIG. 2. Derivative of the rate function, $\Psi'_N(\Lambda)$. The main branch is in blue, and the second (solitonic) branch is in orange. The ordinate is also equal to $z_N(t=1) = e^H$, hence one can read the relation between H and Λ (which is not one-to-one).

C. Limit to the WNT the KPZ equation

As we have obtained our solution for general N, it is natural to study the limit when the polymer sees a large number of lines, i.e., $N \gg 1$. It is known that in the large N limit the OY polymer point-to-point partition sum converges, under the proper rescaling, to the solution of the stochastic heat equation, i.e., to the exponential of the KPZ height, for the so-called droplet initial conditions [28]. Here we can check that Eq. (5) converges to the corresponding equality for the KPZ equation at short time, which was obtained in [1,22]. To this aim we first define the rescaled variables

$$z = -\frac{\Lambda}{\Lambda_c}, \qquad T_{\rm KPZ} = \frac{\varepsilon^2}{2N}, \tag{39}$$

where $T_{\rm KPZ} \ll 1$ is the time in the KPZ equation. We then expand the formula (28) for $\Psi_N(\Lambda)$ at large N around the point $v = N$ on the contour, by setting $v = Ne^{q\sqrt{2/N}}$ which yields for large N,

$$\frac{\Psi_N(\Lambda)}{\varepsilon} \to \frac{\Psi_{\rm KPZ}(z)}{\sqrt{T_{\rm KPZ}}}, \tag{40}$$

where

$$\Psi_{\rm KPZ}(z) = -\int_{\mathbb{R}} \frac{dk}{2\pi}{\rm Li}_2(-ze^{-q^2}) \tag{41}$$

is the rate function for the KPZ equation with droplet initial data. Equation (40) shows the convergence of the right-hand side of (5) and the convergence of the left-hand side of (5) is obtained using Ref. [28], Sec. 5.4.1, which allows us to identify at large N

$$\frac{\Lambda}{\varepsilon}z_N(t=1) \to \frac{ze^{H_{\rm KPZ}}}{\sqrt{T_{\rm KPZ}}}, \tag{42}$$

where $H_{\rm KPZ}$ is the properly shifted KPZ height field, denoted H in [1]. To control the convergence of the solitonic branch, we expand the Lambert functions $W_{0/-1}(x)$ around $x = -1/e$ in (38) and obtain (see Appendix J)

$$\frac{\Delta_N(\Lambda)}{\varepsilon} \simeq \frac{\Delta_{\rm KPZ}(z)}{\sqrt{T_{\rm KPZ}}}, \quad \Delta_{\rm KPZ}(z) = \frac{4}{3}[-\log(-z)]^{\frac{3}{2}}. \quad (43)$$

This shows the convergence to the corresponding solitonic branch of the KPZ equation, obtained in [1] (see also [3]). For completeness, we have also derived the above results for $\Psi_N(\Lambda)$ from an asymptotic analysis of a determinantal representation formula for $\overline{e^{-uZ_N(t)}}$ [28,37] using a first cumulant approximation; see details in Appendix N.

D. Limit to the classical Toda lattice

The OY polymer is also related to the quantum Toda lattice [27]. Here we show that its weak noise theory (10) converges to the *classical* Toda lattice [45] in the small time limit. Indeed, using the Cole-Hopf parametrization $z_n(t) = \alpha e^{h_n(t)+\alpha^2 t}$, $\tilde{z}_n(t)z_n(t) = \alpha[\alpha + p_n(t)]$, and taking $\alpha \to \infty$ we find the Toda dynamics in scaled time $\tau = \alpha t$ (see Appendix M):

$$\partial_\tau h_n = p_n, \ \partial_\tau p_n = e^{h_{n-1}-h_n} - e^{h_n - h_{n+1}}. \quad (44)$$

Hence our results on the scattering theory and on the Fredholm determinants apply, extending known solutions for solitons [46]. Finally, there seems to exist a connection between the weak noise theory of the OY polymer and an integrable spin chain which we discuss in [47], Sec. XVII in the Supp. Material.

V. CONCLUSION

In conclusion we have shown that the weak noise theory of the OY polymer is integrable for any N, obtained its general solution in terms of Fredholm determinants, and computed large deviation rate functions from conserved quantities. The system (10) provides a discretization of the NLS equation with local nonlinearity and converges at large N to the weak noise of the KPZ equation. A distinct limit is the classical Toda chain, which is related to the QR decompositions in linear algebra [48,49], hence the extension to our system may provide applications to linear algebra algorithms. The tools introduced in this work could also find additional applications in the study of tilted elastic lines in the presence of columnar disorder, complementing earlier work [50]. Finally, this work opens the path for a general study of weak noise limit in stochastic integrable systems and their connection to classical integrability. Since semidiscrete integrable systems have found renewed interest given their connections to random matrix theory in the context of generalized equilibrium measures [51–55], the model studied in this work will find additional applications. The central object to study these properties are the dual Lax pairs which are derived for the OY model in [47], Sec. 16 of the Supp. Material. A potential outcome of this general study might lead to a broader classification of stochastic integrable models. Finally, it would be interesting to connect the large deviations in the short-time/weak noise limit to those in the large time limit, which have been much studied recently using Riemann-Hilbert methods and Coulomb gases; see [56–62].

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APPENDIX A: MODEL AND OBSERVABLE

Note that in the main text we use the notation $\mathsf{z}_n(t)$ for the OY partition sums (i.e., a random variable) and $z_n(t)$ for the solution of the weak noise saddle point equations. In the Appendixes we will simplify notations and use the same letter for both, hoping that the context can help determine which is which.

In this Appendix we will consider a slight generalization of the partition sums (1) defined in the text, which corresponds to adding drifts to the Brownian weights $B_j(s) \to B_j(s) + a_j s$. One thus defines for $1 \leqslant n \leqslant N$ the partition function

$$Z_n(t) = \int_{s_0=0<s_1<\cdots<s_n=t} {\rm d}s_1 \cdots {\rm d}s_{n-1} \times e^{\sqrt{\varepsilon}\sum_{j=1}^n [B_j(s_j)-B_j(s_{j-1})]+\sum_{j=1}^n a_j(s_j-s_{j-1})}. \quad (A1)$$

It is convenient to define $z_n(t) = e^{-(1+\frac{\varepsilon}{2})t}Z_n(t)$, so that the first two terms read explicitly

$$z_1(\tau) = e^{-\tau-\frac{1}{2}\varepsilon\tau}e^{\sqrt{\varepsilon}B_1(\tau)+a_1\tau},$$
$$z_2(\tau) = e^{-\tau-\frac{1}{2}\varepsilon\tau}\int_0^\tau {\rm d}s e^{\sqrt{\varepsilon}B_1(s)+a_1 s}e^{\sqrt{\varepsilon}[B_2(t)-B_2(s)]+a_2(\tau-s)}. \quad (A2)$$

Using the rules of Itô calculus it is easy to see that these partition sums satisfy the following coupled stochastic equations in Itô discretization for $1 \leqslant n \leqslant N$:

$$\partial_t z_n(t) = z_{n-1}(t) - z_n(t) + \sqrt{\varepsilon}z_n(t)\eta_n(t) + a_n z_n(t), \quad (A3)$$

where the $\eta_n(t)$ are standard independent white noises, i.e., with correlators $\overline{\eta_n(t)\eta_{n'}(t')} = \delta(t-t')\delta_{nn'}$. One uses the convention $z_0(t) = 0$ and the initial condition is $z_n(0) = \delta_{n,1}$.

The observable of interest is the partition function at one point $z_N(t=1)$ and its probability distribution function $\mathcal{P}_N(z)$. It is useful to also introduce its logarithm $H_N = \log z_N(t=1)$ and its PDF $\mathcal{P}_N(H)$. These PDF's exhibit in the weak noise limit $\varepsilon \to 0$ the following large deviation principle:

$$\mathcal{P}_N(z) \underset{\varepsilon\to 0^+}{\sim} e^{-\frac{1}{\varepsilon}\hat{\Phi}_N(z)}, \quad \mathcal{P}_N(H) \underset{\varepsilon\to 0^+}{\sim} e^{-\frac{1}{\varepsilon}\Phi_N(H)}. \quad (A4)$$

To compute these rate functions, which differ only by a change of variable, $\hat{\Phi}_N(z) = \Phi_N(H = \log z)$ we will first compute the generating function of the cumulants of $z_N(t=1)$

$$\overline{e^{-\frac{\Lambda}{\varepsilon}z_N(t=1)}} \underset{\varepsilon\to 0^+}{\sim} e^{-\frac{1}{\varepsilon}\Psi_N(\Lambda)} \quad (A5)$$

and the rate function $\Psi_N(\Lambda)$. The two rate functions are related by the Legendre transform

$$\Psi_N(\Lambda) = \max_{z\in\mathbb{R}^+}[\Lambda z + \hat{\Phi}_N(z)] = \max_{H\in\mathbb{R}}[\Lambda e^H + \Phi_N(H)]. \quad (A6)$$

APPENDIX B: DYNAMICAL FIELD THEORY AND SADDLE POINT EQUATIONS

We now use the standard path integral representation using source fields $j_1(t), \dots, j_N(t)$,

$$\overline{e^{\frac{1}{\varepsilon}\int_0^{+\infty} \mathrm{d}t \sum_{n=1}^N j_n(t) z_n(t)}} = \iiint \mathcal{D}\eta \mathcal{D}\tilde{z} \mathcal{D}z e^{\int_0^{+\infty} \mathrm{d}t\, \sum_{n=1}^N [-\frac{\tilde{z}_n}{\varepsilon}(\partial_t z_n - z_{n-1} + z_n - \sqrt{\varepsilon} z_n \eta_n - a_n z_n) - \frac{1}{2}\eta_n^2 + j_n z_n]} \quad (B1)$$

$$= \iint \mathcal{D}\tilde{z}\mathcal{D}z e^{-\frac{1}{\varepsilon}S[z,\tilde{z},j]}, \quad (B2)$$

where $\mathcal{D}z = \prod_{n=1}^N \mathcal{D}z_n(t)$ (and similarly for $\tilde{z}$ and $\tilde{\eta}$) is the path integral measure, in terms of the dynamical action (time dependence is implicit)

$$S[z,\tilde{z},j] = S_0[z,\tilde{z}] - \int_0^{+\infty} \mathrm{d}t \sum_{n=1}^N j_n(t) z_n(t),$$

$$S_0[z,\tilde{z}] = \int_0^{+\infty} \mathrm{d}t \sum_{n=1}^N \bigg[\tilde{z}_n(\partial_t z_n - z_{n-1} + z_n - a_n z_n) - \frac{1}{2} z_n^2 \tilde{z}_n^2\bigg], \quad (B3)$$

where in the last line we have integrated over the noises $\eta_n(t)$. We have introduced the response field $\frac{\tilde{z}_n}{\varepsilon}$ to enforce the N equations of motion. To obtain the observable in (A5) we need to choose the source field $j_n(t) = -\Lambda \delta_{n,N}\delta(t-1)$.

In the limit $\varepsilon \to 0$ the path integral (B2) is dominated by the saddle point of the action $S[z,\tilde{z},j]$. The saddle point equations take the form of the following system of equations:

$$\partial_t z_n = z_{n-1} - z_n + \frac{g}{2} z_n^2 \tilde{z}_n + a_n z_n,$$

$$-\partial_t \tilde{z}_n = \tilde{z}_{n+1} - \tilde{z}_n + \frac{g}{2} z_n \tilde{z}_n^2 + a_n \tilde{z}_n. \quad (B4)$$

We have added for convenience the parameter g for the nonlinearity, but for the application here one must set $g = 2$. Note that *a priori* the above saddle point equations hold for $1 \leqslant n \leqslant N$. However, for technical reasons it is useful to extend the system (B4) for all $n \in \mathbb{Z}$, which we do from now on. We then consider the initial and final conditions,

$$\{z_0(t) = 0,\ z_n(0) = \delta_{n,1},\ \tilde{z}_{N+1}(t) = 0,\ \tilde{z}_n(1) = -\Lambda\delta_{N,n}\}, \quad (B5)$$

where imposing the last condition is equivalent to omitting the source term $j_n(t) = -\Lambda\delta_{n,N}\delta(t-1)$ in the equation of motion for $\tilde{z}_n$. Note that one also has $z_n(t) = 0$ for all $n \leqslant 0$, and $\tilde{z}_n(t) = 0$ for all $n \geqslant N+1$. In practice we also take $a_n = 0$ except if $n = 1, \dots, N$. Finally the response field $\tilde{z}_n(t)$ vanishes for $t > 1$ for all n.

It is useful to note the following exact symmetry of the solution of the system (B4) with the boundary conditions (B5). If $a_\ell = a_{N-\ell+1}$ then

$$\tilde{z}_\ell(t) = -\Lambda z_{N-\ell+1}(1-t). \quad (B6)$$

We will obtain below the solution of this system of equations for any N. Once this is done one can insert the solution into the dynamical action to obtain its value at the saddle point. Hence one has

$$\overline{e^{-\frac{\Lambda}{\varepsilon} z_N(t=1)}} \underset{\varepsilon\to 0^+}{\sim} \iint \mathcal{D}\tilde{z}\mathcal{D}z e^{-\frac{1}{\varepsilon}S_0^{\rm sp}[z,\tilde{z}]} e^{-\frac{\Lambda}{\varepsilon} z_N(t=1)}. \quad (B7)$$

Hence the PDF $\mathcal{P}_N(z)$ of $z_N(t=1)$ is given by the optimal action $S_0^{\rm sp}[z,\tilde{z}]$, which using the saddle point equations simplifies to (for $g = 2$)

$$\mathcal{P}_N(z) \sim \exp\left(-\frac{1}{2\varepsilon}\int_0^1 \mathrm{d}t \sum_{n=1}^N z_n^2 \tilde{z}_n^2\right) \quad (B8)$$

evaluated using the solution of (B4) with $z_N(t=1) = z$, this determines $\hat{\Phi}_N(z)$. In practice, however, we will obtain $z = z_N(t=1)$ as a function of Λ by solving (B4). From the Legendre relation $z = \Psi'(\Lambda)$ obtained by taking a derivative of (A6) w.r.t. Λ, this will allow us to determine $\Psi_N(\Lambda)$. The last step will be to obtain $\hat{\Phi}_N(z)$ by an inverse Legendre transform.

We note that the evolution equations (B4) differ in part by the minus sign in front of the time derivative. This reflects that the evolution of z_n can be seen as forward in time and the evolution of $\tilde{z}_n$ as backwards in time so that the whole problems is closer to a shooting problem rather than a dynamical problem.

APPENDIX C: TWO SIMPLE CASES

1. $g = 0$

Let us consider the system (B4) in the simple case $g = 0$ with the boundary conditions (B5). In this case the two equations decouple and are simply linear. In the absence of drifts the solution is readily obtained as

$$z_n(t) = e^{-t}\frac{t^{n-1}}{(n-1)!}, \quad \tilde{z}_n(t) = -\Lambda e^{t-1}\frac{(1-t)^{N-n}}{(N-n)!}. \quad (C1)$$

We plot this solution numerically in Fig. 3. In the presence of drifts it is easier to solve these equations using a Fourier representation which will be useful in the following. Let us write

$$z_n(t) = \int_C \frac{\mathrm{d}\lambda^2}{2\mathbf{i}\pi\lambda^2}\hat{z}_n(\lambda)e^{t(\lambda^2-1)}, \quad (C2)$$

where C is a circle around 0 in the complex plane. Inserting into

$$\partial_t z_n = z_{n-1} - z_n + a_n z_n \quad (C3)$$

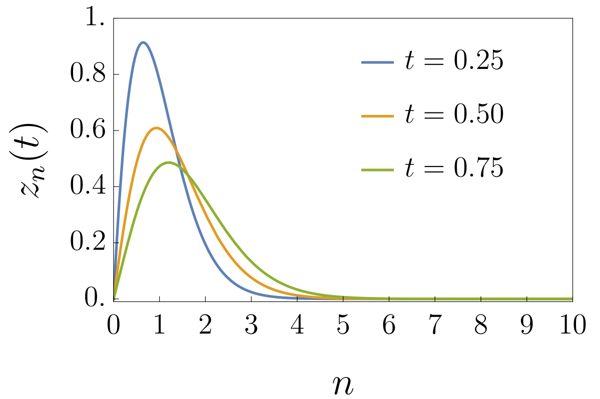

FIG. 3. Solution $z_n(t)$ of the problem without interaction, $g = 0$, as given in Eq. (C1).

for $1 \leqslant n \leqslant N$, it gives

$$\hat{z}_n(\lambda) = \frac{\lambda^{-2}}{1 - \frac{a_n}{\lambda^2}} \hat{z}_{n-1}(\lambda). \tag{C4}$$

For $z_1(t)$ we can solve directly

$$\partial_t z_1 = -z_1 + a_1 z_1, \quad z_1(t) = e^{-t + a_1 t}. \tag{C5}$$

Hence one has

$$\hat{z}_1(\lambda) = \frac{\lambda^2}{\lambda^2 - a_1}, \quad z_1(t) = \int_C \frac{d\lambda^2}{2\mathbf{i}\pi\lambda^2} \frac{\lambda^2}{\lambda^2 - a_1} e^{t(\lambda^2 - 1)}, \tag{C6}$$

and the contour must include a_1. Then we obtain for $1 \leqslant n \leqslant N$

$$\hat{z}_n(\lambda) = \frac{\lambda^{2(1-n)}}{\prod_{k=1}^{n} \left(1 - \frac{a_k}{\lambda^2}\right)}, \tag{C7}$$

which gives for any $n \in \mathbb{Z}$ (using that $a_n = 0$ for $n \geqslant N + 1$)

$$z_n(t) = \int_C \frac{d\lambda^2}{2\mathbf{i}\pi\lambda^2} \frac{\lambda^{2(1-n)}}{\prod_{k=1}^{\min(n,N)} \left(1 - \frac{a_k}{\lambda^2}\right)} e^{(\lambda^2 - 1)t}, \tag{C8}$$

which automatically vanishes for $n \geqslant 0$. The contour C must contain all the a_n. Similarly one has $\tilde{z}_N(t) = -\Lambda e^{(a_N - 1)(1-t)}$ and one obtains

$$\tilde{z}_n(t) = -\Lambda \int_C \frac{d\lambda^2}{2\mathbf{i}\pi\lambda^2} \frac{\lambda^{2(n-N)}}{\prod_{k=1}^{\min(N-n+1,N)} \left(1 - \frac{a_{N-k+1}}{\lambda^2}\right)} e^{(\lambda^2 - 1)(1-t)}. \tag{C9}$$

If we set all $a_n = 0$ one can check that one recovers (C1).

2. $N = 1$

For $N = 1$ there are several ways to obtain the large deviation rate functions.

First method. For $N = 1$ the system (B4) involves only the variables $z_1(t)$ and $\tilde{z}_1(t)$ and becomes

$$\begin{aligned} \partial_t z_1 &= -z_1 + z_1^2 \tilde{z}_1 + a_1 z_1, \\ -\partial_t \tilde{z}_1 &= -\tilde{z}_1 + z_1 \tilde{z}_1^2 + a_1 \tilde{z}_1 \end{aligned} \tag{C10}$$

with $z_1(t = 0) = 1$ and $\tilde{z}_1(t = 1) = -\Lambda$. The quantity $z_1(t)\tilde{z}_1(t)$ is obviously conserved so that $z_1(t)\tilde{z}_1(t) = c_1$ where c_1 is a constant independent of time. Thus the system becomes

$$\begin{aligned} \partial_t z_1 &= (c_1 - 1 + a_1) z_1, \\ -\partial_t \tilde{z}_1 &= (c_1 - 1 + a_1) \tilde{z}_1, \end{aligned} \tag{C11}$$

and we find that $z_1(t) = e^{(c_1 - 1 + a_1)t}$ and $\tilde{z}_1(t) = -\Lambda e^{(c_1 - 1 + a_1)(1-t)}$. The constant c_1 is thus determined by

$$z_1 \tilde{z}_1 = c_1 = -\Lambda e^{c_1 - 1 + a_1}, \quad c_1 = -W(\Lambda e^{a_1 - 1}), \tag{C12}$$

where $W(x)$ is the Lambert function, such that $W(x)e^{W(x)} = x$. From the optimal action (B8) we obtain, parametrically

$$\hat{\Phi}(Z) = \frac{c_1^2}{2}, \quad Z = z_1(1) = e^{c_1 - 1 + a_1}, \tag{C13}$$

which leads to

$$\hat{\Phi}(Z) = \tfrac{1}{2}(1 - a_1 + \log Z)^2. \tag{C14}$$

Second method. Alternatively going back to the original stochastic model, where the partition sum for $N = 1$ is simply $z_1(t) = e^{-t} e^{\sqrt{\varepsilon} B_1(t) + a_1 t}$ [here beware $z_1(t)$ is not the solution of the SP equation] one can compute directly its cumulant-generating function, as sketched in the text. Indeed, denoting $u = \sqrt{\varepsilon} B(1)$ one has for small ε,

$$\overline{e^{-\frac{\Lambda}{\varepsilon} z_1(1)}} = \overline{e^{-\frac{\Lambda}{\varepsilon} e^{-1 + \sqrt{\varepsilon} B(1) + a_1}}} \sim \int_{\mathbb{R}} du \, e^{-\frac{u^2}{2\varepsilon} - \frac{\Lambda}{\varepsilon} e^{u - 1 + a_1}} \sim e^{-\frac{1}{\varepsilon}\Psi(\Lambda)}, \tag{C15}$$

arising from events where the Brownian is anomalously large, i.e., $u = \sqrt{\varepsilon} B(1) = O(1)$. Here we have for $\Lambda > 0$

$$\Psi(\Lambda) = \min_{u \in \mathbb{R}} \left(\frac{u^2}{2} + \Lambda e^{u - 1 + a_1} \right). \tag{C16}$$

There is a single minimum which is reached at

$$u = -W(\Lambda e^{a_1 - 1}), \tag{C17}$$

which is seen to equal the constant c_1 of the first method. This gives

$$\Psi(\Lambda) = \frac{W(\Lambda e^{a_1 - 1})^2}{2} + W(\Lambda e^{a_1 - 1}) \tag{C18}$$

as given in the text for $a_1 = 0$. This gives also $\Lambda \Psi'(\Lambda) = \Lambda e^{u - 1 + a_1} = -u = W(\Lambda e^{a_1 - 1})$. Note that $u = c_1 = \tilde{z}_1 z_1 = -\Lambda \Psi'(\Lambda)$ in agreement with the general formula for conserved quantities (see Appendix F).

Although the function $\Psi(\Lambda)$ was defined and computed for $\Lambda > 0$ the formula (C18) can be continued to negative Λ, on the principal branch $W = W_0$, down to $\Lambda = \Lambda_c(1)e^{a_1} = -1$. This is in agreement with the general formula for arbitrary N obtained below which states that

$$\Lambda_c(N) e^{a_1} = -e^{1-N} N^N. \tag{C19}$$

Let us now discuss the Legendre transform which relates $\hat{\Phi}(Z)$ and $\Psi(\Lambda)$. One has

$$\Psi(\Lambda) = \min_Z [\Lambda Z + \hat{\Phi}(Z)] \tag{C20}$$

leading to

$$Z = \Psi'(\Lambda) = \frac{W(\Lambda e^{a_1 - 1})}{\Lambda}. \tag{C21}$$

The typical value of Z is given by $Z_{\rm typ} = e^{a_1-1}$ using that $W_0'(0) = 1$. Using that $W_0(-1/e) = -1$ we see that the domain $0^+ < Z \leqslant Z_c = e^{a_1}$ corresponds to $\Lambda > \Lambda_c$. This is the principal branch with $W = W_0$.

To obtain the values of $Z \in [Z_c, \infty)$ one needs to consider a second branch. This is achieved using the second real branch of the Lambert function, $W = W_1$. Then one has

$$Z := z_1(t=1) = \Psi'(\Lambda) = \frac{W_{-1}(\Lambda e^{a_1-1})}{\Lambda}, \tag{C22}$$

where Λ increases again from $\Lambda = \Lambda_c$ back to $\Lambda = 0^-$; see Appendix I 3.

One obtains overall

$$\hat{\Phi}(Z) = \max_{\Lambda\in[-1,+\infty)} [-\Lambda Z + \Psi_0(\Lambda)], \quad 0 < Z \leqslant Z_c,$$
$$\hat{\Phi}(Z) = \min_{\Lambda\in[-1,0)} [-\Lambda Z + \Psi_1(\Lambda)], \quad Z \geqslant Z_c \tag{C23}$$

with

$$\Psi_0(\Lambda) = \frac{W_0(\Lambda e^{a_1-1})^2}{2} + W_0(\Lambda e^{a_1-1}), \tag{C24}$$
$$\Psi_1(\Lambda) = \frac{W_1(\Lambda e^{a_1-1})^2}{2} + W_1(\Lambda e^{a_1-1}). \tag{C25}$$

One can verify that this recovers (C14) which is indeed analytic for $Z > 0$.

APPENDIX D: LAX PAIR FOR THE PROBLEM

The aim of this Appendix is to present a semidiscrete Lax pair for the system (B4). The zero curvature relation for a differential-difference model is defined as follows:

$$\partial_t \vec{v}_n = U_n \vec{v}_n,$$
$$\vec{v}_{n+1} = L_n \vec{v}_n. \tag{D1}$$

The original system (B4) should be found back from the compatibility equation

$$\partial_t L_n = U_{n+1} L_n - L_n U_n. \tag{D2}$$

In our case, $\vec{v}_n$ will be a two-component vector

$$\vec{v}_n = \left(v_n^{(1)}, v_n^{(2)}\right)^\intercal, \tag{D3}$$

and the 2×2 matrices U_n, L_n composing the Lax pair read

$$U_n = \begin{pmatrix} \frac{\lambda^2-1}{2} & -z_{n-1} \\ \frac{g}{2}\tilde{z}_n & \frac{1-\lambda^2}{2} \end{pmatrix}, \quad L_n = \begin{pmatrix} \frac{1}{\lambda} & \frac{z_n}{\lambda} \\ -\frac{g}{2\lambda}\tilde{z}_n & \lambda - \frac{g}{2\lambda} z_n \tilde{z}_n - \frac{a_n}{\lambda} \end{pmatrix}, \tag{D4}$$

where λ is the spectral parameter. We emphasize again here that in our problem the drifts $\{a_n\}$ are zero outside the interval $[1, N]$. Quite remarkably, we note that the matrix L_k admits the factorization

$$L_n = \begin{pmatrix} 1 & 0 \\ -\frac{g}{2}\tilde{z}_n & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\lambda} & 0 \\ 0 & \lambda - \frac{a_n}{\lambda} \end{pmatrix} \begin{pmatrix} 1 & z_n \\ 0 & 1 \end{pmatrix} \tag{D5}$$

so that the contribution of the optimal partition function z_n and the response field $\tilde{z}_n$ can be split.

APPENDIX E: DEFINITION OF THE SCATTERING PROBLEM

Assuming the fields $\{z_n, \tilde{z}_n\}$ as well as the drifts $\{a_n\}$ to decay to 0 for $n \to \pm\infty$, the asymptotic Lax matrices (D4) are diagonal so that we can define two sets of independent solutions asymptotically:

$$\phi_n \underset{n\to-\infty}{\sim} \lambda^{-n}\begin{pmatrix}1\\0\end{pmatrix}, \quad \bar{\phi}_n \underset{n\to-\infty}{\sim} \lambda^{n}\begin{pmatrix}0\\-1\end{pmatrix} \tag{E1}$$

and

$$\psi_n \underset{n\to+\infty}{\sim} \lambda^{n}\begin{pmatrix}0\\1\end{pmatrix}, \quad \bar{\psi}_n \underset{n\to+\infty}{\sim} \lambda^{-n}\begin{pmatrix}1\\0\end{pmatrix}. \tag{E2}$$

Since there can be only two independent solutions, there exists a linear combination relating the two sets which defines the scattering amplitudes:

$$\phi_n = a(\lambda,t)\bar{\psi}_n + b(\lambda,t)\psi_n,$$
$$\bar{\phi}_n = -\tilde{a}(\lambda,t)\psi_n + \tilde{b}(\lambda,t)\bar{\psi}_n \tag{E3}$$

equivalent to implying the following asymptotic conditions at $+\infty$:

$$\phi_n \underset{n\to+\infty}{\sim} \begin{pmatrix} a(\lambda,t)\lambda^{-n} \\ b(\lambda,t)\lambda^{n}\end{pmatrix}, \quad \bar{\phi}_n \underset{n\to+\infty}{\sim} \begin{pmatrix} \tilde{b}(\lambda,t)\lambda^{-n} \\ -\tilde{a}(\lambda,t)\lambda^{n}\end{pmatrix}. \tag{E4}$$

In practice, the two independent solutions $\{\vec{v}_n\}$ verifying the time evolution equation will be chosen as

$$\vec{v}_n = e^{\frac{\lambda^2-1}{2}t}\phi_n \quad \text{and} \quad \vec{v}_n = e^{-\frac{\lambda^2-1}{2}t}\bar{\phi}_n. \tag{E5}$$

1. Time dependence of the scattering amplitudes

Inserting the solutions (E5) into the time equation of the Lax pair (D1) and evaluating it at $n = +\infty$, we obtain that

$$\partial_t a(\lambda,t) = 0,$$
$$\partial_t \tilde{a}(\lambda,t) = 0,$$
$$\partial_t b(\lambda,t) = (1-\lambda^2) b(\lambda,t),$$
$$\partial_t \tilde{b}(\lambda,t) = (\lambda^2-1)\tilde{b}(\lambda,t). \tag{E6}$$

Hence the scattering amplitudes are either time-independent [for $a(\lambda)$ and $\tilde{a}(\lambda)$] or have the following simple time dependence:

$$\tilde{b}(\lambda,t) = \tilde{b}(\lambda)e^{(\lambda^2-1)t}, \quad b(\lambda,t) = b(\lambda)e^{(1-\lambda^2)t}. \tag{E7}$$

The opposite sign in the time evolution of $b(\lambda)$ and $\tilde{b}(\lambda)$ reflects the fact that the saddle point equations describe simultaneously forward and backward evolutions in time. This result is universal as long as the fields vanish at $\pm\infty$.

2. Wronskian of the solution

On top of the time evolution of the scattering amplitudes, we now determine a normalization relation using the Wronskian of the problem. For the two solutions of the Lax problem $\{\phi_n, \bar{\phi}_n\}_{n\in\mathbb{Z}}$, we define the Wronskian as

$$W_n = W(\phi_n, \bar{\phi}_n) = \phi_n^{(1)}\bar{\phi}_n^{(2)} - \phi_n^{(2)}\bar{\phi}_n^{(1)}. \tag{E8}$$

From the evolutions (D1), we start with the time derivative and the index increment

$$\partial_t W_n = \mathrm{Tr}(U_n) W_n = 0,$$
$$W_{n+1} = \mathrm{Det}(L_n) W_n = \left(1 - \frac{a_n}{\lambda^2}\right) W_n. \quad \text{(E9)}$$

The Wronskian of $\{\phi_n, \bar{\phi}_n\}_{n\in\mathbb{Z}}$ at both infinities read

$$W_n \underset{n\to-\infty}{\sim} -1, \quad W_n \underset{n\to+\infty}{\sim} -(a\tilde{a} + b\tilde{b}), \quad \text{(E10)}$$

and we have also have from Eq. (E9)

$$W_{+\infty} = \prod_{n=1}^{N} \left(1 - \frac{a_n}{\lambda^2}\right) W_{-\infty}. \quad \text{(E11)}$$

We then deduce that

$$a(\lambda)\tilde{a}(\lambda) + b(\lambda)\tilde{b}(\lambda) = \prod_{n=1}^{N} \left(1 - \frac{a_n}{\lambda^2}\right). \quad \text{(E12)}$$

This normalization relation is universal. Contrary to the Ablowitz-Ladik integrable system [41,63], the normalization of the scattering amplitudes (E12) does not depend on the fields $\{z_n, \tilde{z}_n\}$.

APPENDIX F: CONSERVED QUANTITIES

We derive in this Appendix the conserved quantities of the integrable system (B4). As we shall see, there exist an infinite amount of conserved quantities, which is standard for such models. To obtain the conserved quantities, we require three ingredients:

(1) The Ricatti equation of the Lax pair

(2) A continuity equation arising from the compatibility of the Lax equations expressed with log derivatives and the Ricatti variables and a relation between the continuity equation and the scattering amplitudes

(3) A suitable Taylor expansion of $\log a(\lambda)$ or Laurent expansion of $\log \tilde{a}(\lambda)$ as a function of the spectral parameter.

1. Ricatti equation

Let us first define the Ricatti variable Γ and its inverse $\tilde{\Gamma}$ as

$$\Gamma_n = \frac{v_n^{(2)}}{v_n^{(1)}}, \quad \tilde{\Gamma}_n = \frac{1}{\Gamma_n}. \quad \text{(F1)}$$

Dividing the two equations of the space part of the Lax pair, we obtain the following recursions for the Ricatti variable and its inverse:

$$(\Gamma_{n+1} + \tilde{z}_n)(1 + z_n\Gamma_n) = \Gamma_n(\lambda^2 - a_n) \quad \text{(F2)}$$

and

$$(1 + \tilde{z}_n\tilde{\Gamma}_{n+1})(\tilde{\Gamma}_n + z_n) = \tilde{\Gamma}_{n+1}(\lambda^2 - a_n). \quad \text{(F3)}$$

As we shall see subsequently, we need to expand these equations to obtain the Taylor and Laurent series of the Ricatti variables.

2. Continuity equations

The continuity equations will be obtained as the compatibility of the dynamics of $\log v_n^{(1)}$ and $\log v_n^{(2)}$, respectively.

a. First continuity equation

From the Lax pair system, we obtain the pair of equations for $\log v_n^{(1)}$,

$$\partial_t \log v_n^{(1)} = \frac{\lambda^2 - 1}{2} - z_{n-1}\Gamma_n$$
$$\log\left(\frac{v_{n+1}^{(1)}}{v_n^{(1)}}\right) = \Delta^+ \log v_n^{(1)} = \log\left(\frac{1}{\lambda} + \frac{z_n}{\lambda}\Gamma_n\right), \quad \text{(F4)}$$

where we introduced the following notation for the finite difference $\Delta^+ f_n = f_{n+1} - f_n$. We rewrite the above system for convenience as

$$\partial_t \log\left(v_n^{(1)}\lambda^n e^{-\frac{\lambda^2-1}{2}t}\right) = -z_{n-1}\Gamma_n$$
$$\log\left(\frac{v_{n+1}^{(1)}\lambda^{n+1}e^{-\frac{\lambda^2-1}{2}t}}{v_n^{(1)}\lambda^n e^{-\frac{\lambda^2-1}{2}t}}\right) = \Delta^+ \log\left(v_n^{(1)}\lambda^n e^{-\frac{\lambda^2-1}{2}t}\right)$$
$$= \log(1 + z_n\Gamma_n). \quad \text{(F5)}$$

The compatibility is obtained by the commutation relation

$$\partial_t \Delta^+ = \Delta^+ \partial_t, \quad \text{(F6)}$$

which yields the first compatibility equation

$$\partial_t \log(1 + z_n\Gamma_n) = -\Delta^+(z_{n-1}\Gamma_n). \quad \text{(F7)}$$

We therefore interpret $J_n^{(1)} = z_{n-1}\Gamma_n$ as a generalized current and $\varrho_n^{(1)} = \log(1 + z_n\Gamma_n)$ as a generalized density. In particular, since the potentials $\{z_n, \tilde{z}_n\}$ vanish at infinities, we have the conservation law

$$\partial_t \left(\sum_{n=-\infty}^{+\infty} \varrho_n^{(1)}\right) = 0. \quad \text{(F8)}$$

With the particular choice of $\vec{v}_n = e^{\frac{\lambda^2-1}{2}t}\phi_n$ and summing the second equation of (F5) over integers in $\mathbb{Z}$ we obtain the expected relation between the scattering amplitude and this set of conserved charges as

$$\log a(\lambda) = \sum_{n=-\infty}^{+\infty} \log(1 + z_n\Gamma_n)$$
$$\longleftrightarrow \quad a(\lambda) = \prod_{n=-\infty}^{+\infty} (1 + z_n\Gamma_n). \quad \text{(F9)}$$

b. Second continuity equation

We now repeat the same exercise for $\log v_n^{(2)}$ and first obtain

$$\partial_t \log v_n^{(2)} = \frac{1 - \lambda^2}{2} + \tilde{z}_n\tilde{\Gamma}_n$$
$$\log\left(\frac{v_{n+1}^{(2)}}{v_n^{(2)}}\right) = \Delta^+ \log v_n^{(2)}$$
$$= \log\left(-\tilde{z}_n\tilde{\Gamma}_n + \lambda - \frac{1}{\lambda}z_n\tilde{z}_n - \frac{a_n}{\lambda}\right), \quad \text{(F10)}$$

which is also rewritten for convenience as

$$\partial_t \log\left(-v_n^{(2)}\lambda^{-n}e^{\frac{\lambda^2-1}{2}t}\right)=\tilde{z}_n\tilde{\Gamma}_n$$
$$\log\left(\frac{-v_{n+1}^{(2)}\lambda^{-n-1}e^{\frac{\lambda^2-1}{2}t}}{-v_n^{(2)}\lambda^{-n}e^{\frac{\lambda^2-1}{2}t}}\right)=\log\left[1-\frac{1}{\lambda^2}\tilde{z}_n(\tilde{\Gamma}_n+z_n)-\frac{a_n}{\lambda^2}\right]. \tag{F11}$$

The compatibility in this case reads

$$\partial_t \log\left[1-\frac{1}{\lambda^2}\tilde{z}_n(\tilde{\Gamma}_n+z_n)-\frac{a_n}{\lambda^2}\right]=-\Delta^+(-\tilde{z}_n\tilde{\Gamma}_n). \tag{F12}$$

We also interpret $J_n^{(2)}=-\tilde{z}_n\tilde{\Gamma}_n$ as a generalized current and $\varrho_n^{(2)}=\log[1-\frac{1}{\lambda^2}\tilde{z}_n(\tilde{\Gamma}_n+z_n)-\frac{a_n}{\lambda^2}]$ as a generalized density. In particular, since the potentials $\{z_n,\tilde{z}_n\}$ vanish at infinities, we have the conservation law

$$\partial_t\left(\sum_{n=-\infty}^{+\infty}\varrho_n^{(2)}\right)=0. \tag{F13}$$

With the particular choice of $\vec{v}_n=e^{\frac{1-\lambda^2}{2}t}\bar{\phi}_n$ and summing the second equation of (F11) over integers in $\mathbb{Z}$ we obtain the expected relation between the scattering amplitude and this set of conserved charges as

$$\log\tilde{a}(\lambda)=\sum_{n=-\infty}^{+\infty}\log\left[1-\frac{1}{\lambda^2}\tilde{z}_n(\tilde{\Gamma}_n+z_n)-\frac{a_n}{\lambda^2}\right]$$
$$\longleftrightarrow\quad \tilde{a}(\lambda)=\prod_{n=-\infty}^{+\infty}\left[1-\frac{1}{\lambda^2}\tilde{z}_n(\tilde{\Gamma}_n+z_n)-\frac{a_n}{\lambda^2}\right]. \tag{F14}$$

3. Conserved charges

We now complete the determination of the conserved charges in the system by proceeding to the suitable expansion of the continuity equations. We now take all drifts equal to 0, i.e., $a_n=0$, to simplify the expressions.

a. Taylor expansion of $a(\lambda)$

Since $\log a(\lambda)$ is analytic close to the origin, we choose to expand Eqs. (F2), (F3), and (F9) in a Taylor series

$$\log a(\lambda)=\sum_{\ell=0}^{\infty}\lambda^{2\ell}C_\ell,\;\;\Gamma_n=\sum_{\ell=0}^{\infty}\lambda^{2\ell}\Gamma_n^{(\ell,0)},$$
$$\tilde{\Gamma}_n=\sum_{\ell=0}^{\infty}\lambda^{2\ell}\tilde{\Gamma}_n^{(\ell,0)} \tag{F15}$$

and

$$\varrho_n^{(1)}=\sum_{\ell=0}^{\infty}\lambda^{2\ell}\varrho_{n,\ell}^{(1)},\quad J_n^{(1)}=\sum_{\ell=0}^{\infty}\lambda^{2\ell}J_{n,\ell}^{(1)},\quad C_\ell=\sum_{n\in\mathbb{Z}}\varrho_{n,\ell}^{(1)}. \tag{F16}$$

Formally, there exists two solutions to the Taylor expansion of Eqs. (F2) and (F3):

(1) The first one is physical as it is consistent with a Taylor expansion of $\log a(\lambda)$, and it provides the expected conserved quantities;

(2) The second one is unphysical as it would impose that $\log a(\lambda)$ behaves as $\log\lambda^2$ for small spectral parameter and therefore that the solution would include a zero mode, incompatible with the decay of $z_n,\tilde{z}_n$ at infinities. Nonetheless formally, this expansion yields additional conserved quantities, which conservation can also be checked by hand, and for completeness we will provide them.

The first solution, the "physical" expansion, yields for its first two terms

(1) For $\ell=0$

$$\varrho_{n,0}^{(1)}=\log(1-z_n\tilde{z}_{n-1}),\quad J_{n,0}^{(1)}=-z_{n-1}\tilde{z}_{n-1}. \tag{F17}$$

(2) For $\ell=1$

$$\varrho_{n,1}^{(1)}=-\frac{z_n\tilde{z}_{n-2}}{(1-z_{n-1}\tilde{z}_{n-2})(1-z_n\tilde{z}_{n-1})},$$
$$J_{n,1}^{(1)}=1-\frac{1}{1-z_{n-1}\tilde{z}_{n-2}}. \tag{F18}$$

The conserved charges are then

$$C_0=\sum_{n=-\infty}^{\infty}\log(1-z_n\tilde{z}_{n-1}),$$
$$C_1=\sum_{n=-\infty}^{\infty}-\frac{z_n\tilde{z}_{n-2}}{(1-z_{n-1}\tilde{z}_{n-2})(1-z_n\tilde{z}_{n-1})}, \tag{F19}$$

and one can verify using the dynamical equations for $\{z_n,\tilde{z}_n\}$(10) that for any solution we obtain

$$\partial_t\varrho_{n,\ell}^{(1)}=J_{n,\ell}^{(1)}-J_{n+1,\ell}^{(1)}. \tag{F20}$$

The second solution, the "unphysical" expansion, yields for its first two terms (we use again the same notation for simplicity)

(1) For $\ell=0$

$$\varrho_{n,0}^{(1)}=\log\left(\frac{z_{n+1}}{z_n(1-z_{n+1}\tilde{z}_n)}\right),\quad J_{n,0}^{(1)}=-\frac{z_{n-1}}{z_n}. \tag{F21}$$

(2) For $\ell=1$

$$\varrho_{n,1}^{(1)}=\frac{\frac{z_{n+2}}{z_{n+2}\tilde{z}_{n+1}-1}+\frac{z_{n+1}^2}{z_n}}{z_{n+1}(z_{n+1}\tilde{z}_n-1)},\quad J_{n,1}^{(1)}=-\frac{z_{n-1}z_{n+1}}{z_n^2(z_{n+1}\tilde{z}_n-1)}. \tag{F22}$$

b. Laurent expansion of $\tilde{a}(\lambda)$

Since $\log\tilde{a}(\lambda)$ is analytic for large $|\lambda|$, we choose to expand Eqs. (F2), (F3), and (F14) as Laurent series

$$\log\tilde{a}(\lambda)=\sum_{\ell=1}^{\infty}\frac{\tilde{C}_\ell}{\lambda^{2\ell}},\quad \Gamma_n=\sum_{\ell=1}^{\infty}\frac{\Gamma_n^{(\ell,\infty)}}{\lambda^{2\ell}},\quad \tilde{\Gamma}_n=\sum_{\ell=1}^{\infty}\frac{\tilde{\Gamma}_n^{(\ell,\infty)}}{\lambda^{2\ell}} \tag{F23}$$

and

$$\varrho_n^{(2)}=\sum_{\ell=1}^{\infty}\frac{\varrho_{n,\ell}^{(2)}}{\lambda^{2\ell}},\quad J_n^{(2)}=\sum_{\ell=1}^{\infty}\frac{J_{n,\ell}^{(2)}}{\lambda^{2\ell}},\quad \tilde{C}_\ell=\sum_{n\in\mathbb{Z}}\varrho_{n,\ell}^{(2)}. \tag{F24}$$

Formally, there exists two solutions to the Laurent expansion of Eqs. (F2) and (F3):

(1) The first one is physical as it is consistent with a Laurent expansion of $\log\tilde{a}(\lambda)$ and it provides the expected conserved quantities;

(2) The second one is unphysical as it would impose that $\log \tilde{a}(\lambda)$ behaves as $\log \lambda^2$ for large spectral parameter whereas we seek a solution for which $\log \tilde{a}(\lambda) \to 0$ for $|\lambda| \to \infty$. Nonetheless, formally this expansion yields additional conserved quantities, which conservation can also be checked by hand, and for completeness we will provide them.

The first solution, the "physical" expansion, yields for its first two terms

(1) For $\ell = 1$

$$\varrho^{(2)}_{n,1} = -z_n\tilde{z}_n, \quad J^{(2)}_{n,1} = -z_{n-1}\tilde{z}_n. \tag{F25}$$

(2) For $\ell = 2$

$$\varrho^{(2)}_{n,2} = -\tfrac{1}{2}\tilde{z}_n\big(z_n^2\tilde{z}_n + 2z_{n-1}\big), \quad J^{(2)}_{n,2} = -\tilde{z}_n\big(z_{n-1}^2\tilde{z}_{n-1} + z_{n-2}\big). \tag{F26}$$

The conserved charges are then

$$\tilde{C}_1 = -\sum_{n=-\infty}^{\infty} z_n\tilde{z}_n,$$
$$\tilde{C}_2 = -\sum_{n=-\infty}^{\infty} \frac{1}{2}\tilde{z}_n\big(z_n^2\tilde{z}_n + 2z_{n-1}\big), \tag{F27}$$

and one can verify using the dynamical equations for $\{z_n, \tilde{z}_n\}$(10) that for any solution we obtain

$$\partial_t\varrho^{(2)}_{n,\ell} = J^{(2)}_{n,\ell} - J^{(2)}_{n+1,\ell}. \tag{F28}$$

The second solution, the "unphysical" expansion, yields for its first two terms (we use again the same notation for simplicity)

(1) For $\ell = 1$

$$\varrho^{(2)}_{n,1} = \log\left(\frac{\tilde{z}_{n+1}}{\tilde{z}_n}\right), \quad J^{(2)}_{n,1} = z_n\tilde{z}_n + \frac{\tilde{z}_{n+1}}{\tilde{z}_n}. \tag{F29}$$

(2) For $\ell = 2$

$$\varrho^{(2)}_{n,2} = z_{n+1}\tilde{z}_{n+1} - \frac{\tilde{z}_{n+1}}{\tilde{z}_n} + \frac{\tilde{z}_{n+2}}{\tilde{z}_{n+1}},$$
$$J^{(2)}_{n,2} = \frac{z_{n+1}\tilde{z}_{n+1}^2}{\tilde{z}_n} - \frac{\tilde{z}_{n+1}^2}{\tilde{z}_n^2} + \frac{\tilde{z}_{n+2}}{\tilde{z}_n}. \tag{F30}$$

4. Discussion on generalized Gibbs ensembles and other flows generated by the conserved charges

We first comment on the fact that the quantity $(z_n\tilde{z}_n)$ is at the same a local density and a local current as

$$-z_n\tilde{z}_n = J^{(1)}_{n,0} = \varrho^{(2)}_{n,1}. \tag{F31}$$

Such equality also arises in other models and has *surprising consequences* (see Ref. [54], Sec. 2.1) in the context of the Toda lattice. It would be interesting in the present context to analyze the consequence of this identity.

What is more, from the knowledge of the conserved quantities, it is possible to formally define a generalized Gibbs measure on the fields $z_n, \tilde{z}_n$ that is conserved by the dynamics (10). Such construction was done for a variety of semidiscrete integrable models in recent works; see Refs. [51–55]. To this aim, we select a subset of conserved quantities, e.g., $C_0, \tilde{C}_1, \tilde{C}_2$, and a set of conjugated "temperature" (in this example $\beta_0, \beta_1, \beta_2$) and construct the measure

$$\prod_{n=1}^{N} \mathrm{d}z_n\mathrm{d}\tilde{z}_n \exp(-\beta_0C_0 - \beta_1\tilde{C}_1 - \beta_2\tilde{C}_2). \tag{F32}$$

If the initial conditions of the OY system are chosen randomly according to the measure (F32), we then expect this distribution of $z_n, \tilde{z}_n$ to be stationary as the dynamics will preserve $C_0, \tilde{C}_1, \tilde{C}_2$. Such measure can a priori be extended to arbitrary number of conserved quantities and for the usual semidiscrete integrable models, this measure was related to measures appearing in Random Matrix Theory [51–55], we expect such a connexion to hold in this case as well.

Finally, we address a last comment about the conserved charges which is the flow induced by these. We can recast the original OY system (10) as

$$\partial_t z_n = \frac{\delta}{\delta\tilde{z}_n}(\tilde{C}_1 - \tilde{C}_2),$$
$$-\partial_t\tilde{z}_n = \frac{\delta}{\delta z_n}(\tilde{C}_1 - \tilde{C}_2). \tag{F33}$$

Similarly to Ref. [38], one could define other dynamics by replacing in (F33) the term $\tilde{C}_1 - \tilde{C}_2$ by any linear combination of conserved quantities. The differential system induced would remain integrable with the same space Lax matrix L_n in (D4) but the time Lax matrix U_n would have to be modified.

The Hamiltonian system (F33) can also be rewritten in a symplectic form by introducing the suitable Poisson bracket. To this aim, we introduce the field gradient

$$\nabla_n = \begin{pmatrix} \dfrac{\delta}{\delta z_n} \\ \dfrac{\delta}{\delta\tilde{z}_n}, \end{pmatrix} \tag{F34}$$

and rewrite (F33) as

$$\partial_t\begin{pmatrix} z_n \\ \tilde{z}_n \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\nabla_n(\tilde{C}_1 - \tilde{C}_2). \tag{F35}$$

Then by introducing the Poisson bracket

$$\langle F, G\rangle = \sum_{n=-\infty}^{+\infty} \nabla_n(F)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\nabla_n(G), \tag{F36}$$

we have that if $I[z, \tilde{z}]$ is a functional of $z, \tilde{z}$, its dynamic is given by

$$\partial_t I[z, \tilde{z}] = \langle I[z, \tilde{z}], \tilde{C}_1 - \tilde{C}_2\rangle. \tag{F37}$$

In the present case, the canonical Poisson bracket (F36) is the natural structure behind the dynamics but other Poisson brackets could appear in different semidiscrete integrable models as is the case for continuous systems where the Poisson bracket is not the same for the nonlinear Schrödinger equation [63] and the derivative nonlinear Schrödinger equation [21].

APPENDIX G: SOLUTION OF THE SCATTERING PROBLEM FOR ARBITRARY g

We now obtain the solution of the scattering problem associated to the nonlinear system for arbitrary g, for the specific initial and final conditions considered in this paper, that we now recall,

$$z_n(t=0)=\delta_{n,1},\quad \tilde{z}_n(t=1)=-\Lambda\delta_{N,n}, \tag{G1}$$

and we recall that $z_{n\leqslant 0}(t)=0$ and $\tilde{z}_{n\geqslant N+1}(t)=0$. The scattering problem is defined by the recursion, for $n\in\mathbb{Z}$,

$$\phi_{n+1}=\begin{pmatrix}\phi^{(1)}_{n+1}\\ \phi^{(2)}_{n+1}\end{pmatrix}=\begin{pmatrix}\frac{1}{\lambda} & \frac{z_n}{\lambda}\\ -\frac{g}{2\lambda}\tilde{z}_n & \lambda-\frac{g}{2\lambda}z_n\tilde{z}_n-\frac{a_n}{\lambda}\end{pmatrix}\begin{pmatrix}\phi^{(1)}_n\\ \phi^{(2)}_n\end{pmatrix} \tag{G2}$$

together with the same equation for $\bar{\phi}_n$, with the following boundary conditions at $n\to\pm\infty$:

$$\phi_n\sim\lambda^{-n}\begin{pmatrix}1\\0\end{pmatrix},\quad \bar{\phi}_n\sim\lambda^{n}\begin{pmatrix}0\\-1\end{pmatrix},\quad n\to-\infty, \tag{G3}$$

$$\phi_n\sim\begin{pmatrix}a(\lambda)\lambda^{-n}\\ b(\lambda)e^{(1-\lambda^2)t}\lambda^n\end{pmatrix},\quad \bar{\phi}_n\sim\begin{pmatrix}\tilde{b}(\lambda)e^{(\lambda^2-1)t}\lambda^{-n}\\ -\tilde{a}(\lambda)\lambda^n\end{pmatrix},\quad n\to+\infty. \tag{G4}$$

We will now consider the scattering problem successively at the initial time $t=0$ and at the final time $t=1$. In each case we consider it first for ϕ and then for $\bar{\phi}$.

1. Scattering problem at $t=0$

a. A.1 For ϕ

One must solve, from (G2) and the initial condition (G1),

$$\phi^{(1)}_{n+1}=\frac{1}{\lambda}\phi^{(1)}_n+\frac{1}{\lambda}\delta_{n,1}\phi^{(2)}_n,$$
$$\phi^{(2)}_{n+1}=-\frac{g}{2\lambda}\tilde{z}_n\phi^{(1)}_n+\left(\lambda-\frac{g}{2\lambda}\tilde{z}_n\delta_{n,1}-\frac{a_n}{\lambda}\right)\phi^{(2)}_n. \tag{G5}$$

Using (G3), the first equation implies that

$$\phi^{(1)}_n=\lambda^{-n},\quad n\leqslant 1,$$
$$\phi^{(1)}_n=\lambda^{-n}+\lambda^{1-n}\phi^{(2)}_1,\quad n\geqslant 2. \tag{G6}$$

In particular we obtain from (G4) that

$$a(\lambda)=1+\lambda\phi^{(2)}_1. \tag{G7}$$

The second equation in (G5) then leads to (we recall that $a_n=0$ for $n<1$ and for $n>N$)

$$\phi^{(2)}_{n+1}=-\frac{g}{2}\tilde{z}_n\lambda^{-n-1}+\lambda\phi^{(2)}_n,\quad n<1,$$
$$\phi^{(2)}_{n+1}=-\frac{g}{2}\tilde{z}_n\lambda^{-n-1}\left(1+\lambda\phi^{(2)}_1\right)+\lambda\left(1-\frac{a_n}{\lambda^2}\right)\phi^{(2)}_n,\quad n>1, \tag{G8}$$

which implies using (G3)

$$\phi^{(2)}_n=\sum_{k=-\infty}^{n-1}-\frac{g}{2}\tilde{z}_k\lambda^{n-2-2k},\quad n\leqslant 1, \tag{G9}$$

$$\phi^{(2)}_2=-\frac{g}{2\lambda^2}\tilde{z}_1+\left[\lambda-\frac{g}{2\lambda}\tilde{z}_1(t=0)-\frac{a_1}{\lambda}\right]\phi^{(2)}_1,\quad n=2, \tag{G10}$$

$$\phi^{(2)}_n=\sum_{k=2}^{n-1}\left(-\frac{g}{2}\tilde{z}_k\lambda^{n-2-2k}\left(1+\lambda\phi^{(2)}_1\right)\prod_{\ell=k+1}^{n-1}\left(1-\frac{a_\ell}{\lambda^2}\right)\right)+\lambda^{n-2}\prod_{\ell=2}^{n-1}\left(1-\frac{a_\ell}{\lambda^2}\right)\phi^{(2)}_2,\quad n\geqslant 3. \tag{G11}$$

We can simplify the $n=2$ term as

$$\phi^{(2)}_2=-\frac{g}{2\lambda^2}\tilde{z}_1(t=0)+\left[\lambda-\frac{g}{2\lambda}\tilde{z}_1(t=0)-\frac{a_1}{\lambda}\right]\frac{a(\lambda)-1}{\lambda}$$
$$=a(\lambda)\left[1-\frac{g}{2\lambda^2}\tilde{z}_1(t=0)-\frac{a_1}{\lambda^2}\right]-1+\frac{a_1}{\lambda^2}. \tag{G12}$$

We finally obtain $a(\lambda)$ from (G7) with $\phi^{(2)}_1$ given in (G9), and $b(\lambda)$ from the asymptotic behavior at $n\to+\infty$ of (G11):

$$a(\lambda)=1-\frac{g}{2}\sum_{k=-\infty}^{0}\tilde{z}_k(t=0)\lambda^{-2k}, \tag{G13}$$

$$b(\lambda)=\frac{a(\lambda)}{\lambda^2}\sum_{k=1}^{\infty}\left(-\frac{g}{2}\tilde{z}_k(t=0)\lambda^{-2k}\prod_{\ell=k+1}^{N}\left(1-\frac{a_\ell}{\lambda^2}\right)\right)+\frac{a(\lambda)-1}{\lambda^2}\prod_{\ell=1}^{N}\left(1-\frac{a_\ell}{\lambda^2}\right). \tag{G14}$$

b. A.2 For $\bar{\phi}$

The vector $\bar{\phi}$ satisfies the same equation as ϕ but with different boundary conditions at infinity [see (G3) and (G4)]:

$$\bar{\phi}^{(1)}_{n+1}=\frac{1}{\lambda}\bar{\phi}^{(1)}_n+\frac{1}{\lambda}\delta_{n,1}\bar{\phi}^{(2)}_n,$$
$$\bar{\phi}^{(2)}_{n+1}=-\frac{g}{2\lambda}\tilde{z}_n\bar{\phi}^{(1)}_n+\left(\lambda-\frac{g}{2\lambda}\tilde{z}_n\delta_{n,1}-\frac{a_n}{\lambda}\right)\bar{\phi}^{(2)}_n. \tag{G15}$$

Using (G3) the first equation gives

$$\bar{\phi}^{(1)}_n=0,\quad n\leqslant 1,$$
$$\bar{\phi}^{(1)}_n=\frac{\bar{\phi}^{(2)}_1}{\lambda^{n-1}},\quad n\geqslant 2, \tag{G16}$$

which from (G4) implies that

$$\tilde{b}(\lambda)=\lambda\bar{\phi}^{(2)}_1. \tag{G17}$$

Using (G3) the second equation in (G15) can be solved as

$$\bar{\phi}^{(2)}_n=-\lambda^n,\quad n\leqslant 1,$$
$$\bar{\phi}^{(2)}_2=-\left(\lambda-\frac{g}{2\lambda}\tilde{z}_1(t=0)-\frac{a_1}{\lambda}\right)\lambda,\quad n=2,$$
$$\bar{\phi}^{(2)}_n=\sum_{k=2}^{n-1}\left(\frac{g}{2}\tilde{z}_k\lambda^{n-2k}\prod_{\ell=k+1}^{n-1}\left(1-\frac{a_\ell}{\lambda^2}\right)\right)+\lambda^{n-2}\prod_{\ell=2}^{n-1}\left(1-\frac{a_\ell}{\lambda^2}\right)\bar{\phi}^{(2)}_2,\quad n\geqslant 3. \tag{G18}$$

From the $n \to +\infty$ asymptotics using (G4) we thus obtain

$$\tilde{a}(\lambda) = -\sum_{k=2}^{\infty}\left(\frac{g}{2}\tilde{z}_k\lambda^{-2k}\prod_{\ell=k+1}^{N}\left(1-\frac{a_\ell}{\lambda^2}\right)\right) - \lambda^{-2}\prod_{\ell=2}^{N}\left(1-\frac{a_\ell}{\lambda^2}\right)\bar{\phi}_2^{(2)}, \tag{G19}$$

which, upon the replacement of the value of $\bar{\phi}_1^{(2)} = -\lambda$ and of $\bar{\phi}_2^{(2)}$ leads to

$$\tilde{a}(\lambda) = \prod_{\ell=1}^{N}\left(1-\frac{a_\ell}{\lambda^2}\right) - \sum_{k=1}^{\infty}\left(\frac{g}{2}\tilde{z}_k(t=0)\lambda^{-2k}\prod_{\ell=k+1}^{N}\left(1-\frac{a_\ell}{\lambda^2}\right)\right). \tag{G20}$$

$$\tilde{b}(\lambda) = -\lambda^2 \tag{G21}$$

At this stage it is interesting to verify the normalization relation, by inserting (G20) and (G14),

$$\begin{aligned} a(\lambda)\tilde{a}(\lambda) + b(\lambda)\tilde{b}(\lambda) &= a(\lambda)\prod_{\ell=1}^{N}\left(1-\frac{a_\ell}{\lambda^2}\right) \\ &\quad - a(\lambda)\sum_{k=1}^{\infty}\left(\frac{g}{2}\tilde{z}_k(t=0)\lambda^{-2k}\prod_{\ell=k+1}^{N}\left(1-\frac{a_\ell}{\lambda^2}\right)\right), \\ &\quad - a(\lambda)\sum_{k=1}^{\infty}\left(-\frac{g}{2}\tilde{z}_k(t=0)\lambda^{-2k}\prod_{\ell=k+1}^{N}\left(1-\frac{a_\ell}{\lambda^2}\right)\right) \\ &\quad + [1-a(\lambda)]\prod_{\ell=1}^{N}\left(1-\frac{a_\ell}{\lambda^2}\right) \\ &= \prod_{\ell=1}^{N}\left(1-\frac{a_\ell}{\lambda^2}\right), \end{aligned} \tag{G22}$$

which provides a nontrivial check of our calculations.

2. Scattering problem at $t = 1$

a. B.1 For ϕ

Using that $\tilde{z}_n(t=1) = -\Lambda\delta_{N,\ell}$ one must solve

$$\begin{aligned} \phi_{n+1}^{(1)} &= \frac{1}{\lambda}\phi_n^{(1)} + \frac{1}{\lambda}z_n\phi_n^{(2)}, \\ \phi_{n+1}^{(2)} &= \frac{g\Lambda}{2\lambda}\delta_{n,N}\phi_n^{(1)} + \left(\lambda + \frac{g\Lambda}{2\lambda}z_n\delta_{n,N} - \frac{a_n}{\lambda}\right)\phi_n^{(2)}. \end{aligned} \tag{G23}$$

Let us solve first the second equation, using the boundary condition (G3). with $t = 1$ One finds (recalling that $a_n = 0$ for $n \geqslant N+1$)

$$\begin{aligned} \phi_n^{(2)} &= 0, \quad n \leqslant N, \\ \phi_n^{(2)} &= \frac{g\Lambda}{2}\lambda^{n-N-2}\phi_N^{(1)}, \quad n \geqslant N+1. \end{aligned} \tag{G24}$$

Hence from (G4) we obtain

$$b(\lambda)e^{1-\lambda^2} = \frac{g\Lambda}{2}\lambda^{-N-2}\phi_N^{(1)}. \tag{G25}$$

We can now solve the first equation in (G23) and obtain

$$\begin{aligned} \phi_n^{(1)} &= \lambda^{-n}, \quad n \leqslant N+1, \\ \phi_n^{(1)} &= \lambda^{-n} + \sum_{k=N+1}^{n-1} z_k(t=1)\frac{g\Lambda}{2}\lambda^{2k-n-N-2}\phi_N^{(1)}, \\ &\quad n \geqslant N+2. \end{aligned} \tag{G26}$$

From the asymptotics for $n \to +\infty$ and (G4), inserting $\phi_N^{(1)} = \lambda^{-1}$ one finds

$$a(\lambda) = 1 + \frac{g\Lambda}{2}\sum_{k=N+1}^{\infty} z_k(t=1)\lambda^{2k-2N-2}, \tag{G27}$$

$$b(\lambda)e^{1-\lambda^2} = \frac{g\Lambda}{2}\lambda^{-2N-2}. \tag{G28}$$

b. B.2 For $\bar{\phi}$

One must solve for $\bar{\phi}$

$$\begin{aligned} \bar{\phi}_{n+1}^{(1)} &= \frac{1}{\lambda}\bar{\phi}_n^{(1)} + \frac{1}{\lambda}z_n\bar{\phi}_n^{(2)}, \\ \bar{\phi}_{n+1}^{(2)} &= \frac{g\Lambda}{2\lambda}\delta_{n,N}\bar{\phi}_n^{(1)} + \left(\lambda + \frac{g\Lambda}{2\lambda}z_n\delta_{n,N} - \frac{a_n}{\lambda}\right)\bar{\phi}_n^{(2)}. \end{aligned} \tag{G29}$$

Again we start with the second equation, using the asymptotics in (G3). We find

$$\begin{aligned} \bar{\phi}_n^{(2)} &= -\lambda^n\prod_{\ell=1}^{n-1}\left(1-\frac{a_\ell}{\lambda^2}\right), \quad n \leqslant N, \\ \bar{\phi}_n^{(2)} &= \lambda^{n-N-1}\left[\frac{g\Lambda}{2\lambda}\bar{\phi}_N^{(1)} - \left(\lambda + \frac{g\Lambda}{2\lambda}z_N - \frac{a_N}{\lambda}\right)\lambda^N\prod_{\ell=1}^{N-1}\left(1-\frac{a_\ell}{\lambda^2}\right)\right], \\ &\quad n \geqslant N+1. \end{aligned} \tag{G30}$$

From the $n \to +\infty$ asymptotics and (G4) one finds

$$\tilde{a}(\lambda) = -\lambda^{-N-1}\left[\frac{g\Lambda}{2\lambda}\bar{\phi}_N^{(1)} - \left(\lambda + \frac{g\Lambda}{2\lambda}z_N - \frac{a_N}{\lambda}\right)\lambda^N\prod_{\ell=1}^{N-1}\times\left(1-\frac{a_\ell}{\lambda^2}\right)\right]. \tag{G31}$$

The solution for $\bar{\phi}_n^{(2)}$ can then be inserted in the first equation in (G29). It reads

$$\begin{aligned} \bar{\phi}_{n+1}^{(1)} &= \frac{1}{\lambda}\bar{\phi}_n^{(1)} - \lambda^{n-1}z_n\prod_{\ell=1}^{n-1}\left(1-\frac{a_\ell}{\lambda^2}\right), \quad n \leqslant N, \\ \bar{\phi}_{n+1}^{(1)} &= \frac{1}{\lambda}\bar{\phi}_n^{(1)} + z_n\lambda^{n-N-2}\left(\frac{g\Lambda}{2\lambda}\bar{\phi}_N^{(1)} - \left[\lambda + \frac{g\Lambda}{2\lambda}z_N - \frac{a_N}{\lambda}\right)\right. \\ &\quad \left.\times\lambda^N\prod_{\ell=1}^{N-1}\left(1-\frac{a_\ell}{\lambda^2}\right)\right], \quad n \geqslant N+1. \end{aligned} \tag{G32}$$

Taking into account the boundary condition (G3), its solution is found as

$$\bar{\phi}_n^{(1)} = -\sum_{k=-\infty}^{n-1} z_k \lambda^{2k-n} \prod_{\ell=1}^{k-1}\left(1-\frac{a_\ell}{\lambda^2}\right), \quad n \leqslant N+1,$$

$$\bar{\phi}_n^{(1)} = \lambda^{-n+N+1}\bar{\phi}_{N+1}^{(1)} + \left(\frac{g\Lambda}{2\lambda}\bar{\phi}_N^{(1)} - \left[\lambda + \frac{g\Lambda}{2\lambda}z_N - \frac{a_N}{\lambda}\right)\right.$$
$$\times \lambda^N \prod_{\ell=1}^{N-1}\left(1-\frac{a_\ell}{\lambda^2}\right)\Bigg] \sum_{k=N+1}^{n-1} z_k \lambda^{2k-n-N-1}, \quad n \geqslant N+2. \tag{G33}$$

From the asymptotics at $n \to +\infty$ and (G4) one finds

$$\tilde{b}(\lambda)e^{\lambda^2-1} = \lambda^{N+1}\bar{\phi}_{N+1}^{(1)} + \left[\frac{g\Lambda}{2\lambda}\bar{\phi}_N^{(1)} - \left(\lambda + \frac{g\Lambda}{2\lambda}z_N - \frac{a_N}{\lambda}\right)\right.$$
$$\times \lambda^N \prod_{\ell=1}^{N-1}\left(1-\frac{a_\ell}{\lambda^2}\right)\Bigg] \sum_{k=N+1}^{\infty} z_k \lambda^{2k-N-1}. \tag{G34}$$

Inserting the value of $\bar{\phi}_N^{(1)}$ and of $\bar{\phi}_{N+1}^{(1)}$ in (G31) and in (G34) we obtain

$$\tilde{a}(\lambda) = \prod_{\ell=1}^{N}\left(1-\frac{a_\ell}{\lambda^2}\right) + \frac{g\Lambda}{2}\sum_{k=-\infty}^{N}$$
$$\times \left(z_k(t=1)\lambda^{2k-2N-2}\prod_{\ell=1}^{k-1}\left(1-\frac{a_\ell}{\lambda^2}\right)\right), \tag{G35}$$

$$\tilde{b}(\lambda)e^{\lambda^2-1} = -\sum_{k=-\infty}^{N} z_k(t=1)\lambda^{2k}\prod_{\ell=1}^{k-1}\left(1-\frac{a_\ell}{\lambda^2}\right)$$
$$-\,\tilde{a}(\lambda)\sum_{k=N+1}^{\infty} z_k(t=1)\lambda^{2k}. \tag{G36}$$

3. Summary: Result for the scattering amplitudes

In summary, the scattering amplitudes $b(\lambda)$ and $\tilde{b}(\lambda)$ have been completely determined. They read

$$\tilde{b}(\lambda) = -\lambda^2,$$
$$b(\lambda) = \frac{g\Lambda}{2}\lambda^{-2N-2}e^{\lambda^2-1}. \tag{G37}$$

By contrast, for each of the amplitudes $a(\lambda)$ and $\tilde{a}(\lambda)$ we has obtained only two relations to the two unknown set of variables $z_n(t=1)$ and $\tilde{z}_n(t=0)$. These relations read

$$a(\lambda) = 1 + \frac{g\Lambda}{2}\sum_{n=N+1}^{\infty} z_n(t=1)\lambda^{2n-2N-2}$$
$$= 1 - \frac{g}{2}\sum_{n=-\infty}^{0}\tilde{z}_n(t=0)\lambda^{-2n}. \tag{G38}$$

A priori we have obtained each relation (each line) separately. However, one can check that due to the symmetry

$$\{\tilde{z}_\ell(t) = -\Lambda z_{N-\ell+1}(1-t), \quad a_\ell = a_{N-\ell+1}\} \tag{G39}$$

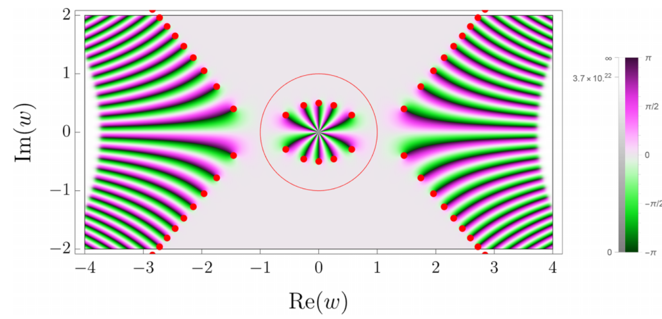

FIG. 4. Representation of arg(G), i.e., the phase (shown by a color code) of the rescaled function $w \mapsto 1 + \frac{\Lambda}{|\Lambda_c|}w^{-2N}e^{N(w^2-1)} = G(\lambda = w\sqrt{N})$ for $N = 5$ and $\frac{\Lambda}{|\Lambda_c|} = 0.5$ in the complex plane. The plot was made using the `ComplexPlot` function of *Mathematica* with the `GreenPinkTones` color function. The red dots represent the zeros of the function G, and we have represented the circle of radius $|w| = 1$, which corresponds to contour $\mathcal{C}$ (equal to the circle $|\lambda| = \mathtt{R} = \sqrt{N}$ mentioned in the text). The black lines can be interpreted as the positions of branch cuts in $\log G$, and we see that the contour $\mathcal{C}$ does not cross them.

these two expressions are identical. Similarly we have found

$$\tilde{a}(\lambda) = \prod_{\ell=1}^{N}\left(1-\frac{a_\ell}{\lambda^2}\right) + \frac{g\Lambda}{2}\sum_{k=-\infty}^{N}$$
$$\times \left(z_k(t=1)\lambda^{2k-2N-2}\prod_{\ell=1}^{k-1}\left(1-\frac{a_\ell}{\lambda^2}\right)\right)$$
$$= \prod_{\ell=1}^{N}\left(1-\frac{a_\ell}{\lambda^2}\right)$$
$$-\sum_{k=1}^{\infty}\left(\frac{g}{2}\tilde{z}_k(t=0)\lambda^{-2k}\prod_{\ell=k+1}^{N}\left(1-\frac{a_\ell}{\lambda^2}\right)\right), \tag{G40}$$

and one can check that the two lines are again identical due to the symmetry (G39).

Although the expressions for $a(\lambda)$ and $\tilde{a}(\lambda)$ seem complicated the product $a(\lambda)\tilde{a}(\lambda)$ has a simple expression, due to the normalization relation (which we have checked to hold)

$$a(\lambda)\tilde{a}(\lambda) = 1 - b(\lambda)\tilde{b}(\lambda) = \prod_{\ell=1}^{N}\left(1-\frac{a_\ell}{\lambda^2}\right) + \frac{g\Lambda}{2}\lambda^{-2N}e^{\lambda^2-1}. \tag{G41}$$

This equation is amenable to a solution using scalar Riemann-Hilbert methods.

APPENDIX H: SOLUTION OF THE RIEMANN-HILBERT PROBLEM FOR THE SCATTERING AMPLITUDES

Here we restrict to the case with zero drift $a_\ell = 0$ and we set the value of the coupling constant $g = 2$. We need to solve

$$a(\lambda)\tilde{a}(\lambda) = G(\lambda) = 1 + \Lambda\lambda^{-2N}e^{\lambda^2-1}. \tag{H1}$$

Let us first study the function $G(\lambda)$ (which is a function of λ^2). It has an infinity of zeros in the complex plane (they are represented by red dots in Figs. 4–7). To obtain their analytical

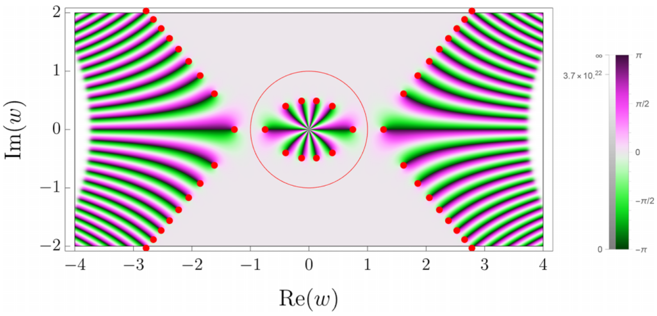

FIG. 5. The same as Fig. 4 with $\frac{\Lambda}{|\Lambda_c|} = -0.5$. Note the four zeros on the real axis, which correspond to (in the order of increasing real parts) to $\{-\lambda_{-1}, -\lambda_0, \lambda_0, \lambda_{-1}\}$. This feature happens for any $\frac{\Lambda}{|\Lambda_c|} \in]-1, 0[$. These zeros are related to the soliton rapidities, for the solution where solitons are present; see discussion in the text. As $\frac{\Lambda}{|\Lambda_c|} \to -1^+$ these zeros get closer and merge pairwise. For $\frac{\Lambda}{|\Lambda_c|} < -1$ the zeros go along a curve in the plane; see Fig. 6.

expressions one writes

$$\lambda^{-2N} e^{\lambda^2} = -\frac{e}{\Lambda}. \tag{H2}$$

Taking the power $1/N$ and multiplying by N we obtain

$$\frac{N}{\lambda^2} e^{\frac{\lambda^2}{N}} = N\left(-\frac{e}{\Lambda}\right)^{1/N} e^{-2\mathrm{i}\pi\frac{n}{N}} \tag{H3}$$

with $n = 0, \ldots, N-1$ and for any $z = |z|e^{i\theta}$, we denote $z^{1/N} = |z|^{1/N} e^{\mathrm{i}\theta/N}$.

Now, since the solution of $e^x/x = y$ is $x = -W_k(-1/y)$, where W_k is any of the branches of the Lambert function with $k \in \mathbb{Z}$ [30], we obtain that the zeros of $G(\lambda)$ can be written as $\pm\lambda_{k,n}$, where

$$\lambda_{k,n}^2 = -NW_k\left(-e^{-1}\left(\frac{\Lambda}{\Lambda_c}\right)^{1/N} e^{2\mathrm{i}\pi\frac{n}{N}}\right), \quad k \in \mathbb{Z},$$
$$n = 0, \ldots, N-1. \tag{H4}$$

We have further defined

$$\Lambda_c = -e^{1-N}N^N < 0. \tag{H5}$$

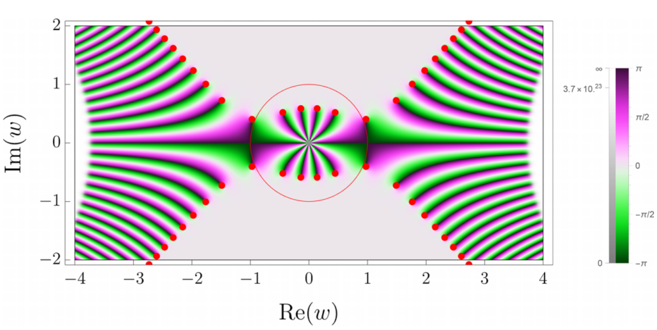

FIG. 6. The same as Fig. 4 with $\frac{\Lambda}{|\Lambda_c|} = -5$. The horizontal branch cuts originating from the origin and from infinity have merged at $\frac{\Lambda}{|\Lambda_c|} = -1$ and expand in lower and upper half planes. We will not study in details the case $\frac{\Lambda}{|\Lambda_c|} < -1$ here since we do not need it for our large deviation problem.

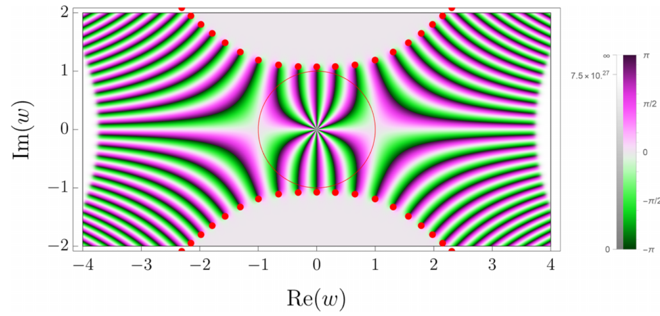

FIG. 7. The same as Fig. 4 with $\frac{\Lambda}{|\Lambda_c|} = 100\,000$. As discussed in the text, in the special case of large positive Λ, one needs to deform the circular contour (in an obvious way) to avoid the branch cuts in $\log G$.

Below we will always denote $\lambda_k = \lambda_{k,0}$. Finally, let us note that $G(\lambda)$ has a pole of order $2N$ at the origin $\lambda = 0$.

Now recall that we expect $a(\lambda)$ to be analytic inside a contour $\mathcal{C}$ (we will call $\mathcal{D}$ the interior of $\mathcal{C}$) and $\tilde{a}(\lambda)$ to be analytic outside the contour $\mathcal{C}$ in the complementary domain $\mathcal{D}^c$. In practice, in most cases one can take the contour $\mathcal{C}$ to be a circle of radius $\mathtt{R} = \sqrt{N}$, in which case $a(\lambda)$ is analytic for for $|\lambda| < \mathtt{R}$ and $\tilde{a}(\lambda)$ is analytic for $|\lambda| > \mathtt{R}$. In some special case however (see below) we will need to deform the circle.

From the knowledge of the zeros of $G(\lambda)$, one can then determine the solution for $a(\lambda)$ and $\tilde{a}(\lambda)$. There are two types of solutions.

Solution without soliton. Let us first assume that $a(\lambda)$ has no zeros for $|\lambda| < \mathtt{R}$ and that $\tilde{a}(\lambda)$ has no zeros for $|\lambda| > \mathtt{R}$, or more generally for $\lambda \in \mathcal{D}$ and $\lambda \in \mathcal{D}^c$, respectively. From Cauchy's theorem, and taking into account that both functions are even functions of λ one has, for $\lambda \in \mathcal{D}$,

$$\log a(\lambda) = \oint_{\mathcal{C}} \frac{\mathrm{d}w}{2\mathrm{i}\pi} \frac{w}{w^2 - \lambda^2} \log a(w),$$
$$0 = \oint_{\mathcal{C}} \frac{\mathrm{d}w}{2\mathrm{i}\pi} \frac{w}{w^2 - \lambda^2} \log \tilde{a}(w), \quad \lambda \in \mathcal{D}, \tag{H6}$$

where in the second equality we have closed the contour at infinity [assuming that $\tilde{a}(\lambda)$ goes to unity for $|\lambda| \to +\infty$]. Similarly one has, for $\lambda \in \mathcal{D}^c$,

$$\log \tilde{a}(\lambda) = -\oint_{\mathcal{C}} \frac{\mathrm{d}w}{2\mathrm{i}\pi} \frac{w}{w^2 - \lambda^2} \log \tilde{a}(w),$$
$$0 = \oint_{\mathcal{C}} \frac{\mathrm{d}w}{2\mathrm{i}\pi} \frac{w}{w^2 - \lambda^2} \log a(w), \quad \lambda \in \mathcal{D}^c. \tag{H7}$$

Subtracting these equations, we find

$$\log \tilde{a}(\lambda) = -\oint_{\mathcal{C}} \frac{\mathrm{d}w}{2\mathrm{i}\pi} \frac{w}{w^2 - \lambda^2} \log a(w)\tilde{a}(w)$$
$$= -\varphi(\lambda), \quad \lambda \in \mathcal{D}^c,$$
$$\log a(\lambda) = \varphi(\lambda), \quad \lambda \in \mathcal{D}, \tag{H8}$$

where everywhere we denote

$$\varphi(\lambda) = \oint_{\mathcal{C}} \frac{\mathrm{d}w}{2\mathrm{i}\pi} \frac{w}{w^2 - \lambda^2} \log(1 + \Lambda w^{-2N} e^{w^2 - 1}), \quad \lambda \notin \mathcal{C}. \tag{H9}$$

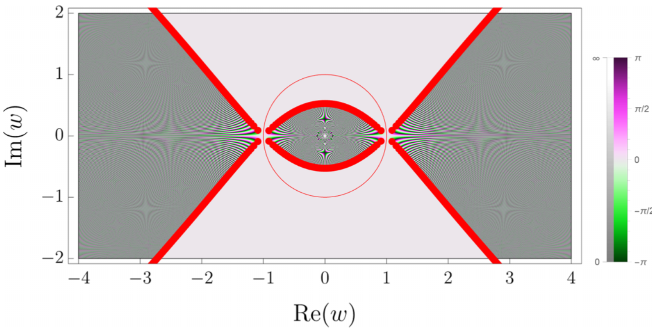

FIG. 8. The same as Fig. 4 with $\frac{\Lambda}{|\Lambda_c|} = 1$ and $N = 100$.

Note that since the number of zeros of $G(\lambda)$ is infinite, $\tilde{a}(\lambda)$ has zeros inside the contour $\mathcal{C}$ (i.e., in $\mathcal{D}$), and $a(\lambda)$ has an infinite number of zeros outside the contour $\mathcal{C}$ (i.e., in $\mathcal{D}^c$). The positions of these zeros are shown as red dots in Figs. 4–7 for $N = 5$. These zeros lead to branch cuts in the function $\log G(\lambda)$ as can be seen in these figures. We have checked that for $\frac{\Lambda}{\Lambda_c} \in (-1, +\infty[$, which is the region that we need for our large deviation problem, there is a choice of the contour $\mathcal{C}$ so that no branch cut is crossed when integrating $\log G(\lambda)$ over the contour $\mathcal{C}$. There exists a threshold value $\Lambda^*(N)$ such that for $\Lambda < \Lambda^*(N)$ the contour $\mathcal{C}$ can be chosen to be the circle of radius $\mathtt{R} = \sqrt{N}$ (represented as a circle of radius unity in the rescaled variables in the figures). For $\Lambda > \Lambda^*(N)$ the contour can be chosen as a deformed circle as can be seen from Fig. 7. The value $\Lambda^*(N)$ increases very fast with N and corresponds to the first time that a zero initially inside the circle hits the circle as Λ increases [we did not attempt to find its analytical value, except for $N = 1$ where one has $\Lambda^*(1) = e^2$].

Finally, we also plotted the zeros for $N = 100$ in Figs. 8 and 9. One can see that they fall on some limit curves that we did not attempt to describe or interpret.

Solution with a soliton. From the considerations discussed in the text, we know that there should be another branch of solutions with solitons. Indeed, for $\Lambda \in [\Lambda_c, 0)$ there are two pairs of real zeros of $G(\lambda)$, $\{\pm\lambda_0, \pm\lambda_{-1}\}$ (we recall that we denote $\lambda_k = \lambda_{k,0}$) as can be seen in Fig. 5. Furthermore one has $0 < \lambda_0 < \mathtt{R} = \sqrt{N}$ and $\mathtt{R} = \sqrt{N} < \lambda_{-1}$. Thus in the interval $\Lambda \in [\Lambda_c, 0[$ one can consider a solution such that $a(\lambda)$ has two zeros at $\lambda = \pm\lambda_0$ inside the $\mathtt{R}$ circle, and $\tilde{a}(\lambda)$ has two zeros at $\lambda = \pm\lambda_1$, outside the $\mathtt{R}$ circle. It is then natural to redefine

$$a(\lambda) = \alpha(\lambda)\frac{\lambda^2 - \lambda_0^2}{\lambda^2 - \lambda_{-1}^2}, \quad \tilde{a}(\lambda) = \tilde{\alpha}(\lambda)\frac{\lambda^2 - \lambda_{-1}^2}{\lambda^2 - \lambda_0^2}. \tag{H10}$$

Since the product $a(\lambda)\tilde{a}(\lambda) = \alpha(\lambda)\tilde{\alpha}(\lambda) = G(\lambda)$ and $\tilde{\alpha}(\lambda)$ also tend to one at infinity, we can apply the same manipulations as above to $\alpha(\lambda)$ and $\tilde{\alpha}(\lambda)$. The solution thus reads

$$a(\lambda) = \frac{\lambda^2 - \lambda_0^2}{\lambda^2 - \lambda_{-1}^2} \times \begin{cases} e^{\varphi(\lambda)} & |\lambda| < \mathtt{R} = \sqrt{N} \\ G(\lambda)^{1/2} e^{\tilde{\varphi}(\lambda)} & |\lambda| = \mathtt{R} \\ G(\lambda) e^{\varphi(\lambda)} & |\lambda| > \mathtt{R} \end{cases}, \tag{H11}$$

$$\tilde{a}(\lambda) = \frac{\lambda^2 - \lambda_{-1}^2}{\lambda^2 - \lambda_0^2} \times \begin{cases} G(\lambda) e^{-\varphi(\lambda)} & |\lambda| < \mathtt{R} = \sqrt{N} \\ G(\lambda)^{1/2} e^{-\tilde{\varphi}(\lambda)} & |\lambda| = \mathtt{R} \\ e^{-\varphi(\lambda)} & |\lambda| > \mathtt{R} \end{cases}, \tag{H12}$$

where $\varphi(\lambda)$ was defined in (H9) and

$$\tilde{\varphi}(\lambda) = \fint_{|w|=\mathtt{R}} \frac{\mathrm{d}w}{2\mathbf{i}\pi} \frac{w}{w^2 - \lambda^2} \log(1 + \Lambda w^{-2N} e^{w^2 - 1}) \quad |\lambda| = \mathtt{R} \tag{H13}$$

is given as a principal value. The discussion above about the contour and branch cut applies identically. This is quite analogous to the situation for the KPZ equation (which as we show is reached for $N \to +\infty$) as found in [1] and confirmed by the rigorous recent work of Ref. [3].

APPENDIX I: CALCULATION OF THE RATE FUNCTIONS

1. Values of the conserved quantities

From Appendix F we can now obtain the values taken by the conserved quantities $\{\tilde{C}_n, C_n\}$ from the coefficients of the Laurent or Taylor series of the scattering amplitudes. One writes

$$\log \tilde{a}(\lambda) = -\varphi(\lambda) = \sum_{n=1}^{\infty} \frac{\tilde{C}_n}{\lambda^{2n}}, \quad |\lambda| > \mathtt{R}, \tag{I1}$$

$$\log a(\lambda) = \varphi(\lambda) = \sum_{n=0}^{\infty} \lambda^{2n} C_n, \quad |\lambda| < \mathtt{R}. \tag{I2}$$

Recalling the definition (H9) of $\varphi(\lambda)$, this leads to

$$C_n = \oint_{|w|=\mathtt{R}} \frac{\mathrm{d}w}{2\mathbf{i}\pi} \frac{1}{w^{2n+1}} \log(1 + \Lambda w^{-2N} e^{w^2 - 1}), \;\; n \in [0, \infty], \tag{I3}$$

$$\tilde{C}_n = \oint_{|w|=\mathtt{R}} \frac{\mathrm{d}w}{2\mathbf{i}\pi} w^{2n-1} \log(1 + \Lambda w^{-2N} e^{w^2 - 1}), \;\; n \in [1, \infty]. \tag{I4}$$

This is the result in the absence of soliton. In the presence of solitons there is an additional additive contribution so that the values of the conserved quantities are now $C_n + \Delta C_n$ and $\tilde{C}_n + \Delta\tilde{C}_n$ respectively, where $C_n, \tilde{C}_n$ are still given by the

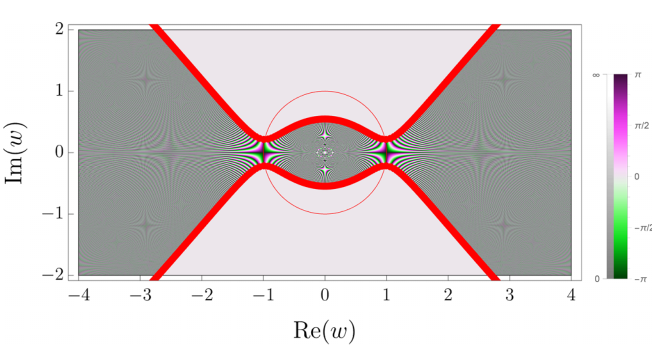

FIG. 9. The same as Fig. 4 with $\frac{\Lambda}{|\Lambda_c|} = -10\,000$ and $N = 100$.

above expressions and, from (H12) for $|\lambda| > \mathtt{R}$,

$$\log\left(\frac{\lambda^2-\lambda_{-1}^2}{\lambda^2-\lambda_0^2}\right)=\sum_{n=1}^{\infty}\frac{1}{\lambda^{2n}}\frac{\lambda_0^{2n}-\lambda_{-1}^{2n}}{n}=\sum_{n=1}^{\infty}\frac{\Delta\tilde{C}_n}{\lambda^{2n}}. \quad (I5)$$

Similarly from (H11) for $|\lambda| < \mathtt{R}$ one obtains

$$\log\left(\frac{\lambda^2-\lambda_0^2}{\lambda^2-\lambda_{-1}^2}\right)=\log\left(\frac{\lambda_0^2}{\lambda_{-1}^2}\right)+\sum_{n=1}^{\infty}\lambda^{2n}\frac{\lambda_{-1}^{-2n}-\lambda_0^{-2n}}{n}$$
$$=\Delta C_0+\sum_{n=1}^{\infty}\lambda^{2n}\Delta C_n. \quad (I6)$$

This leads to

$$\Delta C_0=\log\left(\lambda_0^2\right)-\log\left(\lambda_{-1}^2\right),$$
$$\Delta C_n=\frac{\lambda_{-1}^{-2n}-\lambda_0^{-2n}}{n}\quad \text{for}\quad n\geqslant 1,$$
$$\Delta\tilde{C}_n=\frac{\lambda_0^{2n}-\lambda_{-1}^{2n}}{n}\quad \text{for}\quad n\geqslant 1. \quad (I7)$$

2. Rate function $\Psi(\Lambda)$: Main branch, no soliton

Let us now recall the first conserved quantity from Appendix F. With the value of the coupling constant set to $g=2$, it reads

$$\tilde{C}_1=-\frac{g}{2}\sum_{n=-\infty}^{+\infty}z_n(t)\tilde{z}_n(t), \quad (I8)$$

and it is independent of time t. Let us set and evaluate it at time $t=1$ using the boundary condition $z_n(1)=-\Lambda\delta_{n,N}$. We obtain the relation

$$\tilde{C}_1=\Lambda z_N(t=1). \quad (I9)$$

By definition $z_N(t=1)=e^{H_N}$ is our observable and by taking a derivative of Eq. (A6) w.r.t. the Legendre parameter Λ we see that

$$\Psi'(\Lambda)=z_N(t=1). \quad (I10)$$

Consider here the case without soliton. Hence, using (I4) for $n=1$ we obtain

$$\Lambda\Psi'(\Lambda)=\oint_{|w|=\mathtt{R}}\frac{dw}{2\mathbf{i}\pi}w\log(1+\Lambda w^{-2N}e^{w^2-1})$$
$$=\oint_{|v|=\mathtt{R}^2}\frac{dv}{2\mathbf{i}\pi}\log(1+\Lambda v^{-N}e^{v-1}), \quad (I11)$$

where in the second equality there is no additional factor of 2 since the circle is run twice. We recall that we can choose $\mathtt{R}=\sqrt{N}$.

Remark I.1. For $N=1$ we know from the direct solution [see e.g., (C21)] that

$$\Lambda\Psi'(\Lambda)=W_0\left(\frac{\Lambda}{e}\right). \quad (I12)$$

Hence this provides an integral representation of the Lambert function (which to our knowledge is novel).

Since $\Psi(0)=0$, we can integrate (I11) and obtain

$$\Psi(\Lambda)=-\oint_{|w|=\mathtt{R}}\frac{dw}{2\mathbf{i}\pi}w\mathrm{Li}_2(-\Lambda w^{-2N}e^{w^2-1})$$
$$=-\oint_{|v|=\mathtt{R}^2}\frac{dv}{2\mathbf{i}\pi}\mathrm{Li}_2(-\Lambda v^{-N}e^{v-1}). \quad (I13)$$

Series expansion of $\Psi_N(\Lambda)$ and cumulants of $z_N(t=1)$. Recall the contour representation of the nth Hermite polynomial

$$H_n(x)=n!\oint\frac{dz}{2\mathbf{i}\pi}\frac{e^{2xz-z^2}}{z^{n+1}}, \quad (I14)$$

where the contour encloses the origin. Expanding the logarithm in Eq. (I11) as a series, this leads to

$$\Lambda\Psi'(\Lambda)=\sum_{p\geqslant 1}\frac{(-1)^{p+1}}{p}\oint_{|w|=\mathtt{R}}\frac{dw}{2\mathbf{i}\pi w}\Lambda^p e^{-p}w^{-2Np+2}e^{pw^2}, \quad (I15)$$

$$=\sum_{p\geqslant 1}\frac{(-1)^{p+1}}{p}(\Lambda e^{-1})^p(-p)^{Np-1}\frac{H_{2Np-2}(0)}{(2Np-2)!}, \quad (I16)$$

$$=\sum_{p\geqslant 1}\frac{(-1)^{p+1}}{p^2}(\Lambda e^{-1}p^N)^p\frac{1}{(Np-1)!}, \quad (I17)$$

where we have used the value of the Hermite polynomial of even indices at the origin:

$$\frac{H_{2N}(0)}{(2N)!}=\frac{\cos(\pi N)}{N!}. \quad (I18)$$

Integrating once w.r.t. λ we obtain

$$\Psi_N(\Lambda)=\sum_{p\geqslant 1}b_p(\Lambda e^{-1})^p,\quad b_p=(-1)^{p+1}\frac{p^{Np-3}}{(Np-1)!}. \quad (I19)$$

One finds that the radius of convergence of this series in Λ is precisely $|\Lambda_c|$ defined above $\Lambda_c=-e^{1-N}N^N$, which is consistent with the behavior of the solutions.

Let us note that the typical value is given by the coefficient for $p=1$ i.e., one has

$$\Psi_N'(0)=e^{H_{\rm typ}}=\overline{z_N(t=1)}=b_1e^{-1}=\frac{1}{e(N-1)!}, \quad (I20)$$

which coincides with the prediction from the Poisson kernel. More generally, since $\Psi_N(\Lambda)$ is the generating function of the cumulants of $z_N(t=1)$, from its definition (A5), one has, to leading order in $\varepsilon\ll 1$,

$$\Psi_N(\Lambda)=\sum_{p\geqslant 1}(-1)^{p+1}\frac{\Lambda^p}{p!}\varepsilon^{1-p}\overline{z_N(t=1)^p}^c, \quad (I21)$$

which implies that

$$\overline{z_N(t=1)^p}^c=(-1)^{p+1}p!\varepsilon^{p-1}b_pe^{-p}=p!\frac{p^{Np-3}}{(Np-1)!}e^{-p}\varepsilon^{p-1}. \quad (I22)$$

In particular the variance reads

$$\overline{z_N(t=1)^2}^c = \frac{2^{2N-2}}{(2N-1)!} e^{-2}\,\varepsilon. \tag{I23}$$

One can easily check the first two cumulants for $N=1$. Let us recall that $z_N(t) = e^{-(1+\frac{\varepsilon}{2})t} Z_N(t)$. For $N=1$ one has $z_1(t) = e^{-(1+\frac{\varepsilon}{2})t+\sqrt{\varepsilon}B(t)}$ and one finds $\overline{z_1} \simeq e^{-1}$ and $\overline{z_1^2}^c = e^{-2}\varepsilon$ in agreement with the above formula.

Finally, one can obtain the cumulants of $H = \log z_N(1)$ from the derivatives of the rate function $\Phi(H)$ (see, e.g., in [23], Sec. 4.2.5 of the Supp. Material). Here they scale as $\overline{H^q}^c \sim \varepsilon^{1-q}$. The variance reads

$$\overline{H^2}^c = \frac{\varepsilon}{\Phi''(H_{\rm typ})} = -\varepsilon\frac{\Psi''(0)}{\Psi'(0)^2} = \frac{2^{2N-2}(N-1)!^2}{(2N-1)!}\,\varepsilon. \tag{I24}$$

3. Rate function $\Psi(\Lambda)$: Second branch, with a soliton

In presence of a soliton, from (I7) we see that the value of the conserved quantity $\tilde{C}_1$ is changed by

$$\Delta\tilde{C}_1 = \lambda_0^2 - \lambda_{-1}^2. \tag{I25}$$

On the second (solitonic) branch, the rate function Ψ thus becomes $\Psi_N(\Lambda) = \Psi_N^{\rm main}(\Lambda) + \Delta_N(\Lambda)$ with

$$\Lambda\Delta_N'(\Lambda) = \Delta\tilde{C}_1 = NW_{-1}\left[-e^{-1}\left(\frac{\Lambda}{\Lambda_c}\right)^{1/N}\right] - NW_0\left[-e^{-1}\left(\frac{\Lambda}{\Lambda_c}\right)^{1/N}\right], \tag{I26}$$

where we have replaced $\lambda_{0/-1} = \lambda_{0/-1,0}$ by their explicit values from (H4). Upon integration one finds that

$$\Delta_N(\Lambda) = \frac{N^2}{2}W_{-1}\left[-\frac{1}{e}\left(\frac{\Lambda}{\Lambda_c}\right)^{1/N}\right]\left(W_{-1}\left[-\frac{1}{e}\left(\frac{\Lambda}{\Lambda_c}\right)^{1/N}\right]+2\right) - \frac{N^2}{2}W_0\left[-\frac{1}{e}\left(\frac{\Lambda}{\Lambda_c}\right)^{1/N}\right]\left(W_0\left[-\frac{1}{e}\left(\frac{\Lambda}{\Lambda_c}\right)^{1/N}\right]+2\right), \tag{I27}$$

which vanishes as it should at $\Lambda = \Lambda_c$ which is the point where solitons are spontaneously generated.

APPENDIX J: CONVERGENCE TO KPZ

1. Convergence of the large deviation rate function

We will now show that the large deviation result for the OY polymer

$$\overline{\exp\left(-\frac{\Lambda}{\varepsilon}z_N(\tau=1)\right)} \sim \exp\left(-\frac{1}{\varepsilon}\Psi_N(\Lambda)\right) \tag{J1}$$

converges to the similar result for the short-time KPZ equation as $N \to +\infty$ under a proper rescaling. We will consider successively the right-hand side of (J1) (the large deviation form) and its left-hand side (the observable) and show that each side converges to the corresponding one for the KPZ equation.

Consider the right-hand side of (J1). Let us consider first the main branch. One recalls that

$$\Psi(\Lambda) = -\oint_{|v|=N}\frac{dv}{2\mathbf{i}\pi}{\rm Li}_2(-\Lambda v^{-N}e^{v-1}) \ . \tag{J2}$$

Note that since Λ_N^* grows very fast with N we need only to consider the case $\Lambda < \Lambda_N^*$ where the contour is a circle. We parametrize the circle of radius N as $v = Ne^{\mathbf{i}k}$, and for large N we expand around $k=0$ as

$$v - N\log v = N - N\log N - N\frac{k^2}{2} + O(Nk^3). \tag{J3}$$

Inserting this expansion into (J2) we obtain

$$\Psi(\Lambda) \simeq -N\int_{\mathbb{R}}\frac{dk}{2\pi}e^{\mathbf{i}k}{\rm Li}_2(-\Lambda e^{N-1-N\log N}e^{-Nk^2/2}) \tag{J4}$$

$$\simeq -\sqrt{2N}\int_{\mathbb{R}}\frac{dq}{2\pi}{\rm Li}_2\left(\frac{\Lambda}{\Lambda_c}e^{-q^2}\right), \tag{J5}$$

where we recall that $\Lambda_c = -e^{1-N}N^N$. We can then make the connection on the right-hand side of Eq. (J1) with the known result for the short-time KPZ equation with droplet initial conditions [1,22] as

$$\exp\left(-\frac{\Psi_N(\Lambda)}{\varepsilon}\right) \to \exp\left(-\frac{1}{\sqrt{T_{\rm KPZ}}}\Psi_{\rm KPZ}(z)\right),$$

$$\Psi_{\rm KPZ}(z) = -\int_{\mathbb{R}}\frac{dk}{2\pi}{\rm Li}_2(-ze^{-q^2}) \tag{J6}$$

with the identification

$$z = -\frac{\Lambda}{\Lambda_c}, \quad \frac{\sqrt{2N}}{\varepsilon} = \frac{1}{\sqrt{T_{\rm KPZ}}}. \tag{J7}$$

Hence the "KPZ time" $T_{\rm KPZ}$ must be identified with the weak noise parameter $\varepsilon^2/(2N)$.

In the case of the second (solitonic) branch we need to study the large N limit of $\Delta_N(\Lambda)$ given by Eq. (I27). We recall the expansions of the Lambert functions for $0<x<1$ and $N\to+\infty$:

$$W_0(-e^{-1}x^{1/N}) = -1 + \sqrt{\frac{2}{N}}\sqrt{\log(1/x)} + \frac{2}{3N}\log x + O\left(\frac{1}{N^{3/2}}\right), \tag{J8}$$

$$W_{-1}(-e^{-1}x^{1/N}) = -1 - \sqrt{\frac{2}{N}}\sqrt{\log(1/x)} + \frac{2}{3N}\log x + O\left(\frac{1}{N^{3/2}}\right). \tag{J9}$$

From theses expansions we obtain from (I26)

$$\Lambda\Delta_N'(\Lambda) = -2\sqrt{2N}\sqrt{\log(\Lambda_c/\Lambda)} + O(1/N^{1/2}). \tag{J10}$$

Integrating over Λ one finds, at large N

$$\Delta_N(\Lambda) = \sqrt{2N}\frac{4}{3}[\log(\Lambda_c/\Lambda)]^{3/2} + O(1/N^{1/2}), \tag{J11}$$

which has precisely the form

$$\frac{\Delta_N(\Lambda)}{\varepsilon} \simeq \frac{1}{\sqrt{T_{\rm KPZ}}}\Delta_{\rm KPZ}(z), \quad \Delta_{\rm KPZ}(z) = \frac{4}{3}[-\log(-z)]^{3/2}, \tag{J12}$$

which agrees with the result for the KPZ equation in Refs. [1,22].

Now we need to consider the left-hand side of (J1), and identify the observables in the two models. For this we will use the result from Ref. [28], Sec. 5.4.1 Formula (5.9), which we first state in the notations of that paper. There is it shown that the OY partition sum denoted there $e^{F_N(T)}$ converges for $N \to +\infty$ to the continuum directed polymer partition sum $Z(X,T)$ (in the notations of [28]) as

$$e^{F_N(\sqrt{TN}+X)} \simeq e^{N+\frac{\sqrt{NT}+X}{2}+X\sqrt{\frac{N}{T}}+\frac{N}{2}\log(TN)-N\log N} Z(X,T). \tag{J13}$$

On the other hand the partition sums defined here are related to those in that paper via

$$z_N(t) = e^{-t(1+\frac{\varepsilon}{2})}Z_N(t), \quad Z_N(t) = \varepsilon^{1-N}e^{F_N(\varepsilon t)}. \tag{J14}$$

We can choose $X=0$, and we obtain the convergence

$$\begin{aligned} z_N(t=1) &= e^{-(1+\frac{\varepsilon}{2})}\varepsilon^{1-N}e^{F_N(\varepsilon)} \simeq \varepsilon e^{N-1-N\log N}Z(0,T) \\ &= -\frac{\varepsilon}{\Lambda_c}Z(0,T), \end{aligned} \tag{J15}$$

where we have identified $\varepsilon = \sqrt{TN}$, i.e., we set $T = \varepsilon^2/N$. Let us now recall that in our previous work [1] we define the continuum-directed polymer partition $Z_{\rm KPZ}(X, T_{\rm KPZ})$ as the solution of the SHE

$$\partial_t Z_{\rm KPZ} = \partial_x^2 Z_{\rm KPZ} + \sqrt{2}\eta(x,t)Z_{\rm KPZ}, \tag{J16}$$

where $\eta(x,t)$ is the standard (unit variance) space-time white noise. Comparing with Ref. [28] we see that we need to write

$$Z(X,T) = Z_{\rm KPZ}(X, T_{\rm KPZ}), \quad 2T_{\rm KPZ} = T. \tag{J17}$$

Thus we have the large N behavior

$$z_N(t=1) \simeq -\frac{\varepsilon}{\Lambda_c}Z_{\rm KPZ}\left(0, T_{\rm KPZ} = \frac{\varepsilon^2}{2N}\right). \tag{J18}$$

Hence the left-hand side of (J1) becomes

$$\begin{aligned} \overline{\exp\left(-\frac{\Lambda}{\varepsilon}z_N(t=1)\right)} &\simeq \overline{\exp\left[-zZ_{\rm KPZ}(0,T_{\rm KPZ})\right]} \\ &= \overline{\exp\left(-\frac{ze^{H(T_{\rm KPZ})}}{\sqrt{T_{\rm KPZ}}}\right)}, \end{aligned} \tag{J19}$$

where we recall that $z = -\Lambda/\Lambda_c$ and that for the droplet solution of the KPZ solution we have defined $H(T_{\rm KPZ}) = \log Z_{\rm KPZ}(0,T_{\rm KPZ}) + \frac{1}{2}\log T_{\rm KPZ}$ in Ref. [1].

2. Convergence of the dynamical system (10) to the nonlinear Schrödinger equation

In this section we investigate how to take the limit of the WNT of the O'Connell-Yor polymer (10) to the WNT equations of the Kardar-Parisi-Zhang equation directly. To this aim, we define a scaling variable x_0, which can be interpreted as a lattice length, and introduce the rescaled space and time as

$$X = x_0(t-n), \quad T = \frac{x_0^2}{2}t. \tag{J20}$$

We then define the partition function Z as

$$\begin{aligned} z_n(t) &= Z\left(x_0 t - x_0 n, \frac{x_0^2}{2}t\right) \\ \longleftrightarrow Z(X,T) &= z_{\frac{2T}{x_0^2}-\frac{X}{x_0}}\left(\frac{2T}{x_0^2}\right) \end{aligned} \tag{J21}$$

and similarly for $\tilde{z}_n$ which we relate to a response field $\tilde{Z}$. Definition (J21) implies the differential relations

$$\begin{aligned} \partial_t z_n(t) &= x_0\partial_X Z + \frac{x_0^2}{2}\partial_T Z, \\ z_{n+1} - z_n &= -x_0\partial_X Z + \frac{x_0^2}{2}\partial_X^2 Z + O(x_0^3), \\ z_{n-1} - z_n &= x_0\partial_X Z + \frac{x_0^2}{2}\partial_X^2 Z + O(x_0^3), \end{aligned} \tag{J22}$$

where we have assumed a small x_0 expansion in the last two equations. In order to keep only one dominant term in the expansion, we consider the following combination:

$$\partial_t z_n + z_{n+1} - z_n = \frac{x_0^2}{2}(\partial_T Z + \partial_X^2 Z) + O(x_0^3). \tag{J23}$$

Upon convergence, we find that the original dynamical system (10) reads at leading order

$$\begin{aligned} x_0^2\partial_T Z &= x_0^2\partial_X^2 Z + 2Z^2\tilde{Z}, \\ -x_0^2\partial_T \tilde{Z} &= x_0^2\partial_X^2 \tilde{Z} + 2Z\tilde{Z}^2. \end{aligned} \tag{J24}$$

To obtain a final system independent on x_0, we finally rescale the fields Z and $\tilde{Z}$ as

$$Q_{\rm KPZ}(X,T) = \frac{1}{x_0}Z(X,T), \quad P_{\rm KPZ}(X,T) = \frac{1}{x_0}\tilde{Z}(X,T) \tag{J25}$$

to obtain the $\{P,Q\}$ system studied in the context of the WNT of the KPZ equation [1].

3. Convergence of the Lax pair (13) to the Lax pair of the nonlinear Schrödinger equation

Recalling the driftless Lax pair for the O'Connell-Yor system,

$$\partial_t v_n = U_n v_n, \quad v_{n+1} = L_n v_n, \quad U_n = \begin{pmatrix} \frac{\lambda^2-1}{2} & -z_{n-1} \\ \tilde{z}_n & \frac{1-\lambda^2}{2} \end{pmatrix},$$

$$L_n = \begin{pmatrix} \frac{1}{\lambda} & \frac{z_n}{\lambda} \\ -\frac{1}{\lambda}\tilde{z}_n & \lambda - \frac{1}{\lambda}z_n\tilde{z}_n. \end{pmatrix}. \tag{J26}$$

From (J23) and (J22) we observe that in order to study the convergence of the Lax pair, we need to consider the combination

$$\partial_t v_n + v_{n+1} - v_n = (U_n + L_n - I_2)v_n \tag{J27}$$

as well as the first order in x_0 of the equations $\partial_t v_n = U_n v_n$ and $v_{n+1} - v_n = (L_n - I_2)v_n$ where I_2 is the identity matrix. Indeed, in the continuum we want to transform the Lax pair (J26) into a continuous version:

$$\partial_X V(X,T) = U_1 V(X,T), \quad \partial_T V(X,T) = U_2 V(X,T). \tag{J28}$$

Equation (J23) together with Eq. (J27) provides a convergence to the operator $(\partial_T + \partial_X^2)V$ which in terms of the continuous Lax matrices (J28) reads $(U_2 + U_1^2 + \partial_X U_1)V$.

To study the convergence of the Lax pair, we choose the following scaling at leading order [see also (J22)]:

$$\begin{aligned}
&\lambda(x_0) = e^{\mathbf{i}\frac{\lambda}{2}x_0},\\
&\tilde{z}_n(t) = x_0\tilde{Z}(X,T), \quad \tilde{z}_{n-1}(t) = x_0\tilde{Z}(X,T) + x_0^2\partial_X\tilde{Z}(X,T) + \frac{x_0^3}{2}\partial_X^2\tilde{Z}(X,T),\\
&z_n(t) = x_0 Z(X,T), \quad z_{n-1}(t) = x_0 Z(X,T) + x_0^2\partial_X Z(X,T) + \frac{x_0^3}{2}\partial_X^2 Z(X,T).
\end{aligned} \tag{J29}$$

Under this scaling, a straightforward replacement gives

$$\begin{aligned}
&L_n \to I_2 - x_0U_1 + \frac{x_0^2}{2}\left(U_1^2 + \partial_X U_1\right) + x_0^2(U_2 - W) + O\left(x_0^3\right),\\
&U_n \to x_0U_1 + x_0^2 W + O\left(x_0^3\right),
\end{aligned} \tag{J30}$$

where

$$U_1 = \begin{pmatrix} \frac{\mathbf{i}\lambda}{2} & -Z(X,T) \\ \tilde{Z}(X,T) & -\frac{\mathbf{i}\lambda}{2} \end{pmatrix}, \tag{J31}$$

$$U_2 = \begin{pmatrix} \frac{1}{2}Z(X,T)\tilde{Z}(X,T) - \frac{\lambda^2}{4} & -\frac{1}{2}\partial_X Z(X,T) - \frac{\mathbf{i}\lambda}{2}Z(X,T) \\ -\frac{1}{2}\partial_X\tilde{Z}(X,T) + \frac{\mathbf{i}\lambda}{2}\tilde{Z}(X,T) & \frac{\lambda}{4}^2 - \frac{1}{2}Z(X,T)\tilde{Z}(X,T) \end{pmatrix}, \tag{J32}$$

$$W = \begin{pmatrix} -\frac{\lambda^2}{4} & -\partial_X Z(X,T) \\ 0 & \frac{\lambda^2}{4} \end{pmatrix}. \tag{J33}$$

We recognize in Eqs. (J31) and (J32) the matrices U_1 and U_2 of the continuous nonlinear Schrödinger equation obtained in the weak noise theory of the KPZ equation [1]. In particular we see the convergence

$$U_n + L_n - I_2 \to x_0^2\left(U_2 + U_1^2 + \partial_X U_1\right) \tag{J34}$$

as expected. This concludes the convergence of the Lax pairs of the discrete problem towards the continuous problem.

APPENDIX K: TRIANGULAR REPRESENTATION OF THE SOLUTION TO THE LAX PROBLEM

Here we set $g = 2$ and the drifts $\{a_\ell\} = 0$, and we recall the asymptotics

$$\phi_n \sim \lambda^{-n}\begin{pmatrix}1\\0\end{pmatrix}, \quad \bar{\phi}_n \sim \lambda^n\begin{pmatrix}0\\-1\end{pmatrix}, \quad n \to -\infty, \tag{K1}$$

$$\psi_n \sim \lambda^n\begin{pmatrix}0\\1\end{pmatrix}, \quad \bar{\psi}_n \sim \lambda^{-n}\begin{pmatrix}1\\0\end{pmatrix}, \quad n \to +\infty, \tag{K2}$$

and

$$\phi_n \sim \begin{pmatrix}a(\lambda,t)\lambda^{-n}\\ b(\lambda,t)\lambda^n\end{pmatrix}, \quad \bar{\phi}_n \sim \begin{pmatrix}\tilde{b}(\lambda,t)\lambda^{-n}\\ -\tilde{a}(\lambda,t)\lambda^n\end{pmatrix}, \quad n \to +\infty. \tag{K3}$$

We have that $\lambda^n\phi_n$ and $\lambda^{-n}\psi_n$ are analytic inside the circle $|\lambda| < \mathtt{R}$ and that $\lambda^{-n}\bar{\phi}_n$ and $\lambda^n\bar{\psi}_n$ are analytic outside of the circle $|\lambda| > \mathtt{R}$. (for simplicity we consider the case $\Lambda < \Lambda_N^*$ but the same manipulations extent to $\Lambda > \Lambda_N^*$, the contour being a deformed circle).

One introduces the triangular representations

$$\begin{aligned}
&\psi_n = \lambda^n\sum_{p\geqslant 0}\lambda^p\begin{pmatrix}K_1(n,n+p)\\K_2(n,n+p)\end{pmatrix} = \sum_{p\geqslant 0}\lambda^{n+p}K(n,n+p),\\
&\phi_n = \lambda^{-n}\sum_{p\geqslant 0}\lambda^p\begin{pmatrix}M_1(n,n+p)\\M_2(n,n+p)\end{pmatrix}, \quad |\lambda| < \mathtt{R},
\end{aligned} \tag{K4}$$

$$\begin{aligned}
&\bar{\psi}_n = \lambda^{-n}\sum_{p\geqslant 0}\lambda^{-p}\begin{pmatrix}\bar{K}_1(n,n+p)\\\bar{K}_2(n,n+p)\end{pmatrix} = \sum_{p\geqslant 0}\lambda^{-n-p}\bar{K}(n,n+p),\\
&\bar{\phi}_n = \lambda^n\sum_{p\geqslant 0}\lambda^{-p}\begin{pmatrix}\bar{M}_1(n,n+p)\\\bar{M}_2(n,n+p)\end{pmatrix}, \quad |\lambda| > \mathtt{R}.
\end{aligned} \tag{K5}$$

The goal is to find the values of the first coefficients in the triangular representation. Let us recall the Lax matrix where

TABLE I. Summary of the results for $\bar{K}_1$ and $\bar{K}_2$.

Index	$\bar{K}_1$	$\bar{K}_2$
(n,n)	$\bar{K}_1(n,n)=1$	$\bar{K}_2(n,n)=0$
$(n,n+1)$	$\bar{K}_1(n,n+1)=0$	$\bar{K}_2(n,n+1)=0$
$(n,n+2)$	$\bar{K}_1(n,n+2)=-\sum_{\ell=n}^{\infty} z_\ell\tilde{z}_\ell$	$\bar{K}_2(n,n+2)=\tilde{z}_n$
$(n,n+2k+1)$	$\bar{K}_1(n,n+2k+1)=0$	$\bar{K}_2(n,n+2k+1)=0$

we took the drifts $\{a_\ell\}$ equal to 0:

$$L_n=\begin{pmatrix}\frac{1}{\lambda} & \frac{z_n}{\lambda}\\ -\frac{1}{\lambda}\tilde{z}_n & \lambda-\frac{1}{\lambda}z_n\tilde{z}_n\end{pmatrix}. \tag{K6}$$

1. Triangular decomposition of $\bar{\psi}_n$

Injecting into $\bar{\psi}_{n+1}=L_n\bar{\psi}_n$ one finds the recursion relations for $p\geqslant 0$:

$$\bar{K}_2(n,n)=0,\quad \bar{K}_2(n,n+1)=0, \tag{K7}$$

$$\bar{K}_1(n+1,n+1+p)=\bar{K}_1(n,n+p)+z_n\bar{K}_2(n,n+p), \tag{K8}$$

$$\begin{aligned}\bar{K}_2(n+1,n+1+p)&\\ =-\tilde{z}_n\bar{K}_1(n,n+p)&+\bar{K}_2(n,n+2+p)\\ -\tilde{z}_n z_n\bar{K}_2(n,n+p).&\end{aligned} \tag{K9}$$

In particular, this implies using $p=0$

$$\bar{K}_1(n,n)=c_0=1,\quad \tilde{z}_n=\frac{\bar{K}_2(n,n+2)}{\bar{K}_1(n,n)}=\bar{K}_2(n,n+2), \tag{K10}$$

where $c_0=1$ comes from the asymptotics at large n. Using $p=1$ one has

$$\bar{K}_1(n,n+1)=c_1=0,\quad \bar{K}_2(n,n+3)=0. \tag{K11}$$

After further examination for all p one obtains

$$\begin{aligned}\bar{K}_1(n,n+2k+1)&=c_{2k+1}=0,\\ \bar{K}_2(n,n+2k+1)&=0,\end{aligned} \tag{K12}$$

where all constants c_{2k+1} are determined by the asymptotic $n\to\infty$ condition. We summarize our findings with the following expansion as well as Table I:

$$\bar{\psi}_n=\lambda^{-n}\begin{pmatrix}1\\0\end{pmatrix}+\lambda^{-n-2}\begin{pmatrix}-\sum_{\ell=n}^{\infty}z_\ell\tilde{z}_\ell\\ \tilde{z}_n\end{pmatrix}+O(\lambda^{-n-4}). \tag{K13}$$

2. Triangular decomposition of ψ_n

Injecting into $\psi_{n+1}=L_n\psi_n$ one finds the recursion relations for $p\geqslant 0$:

$$\begin{aligned}K_1(n,n)+z_nK_2(n,n)&=0,\\ K_1(n,n+1)+z_nK_2(n,n+1)&=0,\\ K_1(n+1,n+1+p)&=K_1(n,n+p+2)+z_nK_2(n,n+p+2),\\ K_2(n+1,n+1+p)&=-\tilde{z}_n[K_1(n,n+p+2)+z_nK_2(n,n+p+2)]+K_2(n,n+p).\end{aligned} \tag{K14}$$

In particular, this implies using $p=0$

$$\begin{aligned}K_1(n+1,n+1)&=K_1(n,n+2)+z_nK_2(n,n+2),\\ K_2(n+1,n+1)&=-\tilde{z}_n[K_1(n,n+2)+z_nK_2(n,n+2)]+K_2(n,n),\\ &=-\tilde{z}_nK_1(n+1,n+1)+K_2(n,n),\\ &=\tilde{z}_nz_{n+1}K_2(n+1,n+1)+K_2(n,n).\end{aligned} \tag{K15}$$

With $p=1$ we further obtain the cancellation of all odd terms in the triangular decomposition. Hence we obtain the following result:

$$\psi_n=\lambda^n\prod_{\ell=n}^{\infty}(1-\tilde{z}_\ell z_{\ell+1})\begin{pmatrix}-z_n\\1\end{pmatrix}+O(\lambda^{n+2}), \tag{K16}$$

also summarized in Table II.

TABLE II. Summary of the results for K_1 and K_2.

Index	K_1	K_2
(n, n)	$K_1(n,n) = -z_n \prod_{\ell=n}^{\infty}(1-\tilde{z}_\ell z_{\ell+1})$	$K_2(n,n) = \prod_{\ell=n}^{\infty}(1-\tilde{z}_\ell z_{\ell+1})$
$(n, n+2p+1)$	$K_1(n, n+2p+1) = 0$	$K_2(n, n+2p+1) = 0$

3. Triangular decomposition of $\bar{\phi}_n$

We also give for completeness the results for $\bar{\phi}_n$ and ϕ_n although we will not use them below. Injecting into $\bar{\phi}_{n+1} = L_n\bar{\phi}_n$ one finds the asymptotic expansion

$$\bar{\phi}_n = \lambda^n \begin{pmatrix} 0 \\ -1 \end{pmatrix} + \lambda^{n-2} \begin{pmatrix} -z_{n-1} \\ \sum_{\ell=-\infty}^{n-1} \frac{g}{2} z_\ell \tilde{z}_\ell \end{pmatrix} + O(\lambda^{n-4}). \tag{K17}$$

4. Triangular decomposition of ϕ_n

Injecting into $\phi_{n+1} = L_n\phi_n$ one finds the asymptotic expansion

$$\phi_n = \lambda^{-n} \prod_{\ell=-\infty}^{n-1} \left(1 - \frac{g}{2}\tilde{z}_{\ell-1} z_\ell\right) \begin{pmatrix} 1 \\ -\frac{g}{2}\tilde{z}_{n-1} \end{pmatrix} + O(\lambda^{-n+2}). \tag{K18}$$

5. Additional series expansion for the Lax problem

As in Ref. [38] we provide an additional discrete integral representation of the solutions to the Lax problem. Grouping the solution of the Lax problem into a 2×2 matrix $\Psi_n = \{\phi_n, \bar{\phi}_n\}$, we first define a rescaled matrix W_n so that

$$\Psi_n = \lambda^{-n\sigma_3}\tilde{W}_n, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \lambda^{-n\sigma_3} = \begin{pmatrix} \lambda^{-n} & 0 \\ 0 & \lambda^n \end{pmatrix}, \tag{K19}$$

whose evolution and asymptotic value are given by

$$\tilde{W}_{\ell+1} = \lambda^{(\ell+1)\sigma_3} L_\ell \lambda^{-\ell\sigma_3}\tilde{W}_\ell, \quad \tilde{W}_{+\infty} = \begin{pmatrix} a & \tilde{b} \\ b & -\tilde{a} \end{pmatrix}. \tag{K20}$$

Rewriting the matrix $\tilde{W}_n$ using a telescopic form and using Eq. (K20), we first obtain that

$$\begin{aligned} \tilde{W}_n &= \tilde{W}_{+\infty} - \sum_{\ell=n}^{\infty}(\tilde{W}_{\ell+1} - \tilde{W}_\ell) \\ &= \begin{pmatrix} a & \tilde{b} \\ b & -\tilde{a} \end{pmatrix} - \sum_{\ell=n}^{\infty}(\lambda^{(\ell+1)\sigma_3} L_\ell \lambda^{-\ell\sigma_3} - \mathbb{1}_2)\tilde{W}_\ell. \end{aligned} \tag{K21}$$

Going back to the $\Psi_n = \{\phi_n, \bar{\phi}_n\}$ space, we have

$$\begin{aligned} \Psi_n &= \lambda^{-n\sigma_3}\begin{pmatrix} a & \tilde{b} \\ b & -\tilde{a} \end{pmatrix} - \sum_{\ell=n}^{\infty} \lambda^{(\ell-n)\sigma_3}(\lambda^{\sigma_3}L_\ell - \mathbb{1}_2)\Psi_\ell, \\ &= \lambda^{-n\sigma_3}\begin{pmatrix} a & \tilde{b} \\ b & -\tilde{a} \end{pmatrix} - \sum_{\ell=n}^{\infty} \lambda^{(\ell+1-n)\sigma_3-1}\begin{pmatrix} 0 & z_\ell \\ -\tilde{z}_\ell & -z_\ell\tilde{z}_\ell \end{pmatrix}\Psi_\ell, \\ &= \lambda^{-n\sigma_3}\begin{pmatrix} a & \tilde{b} \\ b & -\tilde{a} \end{pmatrix} - \sum_{\ell=n}^{\infty} \begin{pmatrix} \lambda^{\ell-n} & 0 \\ 0 & \lambda^{-\ell+n-2} \end{pmatrix}\begin{pmatrix} 0 & z_\ell \\ -\tilde{z}_\ell & -z_\ell\tilde{z}_\ell \end{pmatrix}\Psi_\ell. \end{aligned} \tag{K22}$$

Hence we can obtain the recursion for ϕ_n and $\bar{\phi}_n$ by inspection of the columns as

$$\phi_n = \begin{pmatrix} a(\lambda)\lambda^{-n} \\ b(\lambda)\lambda^n \end{pmatrix} - \sum_{\ell=n}^{\infty} \begin{pmatrix} \lambda^{\ell-n} & 0 \\ 0 & \lambda^{-\ell+n-2} \end{pmatrix}\begin{pmatrix} 0 & z_\ell \\ -\tilde{z}_\ell & -z_\ell\tilde{z}_\ell \end{pmatrix}\phi_\ell \tag{K23}$$

and

$$\bar{\phi}_n = \begin{pmatrix} \tilde{b}(\lambda)\lambda^{-n} \\ -\tilde{a}(\lambda)\lambda^n \end{pmatrix} - \sum_{\ell=n}^{\infty} \begin{pmatrix} \lambda^{\ell-n} & 0 \\ 0 & \lambda^{-\ell+n-2} \end{pmatrix}\begin{pmatrix} 0 & z_\ell \\ -\tilde{z}_\ell & -z_\ell\tilde{z}_\ell \end{pmatrix}\bar{\phi}_\ell. \tag{K24}$$

Since ψ_n and $\bar{\psi}_n$ are linear combinations of ϕ_n and $\bar{\phi}_n$, we further obtain that

$$\psi_n = \lambda^n \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \sum_{\ell=n}^{\infty} \begin{pmatrix} \lambda^{\ell-n} & 0 \\ 0 & \lambda^{-\ell+n-2} \end{pmatrix} \begin{pmatrix} 0 & z_\ell \\ -\tilde{z}_\ell & -z_\ell \tilde{z}_\ell \end{pmatrix} \psi_\ell \tag{K25}$$

and

$$\bar{\psi}_n = \lambda^{-n} \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \sum_{\ell=n}^{\infty} \begin{pmatrix} \lambda^{\ell-n} & 0 \\ 0 & \lambda^{-\ell+n-2} \end{pmatrix} \begin{pmatrix} 0 & z_\ell \\ -\tilde{z}_\ell & -z_\ell \tilde{z}_\ell \end{pmatrix} \bar{\psi}_\ell. \tag{K26}$$

Note that to have only higher order terms in the right-hand side of Eq. (K25), i.e., ψ_ℓ with $\ell > n$, it is possible to rewrite the series as

$$\begin{aligned} \lambda^{-n}\psi_n &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \sum_{\ell=n}^{\infty} \begin{pmatrix} \lambda^{2\ell-2n} & 0 \\ 0 & \lambda^{-2} \end{pmatrix} \begin{pmatrix} 0 & z_\ell \\ -\tilde{z}_\ell & -z_\ell \tilde{z}_\ell \end{pmatrix} \lambda^{-\ell} \psi_\ell \\ &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \sum_{\ell=n}^{\infty} \lambda \begin{pmatrix} \lambda^{2\ell-2n} & 0 \\ 0 & \lambda^{-2} \end{pmatrix} \begin{pmatrix} 0 & z_\ell \\ -\tilde{z}_\ell & -z_\ell \tilde{z}_\ell \end{pmatrix} L_\ell^{-1} \lambda^{-(\ell+1)} \psi_{\ell+1} \\ &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \sum_{\ell=n}^{\infty} \begin{pmatrix} \lambda^{2\ell-2n} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z_\ell \tilde{z}_\ell & z_\ell \\ -\tilde{z}_\ell & 0 \end{pmatrix} \lambda^{-(\ell+1)} \psi_{\ell+1}. \end{aligned} \tag{K27}$$

These series representations have been used in Ref. [38] to obtain analyticity results on the solution to the Lax problem. Note further that we also have the matrix factorization:

$$\begin{pmatrix} 0 & z_\ell \\ -\tilde{z}_\ell & -z_\ell \tilde{z}_\ell \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\tilde{z}_\ell & 1 \end{pmatrix} \begin{pmatrix} 1 & z_\ell \\ 0 & 1 \end{pmatrix} - \mathbb{1}_2. \tag{K28}$$

APPENDIX L: GELFAND-LEVITAN-MARCHENKO EQUATIONS

1. Notations and intermediate objects

To obtain the explicit solutions $\{z_n, \tilde{z}_n\}$ as a function of the scattering data, one needs to obtain the Gelfand-Levitan-Marchenko equations of the problem. To this aim, we start by rewriting the linear relation between $\phi, \bar{\phi}$ and $\psi, \bar{\psi}$, and we further insert the triangular representation of ψ_n and $\bar{\psi}_n$ obtained in Eqs. (K4)–(K5):

$$\begin{aligned} \frac{1}{a(\lambda)}\phi_k &= \bar{\psi}_k + \frac{b(\lambda,t)}{a(\lambda)}\psi_k, \\ \frac{1}{\tilde{a}(\lambda)}\bar{\phi}_k &= -\psi_k + \frac{\tilde{b}(\lambda,t)}{\tilde{a}(\lambda)}\bar{\psi}_k. \end{aligned} \tag{L1}$$

The properties we will require are the following:

(1) $\lambda^n\phi_n$, $\lambda^{-n}\psi_n$, and $a(\lambda)$ are analytic inside the circle $|\lambda| < \mathtt{R}$.

(2) $\lambda^{-n}\bar{\phi}_n$, $\lambda^n\bar{\psi}_n$, and $\tilde{a}(\lambda)$ are analytic outside of the circle $|\lambda| > \mathtt{R}$.

The analyticity properties of the various function will be used to proceed to a contour integration of Eq. (L1) using the following:

(1) For a function analytic inside the circle of radius $\mathtt{R}$, we have that

$$\oint_{|\lambda|=\mathtt{R}} \frac{d\lambda}{2\mathbf{i}\pi} \frac{1}{\lambda-\xi_1} f(\lambda) = f(\xi_1)\Theta(|\xi_1| < \mathtt{R}). \tag{L2}$$

(2) For a function analytic outside the circle of radius $\mathtt{R}$, we have that

$$\oint_{|\lambda|=\mathtt{R}} \frac{d\lambda}{2\mathbf{i}\pi} \frac{1}{\lambda-\xi_1} f(\lambda) = \lim_{\lambda\to+\infty} f(\lambda) - f(\xi_1)\Theta(|\xi_1| > \mathtt{R}), \tag{L3}$$

where Θ denotes the Heaviside function. Furthermore, we use a contour integral representation of the Kronecker delta function

$$\delta_{m,n} = \oint_{|\lambda|=\mathtt{R}} \frac{d\lambda}{2\mathbf{i}\pi} \lambda^{m-n-1}. \tag{L4}$$

We define the Fourier transform on the circle of the reflection coefficients. If the scattering data $\{a, \tilde{a}\}$ do not have any zero, it reads

$$\begin{aligned} F(n) &:= \oint_{|\lambda|=\mathtt{R}} \frac{d\lambda}{2\mathbf{i}\pi} \lambda^{n-1} \frac{b(\lambda,t)}{a(\lambda)} = \oint_{|\lambda|=\mathtt{R}} \frac{d\lambda}{2\mathbf{i}\pi} \lambda^{n-1} \frac{b(\lambda)}{a(\lambda)} e^{(1-\lambda^2)t}, \\ \tilde{F}(n) &:= \oint_{|\lambda|=\mathtt{R}} \frac{d\lambda}{2\mathbf{i}\pi} \lambda^{-n-1} \frac{\tilde{b}(\lambda,t)}{\tilde{a}(\lambda)} = \oint_{|\lambda|=\mathtt{R}} \frac{d\lambda}{2\mathbf{i}\pi} \lambda^{-n-1} \frac{\tilde{b}(\lambda)}{\tilde{a}(\lambda)} e^{(\lambda^2-1)t}. \end{aligned} \tag{L5}$$

In the case of the presence of solitons, i.e., the scattering data having zeros as in the current problem, we first proceed to the replacement

$$a(\lambda) \to a(\lambda)\frac{\lambda^2-\lambda_0^2}{\lambda^2-\lambda_{-1}^2}, \quad \tilde{a}(\lambda) \to \tilde{a}(\lambda)\frac{\lambda^2-\lambda_{-1}^2}{\lambda^2-\lambda_0^2}. \tag{L6}$$

If the scattering data have simple zeros, e.g., $a(\pm\lambda_0)=0$ and $\tilde{a}(\pm\lambda_{-1})=0$, then the expressions of $F(n)$ and $\tilde{F}(n)$ are modified to take into account the respective poles as

$$\begin{aligned} F(n) &:= \oint_{|\lambda|=\mathtt{R}} \frac{d\lambda}{2\mathbf{i}\pi}\lambda^{n-1}\frac{b(\lambda)}{a(\lambda)}\frac{\lambda^2-\lambda_{-1}^2}{\lambda^2-\lambda_0^2}e^{(1-\lambda^2)t} + \frac{\lambda_0^2-\lambda_{-1}^2}{2\lambda_0}\left[\lambda^{n-1}\frac{b(\lambda)}{a(\lambda)}e^{(1-\lambda^2)t}\right]_{-\lambda_0}^{\lambda_0}, \\ \tilde{F}(n) &:= \oint_{|\lambda|=\mathtt{R}} \frac{d\lambda}{2\mathbf{i}\pi}\lambda^{-n-1}\frac{\tilde{b}(\lambda)}{\tilde{a}(\lambda)}\frac{\lambda^2-\lambda_0^2}{\lambda^2-\lambda_{-1}^2}e^{(\lambda^2-1)t} + \frac{\lambda_{-1}^2-\lambda_0^2}{2\lambda_{-1}}\left[\lambda^{-n-1}\frac{\tilde{b}(\lambda)}{\tilde{a}(\lambda)}e^{(\lambda^2-1)t}\right]_{-\lambda_{-1}}^{\lambda_{-1}}, \end{aligned} \tag{L7}$$

where the notation $[f(\lambda)]_a^b$ stands for $f(b)-f(a)$.

Inserting the precise value of the scattering data,

$$\tilde{b}(\lambda) = -\lambda^2,\ b(\lambda) = \Lambda\lambda^{-2N-2}e^{\lambda^2-1},\ a(\lambda) = e^{\varphi(\lambda)},\ \tilde{a}(\lambda) = e^{-\varphi(\lambda)}, \tag{L8}$$

we then obtain in the absence of solitons:

$$\begin{aligned} F(n) &= \Lambda \oint_{|\lambda|=\mathtt{R}} \frac{d\lambda}{2\mathbf{i}\pi\lambda}\lambda^{n-2-2N}e^{(\lambda^2-1)(1-t)-\varphi(\lambda)}, \\ \tilde{F}(n) &= -\oint_{|\lambda|=\mathtt{R}} \frac{d\lambda}{2\mathbf{i}\pi\lambda}\lambda^{-(n-2)}e^{(\lambda^2-1)t+\varphi(\lambda)}, \end{aligned} \tag{L9}$$

and in the presence of solitons:

$$\begin{aligned} F(n) &= \Lambda \oint_{|\lambda|=\mathtt{R}} \frac{d\lambda}{2\mathbf{i}\pi\lambda}\lambda^{n-2-2N}\frac{\lambda^2-\lambda_{-1}^2}{\lambda^2-\lambda_0^2}e^{(\lambda^2-1)(1-t)-\varphi(\lambda)} + \Lambda\frac{\lambda_0^2-\lambda_{-1}^2}{2\lambda_0}[\lambda^{n-3-2N}e^{(\lambda^2-1)(1-t)-\varphi(\lambda)}]_{-\lambda_0}^{\lambda_0}, \\ \tilde{F}(n) &= -\oint_{|\lambda|=\mathtt{R}} \frac{d\lambda}{2\mathbf{i}\pi\lambda}\lambda^{-(n-2)}\frac{\lambda^2-\lambda_0^2}{\lambda^2-\lambda_{-1}^2}e^{(\lambda^2-1)t+\varphi(\lambda)} - \frac{\lambda_{-1}^2-\lambda_0^2}{2\lambda_{-1}}[\lambda^{-n+1}e^{(\lambda^2-1)t+\varphi(\lambda)}]_{-\lambda_{-1}}^{\lambda_{-1}}. \end{aligned} \tag{L10}$$

Remark L.1. Note that the two Fourier transforms verify the discrete linear evolution equations:

$$\begin{aligned} \partial_t F(2n) &= F(2n) - F(2(n+1)), \\ \partial_t \tilde{F}(2n) &= \tilde{F}(2(n-1)) - \tilde{F}(2n). \end{aligned} \tag{L11}$$

2. Derivation of the Gelfand-Levitan-Marchenko equations

a. First equation

We start by considering the first equation of (L1), multiply it by $\lambda^n/(\lambda-\xi_1)$ for $\xi_1 \in \mathbb{C}$ and integrate it over λ:

$$\oint_{|\lambda|=\mathtt{R}} \frac{d\lambda}{2\mathbf{i}\pi}\frac{\lambda^n}{\lambda-\xi_1}\frac{\phi_n}{a(\lambda)} = \oint_{|\lambda|=\mathtt{R}} \frac{d\lambda}{2\mathbf{i}\pi}\frac{\lambda^n}{\lambda-\xi_1}\bar{\psi}_n + \oint_{|\lambda|=\mathtt{R}} \frac{d\lambda}{2\mathbf{i}\pi}\frac{\lambda^n}{\lambda-\xi_1}\frac{b(\lambda,t)}{a(\lambda)}\psi_n. \tag{L12}$$

We first assume that we are in the case without soliton, i.e., no zero of $a(\lambda)$ and $\tilde{a}(\lambda)$ in their domain of analyticity. Using the rules (L2) and (L3), we obtain

$$\xi_1^n\frac{\phi_n(\xi_1)}{a(\xi_1)}\Theta(|\xi_1|<1) - \lim_{|\lambda|\to\infty}\lambda^n\bar{\psi}_n = -\xi_1^n\bar{\psi}_n(\xi_1)\Theta(|\xi_1|>1) + \oint_{|\lambda|=\mathtt{R}} \frac{d\lambda}{2\mathbf{i}\pi}\frac{\lambda^n}{\lambda-\xi_1}\frac{b(\lambda,t)}{a(\lambda)}\psi_n. \tag{L13}$$

We then denote the following limit:

$$\mathcal{I}_1 = \lim_{|\lambda|\to\infty}\lambda^n\bar{\psi}_n. \tag{L14}$$

Taking $|\xi_1| = \mathtt{R}^+$, Eq. (L13) reads

$$-\mathcal{I}_1 = -\xi_1^n\bar{\psi}_n(\xi_1) + \oint_{|\lambda|=\mathtt{R}} \frac{d\lambda}{2\mathbf{i}\pi}\frac{\lambda^n}{\lambda-\xi_1}\frac{b(\lambda,t)}{a(\lambda)}\psi_n. \tag{L15}$$

Multiplying the equation by ξ_1^{m-n-1} with $m \geqslant n$ and integrating ξ_1 over the circle of radius R, we further obtain

$$-\oint_{|\xi_1|=\mathtt{R}} \frac{d\xi_1}{2\mathbf{i}\pi}\xi_1^{m-n-1}\mathcal{I}_1 = -\oint_{|\xi_1|=\mathtt{R}} \frac{d\xi_1}{2\mathbf{i}\pi}\xi_1^{m-1}\bar{\psi}_n(\xi_1) + \oint_{|\xi_1|=\mathtt{R}} \frac{d\xi_1}{2\mathbf{i}\pi}\xi_1^{m-n-1}\oint_{|\lambda|=\mathtt{R}} \frac{d\lambda}{2\mathbf{i}\pi}\frac{\lambda^n}{\lambda-\xi_1}\frac{b(\lambda,t)}{a(\lambda)}\psi_n. \tag{L16}$$

Making use of the contour integral representation of the Kronecker function (L4) and expanding the term $1/\lambda - \xi_1$ as a series, we have

$$\begin{aligned}
-\delta_{m,n}\mathcal{I}_1 &= -\oint_{|\xi_1|=\mathtt{R}} \frac{d\xi_1}{2\mathbf{i}\pi}\xi_1^{m-1}\bar{\psi}_n(\xi_1) - \sum_{\ell=0}^{\infty}\oint_{|\xi_1|=\mathtt{R}} \frac{d\xi_1}{2\mathbf{i}\pi}\oint_{|\lambda|=\mathtt{R}} \frac{d\lambda}{2\mathbf{i}\pi}\xi_1^{m-n-1}\frac{\lambda^{n+\ell}}{\xi_1^{\ell+1}}\frac{b(\lambda,t)}{a(\lambda)}\psi_n \\
&= -\oint_{|\xi_1|=\mathtt{R}} \frac{d\xi_1}{2\mathbf{i}\pi}\xi_1^{m-1}\bar{\psi}_n(\xi_1) - \sum_{\ell=0}^{\infty}\delta_{m,n+\ell+1}\oint_{|\lambda|=\mathtt{R}} \frac{d\lambda}{2\mathbf{i}\pi}\lambda^{n+\ell}\frac{b(\lambda,t)}{a(\lambda)}\psi_n \\
&= -\oint_{|\xi_1|=\mathtt{R}} \frac{d\xi_1}{2\mathbf{i}\pi}\xi_1^{m-1}\bar{\psi}_n(\xi_1) - (1-\delta_{m,n})\oint_{|\lambda|=\mathtt{R}} \frac{d\lambda}{2\mathbf{i}\pi}\lambda^{m-1}\frac{b(\lambda,t)}{a(\lambda)}\psi_n.
\end{aligned} \tag{L17}$$

Inserting the triangular representation of ψ_n, $\bar{\psi}_n$ from Eqs. (K4) and (K5) with the Fourier transform of the scattering data (L5) we obtain for all $m \geqslant n$

$$\delta_{m,n}\mathcal{I}_1 = \bar{K}(n,m) + (1-\delta_{m,n})\sum_{p\geqslant 0} K(n,n+2p)F(m+n+2p). \tag{L18}$$

Choosing $m = n$, this implies that

$$\mathcal{I}_1 = \bar{K}(n,n) = \begin{pmatrix}1\\0\end{pmatrix}. \tag{L19}$$

The value of $\mathcal{I}_1$ can be read in Eq. (K13) and the one of $\bar{K}(n,n)$ in Table I. Furthermore, choosing $m > n$ and separating the $p = 0$ term out of the sum, we obtain the first Gelfand-Levitan-Marchenko equation as

$$\bar{K}(n,m) + K(n,n)F(m+n) + \sum_{p>0} K(n,n+2p)F(m+n+2p) = 0, \quad m > n. \tag{L20}$$

b. Second equation

We repeat the procedure to obtain the second GLM equation. To this aim, we consider the second equation of (L1), multiply it by $\lambda^{-n}/(\lambda - \xi_2)$ for $\xi_2 \in \mathbb{C}$ and integrate it over λ,

$$\oint_{|\lambda|=\mathtt{R}} \frac{d\lambda}{2\mathbf{i}\pi}\frac{\lambda^{-n}}{\lambda-\xi_2}\frac{\bar{\phi}_n}{\tilde{a}(\lambda)} = -\oint_{|\lambda|=\mathtt{R}} \frac{d\lambda}{2\mathbf{i}\pi}\frac{\lambda^{-n}}{\lambda-\xi_2}\psi_n + \oint_{|\lambda|=\mathtt{R}} \frac{d\lambda}{2\mathbf{i}\pi}\frac{\lambda^{-n}}{\lambda-\xi_2}\frac{\tilde{b}(\lambda,t)}{\tilde{a}(\lambda)}\bar{\psi}_n. \tag{L21}$$

We first assume that we are in the case without soliton, i.e., no zero of $a(\lambda)$ and $\tilde{a}(\lambda)$ in their domain of analyticity. Using the rules (L2) and (L3), we obtain

$$-\xi_2^{-n}\frac{\bar{\phi}_n(\xi_2)}{\tilde{a}(\xi_2)}\Theta(|\xi_2|>1) + \lim_{|\lambda|\to+\infty}\lambda^{-n}\frac{\bar{\phi}_n}{\tilde{a}(\lambda)} = -\xi_2^{-n}\psi_n(\xi_2)\Theta(|\xi_2|<1) + \oint_{|\lambda|=\mathtt{R}} \frac{d\lambda}{2\mathbf{i}\pi}\frac{\lambda^{-n}}{\lambda-\xi_2}\frac{\tilde{b}(\lambda,t)}{\tilde{a}(\lambda)}\bar{\psi}_n. \tag{L22}$$

We then denote the following limit:

$$\mathcal{I}_2 = \lim_{|\lambda|\to+\infty}\lambda^{-n}\frac{\bar{\phi}_n}{\tilde{a}(\lambda)}. \tag{L23}$$

We now take $|\xi_2| = \mathtt{R}^-$ so that

$$\mathcal{I}_2 = -\xi_2^{-n}\psi_n(\xi_2) + \oint_{|\lambda|=\mathtt{R}} \frac{d\lambda}{2\mathbf{i}\pi}\frac{\lambda^{-n}}{\lambda-\xi_2}\frac{\tilde{b}(\lambda,t)}{\tilde{a}(\lambda)}\bar{\psi}_n. \tag{L24}$$

Multiplying the equation by ξ_2^{n-m-1} with $m \geqslant n$ and integrating ξ_2 over the circle of radius R, we further obtain

$$\oint_{|\xi_2|=\mathtt{R}} \frac{d\xi_2}{2\mathbf{i}\pi}\xi_2^{n-m-1}\mathcal{I}_2 = -\oint_{|\xi_2|=\mathtt{R}} \frac{d\xi_2}{2\mathbf{i}\pi}\xi_2^{-m-1}\psi_n(\xi_2) + \oint_{|\xi_2|=\mathtt{R}} \frac{d\xi_2}{2\mathbf{i}\pi}\xi_2^{n-m-1}\oint_{|\lambda|=\mathtt{R}} \frac{d\lambda}{2\mathbf{i}\pi}\frac{\lambda^{-n}}{\lambda-\xi_2}\frac{\tilde{b}(\lambda,t)}{\tilde{a}(\lambda)}\bar{\psi}_n \tag{L25}$$

equivalent to

$$\begin{aligned}\delta_{m,n}\mathcal{I}_2 &= -\oint_{|\xi_2|=\mathrm{R}}\frac{\mathrm{d}\xi_2}{2\mathbf{i}\pi}\xi_2^{-m-1}\psi_n(\xi_2)+\oint_{|\xi_2|=\mathrm{R}}\frac{\mathrm{d}\xi_2}{2\mathbf{i}\pi}\xi_2^{n-m-1}\oint_{|\lambda|=\mathrm{R}}\frac{\mathrm{d}\lambda}{2\mathbf{i}\pi}\frac{\lambda^{-n}}{\lambda-\xi_2}\frac{\tilde{b}(\lambda,t)}{\tilde{a}(\lambda)}\bar{\psi}_n\\ &= -\oint_{|\xi_2|=\mathrm{R}}\frac{\mathrm{d}\xi_2}{2\mathbf{i}\pi}\xi_2^{-m-1}\psi_n(\xi_2)+\sum_{\ell=0}^{\infty}\oint_{|\xi_2|=\mathrm{R}}\frac{\mathrm{d}\xi_2}{2\mathbf{i}\pi}\xi_2^{\ell+n-m-1}\oint_{|\lambda|=\mathrm{R}}\frac{\mathrm{d}\lambda}{2\mathbf{i}\pi}\lambda^{-n-\ell-1}\frac{\tilde{b}(\lambda,t)}{\tilde{a}(\lambda)}\bar{\psi}_n\\ &= -\oint_{|\xi_2|=\mathrm{R}}\frac{\mathrm{d}\xi_2}{2\mathbf{i}\pi}\xi_2^{-m-1}\psi_n(\xi_2)+\oint_{|\lambda|=\mathrm{R}}\frac{\mathrm{d}\lambda}{2\mathbf{i}\pi}\lambda^{-m-1}\frac{\tilde{b}(\lambda,t)}{\tilde{a}(\lambda)}\bar{\psi}_n.\end{aligned} \tag{L26}$$

Inserting the triangular representation of ψ_n, $\bar{\psi}_n$ from Eqs. (K4) and (K5) with the Fourier transform of the scattering data (L5) we obtain for all $m \geqslant n$

$$\delta_{m,n}\mathcal{I}_2 = -K(n,m)+\sum_{p\geqslant 0}\bar{K}(n,n+2p)\tilde{F}(m+n+2p). \tag{L27}$$

Choosing $n=m$, this implies that

$$\mathcal{I}_2 = -K(n,n)+\sum_{p\geqslant 0}\bar{K}(n,n+2p)\tilde{F}(2n+2p) = \begin{pmatrix}0\\-1\end{pmatrix}. \tag{L28}$$

The value of $\mathcal{I}_2$ can be read from Eq. (K17) alongside $\lim_{\lambda\to\infty}\tilde{a}(\lambda)=1$ obtained for instance from Eq. (H12). Furthermore, choosing $m>n$ and separating the $p=0$ term out of the sum, we obtain the second Gelfand-Levitan-Marchenko equation as

$$K(n,m)-\bar{K}(n,n)\tilde{F}(m+n)-\sum_{p>0}\bar{K}(n,n+2p)\tilde{F}(m+n+2p)=0, \quad m>n. \tag{L29}$$

3. Operator valued equations

The GLM equations obtained in Eqs. (L20)–(L29) involved summations over explicit indices. We rewrite in this section these equations using operators and invert these to obtain explicitly the solution $\{z_n,\tilde{z}_n\}$ of the original system (10).

Let us define four operators $\{F_n,\tilde{F}_n,K_n,\bar{K}_n\}$ indexed by $n\in\mathbb{Z}$ with the following kernels:

$$\begin{aligned}F_n(i,j)&=F(2n+i+j),\\ \tilde{F}_n(i,j)&=\tilde{F}(2n+i+j),\\ K_n(i,j)&=K(i+n,n+j),\\ \bar{K}_n(i,j)&=\bar{K}(i+n,n+j)\end{aligned} \tag{L30}$$

for $i,j\in\mathbb{N}$ and we recall that the operators K_n and $\bar{K}_n$ are upper-triangular operators. Using the four kernels introduced and choosing $m=n+2j$ for $j>0$, we rewrite the two GLM equations (L20)–(L29) as

$$\begin{aligned}&\bar{K}_n(0,2j)+K_n(0,0)F_n(0,2j)+\sum_{p>0}K_n(0,2p)F_n(2p,2j)=0,\\ &K_n(0,2j)-\bar{K}_n(0,0)\tilde{F}_n(0,2j)-\sum_{p>0}\bar{K}_n(0,2p)\tilde{F}_n(2p,2j)=0.\end{aligned} \tag{L31}$$

These two equations suggest to define an operator product as

$$(\mathcal{O}_1\mathcal{O}_2)(i,j)=\sum_{p>0}\mathcal{O}_1(i,2p)\mathcal{O}_2(2p+j) \tag{L32}$$

and to define the left (resp. right) projectors to zero $\langle\delta|$ (resp. $|\delta\rangle$), so that for any operator $\mathcal{O}$ with kernel $\mathcal{O}(i,j)$, we have

$$\begin{aligned}&(\langle\delta|\,\mathcal{O})(j)=\mathcal{O}(0,j), \quad (\mathcal{O}\,|\delta\rangle)(i)=\mathcal{O}(i,0),\\ &\langle\delta|\,\mathcal{O}\,|\delta\rangle=\mathcal{O}(0,0).\end{aligned} \tag{L33}$$

Equipped with the operator product and the projector, we rewrite the diagonal contribution of the GLM equations, Eqs. (L19) and (L28), as

$$\begin{aligned}\mathcal{I}_1&=\langle\delta|\,\bar{K}_n\,|\delta\rangle=\begin{pmatrix}1\\0\end{pmatrix},\\ \mathcal{I}_2&=-\langle\delta|\,K_n\,|\delta\rangle+\langle\delta|\,\bar{K}_n\,|\delta\rangle\,\langle\delta|\,\tilde{F}_n\,|\delta\rangle+\langle\delta|\,\bar{K}_n\tilde{F}_n\,|\delta\rangle\\ &=\begin{pmatrix}0\\-1\end{pmatrix},\end{aligned} \tag{L34}$$

where the diagonal value of the kernel of K_n reads from Table II

$$\langle\delta|\,K_n\,|\delta\rangle=\prod_{\ell=n}^{\infty}\left(1-\frac{g}{2}\tilde{z}_\ell z_{\ell+1}\right)\begin{pmatrix}-z_n\\1\end{pmatrix}. \tag{L35}$$

The nondiagonal GLM equations (L31) now read

$$\langle\delta|\,\bar{K}_n+\langle\delta|\,K_n\,|\delta\rangle\,\langle\delta|\,F_n+\langle\delta|\,K_nF_n=0, \tag{L36}$$

and

$$\langle\delta|\,K_n-\langle\delta|\,\bar{K}_n\,|\delta\rangle\,\langle\delta|\,\tilde{F}_n-\langle\delta|\,\bar{K}_n\tilde{F}_n=0. \tag{L37}$$

4. Inversion of the GLM equations and explicit expressions of $z_n, \tilde{z}_n$

We invert the GLM equations to obtain the kernels of K_n and $\bar{K}_n$ solely in terms of the ones of F_n and $\tilde{F}_n$. This in turn is used to obtain the explicit expressions of $z_n, \tilde{z}_n$ as a function of the operators F_n and $\tilde{F}_n$.

a. Inversion of the GLM equation

In the first approach, we separate all the diagonal terms in (L36) and (L37) from the rest and respectively replace the values of $\langle\delta|\, K_n$ or $\langle\delta|\, \bar{K}_n$ in each equation. From this, one obtains

$$\langle\delta|\, \bar{K}_n = -\langle\delta|\, K_n\,|\delta\rangle\,\langle\delta|\, F_n \frac{1}{1+\tilde{F}_nF_n} - \langle\delta|\, \bar{K}_n\,|\delta\rangle\,\langle\delta|\, \frac{\tilde{F}_nF_n}{1+\tilde{F}_nF_n}, \tag{L38}$$

$$\langle\delta|\, K_n = \langle\delta|\, \bar{K}_n\,|\delta\rangle\,\langle\delta|\, \tilde{F}_n \frac{1}{1+F_n\tilde{F}_n} - \langle\delta|\, K_n\,|\delta\rangle\,\langle\delta|\, \frac{F_n\tilde{F}_n}{1+F_n\tilde{F}_n}, \tag{L39}$$

where in both equations the right index of the kernels have to be strictly greater than n, meaning that, e.g., the operator $(\langle\delta|\, K_n)(j) = K(n, n+j)$ can be evaluated only for $j > 0$.

As a corollary, we can multiply Eq. (L38) to the right by $\tilde{F}_n$ to evaluate the following product, useful for the diagonal contribution of the GLM equations:

$$\begin{aligned}\langle\delta|\, \bar{K}_n\tilde{F}_n\,|\delta\rangle &= -\langle\delta|\, K_n\,|\delta\rangle\,\langle\delta|\, F_n \frac{1}{1+\tilde{F}_nF_n}\tilde{F}_n\,|\delta\rangle - \langle\delta|\, \bar{K}_n\,|\delta\rangle\,\langle\delta|\, \frac{\tilde{F}_nF_n}{1+\tilde{F}_nF_n}\tilde{F}_n|\delta\rangle \\ &= -\langle\delta|\bar{K}_n|\delta\rangle\langle\delta|\tilde{F}_n|\delta\rangle - \langle\delta|K_n|\delta\rangle\langle\delta|\frac{F_n\tilde{F}_n}{1+F_n\tilde{F}_n}|\delta\rangle + \langle\delta|\bar{K}_n|\delta\rangle\langle\delta|\frac{1}{1+\tilde{F}_nF_n}\tilde{F}_n|\delta\rangle.\end{aligned} \tag{L40}$$

b. Exact solution for z_n

We will now prove the following result:

$$z_n = -\langle\delta|\tilde{F}_n\frac{1}{1+F_n\tilde{F}_n}|\delta\rangle. \tag{L41}$$

To prove this, we will use Eqs. (L34) and (L40); indeed,

$$\begin{aligned}\begin{pmatrix}0\\-1\end{pmatrix} &= -\langle\delta|K_n|\delta\rangle + \langle\delta|\bar{K}_n|\delta\rangle\langle\delta|\tilde{F}_n|\delta\rangle + \langle\delta|\bar{K}_n\tilde{F}_n|\delta\rangle \\ &= -\langle\delta|K_n|\delta\rangle\left(1 + \langle\delta|\frac{F_n\tilde{F}_n}{1+F_n\tilde{F}_n}|\delta\rangle\right) + \langle\delta|\bar{K}_n|\delta\rangle\langle\delta|\frac{1}{1+\tilde{F}_nF_n}\tilde{F}_n|\delta\rangle \\ &= -\prod_{\ell=n}^{\infty}(1-\tilde{z}_\ell z_{\ell+1})\begin{pmatrix}-z_n\\1\end{pmatrix}\left(1 + \langle\delta|\frac{F_n\tilde{F}_n}{1+F_n\tilde{F}_n}|\delta\rangle\right) + \begin{pmatrix}1\\0\end{pmatrix}\langle\delta|\frac{1}{1+\tilde{F}_nF_n}\tilde{F}_n|\delta\rangle.\end{aligned} \tag{L42}$$

Isolating the second component of the above vector we find that

$$\prod_{\ell=n}^{\infty}(1-\tilde{z}_\ell z_{\ell+1})\left(1 + \langle\delta|\frac{F_n\tilde{F}_n}{1+F_n\tilde{F}_n}|\delta\rangle\right) = 1. \tag{L43}$$

Then isolating the first component of the above vector and using (L43) we find the desired result, which is

$$z_n = -\langle\delta|\tilde{F}_n\frac{1}{1+F_n\tilde{F}_n}|\delta\rangle. \tag{L44}$$

We present two other representations of the exact expression of z_n which will be useful later in Appendix L 5. Anticipating the result of Eq. (L59) stating that $F_{n+1}\tilde{F}_{n+1} = F_n\tilde{F}_n - F_{n+1}|\delta\rangle\langle\delta|\tilde{F}_{n+1}$, we consider the finite difference of the resolvant and use the Sherman-Morrison formula

$$\frac{1}{1+F_{n+1}\tilde{F}_{n+1}} = \frac{1}{1+F_n\tilde{F}_n} + \frac{1}{1-\langle\delta|\tilde{F}_{n+1}\frac{1}{1+F_n\tilde{F}_n}F_{n+1}|\delta\rangle}\frac{1}{1+F_n\tilde{F}_n}F_{n+1}|\delta\rangle\langle\delta|\tilde{F}_{n+1}\frac{1}{1+F_n\tilde{F}_n}. \tag{L45}$$

We multiply (L45) by $\tilde{F}_{n+1}$ to the left and by $|\delta\rangle$ to the right and obtain

$$\langle\delta|\tilde{F}_{n+1}\frac{1}{1+F_n\tilde{F}_n}|\delta\rangle = (1-\langle\delta|\tilde{F}_{n+1}\frac{1}{1+F_n\tilde{F}_n}F_{n+1}|\delta\rangle)\langle\delta|\tilde{F}_{n+1}\frac{1}{1+F_{n+1}\tilde{F}_{n+1}}|\delta\rangle. \tag{L46}$$

This leads to the second expression for z_n,

$$z_n = -\frac{\langle\delta|\tilde{F}_n \frac{1}{1+F_{n-1}\tilde{F}_{n-1}}|\delta\rangle}{1 - \langle\delta|\tilde{F}_n \frac{1}{1+F_{n-1}\tilde{F}_{n-1}} F_n|\delta\rangle}. \tag{L47}$$

A third expression can be obtained using that $F_n\tilde{F}_n = F_{n+1}\tilde{F}_{n+1} + F_{n+1}|\delta\rangle\langle\delta|\tilde{F}_{n+1}$ and the Sherman-Morrison formula:

$$\frac{1}{1+F_n\tilde{F}_n} = \frac{1}{1+F_{n+1}\tilde{F}_{n+1}} - \frac{1}{1+\langle\delta|\tilde{F}_{n+1}\frac{1}{1+F_{n+1}\tilde{F}_{n+1}}F_{n+1}|\delta\rangle}\frac{1}{1+F_{n+1}\tilde{F}_{n+1}}F_{n+1}|\delta\rangle\langle\delta|\tilde{F}_{n+1}\frac{1}{1+F_{n+1}\tilde{F}_{n+1}}. \tag{L48}$$

We multiply (L48) by $\tilde{F}_{n+1}$ to the left and by $|\delta\rangle$ to the right and obtain

$$\langle\delta|\tilde{F}_{n+1}\frac{1}{1+F_n\tilde{F}_n}|\delta\rangle = \frac{\langle\delta|\tilde{F}_{n+1}\frac{1}{1+F_{n+1}\tilde{F}_{n+1}}|\delta\rangle}{1+\langle\delta|\tilde{F}_{n+1}\frac{1}{1+F_{n+1}\tilde{F}_{n+1}}F_{n+1}|\delta\rangle}. \tag{L49}$$

This leads to the third expression for z_n,

$$z_n = -\langle\delta|\tilde{F}_n\frac{1}{1+F_{n-1}\tilde{F}_{n-1}}|\delta\rangle(1+\langle\delta|\tilde{F}_n\frac{1}{1+F_n\tilde{F}_n}F_n|\delta\rangle). \tag{L50}$$

c. Exact solution for $\tilde{z}_n$

We continue by taking the right index of Eq. (L38) equal to $m = n+2$:

$$\begin{pmatrix} -\sum_{\ell=n}^{\infty} z_\ell\tilde{z}_\ell \\ \tilde{z}_n \end{pmatrix} = -\prod_{\ell=n}^{\infty}(1-\tilde{z}_\ell z_{\ell+1})\begin{pmatrix} -z_n \\ 1 \end{pmatrix}\langle\delta_n|\frac{1}{1+F_n\tilde{F}_n}F_{n+1}|\delta\rangle - \begin{pmatrix} 1 \\ 0 \end{pmatrix}\langle\delta|\tilde{F}_n\frac{1}{1+F_n\tilde{F}_n}F_{n+1}|\delta\rangle. \tag{L51}$$

Isolating the second coefficient, we obtain

$$\tilde{z}_n = -\frac{\langle\delta|\frac{1}{1+F_n\tilde{F}_n}F_{n+1}|\delta\rangle}{1+\langle\delta|\frac{F_n\tilde{F}_n}{1+F_n\tilde{F}_n}|\delta\rangle}. \tag{L52}$$

which we have not been able to further simplify. Isolating the first coefficient, we obtain

$$-\sum_{\ell=n+1}^{\infty} z_\ell\tilde{z}_\ell = \langle\delta|\tilde{F}_n\frac{1}{1+F_n\tilde{F}_n}F_{n+1}|\delta\rangle, \tag{L53}$$

which is nontrivial from the exact expressions of z_n, $\tilde{z}_n$ obtained in Eqs. (L41)–(L52). As a consequence we have that

$$\begin{aligned} -z_n\tilde{z}_n &= \langle\delta|\tilde{F}_n\frac{1}{1+F_n\tilde{F}_n}F_{n+1}|\delta\rangle - \langle\delta|\tilde{F}_{n-1}\frac{1}{1+F_{n-1}\tilde{F}_{n-1}}F_n|\delta\rangle \\ &= \Delta^+\langle\delta|\tilde{F}_{n-1}\frac{1}{1+F_{n-1}\tilde{F}_{n-1}}F_n|\delta\rangle, \end{aligned} \tag{L54}$$

where $\Delta^+ f_n = f_{n+1} - f_n$ is the finite difference operator.

d. Additional identities

If we combine Eqs. (L52) and (L50), we further obtain that

$$z_n\tilde{z}_n = \langle\delta|\frac{1}{1+F_n\tilde{F}_n}F_{n+1}|\delta\rangle\langle\delta|\tilde{F}_n\frac{1}{1+F_{n-1}\tilde{F}_{n-1}}|\delta\rangle, \tag{L55}$$

which is highly nontrivial from Eq. (L54). Furthermore, Eqs. (L41) and (L52) bring the expressions for z_n and $\tilde{z}_n$ on a more symmetric setting, indeed

$$\begin{aligned} \tilde{z}_n &= -\frac{\langle\delta|\frac{1}{1+F_n\tilde{F}_n}F_{n+1}|\delta\rangle}{1+\langle\delta|\frac{F_n\tilde{F}_n}{1+F_n\tilde{F}_n}|\delta\rangle}, \\ z_n &= -\langle\delta|\tilde{F}_n\frac{1}{1+F_{n-1}\tilde{F}_{n-1}}|\delta\rangle\left(1+\langle\delta|\frac{F_n\tilde{F}_n}{1+F_n\tilde{F}_n}|\delta\rangle\right). \end{aligned} \tag{L56}$$

5. Check of the time derivative of z_n

In this section we verify by an algebraic method that z_n indeed verifies the system of Eqs. (10). To this aim, we require three prelimiary results. From Eqs. (L11), we recall the time derivatives of the Fourier operators

$$\partial_t F_n = F_n - F_{n+1},$$
$$\partial_t \tilde{F}_n = \tilde{F}_{n-1} - \tilde{F}_n. \tag{L57}$$

In particular, the combination of the two for any $n, m \in \mathbb{Z}$ leads to

$$\begin{aligned}\partial_t(F_n\tilde{F}_m) &= (F_n - F_{n+1})\tilde{F}_m + F_n(\tilde{F}_{m-1} - \tilde{F}_m)\\ &= F_n\tilde{F}_{m-1} - F_{n+1}\tilde{F}_m\\ &= F_{n+1}|\delta\rangle\langle\delta|\tilde{F}_m,\end{aligned} \tag{L58}$$

which is a rank-one operator. Furthermore, from the definition of the product of the operators F_n and $\tilde{F}_n$ from Eq. (L32), we obtain the following relation for the finite difference:

$$F_n\tilde{F}_n = F_{n+1}\tilde{F}_{n+1} + F_{n+1}|\delta\rangle\langle\delta|\tilde{F}_{n+1} \Longrightarrow \Delta^+ F_n\tilde{F}_n = -F_{n+1}|\delta\rangle\langle\delta|\tilde{F}_{n+1}, \tag{L59}$$

where $\Delta^+ f_n = f_{n+1} - f_n$ so that the finite difference is a rank-one operator. More generally this relation remains valid for different indices

$$F_n\tilde{F}_m = F_{n+1}\tilde{F}_{m+1} + F_{n+1}|\delta\rangle\langle\delta|\tilde{F}_{m+1} \Longrightarrow \Delta^+ F_n\tilde{F}_m = -F_{n+1}|\delta\rangle\langle\delta|\tilde{F}_{m+1}. \tag{L60}$$

The derivative of the resolvant appearing in the expression for z_n is obtained as follows:

$$\begin{aligned}\partial_t \frac{1}{1+F_n\tilde{F}_n} &= -\frac{1}{1+F_n\tilde{F}_n}\partial_t(F_n\tilde{F}_n)\frac{1}{1+F_n\tilde{F}_n}\\ &= -\frac{1}{1+F_n\tilde{F}_n}F_{n+1}|\delta\rangle\langle\delta|\tilde{F}_n\frac{1}{1+F_n\tilde{F}_n}.\end{aligned} \tag{L61}$$

Equipped with these results, we calculate the time derivative of z_n:

$$\begin{aligned}\partial_t z_n &= -\partial_t\langle\delta|\tilde{F}_n\frac{1}{1+F_n\tilde{F}_n}|\delta\rangle\\ &= -\langle\delta|\partial_t\tilde{F}_n\frac{1}{1+F_n\tilde{F}_n}|\delta\rangle - \langle\delta|\tilde{F}_n\partial_t\frac{1}{1+F_n\tilde{F}_n}|\delta\rangle\\ &= -z_n - \langle\delta|\tilde{F}_{n-1}\frac{1}{1+F_n\tilde{F}_n}|\delta\rangle + \langle\delta|\tilde{F}_n\frac{1}{1+F_n\tilde{F}_n}F_{n+1}|\delta\rangle\langle\delta|\tilde{F}_n\frac{1}{1+F_n\tilde{F}_n}|\delta\rangle\\ &= -z_n - \langle\delta|\tilde{F}_{n-1}\frac{1}{1+F_n\tilde{F}_n}|\delta\rangle - \langle\delta|\tilde{F}_n\frac{1}{1+F_n\tilde{F}_n}F_{n+1}|\delta\rangle z_n.\end{aligned} \tag{L62}$$

Using that $F_n\tilde{F}_n = F_{n-1}\tilde{F}_{n-1} - F_n|\delta\rangle\langle\delta|\tilde{F}_n$ alongside the Sherman-Morrison formula on the second term of the right-hand side, we obtain

$$\begin{aligned}\partial_t z_n &= -z_n - \langle\delta|\tilde{F}_{n-1}\frac{1}{1+F_{n-1}\tilde{F}_{n-1} - F_n|\delta\rangle\langle\delta|\tilde{F}_n}|\delta\rangle - \langle\delta|\tilde{F}_n\frac{1}{1+F_n\tilde{F}_n}F_{n+1}|\delta\rangle z_n\\ &= -z_n + z_{n-1} - \frac{\langle\delta|\tilde{F}_{n-1}\frac{1}{1+F_{n-1}\tilde{F}_{n-1}}F_n|\delta\rangle\langle\delta|\tilde{F}_n\frac{1}{1+F_{n-1}\tilde{F}_{n-1}}|\delta\rangle}{1-\langle\delta|\tilde{F}_n\frac{1}{1+F_{n-1}\tilde{F}_{n-1}}F_n|\delta\rangle} - \langle\delta|\tilde{F}_n\frac{1}{1+F_n\tilde{F}_n}F_{n+1}|\delta\rangle z_n.\end{aligned} \tag{L63}$$

Using Eq. (L47), we recognize in the right-hand side the expression of z_n and we use Eq. (L54) to express the remaining difference as the product $z_n\tilde{z}_n$ and obtain

$$\begin{aligned}\partial_t z_n &= -z_n + z_{n-1} + \left(\langle\delta|\tilde{F}_{n-1}\frac{1}{1+F_{n-1}\tilde{F}_{n-1}}F_n|\delta\rangle - \langle\delta|\tilde{F}_n\frac{1}{1+F_n\tilde{F}_n}F_{n+1}|\delta\rangle\right)z_n\\ &= -z_n + z_{n-1} + z_n^2\tilde{z}_n.\end{aligned} \tag{L64}$$

6. Discrete Fredholm framework for the solution of the nonlinear system

The operator manipulations done so far can be related to a more fundamental object which is the Fredholm determinant of the $F_n\tilde{F}_n$ defined as

$$D_n = \mathrm{Det}(I + F_n\tilde{F}_n)_{\ell^2(\mathbb{N}^*)}, \tag{L65}$$

and we expect the operators to live to act on $\ell^2(0,\infty)$. Let us develop some algebraic framework for such object similarly to what was done in Refs. [42,43] for the Fredholm determinant of products of continuous Hankel operators.

The finite difference relation (L59) implies a multiplicative recursion for the Fredholm determinant D_n. Using the matrix determinant lemma, we have that

$$\begin{aligned} D_{n+1} &= \mathrm{Det}(I + F_{n+1}\tilde{F}_{n+1}) = \mathrm{Det}(I + F_n\tilde{F}_n - F_{n+1}|\delta\rangle\langle\delta|\tilde{F}_{n+1}) \\ &= \mathrm{Det}(I + F_n\tilde{F}_n)\Big(1 - \langle\delta|\tilde{F}_{n+1}\frac{I}{I + F_n\tilde{F}_n}F_{n+1}|\delta\rangle\Big) \\ &= D_n\left(1 - \langle\delta|\tilde{F}_{n+1}\frac{I}{I + F_n\tilde{F}_n}F_{n+1}|\delta\rangle\right). \end{aligned} \tag{L66}$$

This recursion can also be written backwards:

$$\begin{aligned} D_n &= \mathrm{Det}(I + F_n\tilde{F}_n) = \mathrm{Det}(I + F_{n+1}\tilde{F}_{n+1} + F_{n+1}|\delta\rangle\langle\delta|\tilde{F}_{n+1}) \\ &= \mathrm{Det}(I + F_{n+1}\tilde{F}_{n+1})\Big(1 + \langle\delta|\tilde{F}_{n+1}\frac{I}{I + F_{n+1}\tilde{F}_{n+1}}F_{n+1}|\delta\rangle\Big) \\ &= D_{n+1}\left(1 + \langle\delta|\tilde{F}_{n+1}\frac{I}{I + F_{n+1}\tilde{F}_{n+1}}F_{n+1}|\delta\rangle\right), \end{aligned} \tag{L67}$$

which implies the identity

$$\left(1 + \langle\delta|\tilde{F}_{n+1}\frac{I}{I + F_{n+1}\tilde{F}_{n+1}}F_{n+1}|\delta\rangle\right)\left(1 - \langle\delta|\tilde{F}_{n+1}\frac{I}{I + F_n\tilde{F}_n}F_{n+1}|\delta\rangle\right) = 1. \tag{L68}$$

The multiplicative recursion between implies that a suitable object to study is actually the logarithm of the Fredholm determinant, and thus we obtain that

$$\begin{aligned} \Delta^+ \log D_n &= \log\left(1 - \langle\delta|\tilde{F}_{n+1}\frac{I}{I + F_n\tilde{F}_n}F_{n+1}|\delta\rangle\right) \\ &= -\log\left(1 + \langle\delta|\tilde{F}_{n+1}\frac{I}{I + F_{n+1}\tilde{F}_{n+1}}F_{n+1}|\delta\rangle\right). \end{aligned} \tag{L69}$$

We also have from Eq. (L43) the relation between the logarithmic finite difference of the Fredholm determinant and the variables $\{z_n, \tilde{z}_n\}$ of our problem:

$$\log\frac{D_{n+1}}{D_n} = \Delta^+ \log D_n = \sum_{\ell=n+1}^{\infty}\log(1 - \tilde{z}_\ell z_{\ell+1}), \tag{L70}$$

which implies that the logarithmic second-order finite difference of the Fredholm determinant reads

$$\begin{aligned} (\Delta^+)^2 \log D_n &= \Delta^+ \sum_{\ell=n+1}^{\infty}\log(1 - \tilde{z}_\ell z_{\ell+1}) \\ &= -\log(1 - \tilde{z}_{n+1}z_{n+2}). \end{aligned} \tag{L71}$$

Exponentiating this identity, we obtain

$$\frac{D_{n+1}^2}{D_{n+2}D_n} = 1 - \tilde{z}_{n+1}z_{n+2}. \tag{L72}$$

This identity is akin to one obtained in the continuous setting of the weak noise theory of KPZ [1], Eq. (13). We can expect as in Refs. [42,43] that a hierarchy of functions can be constructed which will verify a differential recursion. This precise recursion is left for future work. The first step towards this construction would be to study the finite difference of the resolvant of $F_n\tilde{F}_n$; the starting formulas were presented in Appendix L 4 b, and we recall them here for completeness:

$$\begin{aligned} \frac{1}{1 + F_n\tilde{F}_n} &= \frac{1}{1 + F_{n+1}\tilde{F}_{n+1} + F_{n+1}|\delta\rangle\langle\delta|\tilde{F}_{n+1}} \\ &= \frac{1}{1 + F_{n+1}\tilde{F}_{n+1}} - \frac{1}{1 + \langle\delta|\tilde{F}_{n+1}\frac{1}{1 + F_{n+1}\tilde{F}_{n+1}}F_{n+1}|\delta\rangle}\frac{1}{1 + F_{n+1}\tilde{F}_{n+1}}F_{n+1}|\delta\rangle\langle\delta|\tilde{F}_{n+1}\frac{1}{1 + F_{n+1}\tilde{F}_{n+1}} \end{aligned} \tag{L73}$$

and

$$\begin{aligned}\frac{1}{1+F_{n+1}\tilde{F}_{n+1}} &= \frac{1}{1+F_n\tilde{F}_n - F_{n+1}|\delta\rangle\langle\delta|\tilde{F}_{n+1}} \\ &= \frac{1}{1+F_n\tilde{F}_n} + \frac{1}{1-\langle\delta|\tilde{F}_{n+1}\frac{1}{1+F_n\tilde{F}_n}F_{n+1}|\delta\rangle}\frac{1}{1+F_n\tilde{F}_n}F_{n+1}|\delta\rangle\langle\delta|\tilde{F}_{n+1}\frac{1}{1+F_n\tilde{F}_n}.\end{aligned} \tag{L74}$$

APPENDIX M: LIMIT TO THE CLASSICAL TODA SYSTEM

The weak noise theory of the O'Connell-Yor polymer can be seen as a generalization of the classical Toda system [45]. This can be seen from the Lax pair of the model as in Ref. [33] but also directly from the dynamical system (10). To see this, consider the following Cole-Hopf-type change of variable:

$$\begin{aligned}z_n(t) &= \alpha e^{h_n(t)+\alpha^2 t}, \\ \tilde{z}_n(t) &= e^{-h_n(t)-\alpha^2 t}[\alpha + p_n(t)].\end{aligned} \tag{M1}$$

Note that the Jacobian of this change of variable is a constant. Then the dynamics verified by the two fields $\{h_n, p_n\}$ is

$$\begin{aligned}\partial_t h_n &= -1 + e^{h_{n-1}-h_n} + \alpha p_n + a_n, \\ \partial_t p_n &= e^{h_{n-1}-h_n}(\alpha + p_n) - e^{h_n - h_{n+1}}(\alpha + p_{n+1}).\end{aligned} \tag{M2}$$

Rescaling $t = \frac{\tau}{\alpha}$, and $a_n = \alpha\hat{a}_n$, the action S_0 of the O'Connell-Yor polymer (B3) reads in terms of the new variables:

$$\begin{aligned}S_0 = S[h,p] = \int \mathrm{d}\tau \sum_{n=1}^{N}\bigg[&\alpha\bigg(1 - e^{h_{n-1}(\tau)-h_n(\tau)} - \frac{1}{2}p_n(\tau)^2 \\ &+ p_n(\tau)\partial_\tau h_n(\tau) - \hat{a}_n p_n(\tau)\bigg) \\ &+ \alpha^2\partial_\tau h_n(\tau) + (1 - e^{h_{n-1}(\tau)-h_n(\tau)})p_n(\tau) \\ &+ \frac{\alpha^3}{2} - \alpha^2\hat{a}_n\bigg].\end{aligned} \tag{M3}$$

Note that the terms proportional to α^2 and α^3 are, respectively, is a total derivative and a constant, hence they do not contribute to the equations of motion. Taking $\alpha \to \infty$ we find the classical Toda action, and the associated equations of motion (i.e., saddle point equations):

$$\begin{aligned}\partial_\tau h_n &= p_n + \hat{a}_n, \\ \partial_\tau p_n &= e^{h_{n-1}-h_n} - e^{h_n - h_{n+1}}.\end{aligned} \tag{M4}$$

This result brings a few comments:

(1) Since we have solved the weak noise theory of the O'Connell-Yor polymer using the discrete Fredholm determinant framework in Appendix L, our result is also extended for the Toda lattice.

(2) Note that our Lax pair formulation has allowed us to include drifts $\hat{a}_n$ in the Toda system. It would be interesting to understand their role in the dynamics.

(3) Finally, the Toda system has been related historically to the QR decomposition in linear algebra; see, e.g., Refs. [48,49]. It would be extremely interesting to relate the O'Connell-Yor dynamics (10) as an extension of the flow of the QR decomposition. Furthermore, as we expect the existence of an integrable time discretization of the O'Connell-Yor dynamics and thus of the Toda model, understanding the discrete flow could also shed some new light on linear algebra algorithms. We leave these questions open for a future work. Furthermore, the interpretation of the drifts a_n in these algorithms have to be understood.

APPENDIX N: SMALL-TIME LIMIT OF THE FREDHOLM DETERMINANT RESULT FOR THE O'CONNELL-YOR POLYMER

In this Appendix we start from the formula for the generating function of $Z_N^\beta(t)$, the OY partition sum studied in [37] which is is identical to ours if we identify $\beta = \sqrt{\varepsilon}$. We study its weak noise limit $\varepsilon \ll 1$, leading to a conjectural form for $\Psi_N(\Lambda)$ which agrees with the one derived in the text using inverse scattering. The manipulations in this Appendix are quite heuristic but they have the merit to show that the algebraic structure which emerges from the Fredholm determinant is similar to the one derived in the text from first principles by the inverse scattering method. The method applied in this Appendix is similar to the one used in Ref. [6], Sec. X of the Supp. Material in the context of the crossover from the large deviations of the macroscopic fluctuation theory to the weak noise theory of the KPZ equation.

The starting identity, given in Refs. [28] and [37], Proposition 14 reads

$$\mathbb{E}\left[e^{-\frac{e^{-\beta u}Z_N^\beta(t)}{\beta^{2(N-1)}}}\right] = \mathrm{Det}(I+L)_{L^2(C_0)}, \tag{N1}$$

where the kernel is written as

$$L(v,v') = \int_{\mathbf{i}\mathbb{R}+\delta}\frac{\mathrm{d}w}{2\mathbf{i}\pi}\frac{\pi}{\sin[\pi(v'-w)]}\frac{e^{\varepsilon w^2 t/2 - w\tilde{u}}}{e^{\varepsilon v'^2 t/2 - v'\tilde{u}}}\frac{1}{w-v}\frac{\Gamma(v')^N}{\Gamma(w)^N} \tag{N2}$$

with C_0 a contour enclosing the origin with radius $r < 1/2$ and $r < \delta < 1 - r$. The measure over C_0 is $\mathrm{d}v/2\mathbf{i}\pi$, and we further have the constraint that

$$0 < \mathrm{Re}(w - v') < 1. \tag{N3}$$

The identification with our variables reads

$$z_N(t) = e^{-(1+\frac{\varepsilon}{2})t}Z_N(t), \quad Z_N(t) = Z_N^{\beta=\sqrt{\varepsilon}}(t) \tag{N4}$$

as well as

$$\beta = \sqrt{\varepsilon}, \tag{N5}$$

so that the Fredholm determinant identity yields

$$\mathbb{E}[e^{-\varepsilon^{N-1}e^{-\tilde{u}+(1+\frac{\varepsilon}{2})t}z_N(t)}] = \mathrm{Det}(I+L)_{L^2(C_0)}. \tag{N6}$$

To complete our identification, we further introduce the relation

$$\varepsilon^{N-1}e^{-\tilde{u}+(1+\frac{\varepsilon}{2})}=\frac{\Lambda}{\varepsilon}. \tag{N7}$$

Endowed with the identification, we now transform the Fredholm identity (N6) to derive the large deviation function $\Psi_N(\Lambda)$. The first step is the introduction of a factorization of the kernel $L(v,v')$ as

$$L(v,v')=\int_{\mathbf{i}\mathbb{R}+\delta}\frac{\mathrm{d}w}{2\mathbf{i}\pi}A(v,w)\tilde{A}(w,v'), \tag{N8}$$

where have defined the kernels

$$A(v,w)=\frac{1}{v-w},$$
$$\tilde{A}(w,v')=\frac{\pi}{\sin[\pi(w-v')]}\frac{e^{\varepsilon w^2t/2-w\tilde{u}}}{e^{\varepsilon v'^2t/2-v'\tilde{u}}}\frac{\Gamma(v')^N}{\Gamma(w)^N}, \tag{N9}$$

Introducing the identity

$$\frac{\pi}{\sin(\pi s)}z^s=\int_{\mathbb{R}}\mathrm{d}r\,\frac{z}{z+e^{-r}}e^{-sr},\quad 0<\Re(s)<1, \tag{N10}$$

which we apply using

$$s=w-v',\quad z=e^{-\tilde{u}}. \tag{N11}$$

This allows us to rewrite the kernel $\tilde{A}(w,v')$ and to factorize it as

$$\tilde{A}(w,v')=\int_{\mathbb{R}}\mathrm{d}r\,\frac{1}{1+e^{\tilde{u}-r}}\frac{e^{\varepsilon w^2t/2-wr}}{e^{\varepsilon v'^2t/2-v'r}}\frac{\Gamma(v')^N}{\Gamma(w)^N}$$
$$=\int_{\mathbb{R}}\mathrm{d}r\,\sigma(r)A_1(w,r)A_2(r,v'), \tag{N12}$$

where we have defined

$$\sigma(r)=\frac{1}{1+e^{\tilde{u}-r}},\quad A_1(w,r)=\frac{1}{\Gamma(w)^N}e^{\varepsilon w^2t/2-wr},$$
$$A_2(r,v')=e^{-\varepsilon v'^2t/2+v'r}\Gamma(v')^N. \tag{N13}$$

To summarize, the Fredholm manipulations we have done lead to the factorization

$$\mathrm{Det}(1+L)_{L^2(C_0)}=\mathrm{Det}(1+AA_1\sigma A_2)_{L^2(C_0)}$$
$$=\mathrm{Det}(1+\sigma A_2AA_1)_{L^2(\mathbb{R})}, \tag{N14}$$

where from the first to the second line we have used Sylvester's identity $\mathrm{Det}(I+AB)=\mathrm{Det}(I+BA)$. This last Fredholm determinant has the typical structure for which the first cumulant method, developed in [64–66] to study the relevant asymptotics (here small ε), applies. This allows us to formally interpret this Fredholm as an expectation value of a determinantal point process $\{a_\ell\}_{\ell\geqslant 1}$ which correlations are controlled by the kernel A_2AA_1. Hence

$$\mathrm{Det}(1+L)_{L^2(C_0)}=\mathbb{E}\left[\prod_{\ell=1}^{+\infty}[1-\sigma(a_\ell)]\right]$$
$$=\mathbb{E}\left[\prod_{\ell=1}^{+\infty}e^{-\varphi(a_\ell)}\right], \tag{N15}$$

where we have introduced $e^{-\varphi}=1-\sigma$ to interpret (N15) as a linear statistics of the point process over the observable

$$\varphi(r)=\log(1+e^{r-\tilde{u}})=-\mathrm{Li}_1(-e^{r-\tilde{u}}). \tag{N16}$$

The first cumulant approximation applied to the expectation value (N15) asserts [64], Sec. 6 that as some parameter goes to infinity (here it will be $1/\varepsilon$; see below), we expect the point process to self-average,

$$\mathrm{Det}(1+L)_{L^2(C_0)}=\mathrm{Det}(1+\sigma A_2AA_1)_{L^2(\mathbb{R})}$$
$$=\mathbb{E}\left[\prod_{\ell=1}^{+\infty}e^{-\varphi(a_\ell)}\right]\sim e^{\mathrm{Tr}(\varphi A_2AA_1)}. \tag{N17}$$

The quantity of interest only involves the diagonal part of the kernel A_2AA_1 and we have to evaluate

$$\mathrm{Tr}(\varphi A_2AA_1)=\int_{\mathbb{R}}\mathrm{d}r\int_{\mathbf{i}\mathbb{R}+\delta}\frac{\mathrm{d}w}{2\mathbf{i}\pi}\int_{C_0}\frac{\mathrm{d}v'}{2\mathbf{i}\pi}$$
$$\times\,\mathrm{Li}_1(-e^{r-\tilde{u}})e^{\varepsilon w^2t/2-\varepsilon v'^2t/2}\frac{\Gamma(v')^N}{\Gamma(w)^N}\frac{e^{-(w-v')r}}{w-v'}, \tag{N18}$$

taking into account that the measure on the variables v' is $\frac{\mathrm{d}v'}{2\mathbf{i}\pi}$. Since $0<\mathrm{Re}(w-v')<1$, we can proceed to an integration by part with respect to the integration variable r to obtain

$$\mathrm{Tr}(\varphi A_2AA_1)=\int_{\mathbb{R}}\mathrm{d}r\int_{\mathbf{i}\mathbb{R}+\delta}\frac{\mathrm{d}w}{2\mathbf{i}\pi}\int_{C_0}\frac{\mathrm{d}v'}{2\mathbf{i}\pi}$$
$$\times\,\mathrm{Li}_2(-e^{r-\tilde{u}})e^{\varepsilon w^2t/2-\varepsilon v'^2t/2}\frac{\Gamma(v')^N}{\Gamma(w)^N}e^{-(w-v')r}. \tag{N19}$$

The function Li_2 denotes the dilogarithm.

As a summary, after replacing the variable $\tilde{u}$ by the identification (N7), we obtain that

$$\mathbb{E}[e^{-\frac{\Lambda}{\varepsilon}z_N(1)}]\sim e^{-\mathrm{Tr}(-\varphi A_2AA_1)}, \tag{N20}$$

and upon taking $t=1$ in the kernel, we have

$$-\mathrm{Tr}(\varphi A_2AA_1)=-\int_{\mathbb{R}}\mathrm{d}r\int_{\mathbf{i}\mathbb{R}+\delta}\frac{\mathrm{d}w}{2\mathbf{i}\pi}\int_{C_0}\frac{\mathrm{d}v'}{2\mathbf{i}\pi}$$
$$\times\,\mathrm{Li}_2\big(-e^{-(1+\frac{\varepsilon}{2})}\varepsilon^{-N}\Lambda e^r\big)\frac{e^{\varepsilon w^2/2-wr}}{e^{\varepsilon v'^2/2-v'r}}\frac{\Gamma(v')^N}{\Gamma(w)^N}$$
$$=-\int_{\mathbb{R}}\mathrm{d}r\,\mathrm{Li}_2(-\Lambda e^r)I(r+1+\varepsilon/2+N\log\varepsilon) \tag{N21}$$

from the first to the second line, we have shifted the variable r to absorb the additional factors in the dilogarithm. We further perform the change of variable

$$w=\frac{\tilde{w}}{\varepsilon},\quad v'=\frac{\tilde{v}'}{\varepsilon} \tag{N22}$$

and subsequently drop the tilde. Upon this change, the integral I reads

$$I(R=r+1+\varepsilon/2+N\log\varepsilon)$$
$$=\frac{1}{\varepsilon^2}\int_{\mathbf{i}\mathbb{R}+\delta'}\frac{\mathrm{d}w}{2\mathbf{i}\pi}\int_{C_0'}\frac{\mathrm{d}v'}{2\mathbf{i}\pi}\frac{e^{\frac{1}{\varepsilon}(w^2/2-wR)}}{e^{\frac{1}{\varepsilon}(v'^2/2-v'R)}}\frac{\Gamma(v'/\varepsilon)^N}{\Gamma(w/\varepsilon)^N}, \tag{N23}$$

which we formally write in the following form:

$$I(R = r + 1 + \varepsilon/2 + N \log \varepsilon) = \frac{1}{\varepsilon^2} \int_{\mathrm{i}\mathbb{R}+\delta'} \frac{\mathrm{d}w}{2\mathbf{i}\pi} \int_{C_0'} \frac{\mathrm{d}v'}{2\mathbf{i}\pi} \frac{e^{\frac{1}{\varepsilon}\Phi(w)}}{e^{\frac{1}{\varepsilon}\Phi(v')}}. \tag{N24}$$

Since the parameter ε is taken to be small, it is natural to introduce the function $\Phi(w)$ to obtain a rate function. Its expression reads

$$\Phi(w) = \frac{w^2}{2} - wR - \varepsilon N \log \Gamma\left(\frac{w}{\varepsilon}\right). \tag{N25}$$

We now evaluate the two integrals in I using the saddle point method controlled by the rate $1/\varepsilon$. From the asymptotics of the Γ function

$$\log \Gamma(z) = z \log z - z - \frac{1}{2} \log z + \frac{1}{2} \log 2\pi + \frac{1}{12z} - \frac{1}{360z^3} + O(1/z^5) \tag{N26}$$

alongside the expression $R = r + 1 + \varepsilon/2 + N \log \varepsilon$ one obtains in the limit $\varepsilon \to 0$

$$\Phi(w) = \frac{w^2}{2} + (N - 1 - r)w - Nw \log(w) + \frac{1}{2}\varepsilon(N \log(w) - N \log(\varepsilon) - N \log(2\pi) - w) + O(\varepsilon^2). \tag{N27}$$

At leading order, we re-write $\Phi(w)$

$$\frac{1}{\varepsilon}\Phi(w) = \frac{1}{\varepsilon}(\phi(w) - rw) - \frac{1}{2}w + \frac{N}{2}\log(w) - \frac{N}{2}\log(2\pi\varepsilon) + O(\varepsilon) \tag{N28}$$

and obtain the rate function $\phi(w)$ as

$$\phi(w) = \frac{w^2}{2} + (N - 1)w - Nw \log w. \tag{N29}$$

Its derivative reads

$$\phi'(w) = w - 1 - N \log w. \tag{N30}$$

The saddle point equation is therefore

$$\phi'(w) = r \iff e^r = w^{-N} e^{w-1}. \tag{N31}$$

At the saddle point for w and v' the subdominant terms as well as the constants compensates between the two integrals in (N24) and we have the following estimate:

$$I(R) \simeq \frac{1}{2\mathbf{i}\pi\varepsilon} \frac{1}{\phi''[w(r)]}, \tag{N32}$$

so that

$$-\mathrm{Tr}(\varphi A_2 A A_1) = -\frac{1}{2\mathbf{i}\pi\varepsilon} \int_{\gamma} \mathrm{d}r\, \mathrm{Li}_2(-\Lambda e^r) \frac{1}{\phi''[w(r)]}. \tag{N33}$$

To make this saddle point easily attainable, one way is to deform the integration contour of r which is not $\mathbb{R}$ anymore but the image of (N31) as v' varies along C_0', which we call γ. We have also assumed that the integration contour of w could be deformed to be folded around C_0'.

It is possible to further simplify the estimate of the first cumulant in (N33) by proceeding to a change of variable $r \to w$ using the saddle point equation $\phi'(w(r)) = r$. The Jacobian of this change of variable reads

$$\phi''[w(r)]\frac{\mathrm{d}w}{\mathrm{d}r} = 1, \qquad \frac{\mathrm{d}r}{\phi''[w(r)]} = \mathrm{d}w. \tag{N34}$$

To summarize, the first cumulant of the Fredholm determinant reads in the small ε-limit

$$-\mathrm{Tr}(\varphi A_2 A A_1) = -\frac{1}{\varepsilon} \int_{C_0'} \frac{\mathrm{d}w}{2\mathbf{i}\pi} \mathrm{Li}_2(-\Lambda w^{-N} e^{w-1}), \tag{N35}$$

and our calculation have given us the following large deviation principle:

$$\mathbb{E}[e^{-\frac{\Lambda}{\varepsilon} z_N}] \sim e^{-\frac{1}{\varepsilon}\Psi(\Lambda)}, \tag{N36}$$

where the final large deviation function reads

$$\Psi(\Lambda) = -\int_{C_0'} \frac{\mathrm{d}w}{2\mathbf{i}\pi} \mathrm{Li}_2(-\Lambda w^{-N} e^{w-1}). \tag{N37}$$

This agrees with the result of the main text (28) using the inverse scattering method.

[1] A. Krajenbrink and P. Le Doussal, Inverse scattering of the Zakharov-Shabat system solves the weak noise theory of the Kardar-Parisi-Zhang equation. Phys. Rev. Lett. **127**, 064101 (2021).

[2] A. Krajenbrink and P. L. Doussal, Inverse scattering solution of the weak noise theory of the Kardar-Parisi-Zhang equation with flat and Brownian initial conditions, Phys. Rev. E **105**, 054142 (2022).

[3] L.-C. Tsai, Integrability in the weak noise theory, Trans. Am. Math. Soc. **376**, 6521 (2023).

[4] M. Kardar, G. Parisi, and Y.-C. Zhang, Dynamic scaling of growing interfaces, Phys. Rev. Lett. **56**, 889 (1986).

[5] E. Bettelheim, N. R. Smith, and B. Meerson, Inverse scattering method solves the problem of full statistics of nonstationary heat transfer in the Kipnis-Marchioro-Presutti model, Phys. Rev. Lett. **128**, 130602 (2022).

[6] A. Krajenbrink and P. Le Doussal, Crossover from the macroscopic fluctuation theory to the Kardar-Parisi-Zhang equation controls the large deviations beyond Einstein's diffusion, Phys. Rev. E **107**, 014137 (2023).

[7] A. Grabsch, A. Poncet, P. Rizkallah, P. Illien, and O. Bénichou, Exact closure and solution for spatial correlations in single-file diffusion, Sci. Adv. **8**, eabm5043 (2022).

[8] K. Mallick, H. Moriya, and T. Sasamoto, Exact solution of the macroscopic fluctuation theory for the symmetric exclusion process, Phys. Rev. Lett. **129**, 040601 (2022).

[9] E. Bettelheim, N. R. Smith, and B. Meerson, Full statistics of nonstationary heat transfer in the Kipnis-Marchioro-Presutti model, J. Stat. Mech. (2022) 093103.

[10] A. Grabsch, P. Rizkallah, A. Poncet, P. Illien, and O. Bénichou, Exact spatial correlations in single-file diffusion, Phys. Rev. E **107**, 044131 (2023).

[11] L. Bertini, A. De Sole, D. Gabrielli, G. Jona-Lasinio, and C. Landim, Current fluctuations in stochastic lattice gases, Phys. Rev. Lett. **94**, 030601 (2005).

[12] B. Derrida and A. Gerschenfeld, Current fluctuations in one dimensional diffusive systems with a step initial density profile, J. Stat. Phys. **137**, 978 (2009).

[13] B. Derrida, Non equilibrium steady states: Fluctuations and large deviations of the density and of the current J. Stat. Mech. (2007) P07023.

[14] L. Bertini, A. De Sole, D. Gabrielli, G. Jona-Lasinio, and C. Landim, Macroscopic fluctuation theory, Rev. Mod. Phys. **87**, 593 (2015).

[15] I. V. Kolokolov and S. E. Korshunov, Explicit solution of the optimal fluctuation problem for an elastic string in random potential, Phys. Rev. E **80**, 031107 (2009); Universal and non-universal tails of distribution functions in the directed polymer and KPZ problems, Phys. Rev. B **78**, 024206 (2008); Optimal fluctuation approach to a directed polymer in a random medium, **75**, 140201 (2007).

[16] B. Meerson, E. Katzav, and A. Vilenkin, Large deviations of surface height in the Kardar-Parisi-Zhang equation, Phys. Rev. Lett. **116**, 070601 (2016).

[17] Y. Lin and L.-C. Tsai, Short time large deviations of the KPZ equation, Commun. Math. Phys. **386**, 359 (2021).

[18] D. Nickelsen and H. Touchette, Anomalous scaling of dynamical large deviations, Phys. Rev. Lett. **121**, 090602 (2018); H. Touchette, The large deviation approach to statistical mechanics, Phys. Rep. **478**, 1 (2009).

[19] A. Shabat and V. Zakharov, Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media, Sov. Phys. JETP **34**, 62 (1972).

[20] M. J. Ablowitz, D. J. Kaup, A. C. Newell, and H. Segur, The inverse scattering transform-Fourier analysis for nonlinear problems, Stud. Appl. Math. **53**, 249 (1974).

[21] D. J. Kaup and A. C. Newell, An exact solution for a derivative nonlinear Schrödinger equation, J. Math. Phys. **19**, 798 (1978).

[22] P. Le Doussal, S. N. Majumdar, A. Rosso, and G. Schehr, Exact short-time height distribution in 1D KPZ equation and edge fermions at high temperature, Phys. Rev. Lett. **117**, 070403 (2016).

[23] A. Krajenbrink and P. Le Doussal, Exact short-time height distribution in the one-dimensional Kardar-Parisi-Zhang equation with Brownian initial condition, Phys. Rev. E **96**, 020102(R) (2017).

[24] N. R. Smith and B. Meerson, Exact short-time height distribution for the flat Kardar-Parisi-Zhang interface, Phys. Rev. E **97**, 052110 (2018).

[25] N. O'Connell and M. Yor, Brownian analogues of Burke's theorem, Stoch. Proces. Appl. **96**, 285 (2001).

[26] N. O'Connell and M. Yor, A representation for non-colliding random walks, Electron. Commun. Probab. **7**, 1 (2002).

[27] N. O'Connell, Directed polymers and the quantum Toda lattice, Ann. Probab. **40**, 437 (2012).

[28] A. Borodin and I. Corwin, Macdonald processes, Probab. Theory Relat. Fields **158**, 225 (2014).

[29] J.-P. Bouchaud and M. Potters, *Theory of Financial Risk and Derivative Pricing: From Statistical Physics to Risk Management* (Cambridge University Press, Cambridge, 2003).

[30] R. M. Corless, G. H. Gonnet, D. E. Hare, D. J. Jeffrey, and D. E. Knuth, On the Lambert W function, Adv. Comput. Math. **5**329 (1996).

[31] J. Baik, Z. Liu, and G. L. F. Silva, Limiting one-point distribution of periodic TASEP, Ann. Inst. Henri Poincaré B **58** (2022).

[32] J. Baik, A. Prokhorov, and G. L. Silva, Differential equations for the KPZ and periodic KPZ fixed points, Commun. Math. Phys. **401**, 1753 (2023).

[33] V. Kuznetsov, M. Salerno, and E. Sklyanin, Quantum Bäcklund transformation for the integrable DST model, J. Phys. A: Math. Gen. **33**, 171 (2000).

[34] V. Enol'skii, M. Salerno, N. Kostov, and A. Scott, Alternate quantizations of the discrete self-trapping dimer, Phys. Scr. **43**, 229 (1991).

[35] P. L. Christiansen, M. F. Jørgensen, and V. B. Kuznetsov, On integrable systems close to the Toda lattice, Lett. Math. Phys. **29**, 165 (1993).

[36] G. Silva, private communication (2022).

[37] T. Imamura and T. Sasamoto, Determinantal structures in the O'Connell-Yor directed random polymer model, J. Stat. Phys. **163**, 675 (2016).

[38] I. Merola, O. Ragnisco, and T. Gui-Zhang, A novel hierarchy of integrable lattices, Inverse Probl. **10**, 1315 (1994).

[39] E. Sklyanin, Bäcklund transformations and Baxter's Q-operator, in *Integrable Systems from Classical to Quantum (Montreal, QC, 1999)*, CRM Proceedings of the Lecture Notes Vol. 26 (AMS, Providence, RI, 2000), pp. 227–250.

[40] A. Kundu and O. Ragnisco, A simple lattice version of the nonlinear Schrödinger equation and its deformation with an exact quantum solution, J. Phys. A: Math. Gen. **27**, 6335 (1994).

[41] Note that this contrasts with the normalization relation for the scattering amplitudes of the Ablowitz-Ladik which have an explicit dependence on the potentials $\{z, \tilde{z}\}$; see Appendix E. The Ablowitz-Ladik model is another semidiscrete integrable model related to the NLS equation [63].

[42] A. Krajenbrink. From Painlevé to Zakharov-Shabat and beyond: Fredholm determinants and integro-differential hierarchies, J. Phys. A: Math. Theor. **54**, 035001 (2021).

[43] T. Bothner, A Riemann-Hilbert approach to Fredholm determinants of Hankel composition operators: scalar-valued kernels, arXiv:2205.15007.

[44] Recall that for $\Lambda > \Lambda_N^*$ the integration contour must be elongated along the imaginary direction to avoid the branch cuts of the logarithm; see Appendix H.

[45] M. Toda, Vibration of a chain with nonlinear interaction, J. Phys. Soc. Jpn. **22**, 431 (1967).

[46] W. Bauhardt and C. Pöppe, The Fredholm determinant method for discrete integrable evolution equations, Lett. Math. Phys. **13**, 167 (1987).

[47] A. Krajenbrink and P. Le Doussal, The weak noise theory of the O'Connell-Yor polymer as an integrable discretisation of the nonlinear Schrödinger equation, arXiv:2307.01172.

[48] W. Symes, The QR algorithm and scattering for the finite nonperiodic toda lattice, Phys. D: Nonlinear Phenom. **4**, 275 (1982).

[49] P. Deift, T. Nanda, and C. Tomei, Ordinary differential equations and the symmetric eigenvalue problem, SIAM J. Numer. Anal. **20**, 1 (1983).
[50] A. Krajenbrink, P. Le Doussal, and N. O'Connell, Tilted elastic lines with columnar and point disorder, non-Hermitian quantum mechanics, and spiked random matrices: Pinning and localization, Phys. Rev. E **103**, 042120 (2021).
[51] T. Grava, M. Gisonni, G. Gubbiotti, and G. Mazzuca, Discrete integrable systems and random Lax matrices, J. Stat. Phys. **190**, 10 (2023).
[52] G. Mazzuca, T. Grava, T. Kriecherbauer, K. T. McLaughlin, C. B. Mendl, and H. Spohn, Equilibrium spacetime correlations of the Toda lattice on the hydrodynamic scale, J. Stat. Phys. **190**, 149 (2023).
[53] H. Spohn, Generalized Gibbs ensembles of the classical Toda chain, J. Stat. Phys. **180**, 4 (2020).
[54] H. Spohn, Hydrodynamic equations for the Toda lattice, arXiv:2101.06528.
[55] H. Spohn, Hydrodynamic equations for the Ablowitz–Ladik discretization of the nonlinear Schrödinger equation, J. Math. Phys. **63**, 033305 (2022).
[56] L.-C. Tsai, Exact lower-tail large deviations of the KPZ equation, Duke Math. J. **171** (2022).
[57] I. Corwin, P. Ghosal, A. Krajenbrink, P. Le Doussal, and L.-C. Tsai, Coulomb-gas electrostatics controls large fluctuations of the Kardar-Parisi-Zhang equation, Phys. Rev. Lett. **121**, 060201 (2018).
[58] I. Corwin and P. Ghosal, Lower tail of the KPZ equation, Duke Math. J. **169** (2020).
[59] M. Cafasso and T. Claeys, A riemann-hilbert approach to the lower tail of the Kardar-Parisi-Zhang equation, Commun. Pure Appl. Math. **75**, 493 (2022).
[60] S. Das and L.-C. Tsai, Fractional moments of the stochastic heat equation, Ann. Inst. Henri Poincaré Statist. **57**, 778 (2021).
[61] C. Charlier, T. Claeys, and G. Ruzza, Uniform tail asymptotics for airy kernel determinant solutions to KdV and for the narrow wedge solution to KPZ, J. Funct. Anal. **283**, 109608 (2022).
[62] S. Das, Y. Liao, and M. Mucciconi, Large deviations for the q-deformed polynuclear growth, arXiv:2307.01179 (2023).
[63] M. J. Ablowitz, B. Prinari, and A. D. Trubatch, *Discrete and Continuous Nonlinear Schrödinger Systems* (Cambridge University Press, Cambridge, 2004), Vol. 302.
[64] A. Krajenbrink and P. Le Doussal, Simple derivation of the $(-\lambda H)^{5/2}$ large deviation tail for the 1D KPZ equation, J. Stat. Mech. (2018) 063210.
[65] A. Krajenbrink, P. Le Doussal, and S. Prolhac, Systematic time expansion for the Kardar-Parisi-Zhang equation, linear statistics of the GUE at the edge and trapped fermions, Nucl. Phys. B **936**239 (2018).
[66] A. Krajenbrink, Beyond the typical fluctuations: A journey to the large deviations in the Kardar-Parisi-Zhang growth model, PhD. thesis, PSL Research University, 2019.