Thermodynamic geometry of a system with unified quantum statistics

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(Received 8 July 2023; accepted 23 February 2024; published 3 April 2024)

We investigate the thermodynamic characteristics of unified quantum statistics, a framework exhibiting a crossover between Bose-Einstein and Fermi-Dirac statistics by varying a generalization parameter δ. An intrinsic statistical interaction becomes attractive for $\delta \leqslant 0.5$, maintaining positive thermodynamic curvature across the entire physical range. In the range $0.5 < \delta < 1$, the system predominantly displays Fermi-like behavior at high temperatures. Conversely, at low temperatures, the thermodynamic curvature is positive, resembling bosonic behavior. Further temperature reduction induces a transition into the condensate phase. We introduce a critical fugacity ($z = Z^*$) at which the thermodynamic curvature changes sign. Below ($z < Z^*$) and above ($z > Z^*$) this critical point, the statistical behavior mimics fermions and bosons, respectively. We explore the system's statistical behavior for various δ values with respect to temperature, determining the critical fugacity and temperature-dependent condensation. Finally, we analyze specific heat as a function of temperature and condensation phase transition temperature for different δ values in various dimensions.

DOI: 10.1103/PhysRevE.109.044104

I. INTRODUCTION

Quantum distributions play a crucial role in understanding the statistical behavior of particles in diverse physical systems. The Bose-Einstein and Fermi-Dirac distributions are particularly significant and have been extensively researched [1–6]. These distributions offer valuable insights into the unique characteristics of bosons and fermions, fundamental quantum particles. Bosons can occupy quantum levels without constraints on particle numbers, while fermions adhere to the Pauli exclusion principle, allowing only one fermion per quantum level. In quantum field theory, bosons and fermions are associated with commutative and anticommutative algebras for their creation and annihilation operators, respectively.

The field of generalized statistics has seen various developments, including proposals for anyons [1], Haldane fractional exclusion statistics [2], deformed statistics [3,4], and nonextensive statistics [5]. A recent advancement is the introduction of unified quantum statistics by Yan [6], which seamlessly interpolates between Bose-Einstein and Fermi-Dirac statistics based on the intrinsic properties of constituent particles.

The geometric aspects within thermodynamic systems were initiated by Gibbs [7,8], and further developed by Ruppeiner and Weinhold [9,10]. Ruppeiner introduced a formalism grounded in fluctuation theory, establishing a thermodynamic parameter space with a metric tied to second derivatives of entropy concerning intrinsic extensive thermodynamic parameters [9,11]. The Ruppeiner and Weinhold metrics are essentially equivalent, differing primarily by a conformal factor [12,13]. For different thermodynamic systems, alternative metrics can be introduced, as explored by Diósi *et al.*, Janyszek, and Mrugała [11,14,15].

The thermodynamic curvature, a scalar derived from the metric, often referred to as the Ricci scalar, plays a pivotal role in the analysis of thermodynamic systems [12,13].

To date, many thermodynamic systems have been analyzed by inspecting the corresponding Ricci scalar. For example, it's established that the thermodynamic curvature of a single-component ideal gas is zero [9,16]. Janyszek and Mrugała delved into the investigation of the thermodynamic curvature for well-known ideal quantum gases like Bose and Fermi gases [15,17]. Extensive studies reveal that the thermodynamic curvature of an ideal Bose gas consistently displays a positive value across its entire physical range, while for an ideal Fermi gas, it consistently exhibits a negative value. The choice of sign convention for thermodynamic curvature is arbitrary, serving as a means of categorizing inherent statistical interactions among particles within the system.

Information geometry of quantum gases was reconsidered in [18], and some results of [15] about the Fermi gas calculation were modified. Furthermore, the scalar thermodynamic curvature of ideal quantum gases obeying Gentile's statistics has been investigated using information geometric theory [18]. Recently, the thermodynamic geometry of quantum gases, especially the Bose-Einstein fluid, with a focus on the strongly degenerate case, has been considered [19,20].

Thermodynamic curvature exhibits singular behavior at phase transition points, demonstrated by ideal Bose gas, nonextensive boson gas, and deformed boson gas [15,21–23]. This singularity phenomenon has been investigated in various systems, including black holes [24,25], boundary conformal field theory within gauge/gravity duality [26], nonextensive and Kanadiakis statistics [27,28], and trapped ideal quantum gases [29].

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In this paper, we introduce a tailored thermodynamic geometry approach for unified quantum statistics proposed by Yan [6]. The investigation extends to ideal gas systems adhering to both conventional quantum statistics and various generalized statistics [15,17,21,30]. Based on the analysis of thermodynamic curvature, we demonstrate a rich phase space for this model, including BE-FD crossover and critical Bose-Einstein condensation (BEC).

The paper is organized as follows. In Sec. II, we provide a concise introduction to the recently proposed unified quantum statistics. In Sec. III, we derive expressions for various thermodynamic quantities, including internal energy and total particle number, applicable to an ideal gas subject to unified statistics in arbitrary dimensions. Section IV is dedicated to an in-depth examination of thermodynamic geometry, along with a brief discussion on the computation of thermodynamic curvature. Section V is focused on the construction of the thermodynamic parameter space for an ideal gas incorporating quantum unified statistics, along with an investigation into the thermodynamic curvature. In Sec. VI, we conduct a comprehensive exploration of the condensation phenomenon in the context of quantum statistics. Lastly, in Sec. VII, we present concluding remarks to summarize the key findings of this paper.

II. UNIFIED QUANTUM STATISTICS

Over time, both theoretical and experimental evidence have provided substantial motivation for the emergence of intermediate statistics, coinciding with the well-established quantum statistics of Bose-Einstein and Fermi-Dirac distributions. One motivation for exploring intermediate statistics arises from the observation of quantum systems exhibiting mixed properties, wherein a fermionic system displays certain similarities with bosonic ones. A notable example is bosonization in fermionic systems, where pairs of fermions form bound states, giving rise to a new phase of matter effectively described by a bosonic system at low energies. Another intriguing instance is bosonization resulting from the Cooper instability in superconducting systems. Central to this phenomenon is the attractive interaction between fermions, originating from electron-phonon interactions and spin fluctuations. These Cooper pairs display collective behavior, condensing into a state of superfluidity or superconductivity at low temperatures [31,32].

On the contrary, BEC occurs in bosonic systems, such as alkali atoms cooled to extremely low temperatures. In this scenario, a large number of bosonic particles occupy the quantum ground state, leading to macroscopic coherence [33,34]. In a widely accepted framework, the strength of the attractive interaction between Cooper pairs dictates the nature of the BEC transition of the effective bosons. The BCS-BEC crossover phenomenon is associated with this interaction strength. Specifically, the BCS-BEC crossover involves a gradual transformation between these two phenomena as the strength of the attractive interaction varies. During this transition, the system can shift from a BCS-like superfluid phase dominated by Cooper pairs to a BEC-like phase where the particles exhibit BEC behavior [35]. This crossover is frequently observed in ultracold atomic gases, where interparticle interactions can be controlled [36]. Although fermions inherently exhibit repulsive statistical interactions, Cooper pairs, behaving like bosons, result in an intrinsic statistical behavior that is attractive.

Intermediate statistics offer an intriguing avenue, introducing a significant perspective to the discourse and emerging as a promising explanatory framework for bosonization and subsequent BEC. Intermediate statistics delve into the statistical interactions among particles, with a sign change triggering the process of bosonization. In fermionic (bosonic) systems, the statistical interaction is characterized by negativity (positivity), and this alteration fundamentally transforms the nature of matter. Analogous to bosonization stemming from the Cooper instability, the transition in a fermionic system leads to bosonization, where the "strength" of the bosonic states is regulated by the deformation parameter associated with intermediate statistics.

To illustrate this concept, intriguing intermediate statistics have been proposed by extending the statistical weights associated with energy levels [37]. Recently, there has been a growing perspective on understanding the quantum state of systems comprising multiple particles, conceiving it as a functional entity defined within the particles' internal space. Consequently, a unified framework has emerged, encompassing both bosons and fermions under a single exchange statistics paradigm [6]. Various extensions of the algebra for creation and annihilation operators have been explored, such as q and qp-deformed bosons and fermions, f-deformed fermions, Wignons, and deformed Tamn-Dankoff [21,38–41]. A particularly promising generalization was proposed by Yan [6], who introduced the following algebra:

$$[a_i, a_j^\dagger]_q = \mathbf{1}^q \delta_{ij}, \tag{1}$$

$$[a_i^\dagger, a_j^\dagger]_q = [a_i^\dagger, a_j^\dagger]_q = 0, \tag{2}$$

where a_i ($a_i^\dagger$) represents the annihilation (creation) operator. The unite operator is defined as follows [6]:

$$\mathbf{1}^q = \begin{cases} \mathbf{1}, & \text{if all } n_i = 0 \text{ or } 1, \\ \frac{1}{2}(\mathbf{1} + \hat{q}), & \text{otherwise} \end{cases}, \tag{3}$$

with $\hat{q}$ as the exchange statistics factor operator [6]. It has been argued that for completely indistinguishable particles, the exchange statistics factor operators are independent of the pairs of particles and commute with any operator in the system. They neither change the physical configuration nor mix up internal states further. Furthermore, it has been demonstrated that the exchange statistics factor operator must be Hermitian and unitary. The Hamiltonian of an ideal gas of such particles is given by $H(\hat{q}) = \sum_i \epsilon_i \hat{n}_i$, in which ϵ_i denotes the single particle energy levels and $\hat{n}_i$ is the number operator of levels. The grand partition function of the system has been derived as follows:

$$\hat{\Xi}(\hat{q}) = \prod_i \frac{1 - \frac{1-\hat{q}}{2} e^{2\beta(\mu-\epsilon_i)}}{1 - e^{\beta(\mu-\epsilon_i)}}, \tag{4}$$

with $\beta = 1/k_B T$, and μ is the chemical potential. When the particle's internal state is an eigenstate of the operator $\hat{q}$, it manifests itself either as bosonic or fermionic characteristics. Typically, the internal state of such a particle embodies an admixture of attributes akin to both bosons and fermions. Within this composite state, we can derive the grand partition function as follows:

$$\Xi(\delta) = \prod_i \frac{1 - \delta z^2 e^{-2\beta\epsilon_i}}{1 - z e^{-\beta\epsilon_i}}, \tag{5}$$

where $z = e^{\beta\mu}$ denotes the fugacity and δ is a constant value between zero and one. It is evident that when $\delta = 0$ (1), the equation above simplifies to the partition function of an ideal gas of bosons (fermions). Within this scenario, the mean occupation number of a single particle state of energy ϵ_i is expressed as

$$n_i = \frac{1}{\exp(\beta(\epsilon_i - \mu)) - 1} - \frac{2\delta}{\exp(2\beta(\epsilon_i - \mu)) - \delta}. \tag{6}$$

In the limit of $\delta = 0$ and 1, the well-known Bose-Einstein and Fermi-Dirac distribution function would be recovered, respectively.

III. THERMODYNAMIC QUANTITIES

Utilizing the distribution function of unified quantum statistics, as briefly discussed in the preceding section, several thermodynamic quantities can be driven. We consider a D-dimensional ideal gas with the following energy-momentum dispersion relation:

$$\epsilon = \alpha p^\sigma, \tag{7}$$

in which p, σ, and α are particle's momentum, dispersion exponent, and the proportionality constant, respectively. In the nonrelativistic limit ($\epsilon = p^2/2m$), it is obvious that $\alpha = 1/2m$ (m denotes the particle's mass), and $\sigma = 2$. In contrast, in the ultrarelativistic limit ($\epsilon = pc$), one can easily show that $\alpha = c$ and $\sigma = 1$. In D dimensions, the single particle density of states $\Omega(\epsilon)$ is

$$\Omega(\epsilon) = \frac{A^D}{\Gamma(\frac{D}{\sigma})} \epsilon^{\frac{D}{\sigma}-1}, \quad A = \frac{L\sqrt{\pi}}{(a^{\frac{1}{\sigma}} h)}, \tag{8}$$

wherein A is a constant set to unity for the sake of simplicity. Moreover, L^D represents the volume of a D-dimensional box. Using Eq. (6), the internal energy is given by

$$\begin{aligned} U &= \int_0^\infty \epsilon n(\epsilon)\Omega(\epsilon) d\epsilon \\ &= \frac{D}{\sigma} \beta^{-(\frac{D}{\sigma}+1)} \big[\bar{g}_{\frac{D}{\sigma}+1}(z) - 2^{-\frac{D}{\sigma}} \bar{g}_{\frac{D}{\sigma}+1}(z')\big] \\ &= \nu \beta^{-(\nu+1)} [G(\delta, \nu+1; z)], \end{aligned} \tag{9}$$

where $\frac{D}{\sigma} \equiv \nu$, $z' \equiv \delta z^2$, and

$$G(\delta, \nu; z) \equiv [\bar{g}_\nu(z) - 2^{(1-\nu)} \bar{g}_\nu(z')]. \tag{10}$$

In this formula, $\bar{g}_\nu(z)$ denotes the standard Bose-Einstein integrals defined by

$$\bar{g}_\nu(z) = \frac{1}{\Gamma(\nu)} \int_0^\infty \frac{x^{\nu-1}}{z^{-1}\exp(x) - 1} dx. \tag{11}$$

In addition, the total particles number is given by

$$N = \int_0^\infty n(\epsilon)\Omega(\epsilon) d\epsilon = \beta^{-\nu} [G(\delta, \nu; z)]. \tag{12}$$

Combining Eqs. (5) and (9) with $P = (-\frac{\partial F}{\partial V})_{N,T}$ (F representing Helmholtz free energy) demonstrates that, akin to conventional statistics, there exists a standard relationship between pressure (P), volume (V), and internal energy as follows:

$$PV = \frac{U}{\nu}. \tag{13}$$

IV. THERMODYNAMIC GEOMETRY

Ruppeiner and Weinhold introduced thermodynamic geometry as an innovative approach to the study of thermodynamic systems [9,10]. The thermodynamic parameter space can be conceptualized as a Riemannian space, and gives rise to establishment of a suitable metric within this space. The Ruppeiner metric is constructed by calculating the second-order derivatives of entropy concerning the relevant extensive thermodynamic parameters, encompassing internal energy, volume, and the total number of particles. Furthermore, Weinhold introduced an alternative metric in the energy representation, which is defined by evaluating the second-order derivatives of internal energy concerning the pertinent extensive thermodynamic parameters. It has been noted that these metrics exhibit conformal equivalence [13].

Performing a Legendre transformation on either entropy or internal energy with respect to the extensive parameters results in the derivation of various thermodynamic potentials, such as Helmholtz and Gibbs free energy. The Fisher-Rao metric is defined by the second derivatives of the logarithm of the partition function concerning the nonextensive thermodynamic parameters [11,15,42–46], as shown below:

$$g_{ij} = \partial_i \partial_j \ln \mathcal{Z}. \tag{14}$$

Here, ∂_i is a shorthand notation for the derivative with respect to the nonextensive thermodynamic parameter i, and $\mathcal{Z}$ denotes the partition function.

The logarithm of the partition function for an ideal classical or quantum gas is a multivariable function that depends on the system's volume, $\beta = 1/k_B T$ and $\gamma = -\mu/k_B T$. Typically, the volume is treated as a fixed thermodynamic parameter. Consequently, the thermodynamic parameter space of the system becomes a two-dimensional space.

The connection coefficient (Christoffel symbols) is defined using the components of the metric tensor

$$\Gamma^i_{jk} = \tfrac{1}{2} g^{im}(g_{mj,k} + g_{mk,j} - g_{jk,m}), \tag{15}$$

in which g^{mn} represents the elements of the inverse metric tensor and $g_{ij,k} = \partial_k g_{ij}$. The elements of the Riemann tensor are obtained by

$$R^i_{jkl} = \partial_k \Gamma^i{}_{lj} - \partial_l \Gamma^i{}_{kj} + \Gamma^i{}_{km}\Gamma^m{}_{lj} - \Gamma^i{}_{lm}\Gamma^m{}_{kj}. \tag{16}$$

The Riemann tensor provides insights into the curvature of the thermodynamic space. Furthermore, the second rank Ricci tensor is defined as follows:

$$R_{ij} = R^m_{imj}. \tag{17}$$

Consequently, the Ricci scalar is given by

$$R = g^{ij} R_{ij}, \tag{18}$$

which is also known as the thermodynamic curvature. In a two-dimensional parameter space, the Ricci scalar is simplified to

$$R = -\frac{2\begin{vmatrix} g_{\beta\beta} & g_{\beta\gamma} & g_{\gamma\gamma} \\ g_{\beta\beta,\beta} & g_{\beta\gamma,\beta} & g_{\gamma\gamma,\beta} \\ g_{\beta\beta,\gamma} & g_{\beta\gamma,\gamma} & g_{\gamma\gamma,\gamma} \end{vmatrix}}{\begin{vmatrix} g_{\beta\beta} & g_{\beta\gamma} \\ g_{\beta\gamma} & g_{\gamma\gamma} \end{vmatrix}^2}. \tag{19}$$

V. THERMODYNAMIC PARAMETERS SPACE OF QUANTUM UNIFIED STATISTICS

In the preceding sections, we have laid the groundwork for constructing the parameter space governing the thermodynamic properties of an ideal gas consisting of particles following a unified quantum statistics. We will operate assuming that the volume of the system remains constant throughout the analysis. Consequently, the thermodynamic parameters under consideration are β and γ, resulting in a two-dimensional parameter space. Moreover, by introducing $z = e^{-\gamma}$, $z' = \delta z^2 = \delta e^{-2\gamma}$, and using the chain derivative rule and the definition of the Bose-Einstein integrals, one can find

$$\begin{aligned} \frac{\partial}{\partial\gamma}\bar{g}_\nu(z) &= \frac{\partial z}{\partial\gamma}\frac{\partial}{\partial z}\bar{g}_\nu(z) = -z\frac{\partial}{\partial z}\bar{g}_\nu(z) = -\bar{g}_{\nu-1}(z), \\ \frac{\partial}{\partial\gamma}\bar{g}_\nu(z') &= \frac{\partial z'}{\partial\gamma}\frac{\partial}{\partial z'}\bar{g}_\nu(z') = -2z'\frac{\partial}{\partial z'}\bar{g}_\nu(z') = -2\bar{g}_{\nu-1}(z'). \end{aligned} \tag{20}$$

Therefore, the following relation is obtained for $G(\delta, \nu; z)$:

$$\frac{\partial}{\partial\gamma}G(\delta, \nu; z) = -G(\delta, \nu - 1; z). \tag{21}$$

Using Eqs. (14), (9), (12), and (21) the metric elements corresponding to the aforementioned two-dimensional parameter space are given by

$$\begin{aligned} g_{\beta\beta} &= \frac{\partial^2 \ln \mathcal{Z}}{\partial\beta^2} = -\left(\frac{\partial U}{\partial\beta}\right)_\gamma = -\frac{\partial}{\partial\beta}[\beta^{-(\nu+1)}\nu(G(\delta, \nu + 1; z))] = \frac{\beta^{-(\nu+2)}}{\Gamma(\nu)}\Gamma(\nu + 2)G(\delta, \nu + 1; z), \\ g_{\beta\gamma} &= g_{\gamma\beta} = \frac{\partial^2 \ln \mathcal{Z}}{\partial\beta\partial\gamma} = -\left(\frac{\partial N}{\partial\beta}\right)_\gamma = -\frac{\partial}{\partial\beta}[\beta^{-\nu}G(\delta, \nu; z)] = \frac{\beta^{-(\nu+1)}}{\Gamma(\nu)}\Gamma(\nu + 1)G(\delta, \nu; z), \\ g_{\gamma\gamma} &= \frac{\partial^2 \ln \mathcal{Z}}{\partial\gamma^2} = -\left(\frac{\partial N}{\partial\gamma}\right)_\beta = -\frac{\partial}{\partial\gamma}[\beta^{-\nu}G(\delta, \nu; z)] = \beta^{-\nu}G(\delta, \nu - 1; z). \end{aligned} \tag{22}$$

By employing Eqs. (21) and (22), we can express the derivatives of the metric elements as follows:

$$\begin{aligned} g_{\beta\beta,\beta} &= \frac{\partial}{\partial\beta}g_{\beta\beta} = -\frac{\beta^{-(\nu+3)}}{\Gamma(\nu)}\Gamma(\nu + 3)G(\delta, \nu + 1; z), \\ g_{\beta\beta,\gamma} &= g_{\beta\gamma,\beta} = g_{\gamma\beta,\beta} = \frac{\partial}{\partial\gamma}g_{\beta\beta} = -\frac{\beta^{-(\nu+2)}}{\Gamma(\nu)}\Gamma(\nu + 2)G(\delta, \nu; z), \\ g_{\beta\gamma,\gamma} &= g_{\gamma\beta,\gamma} = g_{\gamma\gamma,\beta} = \frac{\partial}{\partial\beta}g_{\gamma\gamma} = -\frac{\beta^{-(\nu+1)}}{\Gamma(\nu)}\Gamma(\nu + 1)G(\delta, \nu - 1; z), \\ g_{\gamma\gamma,\gamma} &= \frac{\partial}{\partial\gamma}g_{\gamma\gamma} = -\beta^{-\nu}G(\delta, \nu - 2; z). \end{aligned} \tag{23}$$

In general, using the matrix elements defined in Eqs. (22) and (23), the thermodynamic curvature given by Eq. (19) can be evaluated as

$$R = \frac{2(\nu + 1)\beta^\nu(-2G(\delta, \nu + 1; z)G(\delta, \nu - 1; z)^2 + G(\delta, \nu; z)^2G(\delta, \nu - 1; z) + G(\delta, \nu - 2; z)G(\delta, \nu; z)G(\delta, \nu + 1; z))}{(\nu G(\delta, \nu; z)^2 - (\nu + 1)G(\delta, \nu - 1; z)G(\delta, \nu + 1; z))^2}. \tag{24}$$

For $\delta = 0$, this equation is reduced to the thermodynamic curvature of Bose gas, which has been already given in [18].

Figure 1 depicts the thermodynamic curvature in a three-dimensional system of nonrelativistic particles under unified quantum statistics, as a function of fugacity. A notable observation is the distinct behavior of thermodynamic curvatures in specific cases: for $\delta = 0$ (bosons), they consistently exhibit positive values, while for $\delta = 1$ (fermions), negative values are consistently displayed throughout the entire physical range.

For all values $\delta \leqslant 0.5$, the thermodynamic curvature remains positive, indicating an attractive statistical interaction among particles—a defining characteristic in a bosonic system. However, for $0.5 < \delta < 1$, we observe that, depending on the specific value of δ, a corresponding fugacity value denoted as $z = Z$ can be identified, where the sign of the thermodynamic curvature changes. This implies that when $z < Z$ ($z > Z^*$), the intrinsic statistical interaction becomes repulsive (attractive). Figure 2 demonstrates the consistency of these principles across different dimensions and dispersion relations.

We note that the thermodynamic curvature exhibits singularity for all values of δ except for $\delta = 1$ when $z = 1$. It is widely acknowledged that, similar to the singularity

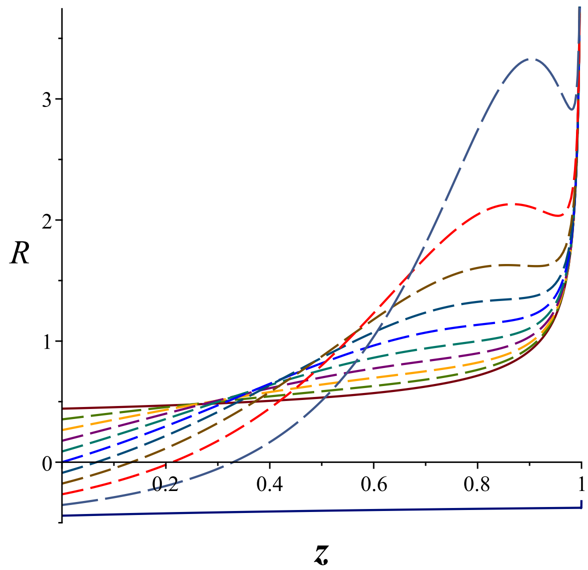

FIG. 1. Thermodynamic curvature of an ideal three dimensional ($D = 3$) gas of particles obeying unified quantum statistics as a function of fugacity for isothermal processes ($\beta = 1$). The particles are supposed to have nonrelativistic dispersion ($\sigma = 2$). The solid (brown) line corresponds to $\delta = 0$ (ideal boson gas) and the lower solid line (blue) represents curvature of $\delta = 1$ (ideal fermion gas). Dashed lines correspond to the values $\delta = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9$ from top to bottom (at $z = 0$), respectively.

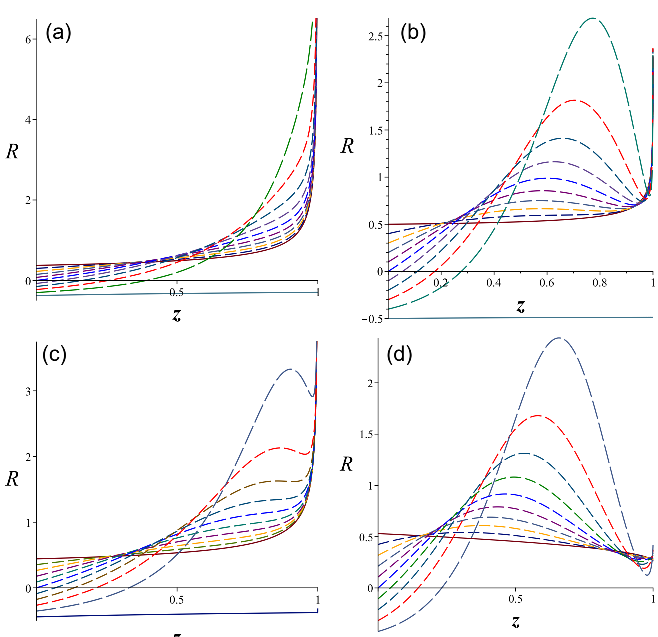

FIG. 2. Thermodynamic curvature of an ideal gas with particles obeying generalized unified quantum statistics as a function of fugacity for isothermal processes ($\beta = 1$). The solid (brown) line corresponds to $\delta = 0$ (ideal boson gas) and the dash dotted line (blue) represents curvature of $\delta = 0.5$. Dashed lines correspond to the values $\delta = 0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1$. (a): ($D = 2, \sigma = 1$), (b): ($D = 2, \sigma = 2$), (c): ($D = 3, \sigma = 2$) and (d): ($D = 1, \sigma = 2$).

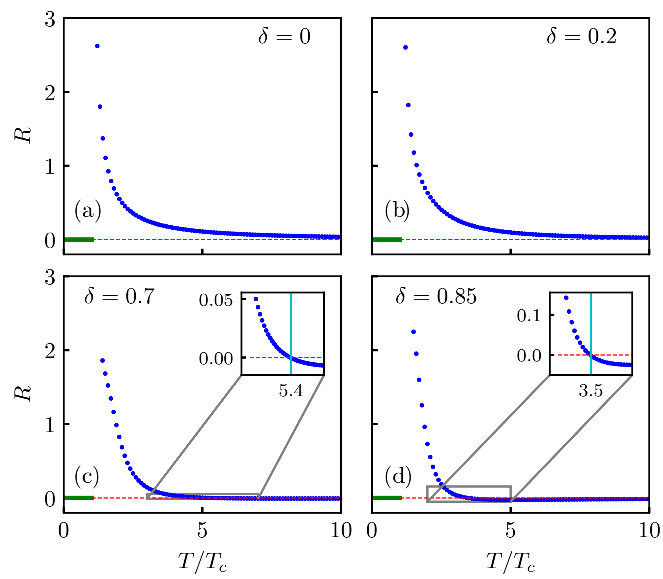

FIG. 3. Thermodynamic curvature of an ideal gas with particles obeying generalized quantum statistics as a function of T/T_c. For $D = 3$ and $\sigma = 2$ and (a) $\delta = 0$, (b) $\delta = 0.2$, (c) $\delta = 0.7$, and (d) $\delta = 0.85$.

observed in the thermodynamic curvature of an ideal Bose gas at the critical fugacity, the highest permissible value for the fugacity allowing for a nonnegative Bose-Einstein distribution function is $z = 1$—corresponding to the critical value of the fugacity at the condensation temperature [47]. Previous studies have underscored that the thermodynamic curvature of a conventional Bose gas displays singularity at the critical fugacity value. Interestingly, the thermodynamic curvature of a unified quantum gas, for any given δ, appears to possess a unique characteristic at $z = 1$. It is well known that for an ideal gas in the condensate phase, the fugacity of the system is fixed at $z = 1$. We postulate that the same behavior for the unified quantum gas occurs at $z = 1$. Hence, the only remaining variable subject to fluctuations in the system is β. Consequently, the thermodynamic parameter space at the condensate phase reduces to a one-dimensional space, which, by definition, is flat. The thermodynamic parameter space for the normal phase is determined by the fugacity and temperature variables. All the quantities examined in this study exhibit this behavior, including the thermodynamic curvature, which can be expressed by $\beta^{\alpha} F(z)$ as functions of β and z. The exponent α and the function $F(z)$ vary across different quantities within distinct dimensions and dispersion relations. Using Eqs. (12) and (18) the thermodynamic curvature of a system can be given at a specific particle number as a function of temperature. Figure 3 reveals that the thermodynamic curvature undergoes a discontinuity at the transition temperature. In the next section, we will consider the condensation temperature in more details. From Figs. 1–3, it is evident that the thermodynamic curvature remains positive across the entire physical range for $\delta \leqslant 0.5$. However, for $\delta > 0.5$, two distinct regimes with both positive and negative curvature can be identified.

VI. TRANSITION TEMPERATURES

In this section, we illustrate that the fugacity in unified quantum statistics must be constrained to ensure a positive distribution function for all values of δ, akin to the behavior seen in ordinary Bose statistics. The fugacity is confined to the interval $0 \leqslant z \leqslant 1$, and the behavior of the thermodynamic curvature exhibits singularity at the upper bound, $z_{\max} = 1$.

The thermodynamic curvature exhibits a discontinuity at $T = T_c$. The singular behavior of the thermodynamic curvature of an ideal Bose gas, occurring at $z_{\max} = 1$, is widely recognized and is associated with the BEC transition.

We anticipate the occurrence of a similar transition for arbitrary values of δ. We determine the phase transition temperature for a constant particle density. Using Eq. (12), we deduce the phase transition temperature for generalized statistics as

$$K_B T_c = \left(\frac{N}{A[\zeta(D/\sigma) - 2^{(1-D/\sigma)} g_{D/\sigma}(\delta)]} \right)^{\sigma/D}. \tag{25}$$

Since $g_{D/\sigma}(0) = 0$, the BEC temperature for special case ($\delta = 0$) can be expressed as

$$K_B T_c^B = \left(\frac{N}{A\zeta(D/\sigma)} \right)^{\sigma/D}, \tag{26}$$

where T_c^B denotes the condensation critical temperature of an ideal Bose gas. We establish a straightforward relationship relating the condensation temperature of generalized unified quantum statistics to the conventional BEC temperature as follows:

$$\frac{T_c}{T_c^B} = \left(\frac{\zeta(D/\sigma)}{\zeta(D/\sigma) - 2^{(1-D/\sigma)} g_{D/\sigma}(\delta)} \right)^{\sigma/D}. \tag{27}$$

Therefore, we calculate the condensation phase transition temperature for unified quantum statistics, regardless of the specific value of δ.

Now, we investigate the temperature at which the thermodynamic curvature undergoes a change in sign.

For a specific parameter value δ, we extract the corresponding value of Z^* using Eq. (12). This allows us to determine the temperature at which the sign change in thermodynamic curvature occurs while maintaining a constant particle density.

In Fig. 4, we observe that the value of Z^* is dependent on the parameters δ and D/σ.

When $\delta < 0.5$, the thermodynamic curvature remains positive across the entire physical range. Conversely, for $\delta > 0.5$, the thermodynamic curvature is positive only for $z > z$, indicating behavior akin to bosons. For $z < z$, the intrinsic statistical interaction within the system becomes repulsive, resembling behavior characteristic of fermions.

We identify the temperature at which the sign change in curvature occurs, considering a fixed density.

The condensation temperature and the temperature at which the sign change occurs are illustrated in Fig. 5.

It is a widely recognized fact that thermodynamic response functions exhibit divergence, discontinuity, or nondifferentiability at phase transition points. Specifically, the heat capacity of an ideal Bose gas displays nondifferentiability at the phase

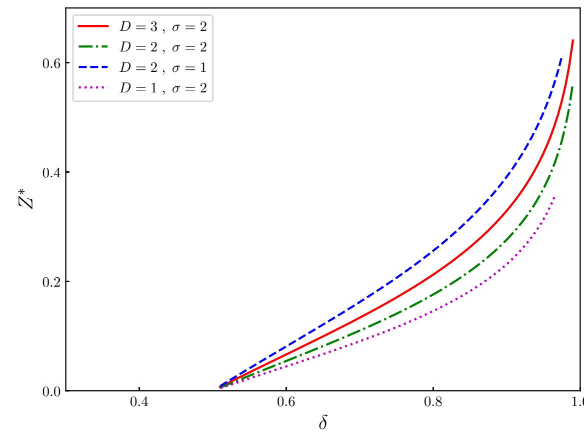

FIG. 4. Z^* in terms at δ for various D and σ values $Z > Z^*(Z < Z^*)$ shows Boson (Fermion) like behavior [dashed line($D = 2, \sigma = 1$), solid line($D = 3, \sigma = 2$), dash dotted line($D = 2, \sigma = 2$), and dotted line($D = 1, \sigma = 2$)].

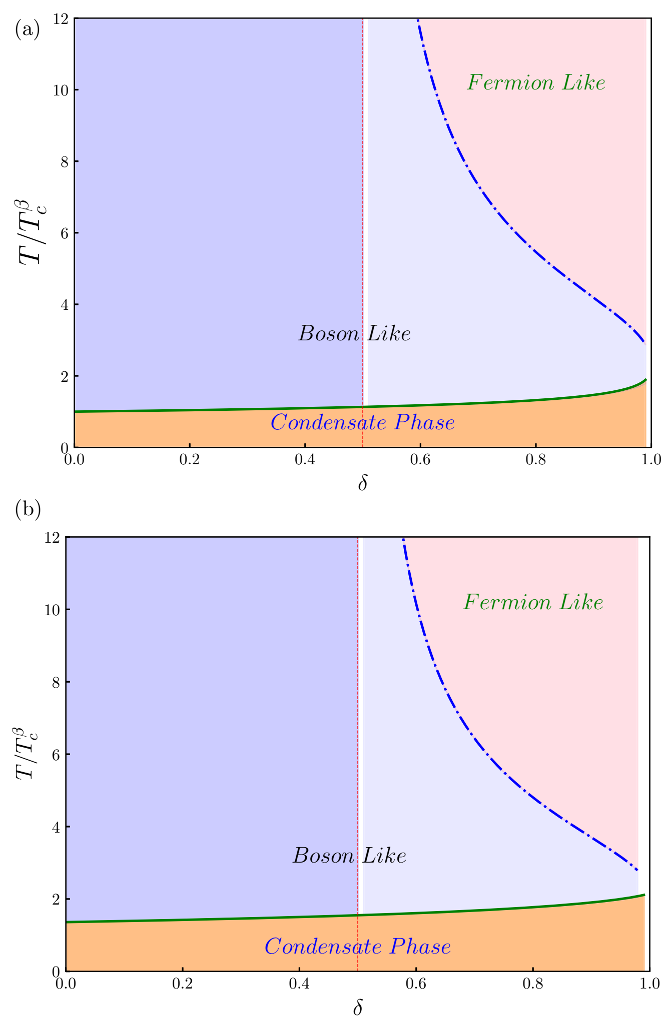

FIG. 5. The T-δ phase diagram of the system for (a) $D = 3, \sigma = 2$ and (b) $D = 2$ and $\sigma = 1$. In both figures the dash dotted line separates the fermionic regime from the bosonic one, while the lower solid line denotes the BEC transition. The vertical dotted line exhibits the asymptotic fermion-boson transition line given by $\delta = 0.5$, so that we do not have a fermionic phase for $\delta < 0.5$.

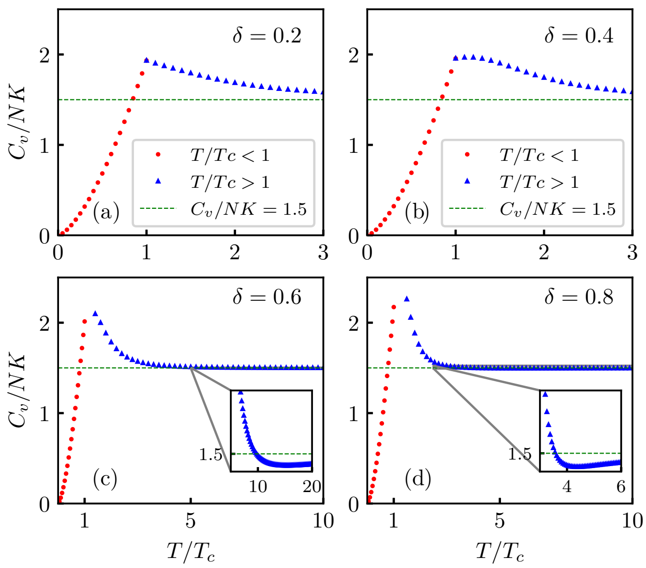

FIG. 6. Heat capacity at fixed volume as a function of temperature. (a) $\delta = 0.2$, (b) $\delta = 0.4$, (c) $\delta = 0.6$, and (d) $\delta = 0.8$.

transition temperature. Using Eqs. (9) and (12), the heat capacity of an ideal unified quantum statistics with an arbitrary value of δ are given by

$$C_V = \left(\frac{\partial U}{\partial T}\right)_V = \frac{15}{4}\frac{\bar{g}_{\frac{5}{2}}(z) - 2^{\frac{-3}{2}}\bar{g}_{\frac{5}{2}}(\delta z^2)}{\bar{g}_{\frac{3}{2}}(z) - 2^{\frac{-1}{2}}\bar{g}_{\frac{3}{2}}(\delta z^2)} - \frac{9}{4}\frac{\bar{g}_{\frac{3}{2}}(z) - 2^{\frac{-1}{2}}\bar{g}_{\frac{3}{2}}(\delta z^2)}{\bar{g}_{\frac{1}{2}}(z) - 2^{\frac{1}{2}}\bar{g}_{\frac{1}{2}}(\delta z^2)}. \tag{28}$$

Figure 6 shows the behavior of heat capacity with respect to temperature for different values of δ. First, we observe that the heat capacity exhibits nondifferentiability at $T = T_c$. Furthermore, at extremely high temperatures, it converges to the heat capacity of an ideal classical gas, denoted by $C_V = 3NK/2$. When $\delta \leqslant 0.5$, the heat capacity exceeds that of an ideal classical gas for temperatures exceeding T_c. As the temperature rises, the heat capacity of the quantum unified gas gradually converges toward the heat capacity of the ideal classical gas in an asymptotic manner. Nevertheless, when $\delta > 0.5$, there exists a finite temperature at which the heat capacity matches that of an ideal classical gas. This temperature coincides with the thermodynamic curvature's sign-change temperature. In fact, there are two distinct regimes: one resembling Bose-like behavior and the other Fermi-like behavior, separated by the curve $C_V = 3NK/2$.

In summary, it is crucial to emphasize that the heat capacity of a quantum unified gas becomes nondifferentiable at $T = T_c$ for any values within the range $0 \leqslant \delta < 1$, marking the transition to the condensate phase. Furthermore, when $0.5 < \delta < 1$, the heat capacity at a specific temperature corresponds to that of a classical gas. This temperature, at which the heat capacity behaves as an analytic function, coincides with the point where the thermodynamic curvature changes sign (zero thermodynamic curvature). The transition from a repulsive intrinsic statistical interaction at higher temperatures (negative thermodynamic curvature) to an attractive one (positive thermodynamic curvature) at lower temperatures occurs due to the change in the sign of the thermodynamic curvature, reminiscent of the BCS transition. However, it is crucial to clarify that we are not suggesting a process analogous to the formation of Cooper pairs.

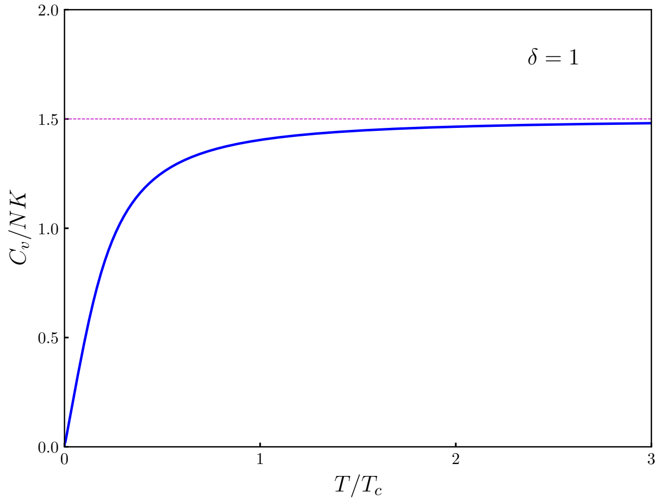

FIG. 7. Heat capacity at fixed volume as a function of temperature for ($\delta = 1$) (fermions).

Using the following relation [47]

$$\bar{g}_\nu(z) - 2^{1-\nu}\bar{g}_\nu(z^2) = -\bar{g}_\nu(-z) = f_\nu(z), \tag{29}$$

we recover the heat capacity of ideal fermion gas for $\delta = 1$ as follows and it is depicted in Fig. 7:

$$C_v = \frac{15}{4}\frac{\bar{g}_{\frac{5}{2}}(z) - 2^{\frac{-3}{2}}\bar{g}_{\frac{5}{2}}(z^2)}{\bar{g}_{\frac{3}{2}}(z) - 2^{\frac{-1}{2}}\bar{g}_{\frac{3}{2}}(z^2)} - \frac{9}{4}\frac{\bar{g}_{\frac{3}{2}}(z) - 2^{\frac{-1}{2}}\bar{g}_{\frac{3}{2}}(z^2)}{\bar{g}_{\frac{1}{2}}(z) - 2^{\frac{1}{2}}\bar{g}_{\frac{1}{2}}(z^2)} = \frac{15}{4}\frac{f_{\frac{5}{2}}(z)}{f_{\frac{3}{2}}(z)} - \frac{9}{4}\frac{f_{\frac{3}{2}}(z)}{f_{\frac{1}{2}}(z)}, \tag{30}$$

where $f_n(x)$ denotes the well-known Fermi-Dirac function which is defined as

$$f_\nu(z) = \frac{1}{\Gamma(\nu)}\int_0^\infty \frac{x^{\nu-1}}{z^{-1}\exp(x) + 1}dx. \tag{31}$$

We investigate the phase transition temperature across different spatial dimensions and diverse dispersion relations. It's worth noting that as $\lim_{x\to 1}\zeta(x) \longrightarrow \infty$, for $D/\sigma \leqslant 1$, the transition temperature approaches absolute zero. This implies that finite-temperature condensation in quantum unified statistics only occurs when $D/\sigma > 1$ similar to ordinary bosons.

VII. CONCLUSION

We conducted an in-depth exploration of thermodynamic geometry within the context of a newly proposed generalized unified quantum statistics. This generalization builds on the premise that the quantum state of a multiparticle system can be expressed as a functional on the internal space of the particles, facilitating a smooth transition between Bose-Einstein and Fermi-Dirac statistics.

We defined the thermodynamic parameter space for an ideal gas composed of particles following unified quantum statistics. Within this framework, we determined the metric elements for the parameter space and subsequently derived the affine connections and the Ricci scalar for this thermodynamic parameter space.

The thermodynamic curvature holds a distinct interpretation closely tied to statistical interactions. The sign of the thermodynamic curvature is directly indicative of intrinsic statistical interactions within the thermodynamic system. The generalization parameter δ in unified quantum statistics, as argued in our study, plays a crucial role in shaping these statistical interactions.

For cases where $\delta \leqslant 0.5$, the thermodynamic curvature remains positive, indicating an attractive statistical interaction, resembling bosonic behavior across the entire physical range. Conversely, for $\delta > 0.5$, the thermodynamic curvature can be either positive or negative, contingent on the system temperature.

We demonstrated that in the high-temperature limit, the dominant statistical interaction aligns with fermionic behavior when $\delta > 0$. As temperature decreases, the repulsive interaction gradually diminishes, vanishing at a specific fugacity value, $z = Z^*$, or equivalently, at a certain temperature threshold. Below this threshold, the thermodynamic curvature remains positive, signifying statistical behavior akin to bosonic statistics.

Singular points in the thermodynamic curvature act as indicators of phase transitions. Notably, the thermodynamic curvature becomes singular at a critical fugacity ($z_c = 1$), indicative of a BEC occurring for quantum unified statistics across all δ values, except when $\delta = 1$ (corresponding to an ordinary fermion gas). Calculating the condensation temperature for a fixed particle density, we observed that the BEC phase transition temperature exceeds the BEC temperature for any arbitrary δ.

Finally, we investigated the heat capacity as a function of temperature, revealing a singular behavior at the critical condensation temperature. Our exploration extended to condensation phenomena in different spatial dimensions and under various dispersion relations. Specifically, for $D/\sigma \leqslant 1$, the phase transition temperature was determined to be zero, akin to an ordinary ideal Bose gas, indicating the absence of finite-temperature condensation in low dimensions [48].

ACKNOWLEDGMENTS

We acknowledge K. Monkman, A. Akbari, and S. Vadnais for helpful conversations.

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