Tuning limit cycles with a noise: Survival and collapse

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We consider a general class of limit cycle oscillators driven by an additive Gaussian white noise. Based on the separation of timescales, we construct the equation of motion for slow dynamics after appropriate averaging over the fast motion. The equation for slow motion whose coefficients are modified by noise characteristics is solved to obtain the analytic solution in the long time limit. We show that with increase of noise strength, the loop area of the limit cycle decreases until a critical value is reached, beyond which the limit cycle collapses. We determine the noise threshold from the condition for removal of secular divergence of the perturbation series and work out two explicit examples of Van der Pol and Duffing-Van der Pol oscillators for corroboration between the theory and numerics.

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I. INTRODUCTION

Interplay of nonlinearity and stochasticity has been an interesting area of investigation in nonlinear dynamics over the last couple decades. The observation of various noise-induced phenomena in nonlinear systems under far-from-equilibrium conditions demonstrates that noise can play a crucial role in reorganizing a system or stabilize it, contrary to our expectations. The examples are abundant, ranging from noise-induced transition [1,2], stochastic resonance [3,4], noise-induced propagation of traveling wavefronts [5], and pattern formation [6–14], etc., to name a few. A key issue in all these studies is the understanding of the problem of instability of a dynamical state acted upon by a noise in additive or multiplicative form. The standard approach to this problem is the linear stability analysis [15]. In the presence of noise, one is therefore restricted to the linear stochastic differential equations within the weak noise approximation [16–18] and the diffusion coefficients are treated as constants. The scheme has found wide applicability in quantum optics [16] and many related fields.

The nonlinear stochastic differential equations [19], on the other hand, pose serious problems since it is difficult to obtain a closed set of equations for moments unless one resorts to mean-field approximations. More importantly, linearization or mean-field approximations fail to capture the generic dynamical behavior of noisy limit cycle oscillators in the long time limit. It is therefore worthwhile to explore how noise affects the stability of a limit cycle and its long time behavior. Because of their ubiquitous presence in living systems [20–28] over a wide ranging timescale of biorhythms from a circadian clock [20,21], Ca^{+2} oscillations [22,23], glycolytic oscillations [24–26], and synchronization of fireflies [27], etc., limit cycles play a crucial role in nonlinear dynamics. Noisy self-sustained oscillatory systems [28–33] therefore naturally emerge as important candidates for many investigations. For example, the study [29] of linear response of a Van der Pol oscillator driven by noise, construction [30] of slow flow based on averaging over the motion of a Van der Pol oscillator with nonlinearity subjected to multiplicative noise, probing [31] noise dependence of the time period and lifetime of a noisy Van der Pol oscillator driven by both additive and multiplicative noise may be mentioned in this regard. The mathematical behavior of the solutions of stochastic and deterministic Van der Pol–Duffing oscillators [32] as well as stochastic P bifurcations [32,33] has been examined. Keeping in mind these developments, a direct method of the analytical solution of noisy limit cycle oscillator and determining its stability is worth pursuing. In what follows, we introduce an explicit separation of timescales to construct the dynamics of slow motion by averaging over the noise in the spirit of Blekhman perturbation theory [34] used in vibrational mechanics [35] and related areas [36–48]. The effect of fast motion is subsumed into the slow dynamics in the form of renormalization of the coefficients of the equation of motion. The slow motion can be solved analytically by Lindstedt-Poincare perturbation technique [49,50] or renormalization group methods [51–53]. An interesting offshoot of the present scheme is the estimate of the threshold for collapse of the limit cycle against a critical noise strength, which emerges from the condition for removal of secular divergence of the perturbation series. We work out two examples, namely, the Van der Pol and Duffing–Van der Pol oscillators. Our theoretical analysis is verified by numerical simulations.

II. THE NOISY NONLINEAR DYNAMICAL SYSTEMS HAVING LIMIT CYCLES—THE ANALYTIC SOLUTION

The outline of the present method is described as follows: We consider a class of nonlinear Langevin equations with additive, Gaussian, and white noise of the following form:

$$\ddot{x} + \epsilon(x^2 - 1)\dot{x} + \omega_0^2 x + cf(x) = \beta\xi(t), \tag{2.1}$$

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where $f(x)$, in general, is a nonlinear function of the system coordinates x and the overdot represents differentiation with respect to time. $\xi(t)$ is a Gaussian white noise with zero mean, i.e.,

$$\langle\xi(t)\rangle = 0,$$
$$\text{and the second moment,}\quad \langle\xi(t)\xi(0)\rangle = 2D\delta(t), \tag{2.2}$$

where D is the noise strength. The physical quantities ω_0 and ϵ represent the natural frequency of the oscillator and the damping parameter, respectively. ϵ has been chosen to be a smallness parameter. The parameter β is a measure of the strength of external noise in the system and considered to be small, and c is a constant. If c becomes zero, Eq. (1) becomes the standard Van der Pol oscillator, driven by a noise, whereas, depending on the nonlinearity of the potential, a nonzero value of c will produce some other nonlinear oscillators. For example, $f(x) = x^3$ corresponds to the Duffing–Van der Pol oscillator, in the presence of a noise. Due to the existence of limit cycles, the Van der Pol and the Duffing–Van der Pol oscillators, have become important systems for studying several aspects of modern day research in nonlinear dynamics [28–33].

For convenience, we discuss these two cases separately in detail. Firstly, in sub-section A, we analyze the $c = 0$ case, i.e., the standard Van der Pol oscillator in its stochastic version. And in the next sub-section B, we consider the dynamics of a noisy Duffing-Van der Pol oscillator. Our method is based on separation of timescales as carried out in vibrational mechanics where the noise is replaced by a high frequency field. All we need to do is to set up the appropriate equations for slow and fast motion. The fast motion is governed by a Langevin equation, a stochastic process in which the inertial term is balanced by the external noise. The effect of the fast motion is subsumed into the slow dynamics which is solved by standard Lindstedt-Poincare perturbation technique.

A. Standard Van der Pol oscillator: Stochastic dynamics

As mentioned earlier, the dynamical equation for this case becomes

$$\ddot{x} + \epsilon(x^2 - 1)\dot{x} + \omega_0^2 x = \beta\xi(t). \tag{2.3}$$

To proceed with our mathematical analysis, we first identify, in the spirit of vibrational mechanics [34,35], two timescales of the dynamics and write x in the form

$$x = X(t, \omega_0 t) + \psi(t, \Omega t), \tag{2.4}$$

where Ω is the inverse of the correlation time τ_c of the noise $\xi(t)$ and $\Omega \gg \omega_0$. For white noise, τ_c tends to zero. Here, $X(t, \omega_0 t)$ and $\psi(t, \Omega t)$ correspond to the *slow* and *fast* components, respectively. By construction, ψ has zero mean as $\langle\psi(t,\eta)\rangle = \lim_{\mathscr{T}\to\infty}\frac{1}{\mathscr{T}}\int_0^{\mathscr{T}}\psi(t,\eta)d\eta = 0$, where $\eta = \Omega t$ refers to the fast timescale and $\mathscr{T}$ is the time over which the averaging on $\psi(t,\eta)$ is done. However, we must note that $\langle\psi(t,\eta)^2\rangle \neq 0$. The variable $X(t)$ therefore refers to the *slow* one with a natural timescale of the system and $\psi(t)$, the *fast* variable with a timescale of the noise. We next make use of the decomposition [Eq. (2.4)] in Eq. (2.3) to write

$$\ddot{X} + \ddot{\psi} + \epsilon[(X+\psi)^2 - 1](\dot{X} + \dot{\psi}) + \omega_0^2(X + \psi) = \beta\xi(t). \tag{2.4a}$$

Adding and subtracting $\langle\psi\rangle$ and $\langle\psi^2\rangle$, we may rewrite the above equation as

$$\begin{aligned}\ddot{X} + \ddot{\psi} + \epsilon[X^2 + \langle\psi^2\rangle + 2X\langle\psi\rangle - 1 + 2X(\psi - \langle\psi\rangle)\\ + (\psi^2 - \langle\psi^2\rangle)](\dot{X} + \dot{\psi}) + \omega_0^2(X+\psi) = \beta\xi(t).\end{aligned} \tag{2.4b}$$

We now split Eq. (2.4b) as follows:

$$\ddot{X} + \epsilon[X^2 + \langle\psi^2\rangle + 2X\langle\psi\rangle - 1]\dot{X} + \omega_0^2 X = 0 \tag{2.4c}$$

and

$$\begin{aligned}\ddot{\psi} + \epsilon[(\psi^2 - \langle\psi^2\rangle) + 2X(\psi - \langle\psi\rangle)]\dot{X}\\ + \epsilon[X^2 + \psi^2 + 2X\psi - 1]\dot{\psi} + \omega_0^2\psi = \beta\xi(t).\end{aligned} \tag{2.4d}$$

Noting that $\langle\psi\rangle = 0$, the above two equations can be put into the following form:

$$\ddot{X} + \epsilon[X^2 + \langle\psi^2\rangle - 1]\dot{X} + \omega_0^2 X = 0 \tag{2.5}$$

and

$$\begin{aligned}\ddot{\psi} + \epsilon[X^2 + \langle\psi^2\rangle - 1]\dot{\psi} + \omega_0^2\psi + \epsilon[2X(\psi - \langle\psi\rangle)\\ + (\psi^2 - \langle\psi^2\rangle)](\dot{X} + \dot{\psi}) = \beta\xi(t),\end{aligned} \tag{2.6}$$

respectively. Here, Eq. (2.5) refers to the slow motion and Eq. (2.6) describes the fast motion. The slow motion is the result of averaging over the noise in Eq. (2.3).

It is easy to verify that the addition of Eqs. (2.5) and (2.6) gives back Eq. (2.3). Equation (2.6) is a nonlinear stochastic differential equation which cannot be solved analytically. To proceed, we note that ψ, being a rapidly changing field, may be assumed further, $\ddot{\psi} \gg \dot{\psi}, \psi, \psi^2$, since $\ddot{\psi} = \Omega^2\frac{d^2\psi}{d\eta^2}$ and $\dot{\psi} = \Omega\frac{d\psi}{d\eta}$ and $\Omega \gg \omega_0$. Therefore, we neglect all terms except the stochastic one on the right-hand side. This is done in vibrational mechanics, where the noise is replaced by a high frequency field. We then arrive at the following equation:

$$\ddot{\psi} = \beta\xi(t). \tag{2.7}$$

This stochastic equation is formally similar [19] to that for the motion of a charged particle in one dimension and has been used earlier in the heating of a plasma in a random electric field [54–56]. Solving this equation, we find (see Appendix)

$$\langle\psi\rangle = 0$$
$$\text{and}\quad \langle\psi^2\rangle = \tfrac{1}{6}\beta^2 D T^3, \tag{2.8}$$

where T is the time period of oscillation in the deterministic limit, i.e., $T = 2\pi/\omega_0$. Now we substitute $\langle\psi^2\rangle$ in the slow dynamics and write the slow dynamics as

$$\ddot{X} + \epsilon(X^2 - \alpha^2)\dot{X} + \omega_0^2 X = 0, \tag{2.9}$$

where $\alpha^2 = (1 - \langle\psi^2\rangle)$. Now we are in a position to solve Eq. (2.9) with the help of standard Lindstedt-Poincare method. First, we introduce a new timescale τ as $\tau = \omega t$, where ω is the new frequency to be determined. Thus, the derivatives of the variable become scaled and we obtain the modified slow dynamics as

$$\omega^2 X'' + \epsilon\omega(X^2 - \alpha^2)X' + \omega_0^2 X = 0. \tag{2.10}$$

Here the prime represents differentiation with respect to the new scaled time τ. Now, let us expand the variables X and the new frequency ω in powers of ϵ as

$$X = X_0 + \epsilon X_1 + \epsilon^2 X_2 + \cdots$$
$$\text{and} \quad \omega = \omega_0 + \epsilon\omega_1 + \epsilon^2\omega_2 + \cdots .$$

Substituting in Eq. (2.10), we find

$$\omega_0^2(X_0'' + X_0) + \epsilon\big[\omega_0^2(X_1'' + X_1) + 2\omega_0\omega_1 X_0'' + \omega_0 X_0'(X_0^2 - \alpha^2)\big] + \epsilon^2\big[\omega_0^2(X_2'' + X_2) + 2\omega_0\omega_1 X_1'' + (\omega_1^2 + 2\omega_0\omega_2)X_0'' + (\omega_0 X_1' + \omega_1 X_0')(X_0^2 - \alpha^2) + 2\omega_0 X_0 X_1 X_0'\big] + \cdots = 0. \tag{2.11}$$

Now we compare the coefficients of various powers of ϵ on both sides. For the zeroth order, we write

$$(X_0'' + X_0) = 0. \tag{2.12a}$$

Similarly, for the first order of ϵ,

$$\omega_0^2(X_1'' + X_1) = -2\omega_0\omega_1 X_0'' - \omega_0 X_0'(X_0^2 - \alpha^2), \tag{2.12b}$$

and for the second order in ϵ:

$$\omega_0^2(X_2'' + X_2) = -2\omega_0\omega_1 X_1'' - (\omega_1^2 + 2\omega_0\omega_2)X_0'' - (\omega_0 X_1' + \omega_1 X_0')(X_0^2 - \alpha^2) - 2\omega_0 X_0 X_1 X_0'. \tag{2.12c}$$

From Eq. (2.12a), we can write the zeroth order solution as

$$X_0 = A\cos\tau + B\sin\tau, \tag{2.13}$$

where A and B are two arbitrary constants. We use the following initial conditions: at $t = 0$, $x(0) = a$, $\dot{x}(0) = 0$ so we can write with respect to the new timescale τ that, at $\tau = 0$, we have $X = a$ and $X' = 0$. It follows that at $\tau = 0$, $X_0(0) = a$; $X_0'(0) = 0$. Also, $X_i(0) = 0$ and $X_i'(0) = 0$, $i \neq 0$. Thus, we obtain the zeroth order solution:

$$X_0 = a\cos\tau. \tag{2.14}$$

Substituting the value of the zeroth order solution in Eq. (2.12b), we find

$$\omega_0^2(X_1'' + X_1) = 2\omega_0\omega_1 a\cos\tau + \omega_0 a\left(\frac{a^2}{4} - \alpha^2\right)\sin\tau + \frac{\omega_0 a^3}{4}\sin 3\tau. \tag{2.15}$$

The coefficients of $\cos\tau$ and $\sin\tau$ in Eq. (2.15) must vanish for removing the singularity, as they correspond to secular or diverging terms. Hence, we must set $\omega_1 = 0$ and

$$\omega_0 a\left(\frac{a^2}{4} - \alpha^2\right) = 0. \tag{2.16a}$$

This leads to $a = \pm 2\alpha$. Let us choose $a = 2\alpha$. This quantifies the radius of the limit cycle. In the deterministic case, i.e., in the absence of noise, α becomes 1 and, hence, the radius of the limit cycle equals exactly 2. However, in the stochastic domain, $\alpha < 1$ and, hence, the radius of the limit cycle becomes less than 2. Therefore, the complete solution for the zeroth order,

$$X_0 = 2\alpha\cos\tau, \tag{2.16b}$$

Eq. (2.15) then reduces to

$$(X_1'' + X_1) = \frac{2\alpha^3}{\omega_0}\sin 3\tau. \tag{2.17a}$$

Its general solution

$$X_1 = \frac{\alpha^3}{\omega_0}\sin^3\tau, \tag{2.17b}$$

where we have used the initial condition that, at $\tau = 0$, $X_1 = 0$ and $\dot{X}_1 = 0$. Thus, the solution X_1 gives us the first or leading order correction to the solution for the slow dynamics X.

Another important observation is that, since we have obtained $\omega_1 = 0$, there is no correction in frequency, in the first order, and therefore we need to analyze the higher orders to get a nonzero correction to the frequency, up to a leading order. With this in mind, we proceed to find the solution for the second order as follows: Substituting X_0 and X_1 from Eqs. (2.16b) and (2.17b) in Eq. (2.12c), we obtain

$$\omega_0^2(X_2'' + X_2) = \left(4\omega_0\omega_2 + \tfrac{1}{4}\alpha^5\right)\cos\tau - \tfrac{3}{2}\alpha^5\cos 3\tau + \tfrac{5}{4}\alpha^5\cos 5\tau. \tag{2.18a}$$

The coefficients of the secular or diverging term must be a vanishing one. Therefore,

$$\omega_2 = -\frac{\alpha^5}{16\omega_0}. \tag{2.18b}$$

Thus, we obtain the corrected frequency up to second order as

$$\omega = \omega_0 + \epsilon\omega_1 + \epsilon^2\omega_2 + \cdots$$
$$\text{or} \quad \omega = \omega_0 - \frac{\epsilon^2\alpha^5}{16\omega_0}. \tag{2.18c}$$

Thus, the solution for X and hence $\langle x\rangle$ in the unscaled time,

$$\langle x\rangle = X(t) = 2\alpha\cos\omega t + \frac{\epsilon\alpha^3}{\omega_0}\sin^3\omega t, \tag{2.19}$$

where the corrected frequency ω used in Eq. (2.19) has been obtained from Eq. (2.18c). The effect of noise is reflected in the expression $\alpha^2 = (1 - \langle\psi^2\rangle)$. Thus, Eq. (2.19) along with the modified frequency as indicated in Eq. (2.18c) describes the complete temporal profile of the Van der Pol oscillator, in the presence of a Gaussian white noise. Our main interest is to examine the existence of a limit cycle for such an oscillator. To this end, the phase plots ($\langle\dot{x}\rangle$ vs $\langle x\rangle$) for the dynamics are examined in Fig. 1. It is clear that the limit

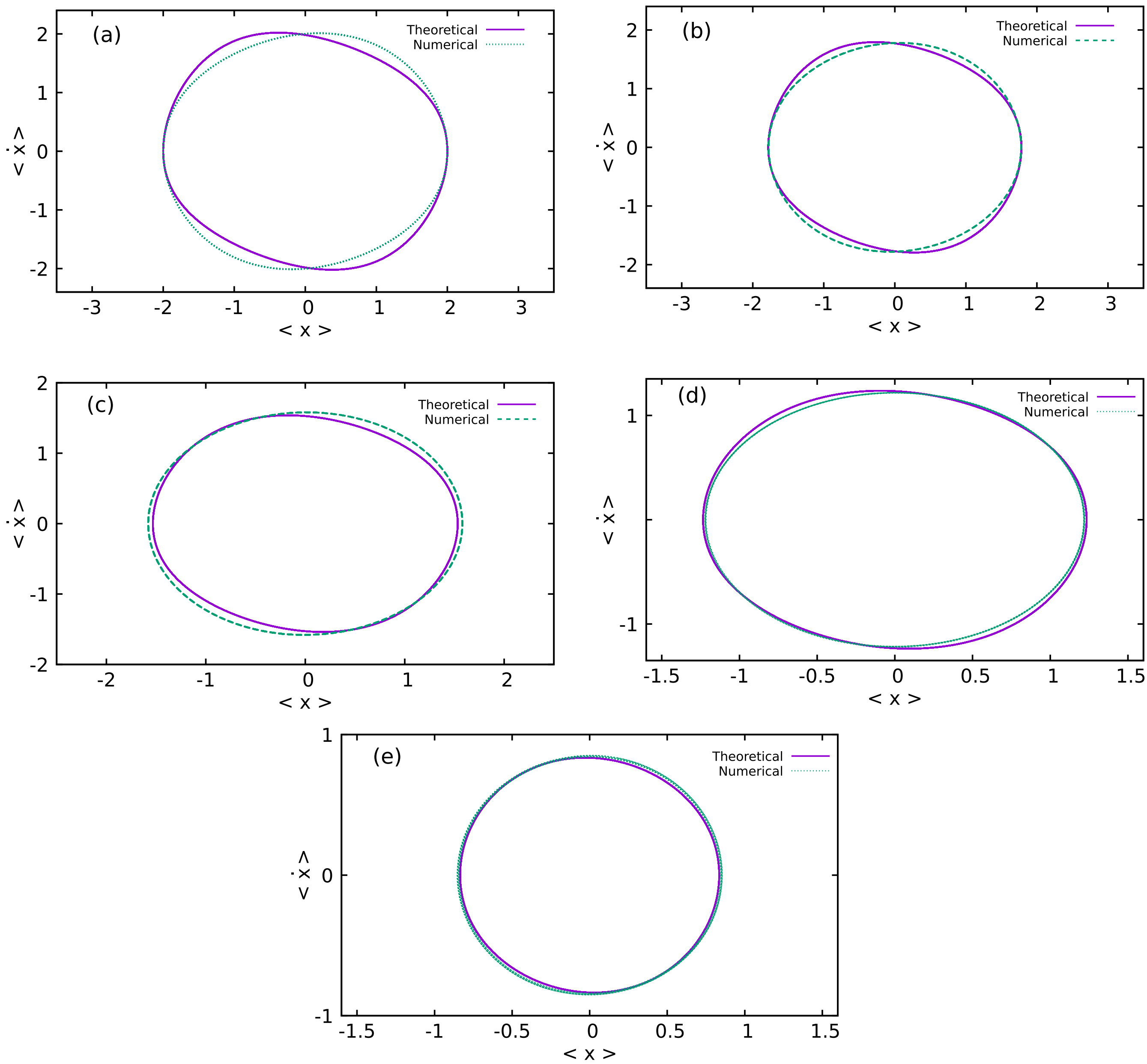

FIG. 1. (a) The phase space plots (i.e., $\langle \dot{x} \rangle$ vs $\langle x \rangle$ plot) of the Van der Pol oscillator for $D = 0$ in the deterministic limit. The purple color loop (limit cycle) is obtained from our theoretical analysis whereas, the green one is from numerical solution. Both graphs have been plotted with the following parameter sets: $\omega_0 = 1.0$, $\epsilon = 0.1$, $\beta = 0.1$ with the initial conditions $x(0) = 1$, $\dot{x}(0) = 0$. In the numerical simulations, averaging has been done over 10^3 trajectories and the step size used is 0.001 (units arbitrary). (b) The phase space plots (i.e., $\langle \dot{x} \rangle$ vs $\langle x \rangle$ plot) of the Van der Pol oscillator for $D = 0.5$. The limit cycle persists, but with a reduced size, having radius 2α. The purple colored loop is obtained from our theoretical analysis whereas the green one is obtained by solving Eq. (2.3) numerically. Both graphs have been plotted with the following parameter sets: $\omega_0 = 1.0$, $\epsilon = 0.1$, $\beta = 0.1$ with the initial conditions $x(0) = 1$, $\dot{x}(0) = 0$. In the numerical simulations averaging has been done over 10^3 trajectories and the step size used is 0.001 (units arbitrary). (c) Same as in (b) but for $D = 1.0$. The size of the limit cycle is further reduced. (d) Same as in (b) but for $D = 1.5$. The size of the limit cycle becomes smaller. (e) Same as in (b) but for $D = 2.0$.

cycle still persists with a reduced radius 2α, depending on the noise strength, as shown in Figs. 1(b)–1(e). Furthermore, our analysis predicts that the radius of the limit cycle tends to zero as D approaches the critical value $D_{\rm cr}$ for which the limit cycle should collapse to a point at $\alpha = 0$ or $\langle \psi^2 \rangle = 1$. For this, the critical value of D is given as $D_{\rm cr} = 3\omega_0^3/4\pi^3\beta^2$. This is shown in Fig. 2 by a continuous line.

B. The noisy dynamics for the Duffing–Van der Pol oscillator

The workability of the present scheme can also be tested for the case of Duffing–Van der Pol oscillators where nonlinearity, in addition to the dissipative term, exists in the potential term and, for this, the leading order correction to the frequency must be calculated up to the second order. Let us begin with the following dynamical equation, driven by a

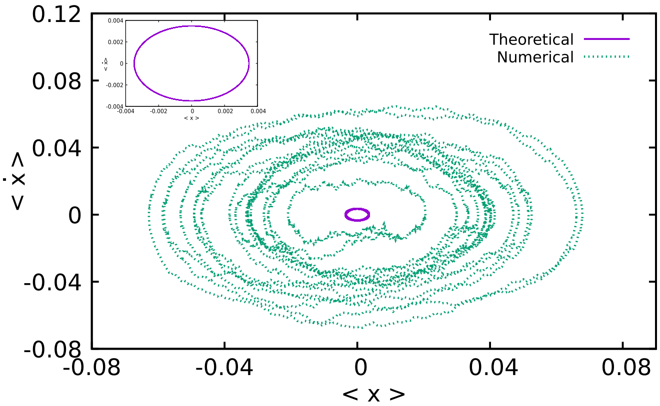

FIG. 2. The phase space plot as obtained from our theoretical analysis (also shown in the inset) at the critical value of $D = D_{\text{cr}} = 2.43$. The other parameters and initial conditions are the same as in Fig. 1(b). At this value of D, the noise effect is so large that the limit cycle virtually collapses to a point as the size of the limit cycle is vanishingly small (units arbitrary).

Gaussian white noise as follows:

$$\ddot{x} + \epsilon(x^2 - 1)\dot{x} + \omega_{01}^2 x + \epsilon\omega_{01}x^3 = \beta\xi(t), \qquad (2.20)$$

where ω_{01} is the natural frequency of the oscillator and the other symbols are the same as mentioned previously. To proceed, let us split, as before, the variable x into a *slow* and a *fast* part and substitute in Eq. (2.20) to separate out the fast and slow dynamics. We write the equation for the fast motion as

$$\begin{aligned}\ddot{\psi} &+ \epsilon(X^2 + \langle\psi^2\rangle - 1)\dot{\psi} + \omega_{01}^2\psi + \epsilon[2X(\psi - \langle\psi\rangle) \\ &+ (\psi^2 - \langle\psi^2\rangle)](\dot{X} + \dot{\psi}) + \epsilon\omega_{01}[3X^2(\psi - \langle\psi\rangle) \\ &+ 3X(\psi^2 - \langle\psi^2\rangle) + \psi^3] = \beta\xi(t). \qquad (2.21a)\end{aligned}$$

We approximate the fast dynamics as before:

$$\ddot{\psi} = \beta\xi(t). \qquad (2.21b)$$

The relevant solution and the averages have been obtained earlier in Eq. (2.8). Also, we obtain the slow dynamics as

$$\ddot{X} + \epsilon(X^2 - \alpha^2)\dot{X} + \omega_0^2 X + \epsilon\omega_{01}X^3 = 0, \qquad (2.22a)$$

where $\alpha^2 = (1 - \langle\psi^2\rangle)$ and $\omega_0^2 = (\omega_{01}^2 + 3\epsilon\omega_{01}\langle\psi^2\rangle)$ or, $\omega_{01} \simeq \omega_0 - 3\epsilon\langle\psi^2\rangle/2$. With a little bit of algebraic manipulation, we rewrite the above equation as

$$\ddot{X} + \epsilon(X^2 - \alpha^2)\dot{X} + \omega_0^2 X + \epsilon\omega_0 X^3 = 0. \qquad (2.22b)$$

By introducing the new timescale $\tau = \omega t$, we express Eq. (2.22b) as follows:

$$\omega^2 X'' + \epsilon\omega(X^2 - \alpha^2)X' + \omega_0^2 X + \epsilon\omega_0 X^3 = 0. \qquad (2.22c)$$

In the next step, we expand X and ω in powers of ϵ as

$$X = X_0 + \epsilon X_1 + \epsilon^2 X_2 + \cdots$$
$$\text{and} \quad \omega = \omega_0 + \epsilon\omega_1 + \epsilon^2\omega_2 + \cdots$$

and substitute in Eq. (2.22c) to obtain

$$\begin{aligned}&\omega_0^2(X_0'' + X_0) + \epsilon\big[\omega_0^2(X_1'' + X_1) + 2\omega_0\omega_1 X_0'' + \omega_0 X_0'\big(X_0^2 - \alpha^2\big) + \omega_0 X_0^3\big] + \epsilon^2\big[\omega_0^2(X_2'' + X_2) \\ &+ 2\omega_0\omega_1 X_1'' + (\omega_1^2 + 2\omega_0\omega_2)X_0'' + (\omega_0 X_1' + \omega_1 X_0')\big(X_0^2 - \alpha^2\big) + 2\omega_0 X_0 X_1 X_0' + 3\omega_0 X_0^2 X_1\big] + \cdots = 0. \qquad (2.23)\end{aligned}$$

We have the zeroth order equation as

$$(X_0'' + X_0) = 0; \qquad (2.24a)$$

for the first order of ϵ,

$$\omega_0^2(X_1'' + X_1) = -2\omega_0\omega_1 X_0'' - \omega_0 X_0'\big(X_0^2 - \alpha^2\big) - \omega_0 X_0^3; \qquad (2.24b)$$

and for the second order,

$$\begin{aligned}\omega_0^2(X_2'' + X_2) = &-2\omega_0\omega_1 X_1'' - \big(\omega_1^2 + 2\omega_0\omega_2\big)X_0'' \\ &- (\omega_0 X_1' + \omega_1 X_0')\big(X_0^2 - \alpha^2\big) \\ &- 2\omega_0 X_0 X_1 X_0' - 3\omega_0 X_0^2 X_1. \qquad (2.24c)\end{aligned}$$

With the zeroth order solution, we obtain

$$\begin{aligned}&\omega_0^2(X_1'' + X_1) \\ &= \omega_0 a\left[2\omega_1 - \frac{3a^2}{4}\right]\cos\tau + \omega_0 a\left[\frac{a^2}{4} - \alpha^2\right]\sin\tau \\ &\quad + \frac{\omega_0 a^3}{4}[\sin 3\tau - \cos 3\tau]. \qquad (2.25)\end{aligned}$$

The conditions for vanishing secular terms lead us to obtain the following relationships:

$$a = 2\alpha$$
$$\text{and} \quad \omega_1 = \frac{3a^2}{8}.$$

While the first one gives us the radius of the limit cycle as in the previous case, the second one reveals the frequency correction at the first order, which is clearly nonzero unlike the previous case. Thus, the frequency corrected up to first order (since nonvanishing contribution) is

$$\omega = \omega_0 + \epsilon\left(\frac{3\alpha^2}{2}\right). \qquad (2.26a)$$

The solution for $\langle x(t)\rangle$ or X in unscaled variable is given by

$$\begin{aligned}\langle x(t)\rangle = X = X_0 + \epsilon X_1 = 2\alpha\cos\omega t + \epsilon\bigg[\frac{\alpha^3}{\omega_0}\sin^3(\omega t) \\ + \frac{\alpha^3}{4\omega_0}(\cos 3\omega t - \cos\omega t)\bigg], \qquad (2.26b)\end{aligned}$$

where the frequency ω is given in Eq. (2.26a). We note that this frequency ω is a result of twofold modification of the natural frequency ω_{01}. First, due to timescale separation, it

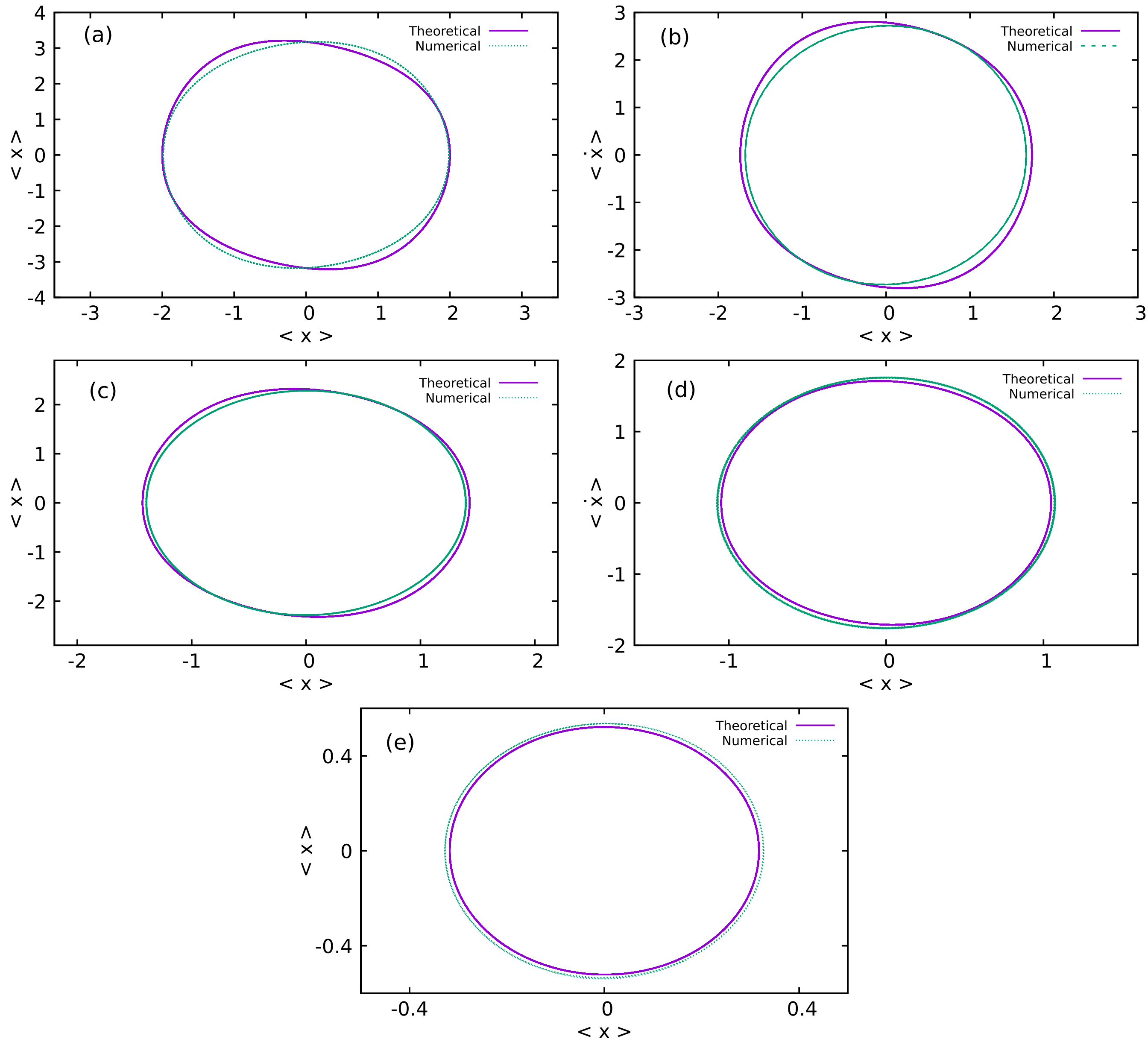

FIG. 3. (a) The phase space plots (i.e., $\langle \dot{x} \rangle$ vs $\langle x \rangle$ plot) of the Duffing–Van der Pol oscillator for $D = 0$ in the deterministic limit. The purple color loop (limit cycle) is obtained from our theoretical analysis whereas the green one is from the numerical solution. Both graphs have been plotted with the following parameter sets: $\omega_{01} = 1.5$, $\epsilon = 0.1$, $\beta = 0.1$ with the initial conditions $x(0) = 1$, $\dot{x}(0) = 0$. In the numerical simulations, averaging has been done over 10^3 trajectories and the step size used is 0.001 (units arbitrary). (b) The phase space plots (i.e., $\langle \dot{x} \rangle$ vs $\langle x \rangle$ plot) of the Duffing–Van der Pol oscillator for $D = 2.0$. The limit cycle persists, but with a reduced size, having radius 2α. The purple colored loop is obtained from our theoretical analysis whereas, the green one is obtained by solving Eq. (2.20) numerically. Both graphs have been plotted with the following parameter sets: $\omega_{01} = 1.5$, $\epsilon = 0.1$, $\beta = 0.1$ with the initial conditions $x(0) = 1$, $\dot{x}(0) = 0$. In the numerical simulations, averaging has been done over 10^3 trajectories and the step size used is 0.001 (units arbitrary). (c) Same as in (b) but for $D = 4.0$. The loop size is further reduced. (d) Same as in (b) but for $D = 6.0$. (e) Same as in (b) but for $D = 8.0$.

is modified as ω_0, incorporating the effect of noise. The next modification in frequency appears after applying the Lindstedt-Poincare method on the modified slow dynamics.

Up to this point, we have found the frequency correction up to first order. To find the second-order corrections to the frequency, we substitute X_0 and X_1 in Eq. (2.24c) and obtain

$$\omega_0^2(X_2'' + X_2) = \left[\frac{7\omega_1 a^3}{16} - \frac{\alpha^5}{16} - \alpha^2\left(\omega_1 - \frac{a^2}{32}\right)a\right]\sin\tau + \left[(\omega_1^2 + 2\omega_0\omega_2)a + \frac{a^5}{32} - \frac{\omega_1 a^3}{16} + \frac{3\alpha^2 a^3}{32}\right]\cos\tau - \left[\frac{5\omega_1 a^3}{16} + \frac{3a^5}{64}\right]\sin 3\tau + \left[\frac{9\omega_1 a^3}{16} - \frac{3\alpha^2 a^3}{32}\right]\cos 3\tau + \left[\frac{a^5}{16}\right]\sin 5\tau + \left[\frac{a^5}{128}\right]\cos 5\tau. \tag{2.27}$$

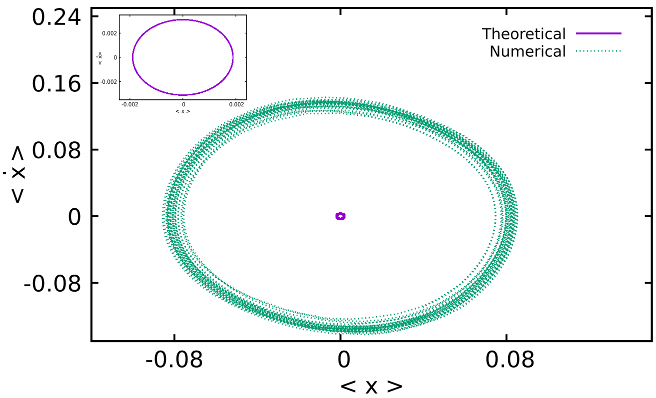

FIG. 4. The phase space plot for the Duffing–Van der Pol oscillator as obtained from our theoretical analysis (also shown in the inset) at the critical value of $D = D_{cr} = 8.17$. The other parameters and initial conditions are same as in Fig. 3(b). At this value of D, the noise effect is so large that the limit cycle virtually collapses to a point as the size of the limit cycle is vanishingly small (units arbitrary).

For finite results, the secular terms must vanish and hence we obtain $\omega_2 = -11\alpha^4/8$. Thus, the frequency corrected up to second order is

$$\omega = \omega_0 + \epsilon\left(\frac{3\alpha^2}{2}\right) + \epsilon^2\left(-\frac{11\alpha^4}{8}\right), \tag{2.28}$$

and the solution for $\langle x(t)\rangle$ or X remains the same as in Eq. (2.26b).

III. NUMERICAL RESULTS

To corroborate the analytical results, we now carry out full numerical simulation of the stochastic differential Eqs. (2.3) and (2.20) for noisy Van der Pol and Duffing–Van der Pol oscillators, respectively. For this purpose, we have used a stochastic algorithm [57] with a step size $h = 0.001$. Unless otherwise stated, the initial conditions are set at $x(0) = 1$, $\dot{x}(0) = 0$ and the stochastic averaging is done over 1000 trajectories for each case. For all numerical simulations, we have further used the parameter set $\epsilon = 0.1$, $\beta = 0.1$, and $\omega_0 = 1.0$ (for Van der Pol) and $\omega_{01} = 1.5$ (for Duffing–Van der Pol). The representative numerical phase space plots ($\langle\dot{x}\rangle$ vs $\langle x\rangle$) shown by the green color are depicted in Figs. 1–4 and compared with those obtained from our corresponding theoretical analysis in the last section shown by purple color for various values of D. For the Van der Pol oscillator, the phase space plots for $D = 0$, $D = 0.5$, $D = 1.0$, $D = 1.5$, and $D = 2.0$ are displayed in Figs. [1(a)–1(e)], respectively. The results clearly demonstrate that as the noise strength D is increased, the limit cycle loop decreases in size till a critical value of D is reached beyond which the limit cycle collapses. The collapse at $D_{cr} = 2.43$ is shown in Fig. 2, where the loop does not close in the asymptomatic limit. The corresponding analytical result shows that the limit cycle reduces almost to a point at $D_{cr} = 2.43$. The agreement between the theory and numerics is therefore quite satisfactory. The numerical results for the Duffing–Van der Pol oscillator are shown in Figs. 3(a)–3(e) and 4. For this case, the calculated value of the critical noise strength is $D_{cr} = 8.17$. We observe that with increase of noise strength D, the loop area decreases till the critical value of D, i.e., D_{cr} is reached beyond which the limit cycle ceases to exist. The phase space plots for $D = 0$, $D = 2.0$, $D = 4.0$, $D = 6.0$, and $D = 8.0$ are shown in Figs. 3(a)–3(e), respectively. The collapse of the loop is shown in Fig. 4 for $D_{cr} = 8.17$ and compared with the analytical result. The theoretical and numerical trends of the loop area reduction and the closeness of the analytically predicted and numerically calculated threshold values of noise strength for the collapse of limit cycle strengthen our claim for applicability of the present analytical scheme for the treatment of the noisy limit cycle oscillators.

IV. CONCLUSION

A noisy self-sustained oscillator is described by a nonlinear stochastic differential equation. Since the limit cycles arise out of a delicate balance between the damping and self-excitation, resulting in isolated closed asymptotic trajectories in phase space, linearization is not an appropriate option for its long time solution. In this paper, we have presented a scheme for analytic treatment of the problem by extending the Blekhman perturbation theory used in deterministic vibrational mechanics to the realm of stochastic dynamics.

By making use of two distinct timescales, we derived the equation of slow motion by stochastic averaging over the fast motion. The equation for the fast motion has a simple generic form that balances the inertial force with the noise force. The effect of fast motion was incorporated into the slow dynamics through renormalization of the coefficients. The equation for slow motion was solved by the Lindstedt-Poincare perturbation technique. We have shown how noise affects the frequency and reduces the amplitude of limit cycle oscillation till the noise strength reaches a critical value beyond which the cycle is destabilized. The analytic noise threshold for this collapse was determined from the condition for removal of secular divergence. Two examples, Van der Pol and Duffing–Van der Pol oscillators have been explored. Our theoretical scheme is in good agreement with numerical simulations.

We now make the following concluding remarks. A closer look at the present scheme reveals that the slow motion even after averaging retains the nonlinear nature of the dynamics, whereas the fast motion is described by a linear stochastic equation analogous to an equation governing the heating of plasma in an electric field. The slow dynamics can be analyzed by the methods of nonlinear dynamics. The fast motion, on the other hand, is purely stochastic with no deterministic component and generic in form for a wide class of limit cycle oscillators so long as the noise is additive in form. The analysis carried out here concerns the single limit cycle and white noise. The extension of treatment to nonlinear systems with multiple limit cycles and to noise processes with finite correlation time may open interesting issues in the realm of dynamical interplay of nonlinearity and stochasticity.

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APPENDIX: CALCULATION OF $\langle\psi^2(t)\rangle$

As described in Eq. (2.7), the dynamics of the fast motion is

$$\ddot{\psi} = \beta\xi(t), \tag{A1}$$

with the noise characteristics as defined earlier. With $\dot{\psi} = v$, we rewrite (A1) as

$$\dot{v} = \beta\xi(t), \tag{A2}$$

On integration, we obtain

$$v(t) = \int_0^{t_1} ds_1 \dot{v}(s_1) \tag{A3}$$

and find $\langle v^2(t)\rangle$ as

$$\langle v^2(t)\rangle = \left\langle \int_0^t ds_2 \int_0^t ds_1 \dot{v}(s_2)\dot{v}(s_1)\right\rangle. \tag{A4}$$

Now,

$$\begin{aligned}\frac{d}{dt}\langle v^2(t)\rangle &= 2\int_0^t ds\langle \dot{v}(t)\dot{v}(s)\rangle \\ &= 2\int_0^t ds\langle \dot{v}(t-s)\dot{v}(0)\rangle = 2\beta^2 D.\end{aligned} \tag{A5}$$

Hence,

$$\langle v^2(t)\rangle_t = 2\beta^2 Dt. \tag{A6}$$

Again, from Eq. (A1), we can write

$$\psi\ddot{\psi} = \beta\psi(t)\xi(t) \tag{A7}$$

and take the ensemble average. Some algebraic manipulations yield

$$\tfrac{1}{2}\langle\ddot{\psi^2}\rangle_t - \langle v^2(t)\rangle_t = \beta\langle\psi(t)\xi(t)\rangle. \tag{A8}$$

The right-hand side of the above equation is zero. Thus,

$$\langle\ddot{\psi^2}\rangle_t = 2\langle v^2(t)\rangle_t = 4\beta^2 Dt. \tag{A9}$$

Integrating once with respect to time:

$$\langle\dot{\psi^2}\rangle_t = 2\beta^2 Dt^2. \tag{A10}$$

Performing one more integration, we have

$$\langle\psi^2\rangle_t = \tfrac{2}{3}\beta^2 Dt^3. \tag{A11}$$

Finally, averaging over the time period of the limit cycle, we arrive at

$$\langle\psi^2\rangle = \frac{1}{T}\int_0^T \langle\psi^2\rangle_t dt = \frac{1}{6}\beta^2 DT^3, \tag{A12}$$

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