Near-integrable dynamics of the Fermi-Pasta-Ulam-Tsingou problem

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It is well known that the classic Fermi-Pasta-Ulam-Tsingou (FPUT) study of a chain of nonlinear oscillators is closely related to a number of completely integrable systems, including the Toda lattice. Here, we present a method that captures the departure of nonintegrable FPUT dynamics from those of a nearby integrable Toda lattice. Using initial long-wave data, we find that the former depart rather sharply from the latter near the predicted shock time of an asymptotic partial differential equation approximation, at which point energy cascades into higher lattice modes. Our method provides an appropriate frame of reference for one to distinguish the short-term dynamics of the two systems, whose macroscopic trajectories diverge noticeably only on a much longer timescale, when the FPUT dynamics thermalize.

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I. INTRODUCTION

The Fermi-Pasta-Ulam-Tsingou (FPUT) problem [1,2] of the 1950s has a well-documented legacy, leading to a number of major discoveries in the field of nonlinear science [3–5]. The aim of the original FPUT experiment was to simulate a one-dimensional Hamiltonian chain of oscillators with nearest-neighbor interactions, including a weakly anharmonic potential. After fixing the boundaries and considering low-mode sinusoidal initial data, the expectation was that the weak anharmonicity would eventually lead to a statistical *equipartition* of energy among all possible lattice modes. Surprisingly, the FPUT simulation results appeared to directly contradict this hypothesis. For the computationally manageable timescales considered, the FPUT experiment never reached equipartition and instead revealed a quasirecurrent sharing of energy among a small subset of lattice modes concentrated around the one initially excited. More recently, experiments have been performed, particularly in fiber optics, in which the quasirecurrent mode sharing simulated in the FPUT lattice, and its eventual long-term route to thermalization [6], can be physically observed [7–9]. There has also been renewed interest in the study of coherent structures on *dimerized* FPUT-like lattices [10,11], motivated by recent advances in the study of topological phenomena in engineered condensed matter and photonic systems.

Coincidentally, the FPUT dynamical system, with the initial data considered in Ref. [1], is quite close to a number of *completely integrable* approximations [12]. Indeed, one of the first explanations of the FPUT experiment was given by Zabusky and Kruskal [13], who studied the long-wave continuum limit of the FPUT lattice and found that, to leading order, the lattice dynamics were well approximated by the integrable Korteweg–de Vries (KdV) equation, a universal partial differential equation (PDE) description of unidirectional nonlinear dispersive waves. In this limit, the formation of KdV solitons near a shock front and their subsequent elastic collisions was put forward as an explanation of the quasirecurrent mode sharing observed by FPUT. It was later shown by Zakharov [14], without making the unidirectional ansatz, that the resulting lowest-order perturbation to the wave equation is also completely integrable. However, the resulting asymptotic PDE has a short-wave instability, evident from the equation's linear dispersion relation. Indeed, the analysis and numerical construction of solutions to this bidirectional description is much more challenging in comparison with the KdV equation [15]. For this reason, recent works have focused on deriving a quasi unidirectional description of FPUT's dynamics in the continuum regime [16,17], including a mathematical justification for using a decoupled pair of KdV equations [18].

Another integrable system closely related to the original FPUT experiment is the discrete Toda lattice [19]. By introducing a noncanonical change of variables, in Ref. [20], Flaschka proved the existence of a sufficient number of conservation laws, showing that Toda's Hamiltonian flow is indeed completely integrable. It was pointed out in Ref. [21] that, in the regime of small specific energy ($\sim E/N$, E being the total energy and N the number of oscillators), which is the regime considered in the original FPUT experiment, the FPUT lattice is actually closer to the Toda lattice than to the linear chain. The authors in Ref. [21] performed extensive numerical simulations of both the FPUT and Toda lattices, showing that in this regime the dynamics of the two systems are virtually indistinguishable over rather long timescales and, only on a much longer timescale, do the FPUT dynamics achieve equipartition, whereas the excited lattice mode distribution of Toda remains localized. Interestingly, both the FPUT and Toda discrete systems relate back to the KdV equation in the small-amplitude and long-wave asymptotic regime. A study of the spectral characteristics of the Toda lattice as $N \to \infty$ and its relation to the scattering problem of the KdV equation has been studied recently in Ref. [22].

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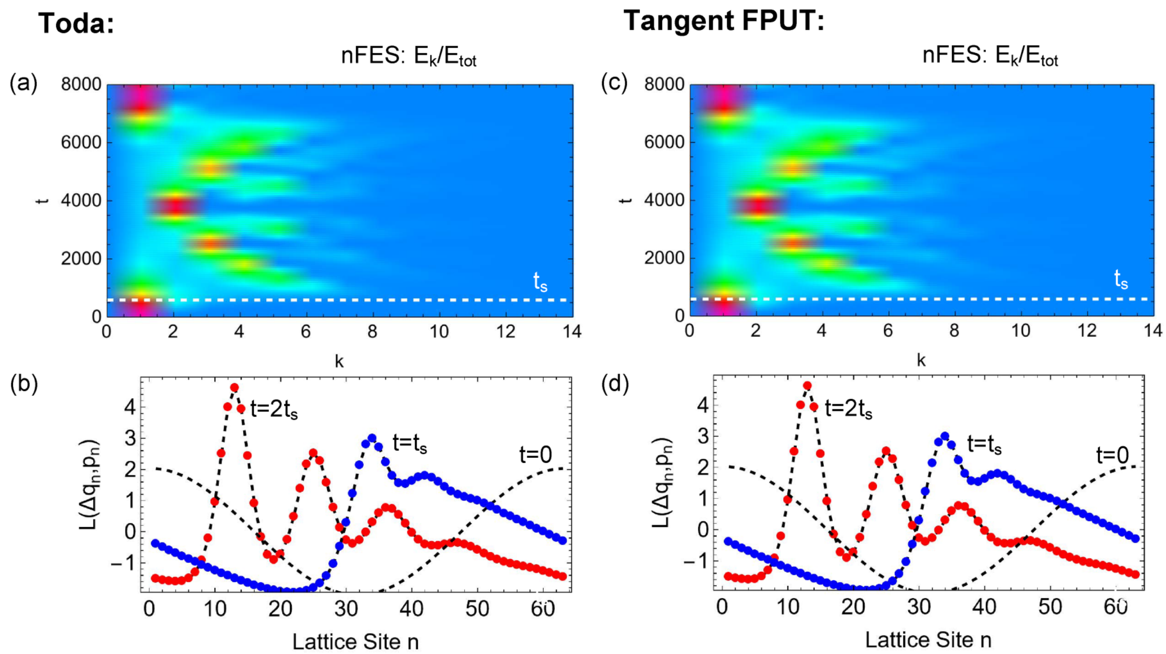

FIG. 1. Simulations of the periodic Toda lattice (left column) and tangent FPUT lattice (right column), where $\delta = 2.45 \times 10^{-2}$ and $N = 64$, of an initially left traveling wave [see (11)]. The top row shows a subset of the normalized Fourier energy spectrum's (nFES) evolution over one quasirecurrent cycle [red (blue) denotes high (low) spectral density]. The white dashed line is the computed Burger's shock time, $t_s \sim 587$. The bottom row shows the spatial profile of the discrete left traveling wave $L(\Delta q_n, p_n)$ at $t = 0, t_s, 2t_s$.

In the present paper we study the dynamics of the FPUT lattice in a neighborhood of the completely integrable systems discussed above. In particular, we study the evolution of unidirectionally traveling, long-wave sinusoidal data on Toda and nearby FPUT chains with periodic boundaries. In this setting, both the FPUT and Toda dynamics can be initially well approximated by a generalized inviscid Burger's equation, as recently derived in Ref. [16], in the so-called thermodynamic limit. Here, we numerically track differences in initially nearby FPUT and Toda systems in appropriate coordinates and show that the two systems actually split on a timescale much shorter than those required for the former system to approach equipartition [21]. Specifically, the two systems diverge sharply at a point in time which agrees well with the initial shock time predicted from the long-wave Burger's approximation. The present paper is motivated by numerous theoretical studies on FPUT's gradual route to thermalization, with direct relevance to ergodic theory in many physical systems [3–5,23]. Our goal is to view FPUT as a perturbation of an integrable system and to provide a means to measure FPUT's initial stages of departure from integrability, highlighted with numerical observations.

We organize this paper as follows: In Sec. II we outline some relevant properties of the FPUT and Toda systems, introduce a frame of reference to distinguish the short-term dynamics of the two systems, and present numerical results; in Sec. III we review the small-amplitude, long-wave asymptotics of FPUT and Toda, including the recent study conducted in Ref. [16], and tie results stemming from the asymptotic description to the numerics of the discrete systems presented in the previous section; finally, in Sec. IV, we give some concluding remarks.

II. TANGENT FPUT AND TODA LATTICES

We consider the so-called $(\alpha + \beta)$-FPUT system, with the equations of motion

$$\ddot{q}_j = q_{j+1} - 2q_j + q_{j-1} + \alpha[(q_{j+1} - q_j)^2 - (q_j - q_{j-1})^2] + \beta[(q_{j+1} - q_j)^3 - (q_j - q_{j-1})^3]. \quad (1)$$

In the case of a chain with N particles ($0 \leqslant j \leqslant N - 1$) and periodic boundaries ($j \in \mathbb{Z}_N$), one can linearize (1) about the zero solution and use the discrete Fourier transform to obtain the following dispersion relation for right and left traveling waves

$$\omega_k = 2\left|\sin\left(\frac{\pi k}{N}\right)\right|, \quad (2)$$

where $k = -N/2 + 1, \ldots, N/2$. Denoting the position and momentum Fourier coefficients as $\hat{q}_k$ and $\hat{p}_k$, the normalized Fourier energy spectrum (nFES) is defined to be

$$\frac{E_k}{E_{\text{tot}}} = \frac{\left(\omega_k^2 \hat{q}_k^2 + \hat{p}_k^2\right)}{\sum_j \left(\omega_j^2 \hat{q}_j^2 + \hat{p}_j^2\right)}. \quad (3)$$

In the linear limit, the initial energy in mode k remains unchanged as the system evolves. The nonlinear terms in (1) allow modal energies to conservatively mix and the nFES's evolution is nontrivial (for instance, see Fig. 1). As an aside, we remark here that a more *physically realistic* nonlinear mechanical mass-spring model and its continuum limit was studied in Ref. [24], where, unlike (1), a potential barrier is introduced so that an infinite compression is required to squash adjacent particles to zero length.

The Toda lattice [19,25] is also a Hamiltonian chain of anharmonic oscillators, but with an exponential potential coupling nearest neighbors. Unlike FPUT, the Toda equations are completely integrable and techniques such as inverse scattering [26] can be applied to obtain exact solutions. Here, we study the periodic Toda system

$$\ddot{q}_j = \frac{1}{2\gamma}\left[e^{2\gamma(q_{j+1}-q_j)} - e^{2\gamma(q_j-q_{j-1})}\right], \quad (4)$$

where γ is a real parameter. The integrability of the Toda equations becomes apparent if we write (4) in Flaschka's variables [20],

$$a_j(t) = \gamma\dot{q}_j, \quad b_j(t) = \tfrac{1}{2}e^{\gamma(q_{j+1}-q_j)}. \quad (5)$$

We then obtain the following first-order system:

$$\dot{a}_j = 2\left(b_j^2 - b_{j-1}^2\right),\ \dot{b}_j = b_j(a_{j+1} - a_j). \quad (6)$$

The equations in (6) can be expressed as a Lax pair, with the defining relation

$$\dot{\mathcal{L}} = [\mathcal{B}, \mathcal{L}], \quad (7)$$

where $[\cdot,\cdot]$ denotes the standard commutator and the real $N \times N$ symmetric and antisymmetric matrices, $\mathcal{L}$ and $\mathcal{B}$, are given by

$$\mathcal{L} = \begin{bmatrix} a_1 & b_1 & 0 & 0 & \cdots & b_N \\ b_1 & a_2 & b_2 & 0 & \cdots & \\ 0 & b_2 & a_3 & b_3 & & \\ \vdots & & \ddots & \ddots & \ddots & \\ & & & & & b_{N-1} \\ b_N & & & & b_{N-1} & a_N \end{bmatrix},$$

$$\mathcal{B} = \begin{bmatrix} 0 & b_1 & 0 & 0 & \cdots & -b_N \\ -b_1 & 0 & b_2 & 0 & \cdots & \\ 0 & -b_2 & 0 & b_3 & & \\ \vdots & & \ddots & \ddots & \ddots & \\ & & & & & b_{N-1} \\ b_N & & & & -b_{N-1} & 0 \end{bmatrix}. \quad (8)$$

We denote the ordered real spectrum of the periodic Jacobi matrix $\mathcal{L}$ by

$$\Gamma_{\mathcal{L},N}(t) = \{\lambda_n\}_{n=1}^N.$$

Using Floquet theory and for N even, it can be shown that the eigenvalues of $\mathcal{L}$ are ordered as [27]

$$\lambda_1 < \lambda_2 \leqslant \lambda_3 < \lambda_4 \cdots < \lambda_N.$$

The Lax formulation in (7) implies that $\mathcal{L}$ is *isospectral* over Toda's evolution, i.e., $\Gamma_{\mathcal{L},N}(t) = \Gamma_{\mathcal{L},N}(0)$ for all $t \geqslant 0$. Using this fact, it can be shown that the traces of $\mathcal{L}^n$ (for $n = 1, 2, \dots, N$) give N independent conserved functionals of the $2N$ ab variables in involution, making the Toda equations completely integrable [20]. The Liouville-Arnold theorem [12] then asserts the existence of action-angle variables, confining Toda phase-space trajectories to wind about N-dimensional invariant tori.

By expanding the right-hand side of Eq. (4) in a Taylor series, one sees that the Toda and FPUT systems agree up to second order in γ if one sets $\alpha = \gamma$ and $\beta = (2/3)\gamma^2$. As the original FPUT experiment, here we consider initial data excited only in the first mode [see (11)]. It follows that for each j,

$$|q_{j+1}(0) - q_j(0)| \leqslant \frac{2\pi A}{N},$$

where A denotes the amplitude of the initial sinusoid. Hence, we define the tangency parameter, $\delta \equiv 2\pi A\gamma/N$, and obtain a family of FPUT chains which are initially $O(\delta^2)$ tangent to the Toda lattice. One expects that the dynamics of the tangent FPUT lattice remain "close" to the Toda system's on a timescale which depends inversely on δ. Indeed, the simulations shown in Fig. 1 show the nearly identical evolutions of both the nFES and real-space lattice dynamics for Toda and a nearby tangent FPUT system (see Sec. III for initial data and definition of t_s). Only on a much longer timescale do the Toda and tangent FPUT dynamics diverge, and, in the latter case, eventually achieve equipartition (not shown here—see Ref. [21]). We note here the clear emergence of KdV-like solitons in both lattices seen in the bottom panels of Fig. 1 for the chosen parameters in the quasirecurrent regime.

To study the short-term departure of the nonintegrable tangent FPUT system from integrable Toda, we express the tangent FPUT equations in the ab coordinates of (5), which leads to

$$\begin{aligned} \frac{da_j}{dt} &= \log\left(\frac{b_j}{b_{j-1}}\right)\{1 + \log(4b_jb_{j-1}) + (2/3)[\log^2(2b_j) \\ &\quad + \log(2b_j)\log(2b_{j-1}) + \log^2(2b_{j-1})]\}, \\ \frac{db_j}{dt} &= b_j(a_{j+1} - a_j). \end{aligned} \quad (9)$$

Note that the b_j terms are strictly positive [see the definition in (5)]. We now reconstruct the matrix $\mathcal{L}$ in (8) with the solutions to the equations of motion in (9). Furthermore, we define the spectral evolution of $\mathcal{L}$ to be

$$\Delta\Gamma_{\mathcal{L},N}(t) = \Gamma_{\mathcal{L},N}(t) - \Gamma_{\mathcal{L},N}(0).$$

Figure 2 shows a side-by-side comparison of the spectral evolutions of the Toda and tangent FPUT systems considered in Fig. 1. Both the relative and absolute tolerances in our numerics are set to $O(10^{-12})$ to properly resolve the spectrum on the scales shown. Interestingly, we observe a sharp departure in the tangent FPUT's spectrum from Toda's isospectral evolution at a particular time, denoted by the vertical dashed line in Fig. 2.

The time t_s in Fig. 2 depicts the initial shock time of the unidirectional hyperbolic PDE long-wave approximation, outlined in the next section. The shock time is also depicted in the top panels of Fig. 1 with a dashed horizontal line, which is seen to accurately predict the excitation of higher lattice modes near the shock front. The bottom panels in Fig. 1 show the nearly identical wave profiles at and after the shock front forms (see Sec. III for a definition of L).

III. CONTINUUM APPROXIMATION

The dispersion relation in (2) makes clear that the linear dynamics of long-wave data can be well approximated by the (bidirectional, nondispersive) wave equation on a chain near the limit $N \to \infty$. However, discrete dispersion effects

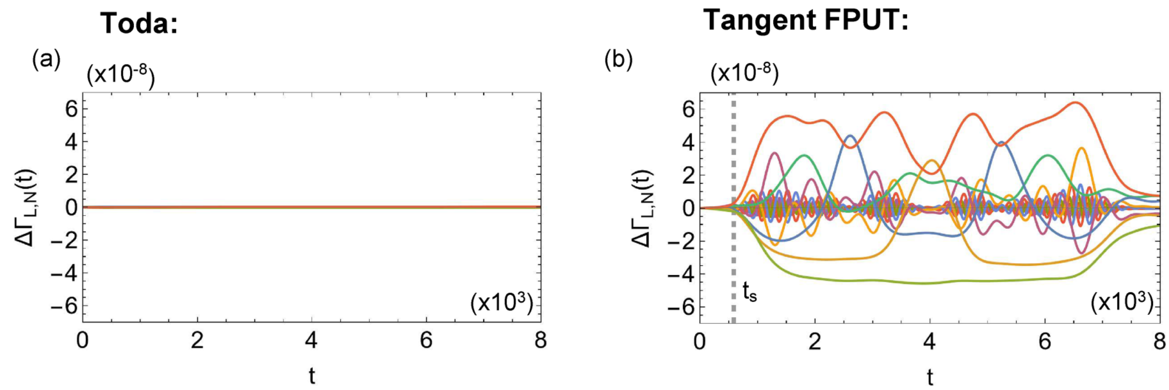

FIG. 2. Simulations of the spectral evolution $\Delta\Gamma_{\mathcal{L},N}(t)$ for the periodic Toda lattice (left column) and tangent FPUT lattice (right column) for the same initial data and parameters in Fig. 1 shown over one quasirecurrent cycle. The first shock time t_s is depicted with a dashed vertical line.

become increasingly noticeable as the wavelength approaches the lattice-spacing and higher-order dispersive terms need to be included in the linear PDE approximation. The continuum asymptotics of the full FPUT chain can be derived by expressing (1) in terms of nearest-neighbor differences or "strain" variables, $r_j = q_j - q_{j-1}$. Introducing a small parameter ϵ, one can make the ansatz $r_j(t) \approx \epsilon u(X, T)$, with long spatiotemporal scales $X = \epsilon j$ and $T = \epsilon t$, and obtain the following asymptotic PDE description of FPUT,

$$\begin{aligned} u_{TT} &= u_{XX} + \epsilon\alpha(u^2)_{XX} + \epsilon^2\left[\frac{1}{12}u^{(4)} + \beta(u^3)_{XX}\right] \\ &\quad + \epsilon^3\frac{1}{12}\alpha(u^2)^{(4)} + \epsilon^4\left[\frac{1}{360}u^{(6)} + \frac{1}{12}\beta(u^3)^{(4)}\right] \\ &\quad + O(\epsilon^5). \end{aligned} \tag{10}$$

The lowest-order approximation in the top line of Eq. (10) with $\beta = 0$, is called the "bad" Boussinesq equation, as its linear dispersion relation implies a short-wave instability. Indeed, it is known that there exist solutions to the bad Boussinesq equation which blow up in finite time [15,28]. The Boussinesq equation is derived in the asymptotic small-amplitude, long-wave regime of the FPUT or Toda lattice, where it is linearly stable. However, due to nonlinear wave mixing, higher wave vectors will eventually be generated, leading to exponential instability. By including the third-order terms in (10), one obtains spectral stability about the zero solution in the regime $k \to \infty$. Thus, the short-wave instability is an artifact of the truncation in the PDE approximation and not the true dynamics of the lattice.

Recent work has focused on a rigorous justification for making the *unidirectional* approximation near the continuum limit of the FPUT lattice, to avoid the stability issues associated with the lowest-order bidirectional description in (10). As mentioned in the Introduction, a pair of decoupled left and right propagating KdV equations was originally proposed in Ref. [13] and later rigorously justified in Ref. [18]. More recently, a quasi unidirectional approximation of FPUT dynamics was given in Refs. [16,17].

To study quasi unidirectional behavior of the FPUT lattice near the continuum limit, the authors in Ref. [16] consider initial data of the form

$$\begin{aligned} q_j(0) &= A\cos\phi\,\sin\left(\frac{2\pi j}{N}\right), \\ p_j(0) &= \omega_1 A\sin\phi\,\cos\left(\frac{2\pi j}{N}\right), \end{aligned} \tag{11}$$

for $0 \leqslant j \leqslant N-1$, where ϕ is a phase parameter. When $\phi = \pi/4$, the initial data propagate purely to the left (left traveling wave or LTW). Here, we use LTW initial data in each figure and, without loss of generality, set $A = 1$ so that the degree of tangency to Toda scales with γ/N. By taking the small parameter $\epsilon = 1/N$ and scaling $q_j(t) \approx \epsilon^{-1}Q(X, T)$ and $p_j(t) \approx P(X, T)$, one can introduce the following left, $L(Q_X, P)$, and right, $R(Q_X, P)$, traveling variables

$$L = \frac{(Q_X + P)\epsilon^{-1}}{\pi\sqrt{2}} \quad \text{and} \quad R = \frac{(Q_X - P)\epsilon^{-1}}{\pi\sqrt{2}}. \tag{12}$$

Plugging into (1), it follows that the equations for L and R decouple at zeroth order into

$$L_T = L_X \quad \text{and} \quad R_T = -R_X,$$

with an $O(\epsilon)$ remainder. At next order, a pair of generalized inviscid Burger's equations can be derived, decoupled at $O(\epsilon^2)$. The Burger's long-wave approximation leads to an explicit expression for the initial shock time, which we use here to compute $t_s \approx 587$ for the data shown in Fig. 1 (see Ref. [16] for details).

Going back to Fig. 1, the bottom panels show the virtually identical evolutions of a LTW for Toda and the tangent FPUT systems at t_s and $2t_s$. The profiles are shown in terms of L in (12), given approximately by $L(\Delta q_j, p_j)$, where $\Delta q_j = q_{j+1} - q_j$. The long-wave Burger's approximation holds well up to the first shock time, but at later times the formation of KdV solitons in Fig. 1 is apparent and higher-order derivatives in the PDE approximation become important [18]. Our numerics clearly depict the connection between the quasirecurrent behavior [Figs. 1(b) and 1(d)] and the formation and subsequent elastic collisions of solitons [Figs. 1(a) and 1(c)], as first proposed in Ref. [13]. Taken together, Figs. 1 and 2 show how the nonintegrable dynamics of the tangent FPUT system split sharply from the integrable dynamics of the nearby Toda system just past the first shock time, where, in both systems,

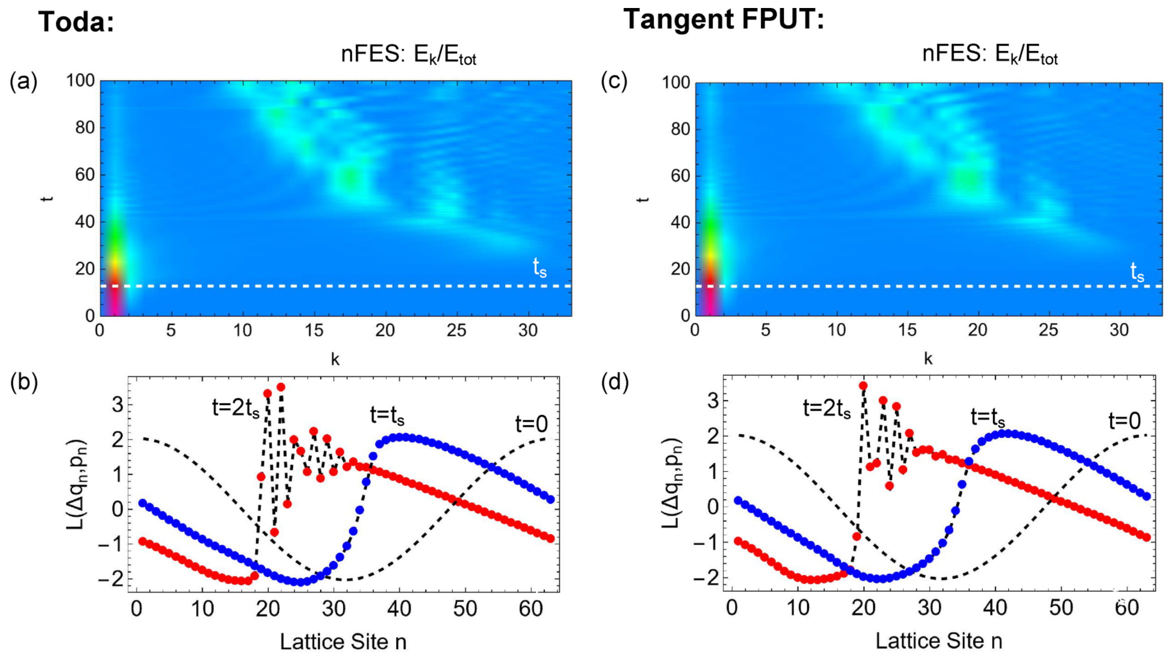

FIG. 3. Simulations of the periodic Toda lattice (left column) and tangent FPUT lattice (right column), where $\delta = 0.884$ and $N = 64$, of an initially left traveling wave [see (11)]. The top row shows the normalized Fourier energy spectrum's (nFES) evolution [red (blue) denotes high (low) spectral density]. The white dashed line is the computed Burger's shock time, $t_s \sim 12.8$. The bottom row shows the spatial profile of the discrete left traveling wave $L(\Delta q_n, p_n)$ at $t = 0, t_s, 2t_s$.

there is a sudden cascade of energy injected into the higher lattice modes. We remark that, due to the relatively high degree of tangency to Toda ($\delta \sim 0.025$), the spectral evolution of FPUT in Fig. 2 exhibits a smooth, quasirecurrent trajectory [with approximately the same quasirecurrent cycle shown in Fig. 1(a)] about Toda's identically zero spectral evolution.

For comparison, Fig. 3 shows the evolution of the nFES and corresponding wave profiles for tangent FPUT and Toda with a much larger tangency parameter, $\delta = 0.884$, and hence a larger nonlinear coefficient γ. Again, the initial shock time is well approximated by the long-wave asymptotics given in Ref. [16]. After the shock forms, and in contrast to the nFES shown in Fig. 1 where only a small subset of higher modes are excited, Fig. 3 shows that energy is suddenly transferred to the highest lattice modes, having wavelengths on the order $1/N$. Differences in the wave profiles of tangent FPUT and Toda are now evident, particularly after the shock forms (compare the bottom panels of Fig. 3 at $t = 2t_s$). Figure 4 again shows a side-by-side comparison of the $\mathcal{L}$-spectral evolution of the Toda and tangent FPUT systems shown in Fig. 3. Clearly the departure is more pronounced and appears from the outset (compare parenthetical scales in Figs. 2 and 4). Figure 4 again shows a sharp divergence in the Toda and tangent FPUT spectral evolutions soon after the initial shock forms. In contrast to the smooth quasirecurrent evolution of tangent FPUT shown in Fig. 2, the postshock evolution of tangent FPUT in Fig. 4 follows a chaoticlike trajectory, whereas the Toda evolution remains isospectral even in the strongly nonlinear regime.

Up to this point we have fixed the number of particles in the lattice at $N = 64$. To validate our assertion further, in the top panels of Fig. 5 we show a sample of results from extensive simulations on larger lattices ($N = 128$ and 512),

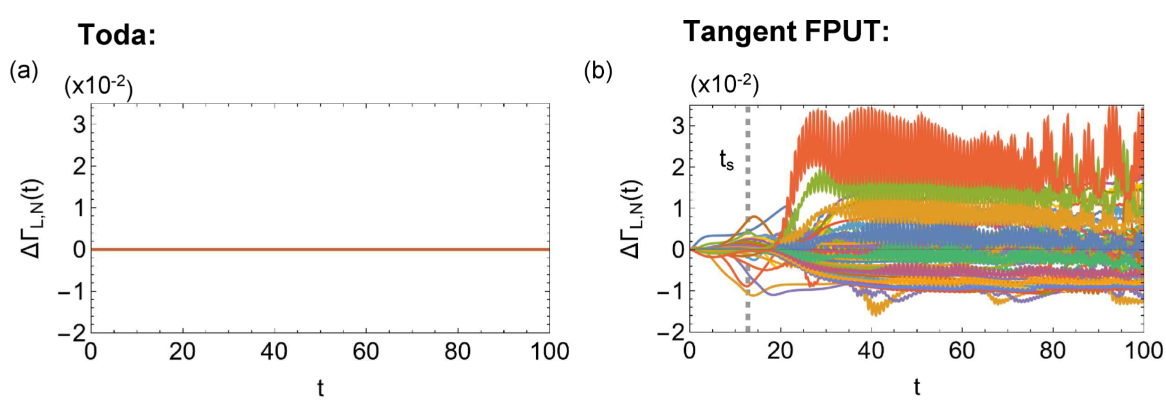

FIG. 4. Simulations of the spectral evolution $\Delta\Gamma_{\mathcal{L},N}(t)$ for the periodic Toda lattice (left column) and tangent FPUT lattice (right column) for the same initial data and parameters in Fig. 3. The first shock time t_s is depicted with a dashed vertical line.

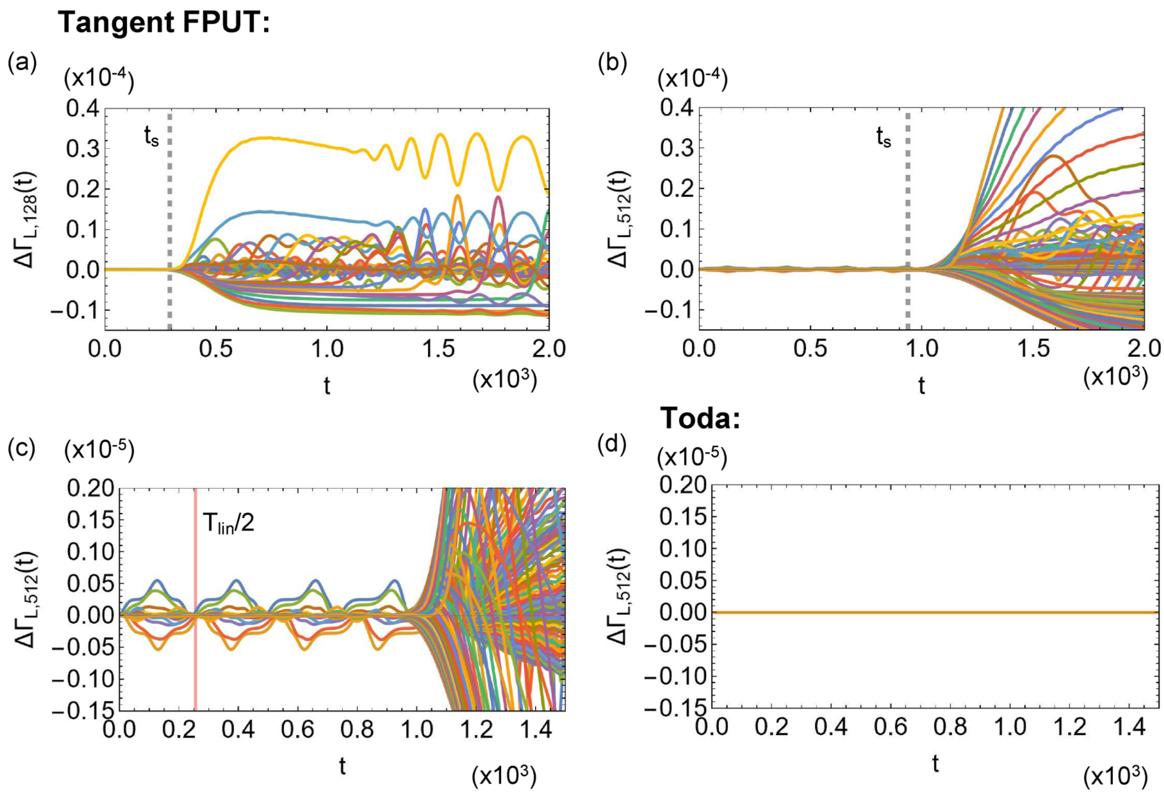

FIG. 5. Simulations of the spectral evolution of the tangent FPUT lattice on a lattice of (a) $N = 128$ ($\delta = 0.1$) and (b) $N = 512$ ($\delta = 0.12$). The first shock time, $t_s \sim 293$ in (a) and $t_s \sim 938$ in (b), is depicted again with a dashed vertical line. (c) is a zoomed-in plot of (b) and the half period of the linear lattice, $T_{\text{lin}}/2 \sim 256$, is depicted with a solid vertical line. For comparison, (d) is the Toda spectral evolution at the same axis scaling as (c).

where again the computed asymptotic shock times t_s mark a localized, pronounced, and short-term departure of tangent FPUT's spectrum from Toda's.

Finally, we remark on the importance of the *tangency parameter* value, $\delta \propto \gamma/N$, in our analysis. Either for a fixed N, as the strength of nonlinearity vanishes, $\gamma \to 0$, or for a fixed γ, as the number of particles grows, $N \to \infty$, the approximate shock -time calculated in Ref. [16] grows $t_s \to \infty$. A natural question is then, in the regime $\delta \ll 1$ and over long timescales, are tangent FPUT's and Toda's spectral evolutions indistinguishable? According to Ref. [6], the α-FPUT lattice eventually reaches equipartition for arbitrarily small nonlinearities on a sufficiently long timescale. Clearly, in the linear limit ($\gamma = 0$), tangent FPUT and Toda are identical: $a(t) = 0$ and $b(t) = 1/2$, and hence $\Delta\Gamma_{\mathcal{L},N}(t) = 0$, for all $t \geqslant 0$. As one tunes $0 < \delta \ll 1$ away from zero, numerically resolving the spectral evolution with sufficient accuracy becomes an issue. However, we note here that in our simulations the spectral evolution of tangent FPUT is *not* identically zero (with respect to the prescribed numerical tolerances) before the first shock formation.

Indeed, Fig. 5(c) shows a zoomed-in plot of Fig. 5(b), where the vertical axis is scaled down by an order of magnitude. Before the computed shock time, Fig. 5(c) shows regular oscillations in tangent FPUT's evolution from the outset, 2–3 orders of magnitude beneath the spectral values to the right side of the shock time. We find that these oscillations have a period well approximated by FPUT's linear dispersion relation (2), i.e., $T_{\text{lin}} = 2\pi/\omega_1$ [see the vertical line in Fig. 5(c)]. We remark that these preshock oscillations in FPUT's spectral evolution are common to all of our simulations for small δ. To confirm that these oscillations are not an artifact of our numerics, we show Toda's trivial evolution again in Fig. 5(d) with the same axis scalings as those in Fig. 5(c).

IV. CONCLUSIONS

We have presented results showing how the virtually indistinguishable macroscopic trajectories of the tangent FPUT and Toda systems actually split abruptly near the first shock time predicted by the long-wave Burger's equation asymptotics, by expressing both systems in the *ab* coordinates and accurately tracking their subsequent spectral evolutions. As the shock time occurs on a much shorter timescale than the one required for FPUT's eventual thermalization, studied in Ref. [21], we provide a means to distinguish FPUT's short-term dynamics from the isospectral evolution of the nearby integrable Toda lattice.

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