Heat transport in an angular-momentum-conserving lattice

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It is expected that the energy-diffusion propagator in a one-dimensional nonlinear lattice with three conserved quantities: energy, momentum, and stretch, consists of a central heat mode and two sound modes. The heat mode follows a Lévy distribution. Consequently, the heat diffusion is super, i.e., the second moment of the diffusion propagator diverges as t^{β} with $\beta > 1$; and the heat conduction is anomalous, i.e., the heat conductivity is size dependent and diverges with size N by N^{α}, with $\alpha > 0$. In this paper, we study a one-dimensional lattice with two-dimensional transverse motions, in which the total angular momentum also conserves. More importantly, the diffusion of this conserved quantity is ballistic. Surprisingly, the above pictures and the values of the mentioned power exponents keep unchanged. The universality of the scalings is then further extended. On the other hand, the detailed strengths of heat transports are largely enhanced. Such a counterintuitive finding can be explained by the change of the phonon mean-free path of the lattices.

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I. INTRODUCTION

The microscopic dynamical origin of heat conduction, in particular the anomalous heat conduction in low-dimensional systems is one of the long-standing tasks in nonequilibrium statistical mechanics and has attracted much interest recently [1–4]. In a macroscopic system, Fourier's law $\vec{j} = -\kappa \nabla T$ is commonly satisfied, where ∇T denotes a small temperature gradient and $\vec{j}$ is its induced stationary-state heat current, and most importantly κ is a system-size-independent heat conductivity. The case is quite different in microscopic low-dimensional systems. The system-size-diverging heat conductivity in a one-dimensional (1D) Fermi-Pasta-Ulam (FPU) lattice was first observed by Lepri *et al.* [5]. Since then, effort has been largely focused on the necessary and sufficient conditions of the Fourier law of heat conduction in 1D systems, and a general consensus has been reached that the conserved quantities play key roles in inducing a normal or anomalous heat conduction. Heat conduction in 1D models with momentum-conservation breaking, such as the Frenkel-Kontorova [6] and ϕ^4 [7] lattices, is generally normal. While in the cases that the momentum is conserved, the heat conduction is generally anomalous, i.e., the heat conductivity becomes size dependent and diverges with system length as $\kappa \sim N^{\alpha}$, with $\alpha > 0$.

On the other hand, in spite of the extensive studies in the past decades, no consensus has been reached yet for the universality and detailed value of the power-exponent α. Early mode-coupling theories (MCT) predict $\alpha = 2/5$ [1] and a renormalization group analysis indicates $\alpha = 1/3$ [8,9]. Later, a self-consistent MCT [10] was proposed, which predicts a two-universality-class scenario that $\alpha = 1/3$ and $1/2$ for models with asymmetric and symmetric interactions, respectively. The prediction for the asymmetric cases agrees with numerical simulations well [11,12]. However, for the symmetric cases like the purely quartic and FPU-β lattices, numerical simulations performed by different groups indicate differently. The values of $\alpha = 2/5$ [12–14], $1/3$ [15], and also $1/2$ [16] have all been reported. The possibility of having a different universality class depending on the number of conserved quantities has also been revealed recently [17].

Among all these theoretical predictions, a newly proposed nonlinear fluctuating hydrodynamic theory (NFHT) has attracted much increasing attention [18–20]. It suggests that for a 1D lattice with three conserved quantities: energy, momentum, and stretch, the value of the power exponent α follows the same two-universality-class scenario, like the above-mentioned self-consistent MCT expects. More importantly, NFHT studies not only the power exponent α, but the scaling properties of various correlation functions of the conserved quantities as well. It predicts that the energy-diffusion propagator consists of a bell-shaped central heat mode and two sound modes extending with speed of sound v_s. For the asymmetric and symmetric interparticle potentials, the heat modes both satisfy Lévy scaling but with different values of index, which induces the different values of α. As for the sound modes, they follow Kardar-Parisi-Zhang scaling [21] and a Gaussian distribution, respectively, in the two cases. Those have been verified numerically [22,23].

To further understand the role of conserved quantities, in this paper, we study the heat transports in a one-dimensional lattice with two-dimensional (2D) transverse motions, i.e., a quasi-2D lattice. Besides the energy, the momentum, and the stretch, the angular momentum is also conserved. More importantly it diffuses ballistically like what the momentum

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does. The rest of the paper is organized as follows. The model and its dynamics will be introduced in Sec. II. In Sec. III, numerical simulations for the energy diffusion and the heat conduction are presented. The phonon mean-free path is then calculated to further understand the change of these. A summary and discussion are given in Sec. IV.

II. MODEL AND DYNAMICS

The model we study is a quasi-2D lattice, i.e., a one-dimensional lattice that consists of particles moving in two transverse directions X and Y. The Hamiltonian reads

$$H = \sum_{i=1}^{N} \left[\frac{|\vec{p}_i|^2}{2} + V(|\vec{q}_{i+1} - \vec{q}_i|) \right], \tag{1}$$

where N denotes the total number of particles, and $\vec{q}_i \equiv x_i\vec{e}_x + y_i\vec{e}_y$ with $\vec{e}_x$ and $\vec{e}_y$ being the unit vectors in the two directions. All the masses have been set to unity. The interparticle interaction V takes the simplest nonlinear form, the purely quartic form, i.e., $V(r) = \frac{1}{4}r^4$. The dynamics that corresponds to the above Hamiltonian follows

$$dv_{xi} = \left(-\Delta x_i^3 - \Delta x_i \Delta y_i^2 + \Delta x_{i+1}^3 + \Delta x_{i+1}\Delta y_{i+1}^2\right)dt, \tag{2}$$

$$dv_{yi} = \left(-\Delta y_i^3 - \Delta y_i \Delta x_i^2 + \Delta y_{i+1}^3 + \Delta y_{i+1}\Delta x_{i+1}^2\right)dt, \tag{3}$$

where Δx_i and Δy_i refer to the relative displacements $x_i - x_{i-1}$ and $y_i - y_{i-1}$, respectively. In a more realistic model with longitudinal and transverse motions as well as bending angle interactions, three types of transport behaviors, i.e., logarithmic, 1/3 power-law, and 2/5 power-law divergences of heat conductivity are all observed in different parameter regimes [24].

In a model that the cross items are omitted [25], i.e.,

$$dv_{xi} = \left(-\Delta x_i^3 + \Delta x_{i+1}^3\right)dt, \tag{4}$$

$$dv_{yi} = \left(-\Delta y_i^3 + \Delta y_{i+1}^3\right)dt, \tag{5}$$

the system reduces to two independent 1D purely quartic lattices, in each of which there exist three locally conserved quantities, i.e., energy, momentum, and stretch. Throughout this paper, we call this model a "simplified" lattice. The 1D purely quartic lattices have been studied extensively and all the known conclusions can be applied to this lattice. In another point of view, the cross items in Eqs. (2) and (3) connect the two 1D independent lattices together, and consequently build the new quasi-2D lattice. Such a connection breaks the separated energy conservations in the two 1D lattices. However, the total energy and the total angular momentum

$$I = \sum_i I_i \equiv \sum_i (v_{xi}y_i - v_{yi}x_i) \tag{6}$$

conserve instead. Recent study has revealed that the angular momentum conservation plays a crucial role in the thermalization process of this lattice [26]. The total number of conserved quantities remains. We will see that the local angular momentum diffuses ballistically, and it will largely enhance heat transports.

III. NUMERICAL SIMULATIONS

A. Diffusion properties of the conserved quantities

First of all, we study the diffusion properties of these conserved quantities systematically by calculating their spatiotemporal correlation functions. The local energy-fluctuation spatiotemporal correlation function was proposed to characterize the energy diffusion processes in a few 1D lattices [27]. It has been applied to various 1D systems [14], 2D systems [28], and has also been extended to other conserved quantities [29–31]. In an equilibrium state, the rescaled energy-fluctuation spatiotemporal correlation is defined as

$$\rho_E(i,t) \equiv \frac{\langle \Delta E_i(t+t')\Delta E_0(t')\rangle_{t'}}{\langle \Delta E_i^2(t')\rangle_{i,t'}}, \tag{7}$$

where $\Delta E_i(t) \equiv E_i(t) - \langle E_i\rangle$, and $E_i(t)$ denotes the energy of the ith particle at time t and $\langle E_i\rangle$ is this long-time average. For a homogeneous lattice with a periodic boundary condition, $\langle E_i\rangle$ is independent of the particle label i. However, the fixed boundaries are applied here, thus the values for the end particles are in fact slightly different. Similarly, the correlations of the local momentum and the local angular momentum are

$$\rho_P(i,t) \equiv \frac{\langle v_{xi}(t+t')v_{x0}(t') + v_{yi}(t+t')v_{y0}(t')\rangle_{t'}}{\langle v_{xi}^2(t') + v_{yi}^2(t')\rangle_{i,t'}}, \tag{8}$$

$$\rho_I(i,t) \equiv \frac{\langle I_i(t+t')I_0(t')\rangle_{t'}}{\langle I_i^2(t')\rangle_{i,t'}}. \tag{9}$$

Due to the reflection symmetry, $\langle v_{xi}\rangle$, $\langle v_{yi}\rangle$, and $\langle I_i\rangle$ must be zero. Therefore, the symbol Δ in Eqs. (8) and (9) have been omitted. Also due to the XY symmetry, we calculate not $\rho_{P_X}(i,t)$ and $\rho_{P_Y}(i,t)$ separately, but their average instead to reduce the thermal fluctuations. Correspondingly, the second moments of these correlations are defined as

$$\sigma_E^2(t) \equiv \sum_i \rho_E(i,t)i^2, \tag{10}$$

$$\sigma_P^2(t) \equiv \sum_i \rho_P(i,t)i^2, \tag{11}$$

$$\sigma_I^2(t) \equiv \sum_i \rho_I(i,t)i^2. \tag{12}$$

The simulations are performed in a lattice with particle number $N = 8001$. Fixed boundary conditions are applied, i.e., $x_0 = y_0 = x_{N+1} = y_{N+1} = 0$. Two Langevin heat baths with identical temperature $T = 1$ are coupled to the left- and right-most particles to keep the whole system in an equilibrium state. The correlations $\rho_E(i,t)$, $\rho_P(i,t)$, and $\rho_I(i,t)$ for various time lag t are plotted in Figs. 1(a) to 1(c). Note that the central particle is labeled as the zeroth one. Similar to the 1D cases [29], the profiles of $\rho_P(i,t)$ form two sound modes moving out with the sound speed v_s [see Fig. 1(b)], which induces a ballistic diffusion, i.e., $\sigma_P^2(t) \sim t^2$, see the corresponding second moments plotted in Fig. 1(d). Interestingly, those for the angular momentum $\rho_I(i,t)$ behave the same [see Fig. 1(c)], and it diffuses ballistically too, i.e., $\sigma_I^2(t) \sim t^2$. We know the energy transport in the corresponding 1D lattice depends only on the momentum transport, thus the change that the extra conserved quantity makes to energy diffusion is of great interest. The pictures of $\rho_E(i,t)$ look similar to those for the 1D lattice, i.e., each forms a central heat mode

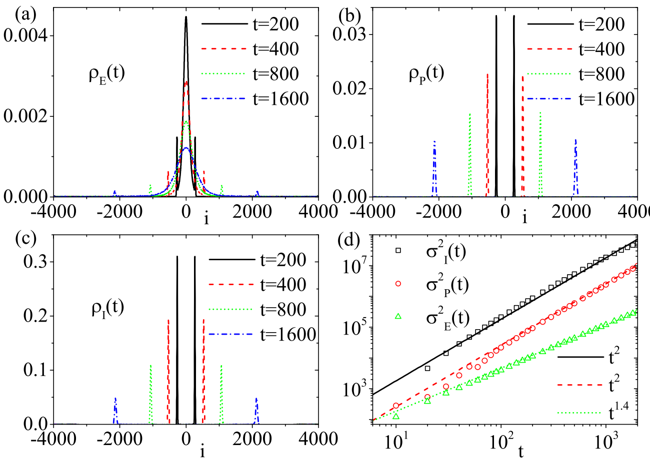

FIG. 1. The diffusions of three conserved quantities, (a) $\rho_E(i,t)$, (b) $\rho_P(i,t)$, and (c) $\rho_I(i,t)$ for various times. (d) The second moments of the diffusions. Canonical simulations with $N = 8001$ and temperature $T = 1$ are applied.

and two sound modes moving out with the sound speed v_s. The overall energy diffusion follows a super diffusion, i.e., $\sigma_E^2(t) \sim t^{\beta}$ with the power exponent $\beta = 1.4$. Although this value of β is basically the same as that we observed in the 1D lattice [14], the detailed value of $\sigma_E^2(t)$ is enlarged noticeably by about 60%. Namely, the heat diffusion has been largely enhanced.

We have also checked the Lévy walk description of the energy diffusion in this quasi-2D lattice. Corresponding to the diffusion power exponent $\beta = 1.4$, the heat mode is expected to follow a Lévy distribution with the index $\mu = 3 - \beta = 1.6$ [32]. Then the height of the heat mode H should decay as $H \sim t^{-\frac{1}{\mu}} = t^{-0.625}$. The numerically calculated H versus t and the above expected decay are plotted together in Fig. 2(a). The agreement is excellent. The heat mode is then expected to satisfy a scaling invariant relation

$$\rho_E(x, ut) \sim u^{-\frac{1}{\mu}} \rho_E\left(u^{-\frac{1}{\mu}} x, t\right). \tag{13}$$

In Fig. 2(b), the rescaled correlations $t^{\frac{1}{\mu}} \rho_E(i,t)$ versus the rescaled location $it^{-\frac{1}{\mu}}$ are plotted. The curves for various time t overlap each other very well. The above scaling-invariant relation is then confirmed. Moreover, the tail of a Lévy distribution should decay as $i^{-(\mu+1)} = i^{-2.6}$. In Fig. 2(c), the tails

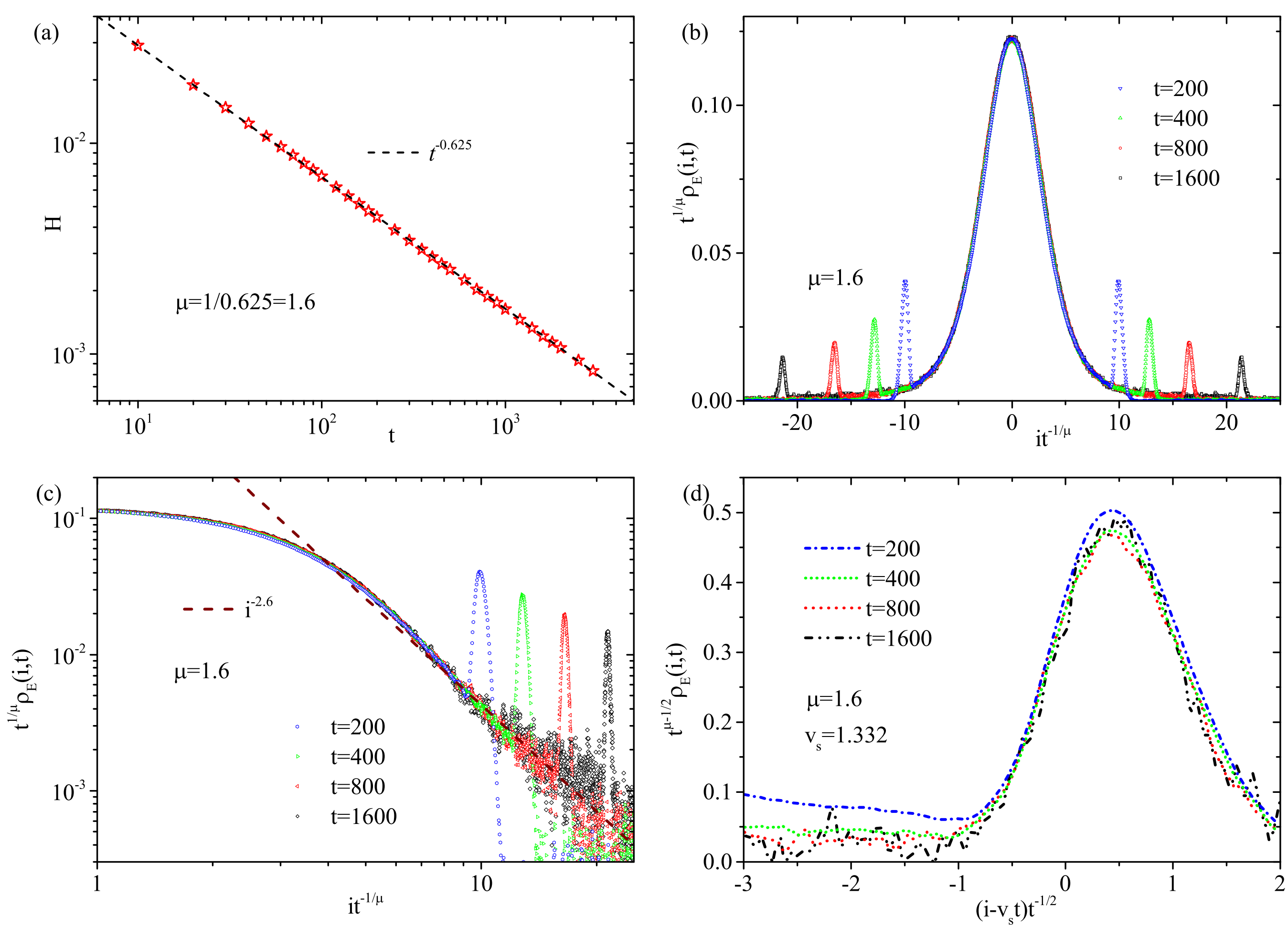

FIG. 2. (a) The height of the heat mode H versus time t. It follows a power-law decay $H \sim t^{-0.625}$ very well, which implies a Lévy scaling with the index $\mu = 1/0.625 = 1.6$. (b) The rescaled profiles of $\rho_E(i,t)$. In the central regimes, curves for various t overlap each other very well. (c) The rescaled profiles in the tail regime. A straight line with slope -2.6 is plotted for reference. (d) The rescaled profiles of the sound modes. The overlap for various t is acceptable.

of the rescaled distributions are plotted in double-logarithmic scale. Clear power-law decay can be confirmed. However, for the detailed power exponent −2.6, due to the large fluctuation, it is hard to make a convincing conclusion. Nevertheless, a power-law decay rather than an exponential decay can be confirmed. As for the sound mode, a random-walk-with-velocity-fluctuation process [33] expects that the dispersion of the humplike sound mode grows as $t^{\frac{1}{2}}$ and its volume follows a power-law decay $t^{-(\mu-1)}$, thus the distribution should satisfy the scaling [34]

$$\rho_E(\bar{x}, ut) \sim u^{-(\mu-\frac{1}{2})}\rho_E\left(u^{-\frac{1}{2}}\bar{x}, t\right), \tag{14}$$

where $\bar{x} \equiv x - v_s t$. In Fig. 2(d), the rescaled correlations $t^{\mu-\frac{1}{2}}\rho_E(i,t)$ versus the rescaled coordinate $(i - v_s t)t^{-\frac{1}{2}}$ are plotted. Curves for various time t basically overlap. The agreement with Eq. (14) is acceptable.

B. Heat conduction of the lattices

Since the extra conserved quantity does not change the scaling properties of heat diffusion but largely enhances the detailed strength, we naturally expect that the heat conduction should be changed in the same way. We will confirm it in this section.

1. Nonequilibrium measure of the heat conductivity

The heat conductivity κ, which measures the strength of heat conduction, is originally defined according to the Fourier law, i.e., $\vec{j} = -\kappa\nabla T$, where ∇T denotes an infinitesimal temperature gradient and $\vec{j}$ denotes its induced stationary-state heat current. This provides a straightforward way of measuring the value of κ, the nonequilibrium heat bath method. To do so, two Langevin heat baths with temperatures $T_L = 1.1$ and $T_R = 0.9$ are coupled to the two end particles of the lattice. Compared with our previous studies [12,14], the temperature difference $\Delta T \equiv T_L - T_R$ we applied here is much smaller, thus the system is even closer to an equilibrium state. The length-dependent heat conductivity $\kappa_{\rm NE}(N)$ is defined as

$$\kappa_{\rm NE}(N) \equiv \frac{JN}{\Delta T}, \tag{15}$$

where J denotes the stationary-state heat current and the subscript "NE" denotes that the calculation is based on the nonequilibrium method. Here the fixed boundary conditions are applied again.

In our numerical calculation, the average of J is taken after enough long transient time so that the local heat currents along the lattices become time t and site i independent and the temperature profiles are well established and also time t independent, see the temperature profiles in the inset of Fig. 3 for the longest lattice $N = 131072$. The so-measured $\kappa_{\rm NE}$ versus N is plotted in Fig. 3. We see it grows with the system length N by N^α, with $\alpha = 2/5$. As a comparison, the result for the simplified model is also plotted. Similar divergence power exponent is observed. However, the detailed heat conductivity is apparently lower. In other words, the cross terms in Eqs. (2) and (3) enhance heat conduction largely, about 38%.

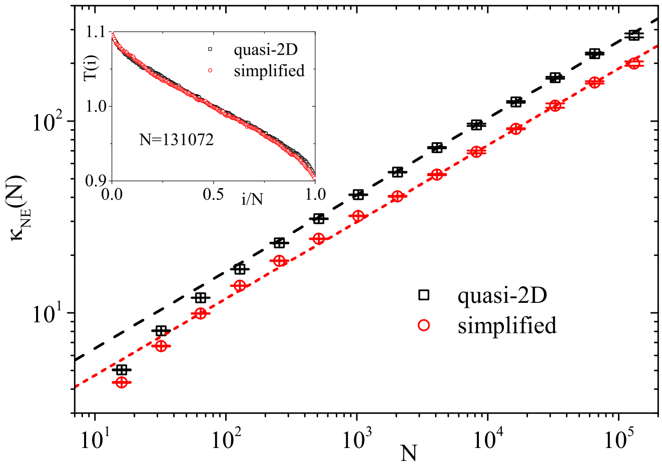

FIG. 3. Heat conductivity $\kappa_{\rm NE}(N)$ versus system length N for the quasi-2D and simplified lattices. Two lines with slope 2/5 are plotted for reference. $\kappa_{\rm NE}(N)$ for the former lattice is apparently (about 38%) higher than that for the latter one. Inset: the temperature profiles for the two lattices with the longest length $N = 131072$.

2. Equilibrium measure of the heat conductivity

An inevasible problem of the above nonequilibrium method is that the applied temperature difference must be neither too small, otherwise the so-induced net heat current cannot be distinguished from the background statistical fluctuations; nor too large, otherwise the system is too far from equilibrium. Numerical difficulties also prevent us from simulating even longer systems, otherwise the consumption of the computational resources increases unacceptably. Meanwhile, the Green-Kubo formula [35] provides a way of calculating the heat conductivity in terms of the autocorrelation function $C_{JJ}(\tau)$ of the fluctuation-induced instantaneous global heat current at equilibrium:

$$C_{JJ}(\tau) \equiv \lim_{N\to\infty} \frac{1}{k_B T^2 N}\langle J_N(t)J_N(t+\tau)\rangle_t, \tag{16}$$

where $J_N(t)$ denotes the global heat current measured in a system with N particles. In the general cases that sound modes exist, the length-dependent heat conductivity is expected as

$$\kappa_{\rm GK}(N) \equiv \int_0^{N/v_s} C_{JJ}(\tau)d\tau, \tag{17}$$

where the subscript "GK" indicates that the calculation is based on the Green-Kubo formula [36]. In such cases, if the autocorrelation $C_{JJ}(\tau)$ decays as $C_{JJ}(\tau) \sim \tau^{-\gamma}$ with $\gamma < 1$ in the long t limit, then $\kappa_{\rm GK}(N)$ is expected to diverge as $\kappa_{\rm GK}(N) \sim N^\alpha$ with $\alpha > 0$ in the thermodynamic limit, and the power exponents α and γ are simply connected by $\alpha = 1 - \gamma$.

To do the simulations, the initial states are randomly extracted from the microcanonical ensemble with zero total momentum, zero total angular momentum, and identical per-particle average energy $\epsilon = 1.5$, which corresponds to the desired temperature $T = 1$. Periodic boundary conditions are applied here, i.e., $x_0 = x_N$, $y_0 = y_N$, $x_{N+1} = x_1$, and $y_{N+1} = y_1$. Theoretically speaking, the system size N should be infinity, which is however practically impossible. Here we use $N = 32768$, which is long enough for the considered regime

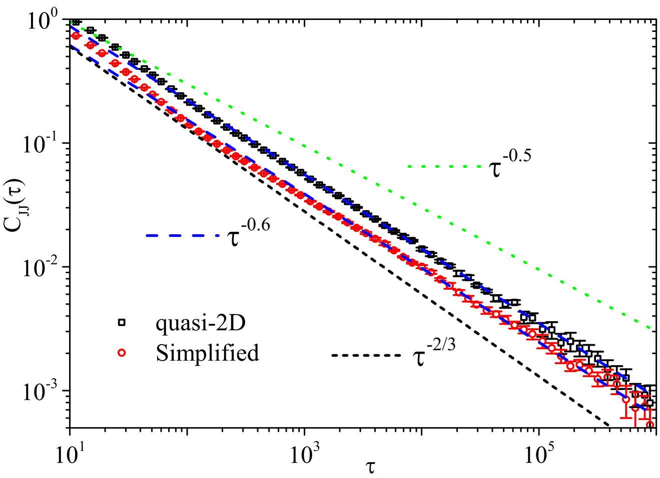

FIG. 4. The global heat-current autocorrelation function $C_{JJ}(\tau)$ for the quasi-2D and simplified lattices. $N = 32768$. The decays of the tails follow $\tau^{-3/5}$ quite well. Lines with the other two commonly expected values of the power exponent -1/2 and -2/3 are also plotted for reference.

of time lag [37]. The results are plotted in Fig. 4. The asymptotic decay of $C_{JJ}(\tau)$ follows a power law $\tau^{-3/5}$ quite well, for both the quasi-2D and the simplified lattices. $\kappa_{\rm GK}(N) \sim N^{2/5}$ is thus expected, which agrees with our nonequilibrium calculation. Furthermore, it has been analytically revealed that for a system without temperature pressure, the super energy diffusion and the anomalous heat conduction are connected by $\frac{d^2}{d\tau^2}\sigma_E^2(\tau) = \frac{2C_{JJ}(\tau)}{k_{\rm B}T^2c_v}$ [38], where c_v denotes the specific volumetric heat capacity. This straightforwardly indicates $\gamma + \beta = 2$. Here our measures agree with it exactly [39].

On the other hand, although $C_{JJ}(\tau)$ decays with the same power exponent for both the quasi-2D and the simplified lattices, the detailed value of $C_{JJ}(\tau)$ for the former one is apparently (about 43%) larger than that for the latter one. The same ratio of heat conductivity difference is expected, which basically agrees with the nonequilibrium calculation.

C. Phonon mean-free path and dispersion relation

Besides the universal decay or divergent power exponents, the detailed values of diffusivity and heat conductivity are also of great practical value. The above studies have revealed that, counterintuitively, the cross terms that connect two independent 1D lattices to a single quasi-2D lattice apparently enhance heat transports. Phenomenologically speaking, the thermal conductivity is determined in terms of a wave-number-dependent phonon mean-free path (MFP) ℓ_k, i.e., $\kappa = \frac{1}{3}\sum_k c_k v_k \ell_k$ [40], where c_k is the specific heat of the phonon mode with wave number k and v_k denotes its group velocity. This provides a perspective to better understand the above enhancement. To this end, we need to study the dispersion relation and measure the phonon MFP of the lattices. An anharmonic phonon (a-ph) approach enables us to do so numerically by calculating the Fourier-transformed susceptibility [41]. Similar to the tuning fork experiment, suppose that the collective response to a periodic external weak force $f(t) = f_1 \cos \omega t$ that is applied to the first particle takes the form of a propagating plane wave, which reads

$$\langle v_n(t)\rangle_f = |A_n| \cos(\omega t + \phi_n) = \mathrm{Re}(|A_n| e^{i(\omega t + \phi_n)}) \quad (18)$$

for the nth particle, and the phase following

$$\phi_n = -kn + \phi_0, \quad (19)$$

where the coefficient k corresponds to the wave number. According to the linear-response theory, the excited motion that is described in Eq. (18) can be expressed as

$$\langle v_n(t)\rangle_f = f_1 \mathrm{Re}[\chi_n(\omega) e^{i\omega t}], \quad (20)$$

and the susceptibility $\chi_n(\omega)$ reads [41]

$$\chi_n(\omega) = \frac{1}{k_B T}\int_0^\infty d\tau \langle v_n(\tau) v_1(0)\rangle e^{-i\omega\tau} \equiv |\chi_n(\omega)| e^{i\phi_n}. \quad (21)$$

We can thus determine the wave number k by calculating the above correlation of velocity $\langle v_n(\tau)v_1(0)\rangle$. Suppose $|\chi_n(\omega)|$ follows an exponential decay with n, i.e.,

$$|\chi_n(\omega)| \propto e^{-n/\ell}, \quad (22)$$

then the MFP of the phonon is naturally the value of ℓ.

In the numerical simulations, the same microcanonical ensemble and periodic boundary conditions that are applied in Sec. III B 2 are applied again. The so-obtained phase $\phi_n(\omega)$ versus n for various $\omega \in (0.058, 1.304)$ is plotted in Fig. 5(a). Quite good linear dependence on n is observed. Accordingly, the dispersion relation is then worked out and plotted in Fig. 5(c). That for the simplified lattice is also plotted for reference. In fact, analytical estimations of the dispersion relations have been worked out for such nonlinear lattices based on the self-consistent phonon theory [42,43] and the renormalized phonon approach [44]. Namely,

$$\widetilde{\omega}_k = \eta\omega_k, \quad (23)$$

where $\omega_k \equiv 2\sin\frac{k}{2}$ denotes the dispersion relation of a 1D linear lattice, and η represents the renormalized factor for the nonlinear lattices. For the 1D FPU lattice (also the simplified lattice) having the interparticle coupling $V(r) = \frac{K}{2}r^2 + \frac{\lambda}{4}r^4$ (the purely quartic lattice corresponds to $K = 0$ and $\lambda = 1$),

$$\eta_{1D} = \sqrt{\frac{K + \sqrt{K^2 + 12\lambda T}}{2}}. \quad (24)$$

As for the quasi-2D lattice, there exist two branches of phonons with opposite sign of angular momentum, and both of the two branches follow the same dispersion relation [26]

$$\eta = \sqrt{\frac{K + \sqrt{K^2 + 16\lambda T}}{2}}. \quad (25)$$

Those analytical estimations are plotted in Fig. 5(c) as curves. In the studied regime, they agree with the numerical results quite well, despite some slight overestimation. For even larger $\omega > 2.675$, similar to the case in Ref. [41], the response decays very fast, yielding a very short phonon MFP. Evaluation of the corresponding wave number k thus becomes quite difficult.

Similarly, the numerically obtained $|\chi_n(\omega)|$ is plotted in single logarithmic scale in Fig. 5(b). The values for each

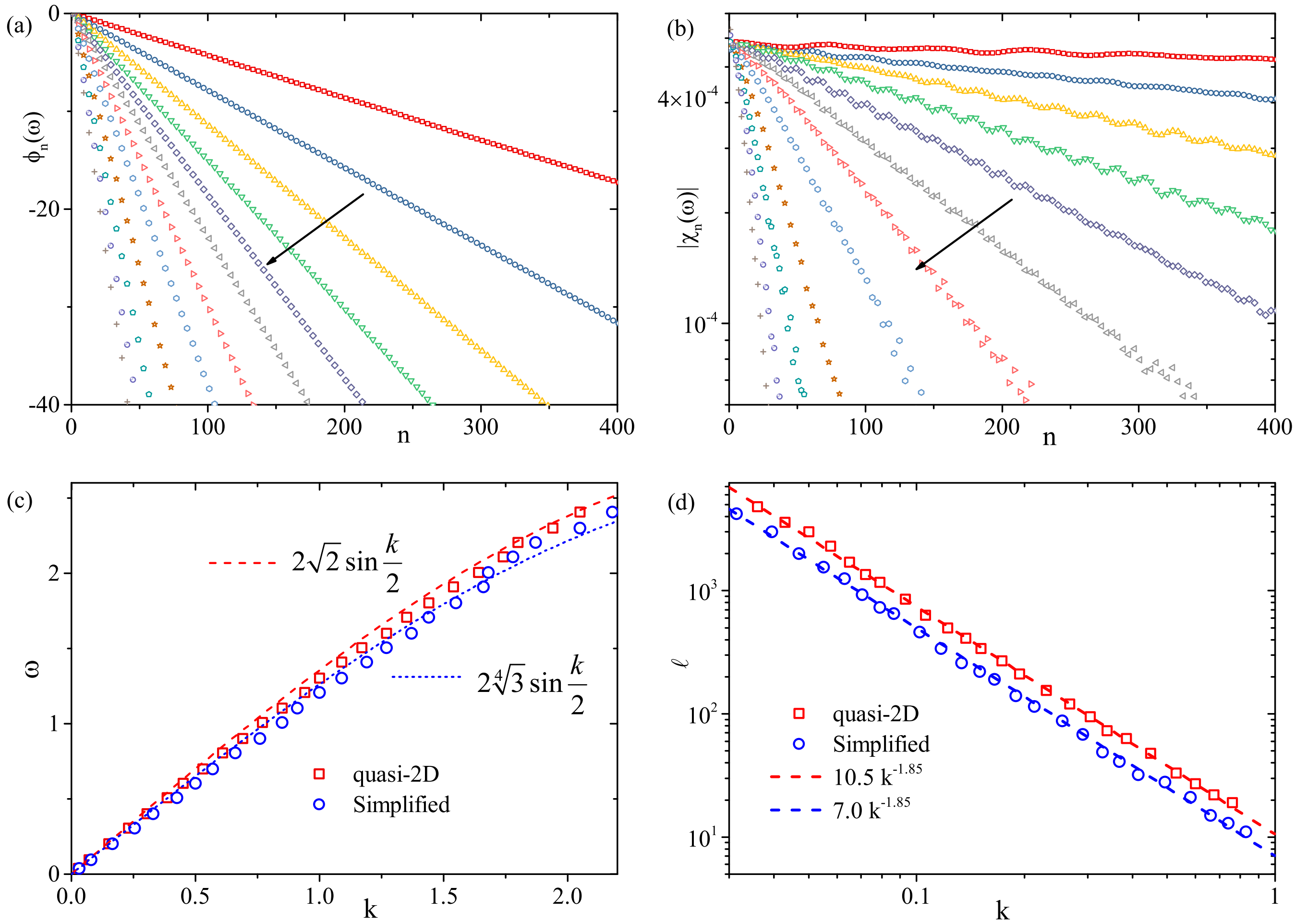

FIG. 5. (a) The phase $\phi_n(\omega)$ and (b) the amplitude $|\chi_n(\omega)|$ for the quasi-2D lattice. The values of ω along the arrow are 0.058, 0.105, 0.153, 0.201, 0.249, 0.307, 0.403, 0.508, 0.699, 0.901, 1.102, and 1.304. The corresponding (c) dispersion relation and (d) phonon MFPs for the two lattices. The curves in (c) denote the renormalized-phonon expectations Eqs. (24) and (25).

ω follow a straight line, indicating exponential dependence on n, which agrees with the expectation of Eq. (22). Based on it and the already obtained dispersion relation, the MFP $\ell(\omega)$ versus the wave number k is worked out and plotted in Fig. 5(d). Those for the simplified lattice are also plotted for comparison. We see that both of them follow power-law decays with the same value of power exponent about -1.85. However, for the detailed values of ℓ, those for the quasi-2D lattice are apparently (about 50%) longer than those for the simplified lattice. This fact well explains the difference of their heat transports in the two lattices.

IV. SUMMARY

To summarize, heat transport properties of a quasi-2D purely quartic lattice are systematically studied. Compared with the corresponding 1D purely quartic lattice, there exists one more conserved quantity that diffuses ballistically, the angular momentum. Since the energy transport in the 1D purely quartic lattice depends only on momentum transport, it is interesting to study the role of the conserved angular momentum. Our numerical simulations show that, similar to the 1D case, the energy diffusion propagator $\rho_E(i,t)$ consists of one central heat mode and two sound modes moving out in opposite directions. The heat mode fits a Lévy distribution with index $\mu = 1.6$ quite well; and the sound modes fit a random-walk-with-velocity-fluctuation process with the same index. Consequently, $\sigma_E^2(t)$, the second moment of $\rho_E(i,t)$, diverges with t by t^β with $\beta = 1.4$. The extra conserved quantity, the angular momentum, which diffuses ballistically like the momentum, does not change the scaling properties of the energy diffusion. The robustness of the universal scalings is clearly presented. Very recently a ballistic transport has been reported in a 1D FPU-like nonintegrable model with long-range interactions having only few conserved quantities. It is quite surprising since ballistic transport is commonly expected exclusively in an integrable system with N conserved quantities [45].

On the other hand, the detailed values of $\sigma_E^2(t)$ are largely enhanced for the quasi-2D lattice. It is then naturally expected that the heat conductivity κ of this lattice changes in the same way. We measure κ by using both nonequilibrium heat bath and equilibrium Green-Kubo methods, and confirm that the length-dependent $\kappa(N)$ diverges with the length N by N^α, with $\alpha = 2/5$. This is consistent with the universal relation $\alpha = \beta - 1$. The detailed value of κ in the quasi-2D lattice is indeed noticeably enlarged. To understand the enhancement of the heat transport, we calculate the phonon mean-free path in the quasi-2D lattice. Clearly, the MFP in the quasi-2D lattice is roughly 50% longer than that in the 1D lattice, which basically explains the change of the heat diffusion and conduction.

For momentum-conserving systems, heat transports commonly much more efficiently in low-dimensional systems than in high-dimensional ones, because in such systems, the long-wavelength phonons contribute the most but transverse connections induce irreversible phonon scattering [46], which badly breaks those phonons. That is why the heat conductivity in a 1D momentum-conserving nonlinear lattice generally follows a power-law divergence with the system length, while in 2D cases the divergence is commonly logarithmic [47], which is much slower. Our study reveals that to form a high-dimensional lattice by properly coupling independent low-dimensional lattices, the overall heat transports can be greatly enhanced. The key is to produce extra conserved quantity with ballistic diffusion. This provides a new idea to control heat transports in microscopic materials.

In the theoretical aspect, the robustness of the universal scalings urges us to conjecture a number-of-conserved-quantity-independent universal scaling for the heat transports. We have tried it in the framework of NFHT. Although the original NFHT studies 1D lattices with three conserved quantities: energy, momentum, and stretch, it is applicable to systems with other numbers of conserved quantities, e.g., the lattice with exchange noise which conserves displacement and energy [48]. In a recently proposed model in which the time-reversal symmetry is broken by a magnetic field [49], NFHT does not apply because the Euler equations for the conserved quantities are not closed due to the expression of the new conserved quantity, the pseudomomentum. Unlike that model, the time-reversal symmetry holds in this quasi-2D model. However, unfortunately, since the angular momentum current depends on not only the relative but also the absolute displacements of the particles, it cannot be expressed by other conversed quantities. The Euler equations cannot be closed either. This mounts an interesting challenge to the application of NFHT and thus provides an interesting open problem awaiting future studies.

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